There does not exist a function $\mathbb{R} \to \mathbb{R}$ that is continuous precisely on $\mathbb{Q}$

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Abstract

We aim to prove that there does not exist a function $\mathbb{R} \to \mathbb{R}$ that is continuous precisely on $\mathbb{Q}$.

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Lemma 1. *If (a_n), (b_n) are sequences in $\mathbb{R}$ such that $a_n \leq b_n$ and*

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$$

then $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

Proof. For convenience, say $S \coloneqq \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Since these intervals are nested, $a_{n+1} \in [a_n, b_n]$ and particularly $a_{n+1} \geq a_n$ for all n, so (a_n) is monotone increasing. Furthermore, $a_n \in [a_1, b_1]$ for all n, so $a_n \leq b_1$ and (a_n) is bounded above. Hence we can set $a = \lim_{n \to \infty} a_n$.

Now to prove $a \in S$ (which is sufficient to show that $S \neq \emptyset$), we need $a \geq a_n$ for all n, which is clear, since if we had some N such that $a < a_N$, then we would have $|a - a_n| \geq \varepsilon$ for all $n \geq N$, where $\varepsilon = a_N - a > 0$, since $a_n \geq a_N$. This would imply $a_n \nrightarrow a$, a contradiction.

We also need $a \leq b_n$ for all n. This is true since if we had an N with $a > b_N$, then by convergence of a_n, we could pick M such that $a - a_M < a - b_N$, so $a_M > b_N$. But then letting $K = \max\{N, M\}$, $K \geq M \implies a_K \geq a_M$, as (a_n) is increasing, and $K \geq N \implies b_K \leq b_N$, as (b_n) is decreasing by a similar argument to earlier. Hence $a_K > b_K$, which is a contradiction.

Hence $a \in S$ and $S \neq \emptyset$. □

Theorem 1. *There does not exist a function $\mathbb{R} \to \mathbb{R}$ that is continuous precisely on $\mathbb{Q}$.*

Proof. Suppose that there is such a function, $f : \mathbb{R} \to \mathbb{R}$.

By the countability of $\mathbb{Q}$, let (x_n) be an enumeration of the rationals.

Let (ε_n) be a sequence with $\varepsilon_n > 0, \varepsilon_n \to 0$, for example $\varepsilon_n = 1/n$.

Now by continuity of f, choose an *irrational* $\delta_1 > 0$ such that

$$|x - x_1| \leq \delta_1 \implies |f(x) - f(x_1)| < \varepsilon_1$$

Note the non-strict inequality - this can be achieved by first picking δ_1' such that this holds for the strict inequality, and then choosing a suitable smaller δ_1 - particularly, any $\delta_1 \in (0, \delta_1') \setminus \mathbb{Q}$.

Now for convenience, discard all the rationals not in $[x_1 - \delta_1, x_1 + \delta_1]$ from our enumeration, so $x_2 \in [x_1 - \delta_1, x_1 + \delta_1]$.

Since we picked $\delta_1 \notin \mathbb{Q}$, and by definition $x_1 \in \mathbb{Q}$, we have $x_1 \pm \delta_1 \notin \mathbb{Q}$, so in fact x_2 is neither of the endpoints and $x_2 \in (x_1 - \delta_1, x_1 + \delta_1)$.

Now pick an irrational $\delta_2 > 0$ such that

$$|x - x_2| \leq \delta_1 \implies |f(x) - f(x_2)| < \varepsilon_2$$

and $x_1, x_1 \pm \delta_1 \notin [x_2 - \delta_2, x_2 + \delta_2]$ - ie the new interval should not contain x_1, or either of the endpoints. This can be done since we can again first use continuity to select a δ_2' and then pick $\delta_2 \in (0, \min\{\delta_2', |x_2 - x_1|, |x_2 - (x_1 \pm \delta_1)|\}) \setminus \mathbb{Q}$.

Repeat this procedure to obtain a sequence (δ_n) with

$$|x - x_n| \leq \delta_n \implies |f(x) - f(x_n)| < \varepsilon_n$$

and $x_n, x_n \pm \delta_n \notin [x_{n+1} - \delta_{n+1}, x_{n+1} + \delta_{n+1}]$.

Particularly, we get a *nested* sequence

$$[x_1 - \delta_1, x_1 + \delta_1] \supsetneq [x_2 - \delta_2, x_2 + \delta_2] \supsetneq [x_3 - \delta_3, x_3 + \delta_3] \supsetneq \cdots$$

where f is getting "more and more continuous" as we go along.

By Lemma 1, we can pick a real $\alpha \in \bigcap_{n=1}^{\infty} [x_n - \delta_n, x_n + \delta_n]$. Note that $\alpha \neq x_n \pm \delta_n$ for any n, since we made the intervals strictly nested.

By the fact that $x_n \notin [x_{n+1} - \delta_{n+1}, x_{n+1} + \delta_{n+1}]$, α must be irrational, since $\mathbb{Q}$ is countable and this construction therefore "dodges" all of the rationals.

But also, given $\varepsilon > 0$, we can pick an n such that $\varepsilon_n < \frac{1}{2}\varepsilon$ (since $\varepsilon_n \to 0$). Now by construction,

$$|x - x_n| < \delta_n \implies |f(x) - f(x_n)| < \varepsilon_n$$

Since $\alpha \in [x_n - \delta_n, x_n + \delta_n]$, we have $|f(\alpha) - f(x_n)| < \varepsilon_n$, so letting $\delta = \min\{|\alpha - (x_n \pm \delta_n)|\}$ (so $\delta > 0$ and $(\alpha - \delta, \alpha + \delta) \subset [x_n - \delta_n, x_n + \delta_n]$),

$$\begin{aligned} |x - \alpha| < \delta \implies |f(x) - f(\alpha)| &= |f(x) - f(x_n) + f(x_n) - f(\alpha)| \\ &\leq |f(x) - f(x_n)| + |f(x_n) - f(\alpha)| \\ &\leq \varepsilon_n + \varepsilon_n \\ &< \varepsilon \end{aligned}$$

So f is continuous at α. $\square$

In fact, this proof shows that a function $\mathbb{R} \to \mathbb{R}$ cannot be continuous at precisely any set that is countable and dense in $\mathbb{R}$.