Existence of Minima

Determine whether there exists a minimum value in the range (set of values) of the following functions defined on the set of real numbers:

1. $f(x) = 2^x + 2^{-x}$
2. $f(x) = \frac{1}{x^2+1}$
3. $f(x) = x^3$
4. $f(x) = x^2$

Solutions

a) $f(x) = 2^x + 2^{-x}$

Analysis:

The function $f(x) = 2^x + 2^{-x}$ is the sum of two exponential functions. To find its minimum value, we can use the properties of exponential functions.

Finding the Minimum:

To analyze the function, rewrite 2^{-x} as $\frac{1}{2^x}$:

$$f(x) = 2^x + \frac{1}{2^x}$$

Let $y = 2^x$. Since $y > 0$, the function can be expressed in terms of y:

$$f(y) = y + \frac{1}{y}$$

To find the minimum value, take the derivative of $f(y)$ with respect to y and set it to zero:

$$f'(y) = 1 - \frac{1}{y^2}$$

Setting $f'(y) = 0$:

$$1 - \frac{1}{y^2} = 0 \implies y^2 = 1 \implies y = 1 \quad (y > 0)$$

Now, evaluate $f(y)$ at this critical point:

$$f(1) = 1 + 1 = 2$$

Conclusion: The minimum value of $f(x)$ is 2, and therefore, there exists a minimum value in the range of the function.

b) $f(x) = \frac{1}{x^2+1}$

Analysis:

The function $f(x) = \frac{1}{x^2+1}$ is defined for all real numbers x. The denominator $x^2 + 1$ is always positive and achieves its minimum value of 1 when $x = 0$.

Finding the Minimum:

The maximum value of $f(x)$ occurs at $x = 0$:

$$f(0) = \frac{1}{0^2 + 1} = 1$$

As $|x|$ increases, $x^2 + 1$ increases, leading to a decrease in $f(x)$:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1}{x^2 + 1} = 0$$

However, $f(x)$ never actually reaches 0; it only approaches it.

Conclusion: Therefore, there is no minimum value (infimum) in the range of the function $f(x)$ because while it can get arbitrarily close to 0, it never actually reaches it.

c) $f(x) = x^3$

Analysis:

The function $f(x) = x^3$ is a cubic polynomial. It is defined for all real x.

Finding the Minimum:

As x approaches negative infinity, $f(x)$ also approaches negative infinity:

$$\lim_{x \to -\infty} f(x) = -\infty$$

As x approaches positive infinity, $f(x)$ approaches positive infinity:

$$\lim_{x \to \infty} f(x) = \infty$$

Since the function can take any real value, there is no minimum value in the range.

Conclusion: There is no minimum value in the range of the function $f(x)$.

d) $f(x) = x^2$

Analysis:

The function $f(x) = x^2$ is a quadratic polynomial. It is defined for all real x and is always non-negative.

Finding the Minimum:

The minimum value occurs at $x = 0$:

$$f(0) = 0^2 = 0$$

As $|x|$ increases, $f(x)$ also increases:

$$\lim_{x \to \pm\infty} f(x) = \infty$$

Conclusion: The minimum value of $f(x)$ is 0, and therefore, there exists a minimum value in the range of the function.

Summary of Results

1. $f(x) = 2^x + 2^{-x}$: Exists minimum value, **2**
2. $f(x) = \frac{1}{x^2+1}$: Does not exist minimum value
3. $f(x) = x^3$: Does not exist minimum value
4. $f(x) = x^2$: Exists minimum value, **0**