Equality Conditions for Positive Real Variables

Is the following equality true for all positive real numbers x and y:

a) $(x + y)^2 = x^2 + 2xy + y^2$;

b) $|x + y - 1| = |x + y| - 1$;

c) $|x + y| = |x| + |y|$;

d) $(x + y)^{-2} = x^{-2} + 2x^{-1}y^{-1} + y^{-2}$?

Solution

a) $(x + y)^2 = x^2 + 2xy + y^2$

This is true. From the definition of the square of a sum, we have:

$$(x + y)^2 = x^2 + 2xy + y^2.$$

b) $|x + y - 1| = |x + y| - 1$

The equality does not always hold. To show when it doesn't hold, let's take $x = 1$ and $y = 1$. Then:

$$|x + y - 1| = |1 + 1 - 1| = |1| = 1$$

and

$$|x + y| - 1 = |1 + 1| - 1 = 2 - 1 = 1.$$

This equality doesn't always hold, especially when $x+y-1 < 0$. For example, if $x = 0.5$ and $y = 0.3$, then:

$$|x + y - 1| = |0.5 + 0.3 - 1| = | - 0.2| = 0.2$$

and

$$|x + y| - 1 = |0.5 + 0.3| - 1 = 0.8 - 1 = -0.2.$$

Thus, the equality doesn't hold.

c) $|x + y| = |x| + |y|$

This equality holds only if both x and y are positive numbers. Therefore, for any positive real numbers x and y, the equality holds:

$$|x + y| = |x| + |y|.$$

d) $(x+y)^{-2} = x^{-2} + 2x^{-1}y^{-1} + y^{-2}$

This equation is false. We can provide a counterexample. Let $x = 1$ and $y = 2$. Let's calculate both sides of the equation.

Left-hand side:

$$(x+y)^{-2} = (1+2)^{-2} = 3^{-2} = \frac{1}{9}.$$

Right-hand side:

$$x^{-2}+2x^{-1}y^{-1}+y^{-2} = 1^{-2}+2\cdot 1^{-1}\cdot 2^{-1}+2^{-2} = 1+\frac{2}{2}+\frac{1}{4} = 1+1+0.25 = 2.25.$$

We see that:

$$\frac{1}{9} \neq 2.25.$$

Thus, the equality does not hold.