Analytical Comparison of Two Polynomial Sequences

1 Problem Statement

Let $f_n = an^2 + bn + 5$ and $g_n = cn^2 + dn + 1000$, where $a, b, c, d \in \mathbb{N}_0$ (the set of non-negative integers). We aim to determine which of the following statements are logically equivalent to the assertion:

There exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, we have $g_n > f_n$.

The candidate statements are:

a) One of the following two substatements holds:

(i) $a < c$

(ii) $a = c$ and it is not true that $b > d$

b) $a \leq c$ and $b \leq d$

c) There are infinitely many natural numbers $n \in \mathbb{N}$ such that $g_n > f_n$

d) The set $\{n \in \mathbb{N} : g_n \leq f_n\}$ has a maximum element

2 Analytical Setup

Define the difference of the sequences:

$$h(n) := g_n - f_n = (c - a)n^2 + (d - b)n + 995$$

We analyze the sign of $h(n)$ as $n \to \infty$. The long-term behavior of $h(n)$ is determined by its leading coefficient $(c - a)$. We consider three cases:

- If $c > a$, then $h(n) \to \infty$ as $n \to \infty$, hence $g_n > f_n$ for all sufficiently large n.
- If $c = a$, then $h(n) = (d - b)n + 995$ becomes a linear function:
  - If $d > b$, then $h(n) \to \infty$, so $g_n > f_n$ eventually.
  - If $d = b$, then $h(n) = 995 > 0$ for all n.
  - If $d < b$, then $h(n) \to -\infty$, so $g_n < f_n$ for large n.
- If $c < a$, then $h(n) \to -\infty$, thus $g_n < f_n$ for large n.

Hence, the condition for $g_n > f_n$ for all $n > n_0$ is:

$$\boxed{c > a \quad \text{or} \quad (c = a \text{ and } d \geq b)}$$

3 Statement Analysis

3.1 Statement a)

This is logically equivalent to:

$$a < c \quad \text{or} \quad (a = c \text{ and } b \leq d)$$

This exactly matches the necessary and sufficient condition derived above.
Answer: Yes.

3.2 Statement b)

This requires:

$$a \leq c \quad \text{and} \quad b \leq d$$

This condition is stronger than needed. For instance, if $a < c$ but $b > d$, the asymptotic inequality still holds. Therefore, this condition is not equivalent.
Answer: No.

3.3 Statement c)

This asserts that $g_n > f_n$ holds for infinitely many natural numbers n.

In the context of integer domains and polynomial functions with integer coefficients, if a quadratic function $h(n)$ is positive for infinitely many natural numbers n, and since $h(n)$ can only change sign finitely many times, it must be eventually always positive. That is, the set $\{n \in \mathbb{N} : h(n) \leq 0\}$ is finite.

Therefore, this condition *is* equivalent to the original statement.
Answer: Yes.

3.4 Statement d)

This states that the set $\{n \in \mathbb{N} : g_n \leq f_n\}$ has a maximum element, i.e., is finite with an upper bound. This is equivalent to $g_n > f_n$ for all sufficiently large n.
Answer: Yes.