Analytical Investigation of Integer Squares

Problem Statement

Determine whether there exists an integer n such that:

(a) The remainder of n^2 divided by 4 is equal to 3.

(b) The sum of the digits of n^2 is equal to 36.

(c) The sum of the digits of n^2 is equal to 21.

(d) The remainder of n^2 divided by 8 is equal to 5.

1 Analysis

1.1 Question (a)

Is there an integer n such that $n^2 \equiv 3 \pmod 4$?

Let us analyze the possible values of $n^2 \mod 4$. For any integer n, it is either even or odd.

$$\begin{aligned} n \equiv 0 \pmod 2 &\Rightarrow n^2 \equiv 0 \pmod 4, \\ n \equiv 1 \pmod 2 &\Rightarrow n^2 \equiv 1 \pmod 4. \end{aligned}$$

Therefore, the only possible values for $n^2 \mod 4$ are 0 and 1. Hence:

Answer: No.

Explanation: A square of an integer cannot be congruent to 3 modulo 4.

1.2 Question (b)

Is there an integer n such that the sum of the digits of n^2 is 36?

When n = 264, then $n^2 = 69696$. The sum of digits is $6 + 9 + 6 + 9 + 6 = 36$; therefore, such a number exists.

Hence:

Answer: Yes.

1.3 Question (c)

Is there an integer n such that the sum of the digits of n^2 is equal to 21?

$$S(n^2) \equiv n^2 \pmod{9}.$$

If $S(n^2) = 21$, then $n^2 \equiv 21 \equiv 3 \pmod{9}$.

We now examine whether 3 can be a quadratic residue modulo 9. The quadratic residues modulo 9 are obtained by squaring all residues modulo 9:

$$\begin{aligned}
0^2 &\equiv 0, \\
1^2 &\equiv 1, \\
2^2 &\equiv 4, \\
3^2 &\equiv 0, \\
4^2 &\equiv 7, \\
5^2 &\equiv 7, \\
6^2 &\equiv 0, \\
7^2 &\equiv 4, \\
8^2 &\equiv 1.
\end{aligned}$$

So the set of quadratic residues modulo 9 is:

$$\{0, 1, 4, 7\}.$$

Since $3 \notin \{0, 1, 4, 7\}$, it follows that $n^2 \not\equiv 3 \pmod{9}$.

Answer: No.

Explanation: Since the sum of the digits of n^2 equals 21, it must be congruent to 3 modulo 9. But 3 is not a quadratic residue modulo 9, so such an n does not exist.

1.4 Question (d)

Is there an integer n such that $n^2 \equiv 5 \pmod{8}$?

We examine the possible quadratic residues modulo 8:

$$\begin{aligned}
n \equiv 0 \pmod{8} &\Rightarrow n^2 \equiv 0 \pmod{8}, \\
n \equiv 1 \pmod{8} &\Rightarrow n^2 \equiv 1 \pmod{8}, \\
n \equiv 2 \pmod{8} &\Rightarrow n^2 \equiv 4 \pmod{8}, \\
n \equiv 3 \pmod{8} &\Rightarrow n^2 \equiv 1 \pmod{8}, \\
n \equiv 4 \pmod{8} &\Rightarrow n^2 \equiv 0 \pmod{8}, \\
n \equiv 5 \pmod{8} &\Rightarrow n^2 \equiv 1 \pmod{8}, \\
n \equiv 6 \pmod{8} &\Rightarrow n^2 \equiv 4 \pmod{8}, \\
n \equiv 7 \pmod{8} &\Rightarrow n^2 \equiv 1 \pmod{8}.
\end{aligned}$$

Thus, possible values of $n^2 \mod 8$ are:

$$\{0, 1, 4\}.$$

Answer: No.

Explanation: Since 5 is not a quadratic residue modulo 8, there does not exist any integer n such that $n^2 \equiv 5 \pmod 8$.

2 Conclusion

(a) No.

(b) Yes.

(c) No.

(d) No.