Analysis of the Existence of Real Solutions for Given Polynomial Equations

1 Problem Statement

Determine whether the following polynomial equations possess at least one real solution.

(a) $x^6 + 5x^3 + 5 = 0$

(b) $x^6 + 5x^2 + 5 = 0$

(c) $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6 = 0$

(d) $x^2 + 127x - 11 = 0$

Each subproblem is examined separately below.

2 Analysis and Solution

2.1 Equation (a): $x^6 + 5x^3 + 5 = 0$

Let us denote $y = x^3$. The equation becomes:

$$y^2 + 5y + 5 = 0$$

Solving this quadratic equation:

$$y = \frac{-5 \pm \sqrt{25 - 4 \cdot 1 \cdot 5}}{2} = \frac{-5 \pm \sqrt{5}}{2}$$

Both roots are real and negative, as the discriminant $\sqrt{5}$ is real and the numerator is negative.

Now, recall that $y = x^3$, so we need to check whether the equation $x^3 = y$ has a real solution for a given negative y. Since the cube root function is defined and continuous for all real numbers and preserves the sign of negative inputs, both values of y correspond to real values of x. Therefore, the original equation has two real solutions.

Answer: Yes.

Explanation: The equation reduces to a quadratic in x^3, which has two real roots. Each of these corresponds to a real cube root, hence two real solutions to the original equation.

2.2 Equation (b): $x^6 + 5x^2 + 5 = 0$

Substitute $y = x^2$, which implies $y \geq 0$. The equation becomes:

$$y^3 + 5y + 5 = 0$$

We study the function $f(y) = y^3 + 5y + 5$ on the interval $[0, \infty)$. Note that:

$$f(0) = 5 > 0, \quad \lim_{y \to \infty} f(y) = \infty$$

Moreover, $f(y)$ is strictly increasing on $[0, \infty)$ since its derivative $f'(y) = 3y^2 + 5 > 0$ for all $y \in \mathbb{R}$. Thus, $f(y) > 0$ for all $y \geq 0$, and the equation $f(y) = 0$ has no solution for $y \geq 0$.

Answer: No.

Explanation: The transformed equation in $y = x^2$ has no non-negative real solution, and since $x^2 \geq 0$, the original equation has no real solution.

2.3 Equation (c): $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6 = 0$

Let us denote this function by:

$$f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6$$

We analyze the behavior of this function. Since all coefficients are positive, we consider:

$$f(x) > 0 \text{ for all } x > 0$$

Now, we evaluate $f(x)$ at several negative values:

$$f(-1) = (-1)^5 + 2(-1)^4 + 3(-1)^3 + 4(-1)^2 + 5(-1) + 6 = -1 + 2 - 3 + 4 - 5 + 6 = 3 > 0$$

$$f(-2) = (-32) + 2(16) - 3(8) + 4(4) - 10 + 6 = -32 + 32 - 24 + 16 - 10 + 6 = -12$$

So:

$$f(-1) > 0, \quad f(-2) < 0$$

Since the function is continuous, and it changes sign in the interval $(-2, -1)$, by the Intermediate Value Theorem, there exists at least one real root in this interval.

Answer: Yes.

Explanation: The function is continuous and changes sign between $x = -2$ and $x = -1$, thus admitting at least one real root.

2.4 Equation (d): $x^2 + 127x - 11 = 0$

This is a standard quadratic equation. Its discriminant is:

$$\Delta = 127^2 + 4 \cdot 1 \cdot 11 = 16129 + 44 = 16173$$

Since $\Delta > 0$, the equation has two distinct real roots.

Answer: Yes.

Explanation: The discriminant is positive, ensuring two distinct real roots.