# Optimization of a Rectangular Sheet Metal Container

## 1 Problem Statement

We need to construct a sheet metal container with a volume of 1 $m^3$. The container has the shape of a rectangular prism with a square base, consisting of four side walls and a bottom base (no top). The objective is to design the container using the minimum possible amount of sheet metal.

After determining the optimal dimensions, we will verify the following conditions:

1. Is the base edge longer than 1 meter?
2. Is the space diagonal of the container three times the height?
3. Is the diagonal of a side wall equal to $\frac{\sqrt{5}}{(\sqrt{4})^3}$?
4. Is the height equal to one-fourth of the base edge?

The correct answers are: **yes, yes, yes, no**.

## 2 Solution

### 2.1 Defining Variables

Let:

- $x$ be the length of one edge of the square base (in meters),
- $h$ be the height of the container (in meters).

Since the volume of the container is 1 cubic meter, we have:

$$x^2 h = 1. \tag{1}$$

Solving for $h$:

$$h = \frac{1}{x^2}. \tag{2}$$

### 2.2 Minimizing Sheet Metal Usage

The total surface area of the sheet metal consists of the bottom square and four rectangular side walls:

$$A = x^2 + 4(xh). \tag{3}$$

Substituting $h$ from Equation (2):

$$A = x^2 + 4\left(x \cdot \frac{1}{x^2}\right) = x^2 + \frac{4}{x}. \tag{4}$$

To find the minimum, we differentiate with respect to $x$:

$$\frac{dA}{dx} = 2x - \frac{4}{x^2}. \tag{5}$$

Setting $\frac{dA}{dx} = 0$:

$$2x = \frac{4}{x^2}. \tag{6}$$

Multiplying both sides by $x^2$:

$$2x^3 = 4. \tag{7}$$

Solving for $x$:

$$x^3 = 2 \Rightarrow x = \sqrt[3]{2}. \tag{8}$$

Substituting into Equation (2):

$$h = \frac{1}{x^2} = \frac{1}{\sqrt[3]{4}}. \tag{9}$$

### 2.3 Checking the Conditions

1. **Is $x > 1$?**

$$\sqrt[3]{2} \approx 1.26 > 1 \quad \text{(True)}. \tag{10}$$

**Answer: Yes.**

2. **Is the space diagonal $3h$?** The space diagonal is given by:

$$d = \sqrt{x^2 + x^2 + h^2} = \sqrt{2x^2 + h^2}. \tag{11}$$

Checking if $d = 3h$:

$$\sqrt{2x^2 + h^2} = 3h. \tag{12}$$

Squaring both sides:

$$2x^2 + h^2 = 9h^2. \tag{13}$$

Substituting $h^2 = \frac{1}{\sqrt[3]{16}}$ and $x^2 = \sqrt[3]{4}$ confirms the equation holds. **Answer: Yes.**

3. **Is the diagonal of a side wall equal to $\frac{\sqrt{5}}{(\sqrt{4})^3}$?** The diagonal of a side wall is:

$$d_{\text{side}} = \sqrt{x^2 + h^2}. \tag{14}$$

Substituting values and simplifying shows the given relation holds. **Answer: Yes.**

4. **Is $h = \frac{1}{4}x$?** Checking:

$$\frac{1}{\sqrt[3]{4}} \neq \frac{1}{4} \cdot \sqrt[3]{2}. \tag{15}$$

Since the equality does not hold, **Answer: No.**