Analysis of the Convergence of Sequences

Task and Solution

We are tasked with answering whether the sequence (a_n) defined by the following formulas converges to a finite limit.

Answer with "yes" or "no" for each case, and provide detailed explanations of the solutions.

1. $a_n = \log_2(n)$
2. $a_n = 2^{-n}$
3. $a_n = \frac{2n^2+3n^3+4}{n^2+1}$
4. $a_n = \log_{n+1}(3)$

For each sequence, we will rigorously analyze its behavior as $n \to \infty$ using exact calculations and formulas.

Solution

1. Sequence $a_n = \log_2(n)$

We are given the sequence $a_n = \log_2(n)$. The logarithmic function $\log_2(n)$ increases without bound as n increases. Specifically:

$$a_n = \log_2(n) = \frac{\ln(n)}{\ln(2)}$$

As $n \to \infty$, the natural logarithm $\ln(n) \to \infty$, so:

$$\lim_{n\to\infty} \log_2(n) = \infty$$

Thus, the sequence $a_n = \log_2(n)$ does not converge to a finite limit.
Answer: No

2. Sequence $a_n = 2^{-n}$

Now, consider the sequence $a_n = 2^{-n}$. This sequence is an exponential decay. We can express it as:

$$a_n = 2^{-n} = \frac{1}{2^n}$$

As $n \to \infty$, the term 2^n grows exponentially, so:

$$\lim_{n\to\infty} 2^{-n} = 0$$

Therefore, the sequence $a_n = 2^{-n}$ converges to 0.
Answer: Yes

3. Sequence $a_n = \frac{2n^2+3n^3+4}{n^2+1}$

Next, we consider the sequence $a_n = \frac{2n^2+3n^3+4}{n^2+1}$. We begin by dividing both the numerator and denominator by n^2:

$$a_n = \frac{2n^2 + 3n^3 + 4}{n^2 + 1} = \frac{n^2(2 + 3n + \frac{4}{n^2})}{n^2(1 + \frac{1}{n^2})}$$

This simplifies to:

$$a_n = \frac{2 + 3n + \frac{4}{n^2}}{1 + \frac{1}{n^2}}$$

As $n \to \infty$, the terms $\frac{4}{n^2}$ and $\frac{1}{n^2}$ approach 0, leaving us with:

$$\lim_{n\to\infty} a_n = 3n + 2$$

Thus, the sequence grows without bound, and it does not converge to a finite limit.
Answer: No

4. Sequence $a_n = \log_{n+1}(3)$

Finally, we analyze the sequence $a_n = \log_{n+1}(3)$. Using the change of base formula for logarithms:

$$a_n = \log_{n+1}(3) = \frac{\ln(3)}{\ln(n+1)}$$

As $n \to \infty$, $\ln(n+1)$ grows indefinitely, so:

$$\lim_{n\to\infty} \frac{\ln(3)}{\ln(n+1)} = 0$$

Thus, the sequence $a_n = \log_{n+1}(3)$ converges to 0.
Answer: Yes