Analytical Problem Solving: Degree Constraints in an Undirected Graph

1 Problem Statement

At a certain party, there were 19 people. Each person knew at least one and at most three of the other attendees. It is known that if person X knows person Y, then Y also knows X. Does this imply that:

a) There was an **odd number** of attendees who knew **only one** other person?

b) There was an **even number** of attendees who knew **three** other people?

c) The number of attendees who knew **only one** other person was the **same** as the number of attendees who knew **three** other people?

d) There was an **odd number** of attendees who knew **exactly two** other people?

2 Reformulation Using Graph Theory

We can model this situation as a simple undirected graph $G = (V, E)$, where:

- Each vertex $v \in V$ represents a person at the party.
- An edge $\{u, v\} \in E$ represents a mutual acquaintance between u and v.

We are told that:

- $|V| = 19$
- The degree $\deg(v) \in \{1, 2, 3\}$ for every vertex $v \in V$

Let us define:

$n_1 =$ number of vertices with degree $1, n_2 =$ number of vertices with degree $2, n_3 =$ number of vertices

Since there are 19 people:

$$n_1 + n_2 + n_3 = 19 \tag{1}$$

The sum of all vertex degrees in a graph equals twice the number of edges:

$$\sum_{v \in V} \deg(v) = 2|E| \Rightarrow n_1 + 2n_2 + 3n_3 = 2|E| \tag{2}$$

We now analyze the individual questions using these relations.

3 Analysis

3.1 Question a: Was there an odd number of attendees who knew only one other person?

We want to determine whether n_1 must be odd.

Consider the sum of all degrees:

$$S = n_1 + 2n_2 + 3n_3$$

This sum must be even, since it equals twice the number of edges.

Let us consider the parity (even or odd) of this sum modulo 2:

$$n_1 + 2n_2 + 3n_3 \equiv n_1 + 3n_3 \pmod 2$$

(since $2n_2 \equiv 0 \mod 2$)

So for the total degree sum to be even:

$$n_1 + 3n_3 \equiv 0 \mod 2 \Rightarrow n_1 + n_3 \equiv 0 \mod 2 \tag{3}$$

This tells us that $n_1 \equiv n_3 \mod 2$. Therefore, n_1 and n_3 must have the same parity.

Since we do not yet know n_3, we cannot conclude definitively that n_1 is odd. For example:

- If $n_1 = 3$, $n_3 = 3$: both odd.
- If $n_1 = 2$, $n_3 = 2$: both even.

Hence, different values of n_1 with different parity are possible under the given constraints.

Answer: No.

3.2 Question b: Was there an even number of attendees who knew three other people?

This is equivalent to asking whether n_3 must be even.

From equation (3) again:

$$n_1 + n_3 \equiv 0 \mod 2 \Rightarrow n_1 \equiv n_3 \mod 2$$

This implies $n_3 \equiv n_1 \mod 2$, so again the parity of n_3 depends on that of n_1.

If n_1 is odd, then n_3 is odd. If n_1 is even, then n_3 is even.

Thus, both even and odd values for n_3 are consistent with the constraints.

Answer: No.

3.3 Question c: Were there as many attendees who knew only one other person as those who knew three?

This asks whether $n_1 = n_3$ must hold.

Equation (3) tells us $n_1 \equiv n_3 \mod 2$, which only implies that their parity is the same, not that they are equal.

For example, the following configurations are valid:

- $n_1 = 2$, $n_2 = 15$, $n_3 = 2$: sum of degrees is $2 + 30 + 6 = 38$, even.
- $n_1 = 4$, $n_2 = 11$, $n_3 = 4$: total 19 nodes, degree sum $4 + 22 + 12 = 38$, even.

In both cases, $n_1 = n_3$, but it is not required by the constraints.
Also:

- $n_1 = 2$, $n_2 = 14$, $n_3 = 3$: total 19 nodes, degree sum $2+28+9 = 39$, odd — invalid.
- $n_1 = 2$, $n_2 = 13$, $n_3 = 4$: total 19 nodes, degree sum $2+26+12 = 40$, even — valid, but now $n_1 \neq n_3$.

Hence, equality is not enforced.
Answer: No.

3.4 Question d: Was there an odd number of attendees who knew exactly two other people?

We now want to determine whether n_2 must be odd.

We return to the degree sum:

$$n_1+2n_2+3n_3 \equiv 0 \mod 2 \Rightarrow n_1+3n_3 \equiv 0 \mod 2 \Rightarrow n_1+n_3 \equiv 0 \mod 2 \Rightarrow n_1+n_3 \text{ even}$$

Since total number of people is 19, we have:

$$n_1 + n_2 + n_3 = 19 \Rightarrow n_2 = 19 - (n_1 + n_3)$$

Then:

$$n_2 \equiv 19 - (n_1 + n_3) \mod 2$$

Because $n_1 + n_3$ is even, this gives:

$$n_2 \equiv 19 - \text{even} \equiv 1 \mod 2 \Rightarrow n_2 \text{ is odd}$$

Therefore, the number of attendees who knew exactly two others **must** be odd.
Answer: Yes.