Checking the greatest common divisor

Task

Is the greatest common divisor given correctly:

- a) $\text{GCD}(42, 84) = 14$;
- b) $\text{GCD}(10^9 + 5, 10^9 + 35) = 15$;
- c) $\text{GCD}(30, 42) = 6$;
- d) $\text{GCD}(10^7 + 14, 10^7 + 21) = 7$?

Solution:

We will evaluate the validity of each of the theorems concerning the greatest common divisor (GCD), also known as the greatest common factor (GCD).

a) GCD$(42, 84) = 14$

To find the GCD of 42 and 84, we will apply prime factorization:

$$42 = 2 \times 3 \times 7$$
$$84 = 2^2 \times 3 \times 7$$

The common factors are: 2^1, 3^1, and 7^1.
GCD is calculated as:

$$\text{GCD}(42, 84) = 2^1 \times 3^1 \times 7^1 = 42.$$

So the statement $\text{GCD}(42, 84) = 14$ is invalid.

b) GCD$(10^9 + 5, 10^9 + 35) = 15$

Let $a = 10^9 + 5$ and $b = 10^9 + 35$.
We can simplify:

$$b - a = (10^9 + 35) - (10^9 + 5) = 30.$$

So we calculate:

$$\text{GCD}(a,b) = \text{GCD}(10^9+5, 30).$$

Now let's find the GCD of 10^9+5 and 30. The prime factorization of 30 is:

$$30 = 2 \times 3 \times 5.$$

Let's calculate $10^9+5 \mod 30$:

$$10^9 \mod 30 \equiv 10 \quad (\text{since } 10 \equiv 10 \mod 30).$$

So,

$$10^9+5 \equiv 10+5 \equiv 15 \mod 30.$$

We calculate the GCD:

$$\text{GCD}(15,30) = 15.$$

So the statement $\text{GCD}(10^9+5, 10^9+35) = 15$ is correct.

c) GCD$(30,42) = 6$

We will apply prime factorization:

$$30 = 2 \times 3 \times 5$$

$$42 = 2 \times 3 \times 7$$

The common factors are 2 and 3:

$$\text{GCD}(30,42) = 2^1 \times 3^1 = 6.$$

Therefore, the statement $\text{GCD}(30,42) = 6$ is correct.

d) GCD$(10^7+14, 10^7+21) = 7$

Let $c = 10^7+14$ and $d = 10^7+21$.

We can simplify:

$$d - c = (10^7+21) - (10^7+14) = 7.$$

Now we calculate:

$$\text{GCD}(c,d) = \text{GCD}(10^7+14, 7).$$

First we calculate $10^7+14 \mod 7$:

$$10^7 \mod 7 \equiv 3 \quad (\text{since } 10 \equiv 3 \mod 7).$$

So,

$$10^7+14 \equiv 3+0 \equiv 3 \mod 7.$$

We calculate the GCD:

$$\text{GCD}(3,7) = 1.$$

So the statement $\text{GCD}(10^7+14, 10^7+21) = 7$ is incorrect.