When A Becomes N

We have a set A that is a subset of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and contains $0 \in A$. We will analyze the following conditions to determine if they imply that $A = \mathbb{N}$. For each question (a, b, c, d), we will respond with "Yes" or "No" and provide detailed explanations of the solutions.

Questions and Answers

Question a

For every $m \in A$, if there exists a natural number less than m in the set A, then $m + 1$ is also in A.

Answer: No

Explanation: The condition states that if there is a natural number $n < m$ which belongs to A, then $m + 1$ must also belong to A. However, this does not guarantee that all natural numbers will be included in A.

For instance, consider the set $A = \{0, 2, 3, 4, 5, \ldots\}$. In this case:

- For $m = 2$, there exists $n = 0 \in A$ (where $n < m$), which implies that $m + 1 = 3$ must be in A, and it is.
- However, $1 \notin A$, which means that $A \neq \mathbb{N}$ since $1 \in \mathbb{N}$.

Thus, this condition does not ensure that A contains all natural numbers.

Question b

For every $m \in \mathbb{N}$, if all natural numbers less than m are in A, then m is also in A.

Answer: Yes

Explanation: This statement resembles the principle of mathematical induction. It asserts that if all natural numbers less than m are included in A, then m must also be in A.

We can illustrate this with mathematical induction:

- **Base Case:** For $m = 1$, since $0 \in A$, we conclude that $1 \in A$.
- **Inductive Step:** Assume the statement is true for all $k < m$. That is, if all numbers less than m are in A, then m must also be in A.

By induction, this means that every natural number must be in A, leading to the conclusion that $A = \mathbb{N}$.

Question c

For every $k \in \mathbb{N}$, there exists $m > k$ such that $m \in A$ and for every $n > 0$, if $n \in A$, then $n - 1 \in A$.

Answer: Yes

Explanation: This condition states that for each natural number k, there exists a number $m > k$ such that $m \in A$ and if any $n \in A$, then $n - 1 \in A$.

This condition implies that A contains an infinite number of elements. If m is included in A and the property that $n - 1 \in A$ holds, it suggests that we can keep finding numbers in A by decreasing from m.

Therefore, this condition can be interpreted as a strong form of downward closure, which implies that if there are numbers in A greater than any k, we can deduce that eventually all smaller natural numbers will also be included in A. Thus, this condition does lead to the conclusion that $A = \mathbb{N}$.

Question d

There exists $n \in \mathbb{N}$ such that $n \in A$ implies $n + 1 \in A$.

Answer: No

Explanation: The statement suggests that there is at least one natural number $n \in A$ such that if n is in A, then $n + 1$ must also be in A. However, this condition alone does not guarantee that all natural numbers are included in A.

For example, consider the set $A = \{0, 1, 3\}$. Here, we can take $n = 1$, which is in A and implies that 2 should also be in A. However, $2 \notin A$, and therefore not all natural numbers are in A. This demonstrates that the condition does not ensure that $A = \mathbb{N}$.