Conditions for Prime Numbers

We investigate whether for any integer $p > 1$, the number p is prime if and only if the following conditions hold:

(a) p is not divisible by any integer k such that $1 < k < \sqrt{p}$.

(b) p is not divisible by any integer k such that $1 < k \leq \frac{p}{2}$.

(c) p is not divisible by any integer k such that $1 < k \leq p$.

(d) p is not divisible by any odd integer k such that $1 < k < p$.

We now analyze each of these conditions.

Condition (a): No divisibility by $1 < k < \sqrt{p}$

A prime number is defined as having no divisors other than 1 and itself. If p is composite, then it can be expressed as $p = k \cdot m$, where k and m are proper divisors of p.

If both k and m were greater than $\sqrt{p}$, then their product would exceed p, which is a contradiction. Thus, at least one of the divisors must be $\leq \sqrt{p}$.

However, this condition alone is not sufficient to determine primality. For instance, the number 9 satisfies this condition but is not prime. Hence, condition (a) is incorrect.

Condition (b): No divisibility by $1 < k \leq \frac{p}{2}$

If p is composite, then it has a divisor k with $1 < k \leq \frac{p}{2}$. If such a divisor exists, there is some $m > 1$ with $p = k \cdot m$. Since $k \leq \frac{p}{2}$, it follows that $m \geq 2$, meaning k is a proper divisor of p.

If p is prime, it has no proper divisors other than itself and 1, meaning no k in the range $1 < k \leq \frac{p}{2}$ can divide p. Therefore, condition (b) is correct.

Condition (c): No divisibility by $1 < k \leq p$

This condition would imply that p is not divisible by any integer k in the range $1 < k \leq p$. However, every number is divisible by itself, making this condition incorrect. Thus, condition (c) is false.

Condition (d): No divisibility by odd numbers $1 < k < p$

This condition suggests that p is prime if it is not divisible by any odd number k in the range $1 < k < p$. This is incorrect because it fails to account for even composite numbers like $p = 4$, which is only divisible by 2 (an even number). Thus, condition (d) is incorrect.