# Inequality Logic

Real numbers $x$ and $y$ satisfy the inequalities:

$$x^2 + y^2 \leq 50 \quad \text{and} \quad 2x + y \geq 15.$$

Determine whether the following statements necessarily follow from these conditions:

(a) $x \leq 5$,

(b) $x^2 + y^2 \geq 45$,

(c) $x + y \leq 10$,

(d) $y \leq 5$.

# Solution

## Step 1: Analyzing the inequalities

1. $x^2 + y^2 \leq 50$: This inequality defines a disk (or circle) of radius $\sqrt{50}$ centered at the origin $(0, 0)$.

2. $2x+y \geq 15$: This inequality represents a half-plane. The boundary line $2x+y = 15$ has a slope of $-2$ and intercept $y = 15$ when $x = 0$. The solution set is the intersection of the disk $x^2 + y^2 \leq 50$ and the half-plane $2x + y \geq 15$.

## Step 2: Checking each statement

**(a)** $x \leq 5$**:** The largest possible value of $x$ occurs along the line $2x + y = 15$ where it intersects the boundary of the circle $x^2 + y^2 = 50$. Substituting $y = 15 - 2x$ into the circle equation:

$$x^2 + (15 - 2x)^2 = 50.$$

Expanding:

$$x^2 + 225 - 60x + 4x^2 = 50 \implies 5x^2 - 60x + 225 = 50.$$

Simplify:

$$5x^2 - 60x + 175 = 0 \implies x^2 - 12x + 35 = 0.$$

Factoring:

$$(x - 7)(x - 5) = 0.$$

The solutions are $x = 7$ and $x = 5$. To determine if $x = 7$ is valid, calculate $y = 15 - 2(7) = 1$. Substituting $x = 7, y = 1$ into the circle equation:

$$x^2 + y^2 = 7^2 + 1^2 = 49 + 1 = 50,$$

which satisfies both inequalities. Hence, $x = 7$ is valid, and $x \leq 5$ does not necessarily follow. **Answer: False.**

**(b)** $x^2+y^2 \geq 45$**:** The minimum value of $x^2+y^2$ occurs at the intersection of $2x+y = 15$ and $x^2 + y^2 = 50$. From the solution to (a), all valid points lie on the circle boundary, where $x^2 + y^2 = 50$, which is always greater than or equal to 45. **Answer: True.**

**(c)** $x + y \leq 10$**:** The maximum value of $x + y$ occurs at the boundary points. From (a), the valid points are:

- At $x = 5, y = 5$: $x + y = 5 + 5 = 10$,
- At $x = 7, y = 1$: $x + y = 7 + 1 = 8$.

The maximum $x + y = 10$, so $x + y \leq 10$ holds for all points in the region. **Answer: True.**

**(d)** $y \leq 5$**:** The maximum value of $y$ occurs along the line $2x + y = 15$. At $x = 5$, $y = 15 - 2(5) = 5$. Since no point in the solution region has $y > 5$, the statement $y \leq 5$ is valid. **Answer: True.**