1 Counting Real Solutions of a Quartic Equation

For a given real number a, let $R(a)$ denote the number of real numbers x such that $x^4 - 2x^2 = a$. Then:

a) $R(-1) = 2$;

b) $R(0) = 3$;

c) $R(1) = 4$;

d) $R(-2) = 1$.

2 Solution

To find $R(a)$, we transform the equation into a function in the variable $y = x^2$. We have:

$$x^4 - 2x^2 = y^2 - 2y = a \implies y^2 - 2y - a = 0$$

Solving this quadratic equation, we use the quadratic formula:

$$y = \frac{2 \pm \sqrt{(2)^2 - 4 \cdot 1 \cdot (-a)}}{2 \cdot 1} = 1 \pm \sqrt{1 + a}$$

The real numbers x corresponding to a given y are such that $y = x^2 \geq 0$. Thus, we need to investigate the conditions for y:

$$1 + a \geq 0 \implies a \geq -1$$

Now we analyse specific values of a:
1. For $a = -1$:

$$y = 1 \pm \sqrt{0} = 1$$

There are two solutions: $x^2 = 1 \implies x = 1$ or $x = -1$.

$$R(-1) = 2 \quad \text{(correct)}$$

2. For $a = 0$:

$$y = 1 \pm \sqrt{1} \implies y = 2 \text{ or } 0$$

For $y = 2 \implies x^2 = 2 \implies x = \sqrt{2}, -\sqrt{2}$. For $y = 0 \implies x = 0$.

$$R(0) = 3 \quad \text{(correct)}$$

3. For $a = 1$:

$$y = 1 \pm \sqrt{2}$$

For $y = 1 + \sqrt{2} \implies x^2 = 1 + \sqrt{2} \implies x = \sqrt{1 + \sqrt{2}}, -\sqrt{1 + \sqrt{2}}$. For $y = 1 - \sqrt{2}$ (no solutions).

$$R(1) = 2 \quad \text{(incorrect)}$$

4. For $a = -2$:

$$y = 1 \pm \sqrt{-1} \text{ (no solutions)}$$

$$R(-2) = 0 \quad \text{(incorrect)}$$