Spanning Multi-Asset Payoffs With ReLUs*

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Consider reducing margins

Abstract

Multi-asset payoff functions decomposed as continuum portfolios of vanilla basket calls or puts can also be approximated with one hidden-layer feedforward ReLU neural networks. We propose a weak, distributional formulation of the continuum spanning problem which has a unique solution if and only if the payoff function is even and absolutely homogeneous, and we establish a Fourier-based formula to calculate the solution. We apply the formula to derive explicit function-, measure- and distribution-type solutions for specific example payoffs. We study the practicality of neural network training schemes to numerically solve the corresponding discrete spanning problem for a selection of archetypal payoffs, for which we obtain better hedging results with vanilla basket calls compared to industry-favored approaches based on single-asset vanilla hedges.

Keywords: static hedging, Carr-Madan spanning formula, multi-asset options, basket options, measures, Schwartz distributions, Fourier transform, one-hidden-layer feedforward ReLU neural network, dispersion call, Radon transform.

Mathematics Subject Classification: 91G20, 62G08, 62M45, 42B10, 46A11.

1 Introduction

The popular Carr & Madan (1998) *spanning formula* shows that perfect replication may be achieved for single-asset options with twice differentiable payoffs, using an

*python code available on https://github.com/hoangdungnguyen/Spanning_multi_asset_payoffs.

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infinite number of vanilla calls whose strikes span a one-dimensional continuum. In honor of this formula, we refer to the practice of hedging a target payoff with a static portfolio of vanilla payoffs as *spanning* throughout this paper. Bossu, Carr & Papanicolaou (2022) and Bossu (2022) spearheaded similar perfect spanning formulas for ℓ^2 dispersion options and more general multi-asset absolutely homogeneous payoffs, using an infinite number of vanilla basket calls whose weights span a multidimensional continuum. Specifically, given a target European payoff function $F(x_1, x_2, \ldots, x_d, k)$ of d underlying asset performances x_j (terminal price ratios or returns) and moneyness parameter k (> 0 inconsistent with some areas in the paper, e.g. prop 2.4), one possible formulation of the continuum spanning problem in strong form is to find a combination of cash, underlying asset and vanilla basket options in respective quantities $\alpha, \mu_j, \nu(\ldots)$, such that

$$\begin{aligned} F(x_1, \ldots, x_d, k) &= \alpha + \sum_{j=1}^{d} \mu_j x_j \\ &+ \int \cdots \int \nu(w_1, \ldots, w_d) \left(\sum_{j=1}^{d} w_j x_j - k \right)^+ \mathrm{d}w_1 \ldots \mathrm{d}w_d, \qquad \text{for all } x_1, \ldots, x_d, \end{aligned} \tag{1.1}$$

wherein $w_1, \ldots, w_d$ are the asset weights of the basket call payoff $\left(\sum_{j=1}^{d} w_j x_j - k \right)^+$. The primary aim of this paper is to deal with the delicate regularization aspects identified by the aforementioned authors. To this end, we introduce a weak formulation of the continuum spanning problem to allow for distributional solutions in a mathematically rigorous setting.

In practice, only a finite number of basket calls and puts may be traded and spanning is imperfect. On top of contributing to the above theory, we study how a European multi-asset payoff may be partially hedged with a finite combination of cash, underlying assets, and vanilla basket calls and puts, corresponding to a discretization of the right-hand side of (1.1). For the selection of archetypal payoffs listed in Table 1 below, we use neural network training techniques to empirically investigate how the discrete spanning error

$$\varepsilon(x_1, \ldots, x_d, k) = F(x_1, \ldots, x_d, k) - \alpha - \sum_{j=1}^{d} \mu_j x_j - \sum_{i=1}^{l} \nu_i \left(w_1^{(i)} x_1 + \cdots + w_d^{(i)} x_d - k_i \right)^+ \tag{1.2}$$

may be minimized with respect to some error metric such as MSE (mean squared error) for optimal quantities α of cash, μ_j of underlying assets $1 \leqslant j \leqslant d$, and ν_i of basket option payoffs $1 \leqslant i \leqslant l$ with associated basket weights $w_1^{(i)}, \ldots, w_d^{(i)}$ and moneyness parameter $k_i > 0$ (I would leave sign unspecified here). The expressiveness power of the neural network family is well known in theory and in practice, and constitutes a particularly convincing paradigm for discrete spanning since (1.2) corresponds to the error function of a one-hidden-layer feedforward neural network[1].

[1] Remarkably enough, the Carr & Madan (1998) spanning formula has an infinite-width neural network counterpart (Savarese, Evron, Soudry & Srebro 2019)

Target option[*]	**Notation**	**Payoff** $F(\mathbf{x}, k)$	**AH variation**[†]
Dispersion call/put	DC/DP	$(\sum_j \lvert x_j\rvert - k)^{\pm}$	$(\sum_j \lvert x_j\rvert - \lvert k\rvert)^{\pm}$
Best-of call/put	BOC/BOP	$(\max_j x_j - k)^{\pm}$	$(\max_j \lvert x_j\rvert - \lvert k\rvert)^{\pm}$
Worst-of call/put	WOC/WOP	$(\min_j x_j - k)^{\pm}$	$(\min_j \lvert x_j\rvert - \lvert k\rvert)^{\pm}$
Best-of-binary call/put	BOBC/BOBP	$\mathbf{1}_{\pm(\max_j x_j - k)>0}$	n/a
Worst-of-binary call/put	WOBC/WOBP	$\mathbf{1}_{\pm(\min_j x_j - k)>0}$	n/a

Table 1: Option payoffs covered in our numerics. k is the strike price, $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ is the vector of asset performances (price ratios or returns), $z^{\pm} = \max(0, \pm z)$ is the positive or negative part of z.

Notes: [*]All options trade on over-the-counter financial markets, either directly or as building blocks for structured products (Bossu 2014, Schofield 2017). [†]Absolutely homogeneous (AH) variation of the payoff formula, provided in view of Theorem 3.3.

1.1 Background and Review

The static hedging of a complex and illiquid payoff with more liquid instruments is useful to derivatives practitioners for price discovery, trading and risk management applications. The topic of multi-asset payoffs has gained in popularity over the past decade in the academic literature. İlhan, Jonsson & Sircar (2009) study the problem of optimally hedging exotic derivatives positions with dynamic and static trading strategies when the performance is quantified by a convex risk measure. Carr & Laurence (2011) express the joint implied distribution of several underlying assets $x_1, \ldots, x_d$ at time T as an inverse Radon transform of basket call prices with maturity T and derive a multi-asset version of Dupire's formula. Alexander & Venkatramanan (2012) derive analytic pricing approximations for European basket and rainbow payoffs using a decomposition as sum of compound payoffs. Molchanov & Schmutz (2014) show how the joint implied distribution may be recovered from best-of option prices, and investigate various symmetries of multi-asset derivatives. Cui & Xu (2022) derive a multi-asset extension of the Carr & Madan (1998) static replication formula in the form of multiple integral of products of call payoffs. Chiu & Cont (2023) propose a model-free approach to determine the superhedging cost of path-dependent payoffs, including Asian options.

In the machine learning literature for finance, Lokeshwar, Bhardawaj & Jain (2022) propose the use of neural network techniques for semi-static hedging of path-dependent exotics with short-term options. Lyons, Nejad & Arribas (2020) use "signature payoffs" to approximate path-dependent exotic derivatives payoffs. Antonov & Piterbarg (2022) discuss alternatives to neural networks for financial function approximation with an emphasis on linear regression concepts.

1.2 Standing Notation & Organization of this Paper

We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. We write $\lim_{x \to 0+}$ for the limit when x goes to 0 in $\mathbb{R}_+^*$. For any vector $\mathbf{z} \in \mathbb{R}^d$, $\|\mathbf{z}\|_1, \|\mathbf{z}\|_2 = |\mathbf{z}|$, and $\|\mathbf{z}\|_\infty$ are respectively the ℓ^1, ℓ^2 (Euclidean) and ℓ^∞ (maximum) norms, and for any other vector $\mathbf{z}' \in \mathbb{R}^d, \mathbf{z} \cdot \mathbf{z}'$ is the dot product. We denote by $\mathcal{F}$ the Fourier transform operator (see Appendix C for further notations and details). Unless stated otherwise, we work with real-valued functions, measures, and distributions, and their corresponding vector spaces and linear forms; in particular:

- $L^p(\mathbb{R}^q)$, for $p \geqslant 1$, the space of Lebesgue-measurable functions on $\mathbb{R}^q$ such that $\|f\|_{L^p} := \left(\int_{\mathbb{R}^q} |f(\mathbf{x})|^p \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} < \infty$;
- $\mathcal{C}^\infty(\mathbb{R}^q)$, the space of infinitely differentiable functions over $\mathbb{R}^q$;
- $\mathcal{S}(\mathbb{R}^q) \subset \mathcal{C}^\infty(\mathbb{R}^q)$, the space of Schwartz functions over $\mathbb{R}^q$, i.e. $\varphi \in \mathcal{C}^\infty(\mathbb{R}^q)$ such that $\varphi(\mathbf{z})$ and each of its derivatives vanish faster than any inverse power of $|\mathbf{z}|$ as $\mathbf{z} \to \infty$ (Ramm & Katsevich 1996, page 395), (King 2009, page 480);
- $\mathcal{D}(\mathbb{R}^q) \subset \mathcal{S}(\mathbb{R}^q)$, the subspace of Schwartz functions over $\mathbb{R}^q$ with compact support;
- $\langle T, \varphi \rangle$, or $\langle T_{\mathbf{z}}, \varphi(\mathbf{z}) \rangle_{\mathbf{z}}$ when there may be ambiguity about the variable, the action of a distribution T over a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^q)$ (generally, the action of a linear form on a test function in an appropriate functional space);
- $\langle \nu(\mathrm{d}\mathbf{z}), \varphi(\mathbf{z}) \rangle_{\mathbf{z}} := \int \varphi(\mathbf{z})\nu(\mathrm{d}\mathbf{z})$, the integral of $\varphi \in \mathcal{S}(\mathbb{R}^q)$ with respect to a measure ν on $\mathbb{R}^q$, if it exists, and $\nu(\mathrm{d}\mathbf{z})$ the corresponding linear form $\varphi \mapsto \langle \nu(\mathrm{d}\mathbf{z}), \varphi(\mathbf{z}) \rangle_{\mathbf{z}}$; $\delta_{\boldsymbol{a}}(\mathrm{d}\mathbf{z})$, a Dirac mass at point $\boldsymbol{a} \in \mathbb{R}^q$.

The paper is organized as follows. Section 2 poses a weak, distributional formulation of the continuum spanning problem and shows its correspondence with strong formulations under general conditions. Theorem 3.3 in Section 3, which is our main theoretical contribution, shows that a unique solution exists if and only if the payoff function is even and absolutely homogeneous, and provides a Fourier-based transform formula (3.3) for its calculation. Section 4 derives explicit solutions of function-, measure-, and distribution-type for specific payoff examples.

The delicate distributional nature of some of these explicit solutions implies that formula (3.3) may not always be numerically tractable, which motivates the use of other numerical approaches. Section 5 showcases how feedforward neural networks, which have a natural interpretation for payoff spanning, may constitute an effective numerical method in comparison to other restricted discrete spanning strategies. Section 6 provides our conclusions and perspectives for future research.

2 Distributional Approach to Continuum Spanning

Our continuum spanning results are established for a class of distributions which are not necessarily of measure-type. This theoretical extension is necessary to give precise mathematical meaning to spanning solutions found for even simple financial payoffs, see e.g. formula (4.5) for the dispersion call. To this end, we define the function set

$$\mathcal{S}_0 = \{\phi_0 \in \mathcal{C}^\infty(\mathbb{R}^{d+1});\ \phi_0(\mathbf{x}, k) = e^{-irk} h(\mathbf{x}) \text{ for some } r \in \mathbb{R}^* \text{ and } h \text{ in } \mathcal{S}(\mathbb{R}^d)\},$$

and vector spaces

$$\begin{aligned} \mathcal{S}_e &= \{\varphi_e \text{ even in } \mathcal{S}(\mathbb{R}^d)\}, \\ \Sigma_e &= \{\psi_e \in \mathcal{C}^\infty(\mathbb{R}^d);\ \psi_e = \varphi_e - \varphi_e(\mathbf{0}) \text{ for some } \varphi_e \in \mathcal{S}_e\}. \end{aligned}$$

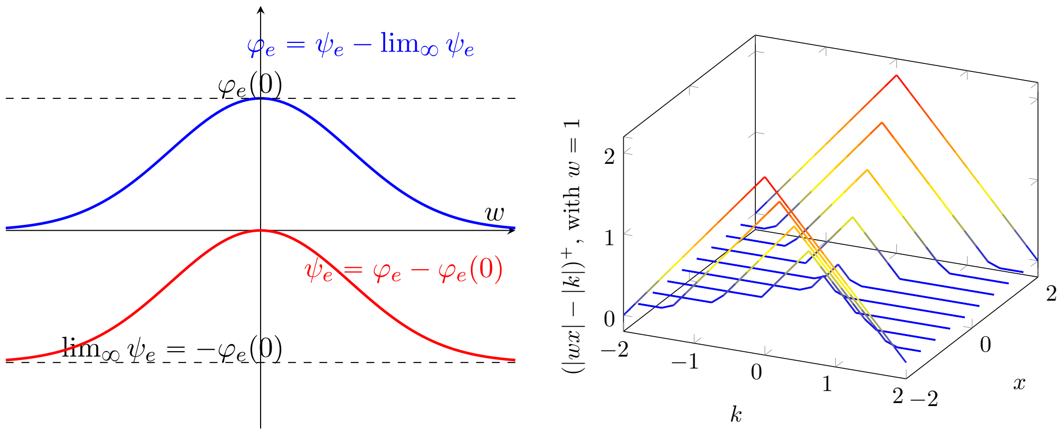


Figure 1: $d = 1$. (*Left*) One-to-one correspondence (2.1) between the example functions $\varphi_e(w) = e^{-w^2}$ and $\psi_e(w) = e^{-w^2} - 1$. (*Right*) "Basket call" payoff $F(x, k) = (|wx| - |k|)^+$ in dimension $d = 1$ used in spanning formula (2.5) with $w = 1$.

Note that $\mathcal{S}_e$ and Σ_e are isomorphic via the maps

$$\begin{aligned} \mathcal{S}_e &\longleftrightarrow \Sigma_e \\ \varphi_e &\longmapsto \varphi_e - \varphi_e(\mathbf{0}) \\ \psi_e - \lim_{\boldsymbol{\infty}} \psi_e &\longleftarrow\!\shortmid\ \psi_e \end{aligned} \tag{2.1}$$

wherein $\lim_{\boldsymbol{\infty}} \psi_e := \lim_{|\mathbf{w}|\to\infty} \psi_e(\mathbf{w})$. This one-to-one correspondence is illustrated in the left panel of Figure 1.

2.1 Weak Formulation of the Continuum Spanning Problem

Lemma 2.1. **(i)** *For any $\phi_0(\mathbf{x}, k) = e^{-ikr} h(\mathbf{x}) \in \mathcal{S}_0$ (with $r \neq 0$ and $h \in \mathcal{S}(\mathbb{R}^d)$), the function*

$$\psi(\mathbf{w}) := -r^2 \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ \phi_0(\mathbf{x}, k) = 2 \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \big(\cos(\mathbf{w}\cdot\mathbf{x}) - 1\big) \tag{2.2}$$

is in Σ_e.
(ii) *The linear operator*

$$\begin{aligned} \mathcal{S}_0 &\longrightarrow \Sigma_e \\ \phi_0 &\longmapsto \psi_e : \mathbf{w} \mapsto \int_{\mathbb{R}^{d+1}} \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ \phi_0(\mathbf{x}, k)\, \mathrm{d}k\, \mathrm{d}\mathbf{x} \end{aligned} \tag{2.3}$$

is well defined.

Proof. **(i)** For any $\phi_0(\mathbf{x}, k) = e^{-ikr} h(\mathbf{x}) \in \mathcal{S}_0$ with $r \neq 0$ and $h \in \mathcal{S}(\mathbb{R}^d)$, by definition,

$$-\frac{1}{r^2}\psi(\mathbf{w}) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} \mathrm{d}k e^{-irk} \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \mathcal{F}_k\big[\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\big](r)$$

By Example C.3, the Fourier transform of the basket call is equal to

$$\begin{aligned}\psi(\mathbf{w}) &= 2\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x})\big(\cos(\mathbf{w}\cdot\mathbf{x}) - 1\big) \\ &= \int_{\mathbb{R}^d} e^{ir\mathbf{w}\cdot\mathbf{x}} h(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} e^{-ir\mathbf{w}\cdot\mathbf{x}} h(\mathbf{x})\,\mathrm{d}\mathbf{x} - 2\int_{\mathbb{R}^d} h(\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

This shows (2.2). As $\lim_{|\mathbf{w}|\to\infty}\int_{\mathbb{R}^d} e^{\pm ir\mathbf{w}\cdot\mathbf{x}} h(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$ (Kanwal 2004, Riemann Lebesgue lemma, page 455), $\lim_{\infty}\psi = -2\int_{\mathbb{R}^d} h(\mathbf{x})\,\mathrm{d}\mathbf{x}$. Let $\varphi = \psi - \lim_{\infty}\psi$. Then $\varphi = \int_{\mathbb{R}^d} e^{ir\mathbf{w}\cdot\mathbf{x}} h(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d} e^{-ir\mathbf{w}\cdot\mathbf{x}} h(\mathbf{x})\,\mathrm{d}\mathbf{x}$ is even, and in $\mathcal{S}(\mathbb{R}^d)$ as a property of Fourier transforms (Kanwal 2004, Theorem 2 page 143). So φ belongs to $\mathcal{S}_e$. Therefore, in view of the bijection (2.1), ψ belongs to Σ_e.
(ii) Let $\phi_0 = e^{-ikr}h(\mathbf{x}) \in \mathcal{S}_0$ (with $r \neq 0$ and $h \in \mathcal{S}(\mathbb{R}^d)$). By (i), the corresponding function ψ in (2.2) is well defined and belongs to Σ_e, hence so does the corresponding function $\psi_e = -r^{-2}\psi$ in (2.3). ∎

Throughout Sections 2 to 4, we assume the target payoff function F satisfies the following mild regularity condition:

Assumption 2.1. The payoff function $F(\mathbf{x}, k)$ is continuous in $\mathbf{x}$, in k, and such that

$$\int_{\mathbb{R}^{d+1}} \mathrm{d}\mathbf{x}\,\mathrm{d}k |F(\mathbf{x}, k)h(\mathbf{x})| < \infty, \qquad h \in \mathcal{S}(\mathbb{R}^d). \tag{2.4}$$

Under this assumption, the following well-defined distributional equation is a weak formulation of the continuum spanning problem:

Problem 2.1. Given a target payoff function $F(\mathbf{x}, k)$ satisfying Assumption 2.1, find a linear form $N : \Sigma_e \to \mathbb{R}$ such that

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k F(\mathbf{x}, k)\phi_0(\mathbf{x}, k) = \\ &\qquad \left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ \phi_0(\mathbf{x}, k)\right\rangle_{\mathbf{w}}, \qquad \phi_0 \in \mathcal{S}_0.\end{aligned} \tag{2.5}$$

Equivalently by correspondence (2.1), find a linear form $T : \mathcal{S}_e \to \mathbb{R}$ such that

$$\langle N_{\mathbf{w}}, \psi_e\rangle_{\mathbf{w}} := \langle T_{\mathbf{w}}, \varphi_e\rangle_{\mathbf{w}}, \quad \text{where } \varphi_e = \psi_e - \lim_{\infty}\psi_e, \tag{2.6}$$

solves (2.5).

Remark 2.1. (i) If such T is continuous for the topology of $\mathcal{S}(\mathbb{R}^d)$, it defines a distribution on $\mathcal{S}_e$ (Kanwal 2004, Definitions page 22). (ii) By Lemma 2.1(ii), for any $\phi_0(\mathbf{x}, k) \in \mathcal{S}_0$ the function $\mathbf{w} \mapsto \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\int_{\mathbb{R}} \mathrm{d}k\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\phi_0(\mathbf{x}, k)$ is indeed in Σ_e. (iii) The operator $N_{\mathbf{w}}$ corresponds to the quantity of basket call payoffs $\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+$ for each vector $\mathbf{w}$ of basket weights across the $\mathbb{R}^d$ continuum. Note that $\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ = \big(\mathbf{w}\cdot\mathbf{x} - |k|\big)^+ + \big(-\mathbf{w}\cdot\mathbf{x} - |k|\big)^+$, which actually corresponds to two "proper basket calls" with opposite basket weights $\mathbf{w}$ and $-\mathbf{w}$ and identical nonnegative-strike $|k|$ (see the right panel in Figure 1).

2.2 Strong Formulations

We now discuss how our weak formulation coincides with a strong formulation of the continuum spanning problem, such as stated in (1.1), when N is a measure-type distribution. In dimension $d = 1$, the Carr & Madan (1998) spanning formula states that any twice differentiable European payoff function $F(x), x \in \mathbb{R}_+$ can be perfectly replicated by a "continuous portfolio" of vanilla calls in quantities $F''(K)\,\mathrm{d}K$ with strike prices spanning the continuum $K \in \mathbb{R}_+$, together with fixed cash and underlying asset positions in ad-hoc quantities, as

$$F(x) = F(0) + F'(0)x + \int_0^\infty (x-K)^+ F''(K)\,\mathrm{d}K, \qquad x \in \mathbb{R}_+. \tag{2.7}$$

Here, it is worth observing that the regularity condition on F is more restrictive than Assumption 2.1. As such, strictly speaking the Carr-Madan formula cannot be applied to replicate e.g. a straddle payoff $F(x) = |x-1|$ which is not differentiable at $x = 1$. In order to accommodate the trivial replication identity "long straddle = long 2 calls, short 1 forward contract",

$$|x-1| = 1 - x + 2(x-1)^+ = 1 - x + 2\int_0^\infty (x-K)^+ \delta_1(\mathrm{d}K), \qquad \mathbf{x} \in \mathbb{R}_+,$$

a theoretical extension is needed to give definite meaning to such statements as "$F''(K)\,\mathrm{d}K = 2\delta_1(\mathrm{d}K)$".

This is precisely the purpose of this paper, in general dimension. Bossu, Carr & Papanicolaou (2022) and Bossu (2022) explored multi-asset replication identities for European multi-asset options with absolutely homogeneous payoffs $F(\lambda\mathbf{x}, \lambda k) = |\lambda|\,F(\mathbf{x}, k)$ for $\lambda \in \mathbb{R}^*$, such as[2]

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (\mathbf{w}\cdot\mathbf{x} - k)^+ \nu(\mathrm{d}\mathbf{w}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{2.8}$$

where k is a real parameter and $\nu(\mathrm{d}\mathbf{w})$ is the quantity of vanilla basket calls with basket weights $\mathbf{w}$ spanning the continuum $\mathbf{w} \in \mathbb{R}^d$. Appendix A shows how (2.7) may be rewritten as (2.8) for $d = 1$ under additional conditions. The correspondence between the strong formulation (2.8) of the continuum spanning problem and our weak formulation (2.5) is best seen in the following proposition beginning with a Dirac approximation lemma:

Lemma 2.2. *Let $h_n \in \mathcal{D}(\mathbb{R}^d)$ be the sequence of multiplicatively separable functions with support inside $\{\|\mathbf{x}\|_\infty < \frac{1}{n}\}$ given as $h_n(\mathbf{x}) = c_n \prod_{i=1}^d f_n(x_i)$, where*

$$f_n(x) = \begin{cases} \exp\Big(-\frac{1}{1-n^2x^2}\Big), & |x| < \frac{1}{n}, \\ 0, & |x| \geqslant \frac{1}{n}, \end{cases} \tag{2.9}$$

and c_n is a normalizing factor such that $\int_{\mathbb{R}^d} h_n(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. Then $\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, h_n(\mathbf{x}) g(\mathbf{x}) \to$

[2] In this version of the spanning problem, the cash and asset terms $\alpha + \boldsymbol{\mu}\cdot\mathbf{x}$ are moved to the left-hand side and inside the arbitrary target payoff function $F(\mathbf{x}, k)$ for ease of writing.

$g(\mathbf{0})$ as $n \to \infty$, for any function $g : \mathbb{R}^d \to \mathbb{C}$ which is continuous at the origin.

Proof. $h_n(\mathbf{x})$ is in $\mathcal{D}(\mathbb{R}^d)$ as (normalized) product of $f_n(x_i) \in \mathcal{D}(\mathbb{R})$ (Kanwal 2004, Lemma 2 page 181), and the support of h_n is inside $\{\|\mathbf{x}\|_\infty < \frac{1}{n}\}$ by (2.9). By continuity of g at the origin, for any $\epsilon > 0$, there exists n_ϵ such that $|g(\mathbf{x})-g(\mathbf{0})| \leqslant \epsilon$ holds on $\{\|\mathbf{x}\|_\infty < \frac{1}{n_\epsilon}\}$. Hence, for $n \geqslant n_\epsilon$, $\left|\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\, h_n(\mathbf{x})(g(\mathbf{x}) - g(\mathbf{0}))\right| \leqslant \int_{\{\|\mathbf{x}\|_\infty < \frac{1}{n_\epsilon}\}} \mathrm{d}\mathbf{x}\, h_n(\mathbf{x})|g(\mathbf{x})-g(\mathbf{0}))| \leqslant \epsilon$ since h_n integrates to 1. ∎

Proposition 2.3. *If a solution N to the weak spanning problem* (2.5) *exists in the form of an integration operator on Σ_e against an even measure ν with finite second moment $\int_{\mathbb{R}^d} |\mathbf{w}|^2\nu(\mathrm{d}\mathbf{w}) < \infty$, then the following strong, pointwise representation holds:*

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \nu(\mathrm{d}\mathbf{w}), \qquad (\mathbf{x}, k) \in \mathbb{R}^d \times \mathbb{R}. \tag{2.10}$$

Remark 2.2. For fixed $k \geqslant 0$ the above can be seen as a variation of (2.8) up to a doubling factor. Indeed, substituting $(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ = (\mathbf{w} \cdot \mathbf{x} - k)^+ + (-\mathbf{w} \cdot \mathbf{x} - k)^+$ and splitting the integrand,

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(\mathrm{d}\mathbf{w}) + \int_{\mathbb{R}^d} (-\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(\mathrm{d}\mathbf{w})$$

Substituting $\mathbf{w} \mapsto -\mathbf{w}$ then $\nu(-\mathrm{d}\mathbf{w}) = \nu(\mathrm{d}\mathbf{w})$ into the second integral above,

$$F(\mathbf{x}, k) = 2\int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(\mathrm{d}\mathbf{w}), \qquad \mathbf{x} \in \mathbb{R}^d, k \in \mathbb{R}_+.$$

Proof of Proposition 2.3. Let $\phi_0 = e^{-irk}h(\mathbf{x}) \in \mathcal{S}_0$ (with $r \neq 0$). Substituting $N_{\mathbf{w}} = \nu(\mathrm{d}\mathbf{w})$ into (2.5),

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k F(\mathbf{x}, k)\phi_0(\mathbf{x}, k) = \int_{\mathbb{R}^d} \nu(\mathrm{d}\mathbf{w}) \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k \big(|\mathbf{w} \cdot \mathbf{x}| - |k|\big)^+ \phi_0(\mathbf{x}, k). \tag{2.11}$$

The second moment condition on ν implies

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}|h(\mathbf{x})| \int_{\mathbb{R}^{d+1}} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \nu(\mathrm{d}\mathbf{w})\, \mathrm{d}k &\leqslant \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}|h(\mathbf{x})| \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x})^2 \nu(\mathrm{d}\mathbf{w}) \\ &\leqslant \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}|h(\mathbf{x})||\mathbf{x}|^2 \int_{\mathbb{R}^d} |\mathbf{w}|^2 \nu(\mathrm{d}\mathbf{w}),\end{aligned}$$

which is finite for all $h \in \mathcal{S}(\mathbb{R}^d)$ (Kanwal 2004, Theorem page 141). By dominated convergence left and right in (2.11) (due respectively to (2.4) and the above inequality), (2.11) also holds for r formally set to 0 in ϕ_0. Replacing $\phi_0(\mathbf{x}, k)$ with $e^{-irk}h(\mathbf{x})$ in (2.11), where $r \in \mathbb{R}$ may now be 0, and reordering integrals,

$$\int_{\mathbb{R}} \mathrm{d}k e^{-irk} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} F(\mathbf{x}, k)h(\mathbf{x}) = \int_{\mathbb{R}} \mathrm{d}k e^{-irk} \int_{\mathbb{R}^d} \nu(\mathrm{d}\mathbf{w}) \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \big(|\mathbf{w} \cdot \mathbf{x}| - |k|\big)^+ h(\mathbf{x}).$$

Recognizing Fourier transforms over k,

$$\mathcal{F}_k\Big[\int_{\mathbb{R}^d} F(\mathbf{x},k)h(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big] = \mathcal{F}_k\Big[\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}^d} \nu(\mathrm{d}\mathbf{w})\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\Big].$$

Since the Fourier transform with respect to k is injective on $L^1(\mathbb{R})$ (see Remark C.1), we recover for all $h \in \mathcal{S}(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} F(\mathbf{x},k)h(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}^d} \nu(\mathrm{d}\mathbf{w})\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+, \tag{2.12}$$

$\mathrm{d}k$-almost everywhere. For each k satisfying (2.12), by continuity of $F(\mathbf{x},k)$ and $\int_{\mathbb{R}^d} \nu(\mathrm{d}\mathbf{w})\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+$ in $\mathbf{x}$, taking $h(\mathbf{x}) = h_n(\mathbf{x}-\mathbf{x}_0)$ and $n \to \infty$ by Lemma 2.2, we recover $F(\mathbf{x}_0,k) = \int_{\mathbb{R}^d}(|\mathbf{w}\cdot\mathbf{x}_0| - |k|)^+\nu(\mathrm{d}\mathbf{w})$ for any $\mathbf{x}_0 \in \mathbb{R}^d$. The validity of (2.10) then follows by continuity in k applied to both sides. ∎

If N is not induced by a measure, then integration over k and the action of N do not necessarily commute and the above proof no longer works. Hence, in general, strong representations of the form "$F(\mathbf{x},k) = \big\langle N_{\mathbf{w}}, \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\big\rangle_{\mathbf{w}}, (\mathbf{x},k) \in \mathbb{R}^{d+1}$" cannot be readily obtained from (2.5). Nevertheless, we have the following regularized representation.

Proposition 2.4. *For any linear form N on Σ_e solving (2.5), if $k \mapsto F(\mathbf{y},k)$ and its Fourier transform $r \mapsto \mathcal{F}_k\big[F(\mathbf{y},k)\big](r)$ are both in $L^1(\mathbb{R})$ for each $\mathbf{y} \in \mathbb{R}^d$, and $\mathbf{y} \mapsto \mathcal{F}_k\big[F(\mathbf{y},k)\big](r)$ is continuous on $\mathbb{R}^d$ for each $r \in \mathbb{R}^*$, the following strong, pointwise representation holds $\mathrm{d}k$-almost everywhere for each $\mathbf{x} \in \mathbb{R}^d$:*

$$F(\mathbf{x},k) = \mathcal{F}_r^{-1}\left[\lim_{n\to\infty} \mathbf{1}_{r\neq 0}\left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r)\right\rangle_{\mathbf{w}}\right](k),$$

where h_n is the sequence of Dirac approximation functions given in (2.9), and $\mathbf{1}_{r\neq 0}\langle\ldots(r)\rangle_{\mathbf{w}}$ vanishes at the point $r = 0$ for which the bracket may fail to be defined.

Proof. Let $\phi_0 = e^{-irk}h(\mathbf{x}) \in \mathcal{S}_0$ (with $r \neq 0$). Substituting into (2.5),

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} \mathrm{d}k F(\mathbf{x},k) e^{-irk} = \left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} \mathrm{d}k \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ e^{-irk}\right\rangle_{\mathbf{w}}.$$

Recognizing Fourier transforms and replacing $\mathbf{x}$ with $\mathbf{y}$ on both sides, and k with κ on the right-hand side,

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h(\mathbf{y})\mathcal{F}_k\big[F(\mathbf{y},k)\big](r) = \left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h(\mathbf{y})\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r)\right\rangle_{\mathbf{w}}.$$

Let $\mathbf{x} \in \mathbb{R}^d$. Substituting $h(\mathbf{y}) \equiv h_n(\mathbf{y}-\mathbf{x})$,

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_k\big[F(\mathbf{y},k)\big](r) = \left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r)\right\rangle_{\mathbf{w}},$$

Consequently, taking $n \to +\infty$ in accordance with Lemma 2.2, the limit of the left-hand side exists as $\mathcal{F}_k\big[F(\mathbf{x},k)\big](r)$, and thus on both sides as

$$\mathcal{F}_k\big[F(\mathbf{x},k)\big](r) = \lim_{n\to+\infty}\left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r)\right\rangle_{\mathbf{w}}, \; r \neq 0.$$

As such, the $\mathbb{R}\to\mathbb{R}$ functions $r \mapsto \mathcal{F}_k\big[F(\mathbf{x},k)\big](r)$ and

$$r \mapsto \mathbf{1}_{r\neq 0} \lim_{n\to+\infty}\left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r)\right\rangle_{\mathbf{w}} \tag{2.13}$$

coincide on $\mathbb{R}^*$. By Remark C.1, the inverse Fourier transform of $\mathbb{R} \ni r \mapsto \mathcal{F}_k\big[F(\mathbf{x},k)\big](r)$ exists as $k \mapsto F(\mathbf{x},k)$, hence both functions admit inverse Fourier transforms, which coincide as functions in $L^1(\mathbb{R})$ do we want to mention this in the proposition? and therefore $\mathrm{d}k$-almost everywhere. ∎

3 Solution

Lemma 3.1. *The following identity holds between any two bijectively related test functions φ_e and $h_e = -\frac{1}{2}\mathcal{F}^{-1}\varphi_e$ that are both in $\mathcal{S}_e$:*

$$\psi_e(\mathbf{w}) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h_e(\mathbf{x}) \int_{\mathbb{R}} \mathrm{d}k\, e^{-ik}\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+, \text{ where } \psi_e = \varphi_e - \varphi_e(\mathbf{0}). \tag{3.1}$$

Proof. Let $\varphi_e \in \mathcal{S}_e$. By the Fourier inversion theorem (see Remark C.2), $\varphi_e(\mathbf{w}) = \mathcal{F}[\mathcal{F}^{-1}\varphi_e(\mathbf{x})](\mathbf{w}) = -2\int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\mathbf{w}} h_e(\mathbf{x})\,\mathrm{d}\mathbf{x}$. Because h_e is even,

$$\begin{aligned}\varphi_e(\mathbf{w}) &= -\int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\mathbf{w}} h_e(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{w}} h_e(\mathbf{x})\,\mathrm{d}\mathbf{x} = -2\int_{\mathbb{R}^d} \cos(\mathbf{x}\cdot\mathbf{w}) h_e(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k e^{ik} h_e(\mathbf{x})\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ - 2\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h_e(\mathbf{x}),\end{aligned} \tag{3.2}$$

where we used (2.2) with $r = -1$ in the last step. Substituting $k \mapsto -k$ and

$$-2\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} h_e(\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} (\mathcal{F}^{-1}\varphi_e)(\mathbf{x}) = \mathcal{F}\mathcal{F}^{-1}\varphi_e(\mathbf{0}) = \varphi_e(\mathbf{0}),$$

then rearranging terms yields the required result. ∎

Corollary 3.2. *The function set* $\{\mathbf{w} \mapsto \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ \phi_0(\mathbf{x},k)\ ;\ \phi_0 \in \mathcal{S}_0\}$ *coincides with Σ_e and Problem 2.1 admits at most one solution.*

Proof. The function set is a subset of Σ_e by Lemma 2.1(ii), whereas the converse is provided by Lemma 3.1. Hence any two solutions N and N' to Problem 2.1 coincide on their entire domain Σ_e, i.e. $N = N'$. ∎

Definition 3.1. The payoff function $F(\mathbf{x},k)$ is said to be of class (AH) if it is absolutely homogeneous in $(\mathbf{x},k)$ and even in both $\mathbf{x}$ and k, i.e.

$$F(\lambda\mathbf{x},\lambda k) = |\lambda|\, F(\mathbf{x},k),\ \lambda \in \mathbb{R}^* \text{ , and } F(\mathbf{x},k) = F(\mathbf{x},-k) = F(-\mathbf{x},k),\ (\mathbf{x},k) \in \mathbb{R}^{d+1}.$$

Remark 3.1. Examples of absolutely homogeneous payoffs can be found in the last column in Table 1. By continuity of F in $\mathbf{x}$ or k postulated in Assumption 2.1, the payoff must vanish at the origin, since e.g. $F(\mathbf{0}, 0) = \lim_{n\to\infty} F(\mathbf{0}, 1/n) = \lim_{n\to\infty} 1/n\, F(\mathbf{0}, 1) = 0$.

3.1 General Weak Solution

Theorem 3.3. *Problem 2.1 admits a solution N, or equivalently T via* (2.6)*, if and only if the payoff $F(\mathbf{x}, k)$ is of class* (AH)*, in which case the solution is unique and given as*

$$\begin{aligned}&\left\langle N^F_{\mathbf{w}}, \psi_e(\mathbf{w})\right\rangle_{\mathbf{w}} = \left\langle T^F_{\mathbf{w}}, \varphi_e(\mathbf{w})\right\rangle_{\mathbf{w}} = -\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \mathcal{F}^{-1}\varphi_e(\mathbf{x})\mathcal{F}_k[F(\mathbf{x}, k)](1) \\ &= -\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{w} e^{i\mathbf{w}\cdot\mathbf{x}}\varphi_e(\mathbf{w})\int_{\mathbb{R}}\mathrm{d}k e^{-ik}F(\mathbf{x}, k),\ \varphi_e \in \mathcal{S}_e, \psi_e = \varphi_e - \varphi_e(\mathbf{0}) \in \Sigma_e.\end{aligned} \tag{3.3}$$

Remark 3.2. In general, the integrals above may not commute, so that the solution T (or N) is not necessarily "of function-type": see Section 3.2.

Proof of Theorem 3.3. The uniqueness of a solution to Problem 2.1 was established in Corollary 3.2. In addition, observe that $\mathcal{F}^{-1}\varphi_e \in \mathcal{S}_e$ for any $\varphi_e \in \mathcal{S}_e$ (Milton 1974, Theorem 3.3 and 4.1). By Assumption 2.1, the right-hand side of (3.3) is therefore well defined for any $\varphi_e \in \mathcal{S}_e$, and it is linear in φ_e.

($\Rightarrow$) Suppose that a solution T to (2.5)-(2.6) exists, and let $\phi_0 \in \mathcal{S}_0$ and $\lambda \in \mathbb{R}^*$. The function $(\mathbf{x}, k) \mapsto \phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k)$ is again in $\mathcal{S}_0$ and may be substituted into (2.5) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k F(\mathbf{x}, k)\phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k) = \\ &\qquad\left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k)\right\rangle_{\mathbf{w}}.\end{aligned} \tag{3.4}$$

By linear change of variable $(\mathbf{x}, k) \mapsto (\lambda\mathbf{x}, \lambda k)$ on both sides, and absolute homogeneity of $\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+$ in $(\mathbf{x}, k)$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k|\lambda|^{-d-1}F(\lambda\mathbf{x}, \lambda k)\phi_0(\mathbf{x}, k) \\ &\qquad= \left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k|\lambda|^{-d-1}\big(|\mathbf{w}\cdot\lambda\mathbf{x}| - |\lambda k|\big)^+\phi_0(\mathbf{x}, k)\right\rangle_{\mathbf{w}} \\ &\qquad= |\lambda|^{-d}\left\langle N_{\mathbf{w}}, \int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k\big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+\phi_0(\mathbf{x}, k)\right\rangle_{\mathbf{w}} \\ &\qquad= |\lambda|^{-d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}k F(\mathbf{x}, k)\phi_0(\mathbf{x}, k),\end{aligned}$$

where we substituted (2.5) in the last step. Multiplying both sides of the above by

$|\lambda|^{d+1}$,

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k F(\lambda\mathbf{x}\lambda k)\phi_0(\mathbf{x}, k) = |\lambda| \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k F(\mathbf{x}, k)\phi_0(\mathbf{x}, k), \tag{3.5}$$

which holds for any $\phi_0 \in \mathcal{S}_0$, and also for r formally set to 0 in ϕ_0 by dominated convergence in (3.5). By the same reasoning that concluded the proof of Proposition 2.3, we obtain $F(\lambda\mathbf{x}, \lambda k) = |\lambda| F(\mathbf{x}, k)$ for all $(\mathbf{x}, k) \in \mathbb{R}^{d+1}$. In addition, $(\mathbf{x}, k) \mapsto \phi_0(\mathbf{x}, -k)$ and $(\mathbf{x}, k) \mapsto \phi_0(-\mathbf{x}, k)$ are also in $\mathcal{S}_0$, and we similarly obtain $F(\mathbf{x}, k) = F(\mathbf{x}, -k) = F(-\mathbf{x}, k)$ for $(\mathbf{x}, k) \in \mathbb{R}^{d+1}$. Therefore, F is of class (AH) as required.
($\Leftarrow$) Suppose that F is of class (AH), and let $\psi_e \in \Sigma_e$, $\varphi_e = \psi_e - \lim_{\infty} \psi$, and $r \in \mathbb{R}^*$. Substituting $F(\mathbf{x}, k) = |r|\, F(r^{-1}\mathbf{x}, r^{-1}k)$ by absolute homogeneity in (3.3),

$$\left\langle N_{\mathbf{w}}^F, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} = -\frac{1}{2} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k |r| F(r^{-1}\mathbf{x}, r^{-1}k) e^{-ik} \mathcal{F}^{-1}\varphi_e(\mathbf{x}).$$

By linear change of variables $(\mathbf{y}, \kappa) = (r^{-1}\mathbf{x}, r^{-1}k)$,

$$\left\langle N_{\mathbf{w}}^F, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} = -\frac{1}{2} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} \int_{\mathbb{R}} \mathrm{d}\kappa |r|^{d+2} F(\mathbf{y}, \kappa) e^{-ir\kappa} \mathcal{F}^{-1}\varphi_e(r\mathbf{y}). \tag{3.6}$$

By Lemma 3.1 with $h_e = -\frac{1}{2}\mathcal{F}^{-1}\varphi_e$, the left-hand side satisfies

$$\begin{aligned} \left\langle N_{\mathbf{w}}^F, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} &= \left\langle N_{\mathbf{w}}^F, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \mathrm{d}k\, e^{-ik} h_e(\mathbf{x}) \big(|\mathbf{w}\cdot\mathbf{x}| - |k|\big)^+ \right\rangle_{\mathbf{w}} \\ &= \left\langle N_{\mathbf{w}}^F, |r|^{d+2} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} \int_{\mathbb{R}} \mathrm{d}\kappa\, e^{-ir\kappa} h_e(r\mathbf{y}) \big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+ \right\rangle_{\mathbf{w}}, \end{aligned} \tag{3.7}$$

where we applied the change of variable $(\mathbf{x}, k) \mapsto (\mathbf{y}, \kappa) = (r^{-1}\mathbf{x}, r^{-1}k)$ in the last step. Connecting with (3.6) and dividing both sides by $|r|^{d+2}$,

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} \int_{\mathbb{R}} \mathrm{d}\kappa F(\mathbf{y}, \kappa) e^{-ir\kappa} h_e(r\mathbf{y}) = \left\langle N_{\mathbf{w}}^F, \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} \int_{\mathbb{R}} \mathrm{d}\kappa e^{-ir\kappa} h_e(r\mathbf{y}) \big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+ \right\rangle_{\mathbf{w}}, \tag{3.8}$$

which holds for any $h_e(r\mathbf{y}) = -\frac{1}{2}\mathcal{F}^{-1}\varphi_e(r\mathbf{y}) \in \mathcal{S}_e$. In addition, for any h_o odd in $\mathcal{S}(\mathbb{R}^d)$, the even-odd product $F(\mathbf{y}, \kappa) h_o(r\mathbf{y})$ is odd in $\mathbf{y}$, so that

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} \int_{\mathbb{R}} \mathrm{d}\kappa F(\mathbf{y}, \kappa) e^{-ir\kappa} h_o(r\mathbf{y}) = \int_{\mathbb{R}} \mathrm{d}\kappa e^{-ir\kappa} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y} F(\mathbf{y}, \kappa) h_o(r\mathbf{y}) = 0$$

as the d$\mathbf{y}$ integral over $\mathbb{R}^d$ of an odd function vanishes. Consequently, replacing h_e by any odd function h_o to the left-hand side of (3.8) yields 0, and the equation thus holds for any $h \in \mathcal{S}(\mathbb{R}^d)$, so that N^F solves (2.5) as required. ∎

Corollary 3.4. *If F is of class* (AH) *and satisfies Assumption 2.1 with condition* (2.4) *strengthened into*

$$\left(\int_{\mathbb{R}} \mathrm{d}k F(\mathbf{x}, k) e^{-ik} \right) \mathrm{d}\mathbf{x} \textit{ is a distribution on } \mathcal{S}(\mathbb{R}^d), \tag{3.9}$$

then T^F is a distribution on $\mathcal{S}_e$.

Proof. If $(\int \mathrm{d}kF(\mathbf{x},k)e^{-ik})\,\mathrm{d}\mathbf{x}$ is a distribution on $\mathcal{S}(\mathbb{R}^d)$ and hence on $\mathcal{S}_e$ (Milton 1974, Theorem 4.4), because $\mathcal{F}^{-1}$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^d) \supset \mathcal{S}_e$ (see Remark C.2), the right-hand side of (3.3) is continuous with respect to $\varphi_e \in \mathcal{S}_e$, hence T^F given by (3.3) defines a distribution on $\mathcal{S}_e$. ∎

3.2 Connection With Radon Transform

The following proposition and its corollary reconciles our Theorem 3.3 with the approach of Bossu (2022) based on Radon transforms and the Fourier slice inversion formula.

Proposition 3.5. **(i)** *If F of class* (AH) *is $L^1(\mathbb{R})$ in k and such that $\mathcal{F}_k[F(\mathbf{x},k)](1)$ is $L^1(\mathbb{R}^d)$ in $\mathbf{x}$, then the corresponding solution T^F to Problem 2.1 is of function-type $T^F_{\mathbf{w}} = f(\mathbf{w})\,\mathrm{d}\mathbf{w}$, where*

$$f(\mathbf{w}) = -\frac{1}{2}\mathcal{F}_{\mathbf{x}}^{-1}\big[\mathcal{F}_k[F(\mathbf{x},k)](1)\big](\mathbf{w}). \tag{3.10}$$

(ii) *If, in addition, $F(\mathbf{x},k)$ is n times continuously differentiable against k and all the derivatives are $L^1(\mathbb{R})$ in k, then we may replace F by $(-i)^n\partial^n_{k^n}F$ in* (3.10)*, e.g., for $n=2$,*

$$f(\mathbf{w}) = \frac{1}{2}\mathcal{F}_{\mathbf{x}}^{-1}\big[\mathcal{F}_k[\partial^2_{k^2}F(\mathbf{x},k)](1)\big](\mathbf{w}). \tag{3.11}$$

Proof. All integrands being now in L^1, for any $\psi_e \in \Sigma_e$ and φ_e such that $\psi_e = \varphi_e - \varphi_e(\mathbf{0})$, we may reorder integrals in (3.3) as

$$\begin{aligned}\big\langle T^F_{\mathbf{w}}, \varphi_e(\mathbf{w})\big\rangle_{\mathbf{w}} &= -\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{w}\varphi_e(\mathbf{w})\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}e^{i\mathbf{w}\cdot\mathbf{x}}\int_{\mathbb{R}}\mathrm{d}ke^{-ik}F(\mathbf{x},k)\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{w}\varphi_e(\mathbf{w})\mathcal{F}_{\mathbf{x}}^{-1}\big[\mathcal{F}_k[F(\mathbf{x},k)](1)\big](\mathbf{w}).\end{aligned}$$

This proves (i), while (ii) follows by derivative rule of Fourier transforms applied to $\mathcal{F}_k[F(\mathbf{x},k)](1)$. ∎

Remark 3.3. If, on top of the assumptions of Proposition 3.5(ii), $\frac{1}{2}\partial^2_{k^2}F(\mathbf{x},k)$ is the Radon transform $\int_{\mathbb{R}^d}\delta_{\mathbf{x}\cdot\mathbf{y}=k}(\mathrm{d}\mathbf{y})f(\mathbf{y})$ of some function $f \in L^1(\mathbb{R}^d)$ along Cartesian parameters[3] $(\mathbf{x},k) \in \mathbb{R}^{d+1}$, then

$$\begin{aligned}\frac{1}{2}\mathcal{F}_{\mathbf{x}}^{-1}\big[\mathcal{F}_k[\partial^2_{k^2}F(\mathbf{x},k)](1)\big](\mathbf{w}) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}e^{i\mathbf{w}\cdot\mathbf{x}}\int_{\mathbb{R}}\mathrm{d}ke^{-ik}\int_{\mathbb{R}^d}\delta_{\mathbf{x}\cdot\mathbf{y}=k}(\mathrm{d}\mathbf{y})f(\mathbf{y})\\ = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}e^{i\mathbf{w}\cdot\mathbf{x}}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{y}e^{-i\mathbf{x}\cdot\mathbf{y}}f(\mathbf{y}) &= \mathcal{F}_{\mathbf{x}}^{-1}\big[\mathcal{F}f(\mathbf{x})\big](\mathbf{w}) = f(\mathbf{w}),\end{aligned}$$

which is (3.11). Hence in this case (3.11) is tantamount to saying that $f(\mathbf{w})$ is the inverse Radon transform of $\frac{1}{2}\partial^2_{k^2}F(\mathbf{x},k)$, in agreement with the solution derived in

[3] See (Rubin 2015, page 156–157) for a similar calculation along cylindrical parameters.

(Bossu 2022, Section 3) for the related strong spanning problem (2.8) when $\nu(\mathrm{d}\mathbf{w}) = f(\mathbf{w})\,\mathrm{d}\mathbf{w}$.

4 Explicit Examples

In this section, we provide explicit solutions to the continuum distributional spanning equation (2.5) (i.e. Problem 2.1), encompassing the function, measure and distribution types that may be encountered based on the regularity of the target payoff F.

4.1 Smooth Payoff Function

Let $G_d(\mathbf{x}, k) = |\mathbf{x}|e^{-\frac{k^2}{|\mathbf{x}|^2}}$, for $\mathbf{x} \neq \mathbf{0}$, and $G_d(\mathbf{0}, k) = 0$, a smooth payoff[4] function of joint class $\mathcal{C}^\infty(\mathbb{R}^{d+1})$, (AH), and $L^1(\mathbb{R})$ in k.

Proposition 4.1. *For $F = G_d$, Problem 2.1 admits a unique solution $T^{G_d}_{\mathbf{w}} = g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$ on $\mathcal{S}_e$ and $N^{G_d}_{\mathbf{w}} = g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$ on Σ_e, where $g_d(\mathbf{w}) = -\frac{\sqrt{\pi}}{2}\mathcal{F}^{-1}_{\mathbf{x}}\big[|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}\big](\mathbf{w})$. In particular,*

$$\begin{gathered} g_1(w) = e^{-w^2}(2w^2 - 1), \qquad g_2(w_1, w_2) = \frac{2}{\sqrt{\pi}} e^{-|\mathbf{w}|^2}(|\mathbf{w}|^2 - 1), \\ g_3(w_1, w_2, w_3) = \frac{1}{\pi} e^{-|\mathbf{w}|^2}(2|\mathbf{w}|^2 - 3). \end{gathered} \tag{4.1}$$

Proof. By Kammler (2008, Example page 139), $\mathcal{F}_k[G_d(\mathbf{x}, k)](1) = \sqrt{\pi}|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}$, which is $L^1(\mathbb{R}^d)$ in $\mathbf{x}$. Hence $F = G_d$ satisfies the assumptions of Proposition 3.5, and the corresponding solution T^{G_d} is of function-type $T^{G_d}_{\mathbf{w}} = g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$ with

$$g_d(\mathbf{w}) = -\frac{1}{2}\mathcal{F}^{-1}_{\mathbf{x}}\big[\mathcal{F}_k[G_d(\mathbf{x}, k)](1)\big](\mathbf{w}) = -\frac{\sqrt{\pi}}{2}\mathcal{F}^{-1}_{\mathbf{x}}\left[|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}\right](\mathbf{w}).$$

The expressions for g_1, g_2 and g_3 given in (4.1) can be obtained by Hankel transform computations (Iosevich & Liflyand 2014, Theorem 4.1 page 93). In addition,

$$\int_{\mathbb{R}^d} g_d(\mathbf{w})\,\mathrm{d}\mathbf{w} = \mathcal{F}g_d(\mathbf{0}) = -\frac{\sqrt{\pi}}{2}|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}\Big|_{\mathbf{x}=\mathbf{0}} = 0. \tag{4.2}$$

Consequently, for any pair (φ_e, ψ_e) in correspondence (2.1), we have $\int_{\mathbb{R}} \varphi_e(\mathbf{w}) g_d(\mathbf{w})\,\mathrm{d}\mathbf{w} = \int_{\mathbb{R}} \psi_e(\mathbf{w}) g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$, wherefore $N^{G_d}_{\mathbf{w}} = g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$ holds on Σ_e as required. ∎

Corollary 4.2. *The following spanning formula holds, for the solution g_d given in Proposition 4.1:*

$$G_d(\mathbf{x}, k) = \int_{\mathbb{R}^d} (|\mathbf{x} \cdot \mathbf{w}| - |k|)^+ g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}, \ (\mathbf{x}, k) \in \mathbb{R}^{d+1}. \tag{4.3}$$

[4]While this payoff does not trade, it provides a valuable example where the solution is of function-type and available in closed form.

Proof. Since $|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}} \in \mathcal{S}(\mathbb{R}^d)$, $g_d(\mathbf{w})$ is also in $\mathcal{S}(\mathbb{R}^d)$ and thus $\int_{\mathbb{R}^d} |\mathbf{w}|^2 |g_d(\mathbf{w})|\,\mathrm{d}\mathbf{w} < \infty$ (Kanwal 2004, Theorem page 141), yielding that the even measure $N_{\mathbf{w}} = g_d(\mathbf{w})\,\mathrm{d}\mathbf{w}$ has finite second moment. (4.3) then follows from Proposition 2.3. ∎

Remark 4.1 (sanity check). For $(x,k) \in \mathbb{R}^* \times \mathbb{R}$, we may verify for $d=1$ that

$$\begin{aligned}
\int_{\mathbb{R}} (|xw| - |k|)^+ g_1(w)\,\mathrm{d}w &= \int_{\frac{|k|}{|x|}}^{\infty} (|x|w - |k|) g_1(w)\,\mathrm{d}w + \int_{-\infty}^{-\frac{|k|}{|x|}} (-|x|w - |k|) g_1(w)\,\mathrm{d}w \\
&= 2\int_{\frac{|k|}{|x|}}^{\infty} (|x|w - |k|) g_1(w)\,\mathrm{d}w = \int_{\frac{|k|}{|x|}}^{\infty} (|x|w - |k|) e^{-w^2}(2w^2-1)\,\mathrm{d}w \\
&= -e^{-w^2}(2w^2|x| + |x| - 2kw)\Big|_{\frac{|k|}{|x|}}^{\infty} = |x|e^{-\frac{k^2}{x^2}} = G_1(x,k),
\end{aligned}$$

while $\int_{\mathbb{R}} (|xw| - |k|)^+ g_1(w)\,\mathrm{d}w = G_1(x,k) = 0$ for $x = 0$. In dimension $d=2$ and $d=3$, (4.3) may be verified by change of variables to polar coordinates.

4.2 Dispersion Call

For $d \geqslant 1$, let

$$C_d(\mathbf{x},k) = \left(\sum_{j=1}^{d} |x_j| - |k|\right)^+ = \left(\,\|\mathbf{x}\|_1 - |k|\,\right)^+, \qquad (\mathbf{x},k) \in \mathbb{R}^{d+1} \tag{4.4}$$

denote the absolutely homogeneous variation of the ℓ^1 dispersion call payoff listed in Table 1 page 3.

Definition 4.1. Let $S_e \subset \mathcal{S}_e$ be the Schwartz subspace generated by functions of the form $\varphi(\mathbf{w}) + \varphi(-\mathbf{w})$ where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is multiplicatively separable as $\varphi(\mathbf{w}) = \prod_{i=1}^{d} \varphi_i(w_i)$, $\varphi_i \in \mathcal{S}(\mathbb{R}), i = 1, \ldots, d$. (For $d = 1, S_e = \mathcal{S}_e$.)
(i) By Proposition B.2 and linearity, we define the following maps consider: linear forms:

$$\left\langle \mathrm{p.v.}_{w_j}\left(\frac{1}{w_j - c}\prod_{j'\neq j, j'=1}^{d} \mathrm{p.v.}_{w_{j'}}\frac{1}{w_{j'} - w_j}\right), \prod_{i=1}^{d}\varphi_i(w_i)\right\rangle_{\mathbf{w}} := \\ \fint_{-\infty}^{\infty} \frac{\varphi_j(w_j)\,\mathrm{d}x_j}{w_j - c} \prod_{j'\neq j, j'=1}^{d} \fint_{-\infty}^{\infty} \frac{\varphi_{j'}(w_{j'})\,\mathrm{d}w_j}{w_{j'} - w_j}, \qquad j = 1, \ldots, d,$$

extended by linearity to S_e;
(ii) We denote by T^d the following linear form on S_e:

- for $d = 1$, $T^1_w = \frac{1}{2}\delta_1(\mathrm{d}w) + \frac{1}{2}\delta_{-1}(\mathrm{d}w)$;
- for $d = 2$,

$$
\begin{aligned}
T^2_{\mathbf{w}} = \frac{1}{4\pi^2}\,\mathrm{p.v.}_{w_1}\left(\left(\frac{1}{w_1+1}+\frac{1}{w_1-1}\right)\mathrm{p.v.}_{w_2}\left(\frac{1}{w_1+w_2}+\frac{1}{w_1-w_2}\right)\right)\\
+\frac{1}{4\pi^2}\,\mathrm{p.v.}_{w_2}\left(\left(\frac{1}{w_2+1}+\frac{1}{w_2-1}\right)\mathrm{p.v.}_{w_1}\left(\frac{1}{w_2+w_1}+\frac{1}{w_2-w_1}\right)\right);
\end{aligned}
\tag{4.5}
$$

- for $d \geqslant 3$ odd,

$$
T^d_{\mathbf{w}} = \frac{(-1)^{(d-1)/2}}{2^d\pi^{d-1}}\sum_{j=1}^{d}\left(\delta_1(\mathrm{d}w_j)+\delta_{-1}(\mathrm{d}w_j)\right)\prod_{j'\neq j,j'=1}^{d}\mathrm{p.v.}_{w_{j'}}\left(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\right);
$$

- for $d \geqslant 2$ even,

$$
T^d_{\mathbf{w}} = \frac{(-1)^{d/2-1}}{(2\pi)^d}\sum_{j=1}^{d}\mathrm{p.v.}_{w_j}\left(\left(\frac{1}{w_j-1}+\frac{1}{w_j+1}\right)\prod_{j'\neq j,j'=1}^{d}\mathrm{p.v.}_{w_{j'}}\left(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\right)\right).
$$

Proposition 4.3. *For $F = C_d$, Problem 2.1 admits a unique solution T^{C_d}, which is a distribution on $\mathcal{S}_e$, namely the continuous extension of $T^d - \delta_{\mathbf{0}}$ to $\mathcal{S}_e$. In particular,*

$$
\left\langle T^{C_d}_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}} = \left\langle T^d_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}} - \phi_e(\mathbf{0}), \qquad \phi_e \in S_e. \tag{4.6}
$$

Proof. By equivalence of finite-dimensional norms, there exist positive constants B, M such that

$$
\left|\int_{\mathbb{R}}\mathrm{d}k C_d(\mathbf{x},k)e^{-ik}\right| \leqslant \int_{\mathbb{R}}\mathrm{d}k(\|\mathbf{x}\|_1-|k|)^+ = 2\,\|\mathbf{x}\|_1^2 \leqslant B(1+|\mathbf{x}|)^M,\ \mathbf{x}\in\mathbb{R}^d. \tag{4.7}
$$

Consequently, the function $\mathbf{x} \mapsto \int \mathrm{d}kC_d(\mathbf{x},k)e^{-ik}$ induces a distribution on $\mathcal{S}(\mathbb{R}^d)$ (Friedlander 1998, Eqn. (8.3.2) page 97), and by Corollary 3.4 the corresponding unique solution T^{C_d} given by (3.3) defines a distribution on $\mathcal{S}_e$. The subspace of $\mathcal{D}(\mathbb{R}^d)$ generated by functions of the form $\prod_{i=1}^d \varphi_i(x_i)$, $\varphi_i \in \mathcal{D}(\mathbb{R}), i = 1,\ldots,d$, is dense in $\mathcal{D}(\mathbb{R}^d)$ (Kanwal 2004, Lemma 2 page 181), which itself is dense in $\mathcal{S}(\mathbb{R}^d)$ (Kanwal 2004, Remark page 140). As a result, S_e is dense in $\mathcal{S}_e$. Hence it only remains to prove (4.6).

For $\psi_e \in \Sigma_e$ and φ_e such that $\psi_e = \varphi_e - \varphi_e(\mathbf{0})$, (3.3) yields

$$
\begin{aligned}
\left\langle N^{C_d}_{\mathbf{w}}, \psi_e(\mathbf{w})\right\rangle_{\mathbf{w}} &= -\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}}\mathrm{d}kC_d(\mathbf{x},k)e^{-ik}\mathcal{F}^{-1}\varphi_e(\mathbf{x})\\
&= -\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\mathcal{F}^{-1}\varphi_e(\mathbf{x})\mathcal{F}_k\big[\,\|\mathbf{x}\|_1-|k|\,\big](1).
\end{aligned}
\tag{4.8}
$$

Substituting (C.2) with $c = \|\mathbf{x}\|_1$ and $r = 1$,

$$\begin{aligned}\left\langle N^{C_d}_{\mathbf{w}}, \psi_e(\mathbf{w})\right\rangle_{\mathbf{w}} &= \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\mathcal{F}^{-1}\varphi_e(\mathbf{x})(\cos\|\mathbf{x}\|_1 - 1) \\ = -\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\mathcal{F}^{-1}\varphi_e(\mathbf{x}) &+ \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}e^{-i\|\mathbf{x}\|_1}\mathcal{F}^{-1}\varphi_e(\mathbf{x}) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}e^{i\|\mathbf{x}\|_1}\mathcal{F}^{-1}\varphi_e(\mathbf{x}).\end{aligned} \tag{4.9}$$

As shown in Section D, for $\varphi_e = \phi_e \in S_e$,

$$\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}e^{-i\|\mathbf{x}\|_1}\mathcal{F}^{-1}\phi_e(\mathbf{x}) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}e^{i\|\mathbf{x}\|_1}\mathcal{F}^{-1}\phi_e(\mathbf{x}) = \left\langle T^d_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}}. \tag{4.10}$$

Plugging $\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\mathcal{F}^{-1}\varphi_e(\mathbf{x}) = \varphi_e(\mathbf{0})$ and (4.10) into (4.9),

$$\left\langle T^d_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}} - \phi_e(\mathbf{0}) = \left\langle N^{C_d}_{\mathbf{w}}, \psi_e(\mathbf{w})\right\rangle_{\mathbf{w}} = \left\langle T^{C_d}_{\mathbf{w}}, \varphi_e(\mathbf{w})\right\rangle_{\mathbf{w}} = \left\langle T^{C_d}_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}}, \tag{4.11}$$

which proves (4.6). ∎

Remark 4.2 (sanity check). For $d = 1$ and $T^1 = \frac{1}{2}(\delta_1 + \delta_{-1})$, (4.11) yields

$$\begin{aligned}\left\langle N^{C_1}_w, \psi_e(w)\right\rangle_w &= \left\langle T^1_w, \phi_e(w)\right\rangle_w - \phi_e(0) = \left\langle \frac{1}{2}\big(\delta_1(\mathrm{d}w) + \delta_{-1}(\mathrm{d}w)\big), \phi_e\right\rangle_w - \phi_e(0) \\ &= \left\langle \frac{1}{2}\big(\delta_1(\mathrm{d}w) + \delta_{-1}(\mathrm{d}w)\big), \phi_e - \phi_e(0)\right\rangle_w = \frac{1}{2}\big(\psi_e(1) + \psi_e(-1)\big),\end{aligned}$$

hence $N^{C_1}_w = \frac{1}{2}\big(\delta_1(\mathrm{d}w) + \delta_{-1}(\mathrm{d}w)\big)$ on Σ_e (recalling that $S_e = \mathcal{S}_e$ for $d = 1$). The corresponding trivial spanning identity

$$\begin{aligned}(|x| - |k|)^+ &= \left\langle N^{C_1}_w, (|wx| - |k|)^+\right\rangle_w \\ &= \frac{1}{2}\big((|x| - |k|)^+ + (|-x| - |k|)^+\big),\ (x, k) \in \mathbb{R}^2,\end{aligned} \tag{4.12}$$

is also a consequence of Proposition 2.3 since N^{C_1} is a measure-type distribution with finite second moment $\frac{1}{2}\int_{\mathbb{R}} w^2\big(\delta_1(\mathrm{d}w) + \delta_{-1}(\mathrm{d}w)\big) = 1$.

In higher dimension $d \geqslant 2$, the distribution T^{C_d} is not of measure-type and there is no strong spanning representation readily following from (2.5). However, by (C.2), $\mathcal{F}_\kappa\left[\big(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\big)^+\right](r) = \frac{2-2\cos(r\mathbf{w}\cdot\mathbf{y})}{r^2}$ which is continuous in $\mathbf{y}$, hence the strong representation from Proposition 2.4 is applicable and can be turned into the following more explicit representation derived for $d = 2$ only for ease of writing:

Proposition 4.4. *In dimension $d = 2$ the dispersion call payoff has the strong repre-*

sentation

$$
\begin{aligned}
&(|x_1| + |x_2| - |k|)^+ \\
&= -\mathcal{F}_r^{-1}\Bigg[\lim_{n\to\infty} \mathbf{1}_{r\neq 0}\Bigg(\frac{2}{r^2} + \frac{1}{4\pi^2}\fint_{-\infty}^{\infty}\Big(\frac{\mathrm{d}w_1}{w_1+1} + \frac{\mathrm{d}w_1}{w_1-1}\Big)\fint_{-\infty}^{\infty}\Big(\frac{\mathrm{d}w_2}{w_1+w_2} + \frac{\mathrm{d}w_2}{w_1-w_2}\Big)\times \\
&\qquad\qquad \Big(\mathcal{F}_\kappa\Big[\int_{\mathbb{R}^2}\mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\left(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\right)^+\Big](r) - \frac{2}{r^2}\Big) \\
&\quad + \frac{1}{4\pi^2}\fint_{-\infty}^{\infty}\Big(\frac{\mathrm{d}w_2}{w_2+1} + \frac{\mathrm{d}w_2}{w_2-1}\Big)\fint_{-\infty}^{\infty}\Big(\frac{\mathrm{d}w_1}{w_2+w_1} + \frac{\mathrm{d}w_1}{w_2-w_1}\Big)\times \\
&\qquad\qquad \Big(\mathcal{F}_\kappa\Big[\int_{\mathbb{R}^2}\mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\left(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\right)^+\Big](r) - \frac{2}{r^2}\Big)\Bigg)\Bigg](k),
\end{aligned}
\tag{4.13}
$$

$\mathrm{d}k$*-almost everywhere for each* $\mathbf{x} \in \mathbb{R}^2$*, where* h_n *is the sequence of Dirac approximation functions given in* (2.9)*.*

Proof. For each $\mathbf{y} \in \mathbb{R}^2$, the function $k \mapsto C_2(\mathbf{y}, k)$ has compact support and is thus in $L^1(\mathbb{R})$, while $r \mapsto \mathcal{F}_k\big[C_2(\mathbf{y}, k)\big](r)$ is in $L^1(\mathbb{R})$ by (C.2) and is continuous in $\mathbf{y} \in \mathbb{R}^2$ for each $r \in \mathbb{R}^*$. Therefore, by Proposition 2.4,

$$
C_2(\mathbf{x}, k) = \mathcal{F}_r^{-1}\left[\lim_{n\to\infty} \mathbf{1}_{r\neq 0}\left\langle N_{\mathbf{w}}^{C_2}, \int_{\mathbb{R}^d}\mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x})\mathcal{F}_\kappa\left[\left(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|\right)^+\right](r)\right\rangle_{\mathbf{w}}\right](k),
\tag{4.14}
$$

where $\langle N_{\mathbf{w}}^{C_2}, \psi_e\rangle = \langle T_{\mathbf{w}}^{C_2}, \varphi_e\rangle$ for any $\psi_e \in \Sigma_e$ and $\varphi_e \in \mathcal{S}_e$ such that $\psi_e = \varphi_e - \varphi_e(\mathbf{0})$. Let $r \in \mathbb{R}^*, h \in \mathcal{S}(\mathbb{R}^2)$, and $\psi_e(\mathbf{w}) = \int_{\mathbb{R}^2}\mathrm{d}\mathbf{y}h(\mathbf{y})\mathcal{F}_\kappa\left[(|\mathbf{w}\cdot\mathbf{y}| - |\kappa|)^+\right](r)$ which is in Σ_e by Lemma 2.1. Substituting (C.2), we have $\psi_e(\mathbf{w}) = \frac{2}{r^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{y}h(\mathbf{y})(1 - \cos(r\mathbf{w}\cdot\mathbf{y}))$. By Riemann-Lebesgue's lemma, $\lim_{\boldsymbol{\infty}}\psi_e = \frac{2}{r^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{y}h(\mathbf{y})$, so that $\varphi_e(\mathbf{w}) = \frac{2}{r^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{y}h(\mathbf{y})\cos(r\mathbf{w}\cdot\mathbf{y})$, which is in $\mathcal{S}(\mathbb{R}^2)$ as Fourier transform of an $\mathcal{S}(\mathbb{R}^2)$ function. If $h(\mathbf{y}) = f_1(y_1)f_2(y_2)$ is multiplicatively separable, expanding $\cos(r\mathbf{w}\cdot\mathbf{y}) = \cos(rw_1y_1)\cos(rw_2y_2) - \sin(rw_1y_1)\sin(rw_2y_2)$ and separating integrals yields

$$
\begin{aligned}
\varphi_e(\mathbf{w}) = \frac{1}{r^2}&\int_{\mathbb{R}}\mathrm{d}y_1 f_1(y_1)\cos(rw_1y_1)\int_{\mathbb{R}}\mathrm{d}y_2 f_2(y_2)\cos(rw_2y_2) \\
&- \frac{1}{r^2}\int_{\mathbb{R}}\mathrm{d}y_1 f_1(y_1)\sin(rw_1y_1)\int_{\mathbb{R}}\mathrm{d}y_2 f_2(y_2)\sin(rw_2y_2),
\end{aligned}
$$

which proves that $\varphi_e \in S_e$. By Proposition 4.3 with $\phi_e \equiv \varphi_e$,

$$\begin{aligned}
&\left\langle N^{C_2}, \psi_e \right\rangle = \left\langle T^{C_2}_{\mathbf{w}}, \varphi_e \right\rangle = \left\langle T^2, \phi_e \right\rangle - \phi_e(0) \\
&= \left\langle T^2_{\mathbf{w}}, \frac{2}{r^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w}\cdot\mathbf{y}) \right\rangle_{\mathbf{w}} - \frac{2}{r^2}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h(\mathbf{y}) \quad = -\frac{2}{r^2}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h(\mathbf{y}) \\
&\quad + \frac{1}{4\pi^2} \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_1}{w_1+1} + \frac{\mathrm{d}w_1}{w_1-1} \right) \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_2}{w_1+w_2} + \frac{\mathrm{d}w_2}{w_1-w_2} \right) \frac{2}{r^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w}\cdot\mathbf{y}) \\
&\quad + \frac{1}{4\pi^2} \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_2}{w_2+1} + \frac{\mathrm{d}w_2}{w_2-1} \right) \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_1}{w_2+w_1} + \frac{\mathrm{d}w_1}{w_2-w_1} \right) \frac{2}{r^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w}\cdot\mathbf{y}).
\end{aligned}$$

Substituting $\frac{2}{r^2}\cos(r\mathbf{w}\cdot\mathbf{y}) = \frac{2}{r^2}\big(1-(1-\cos(r\mathbf{w}\cdot\mathbf{y}))\big) = \frac{2}{r^2} - \mathcal{F}_\kappa\left[(|\mathbf{w}\cdot\mathbf{y}|-|\kappa|)^+\right](r)$ above, and plugging the result into (4.14) with $h(\mathbf{y}) = h_n(\mathbf{y}-\mathbf{x})$,

$$\begin{aligned}
C_2(\mathbf{x},k) =& \mathcal{F}_r^{-1}\Bigg[\lim_{n\to\infty} \mathbf{1}_{r\neq 0} \Bigg(-\frac{2}{r^2}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x}) \\
&+ \frac{1}{4\pi^2} \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_1}{w_1+1} + \frac{\mathrm{d}w_1}{w_1-1} \right) \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_2}{w_1+w_2} + \frac{\mathrm{d}w_2}{w_1-w_2} \right) \times \\
&\qquad \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x}) \left(\frac{2}{r^2} - \mathcal{F}_\kappa\left[(|\mathbf{w}\cdot\mathbf{y}|-|\kappa|)^+\right](r) \right) \\
&+ \frac{1}{4\pi^2} \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_2}{w_2+1} + \frac{\mathrm{d}w_2}{w_2-1} \right) \fint_{-\infty}^{\infty} \left(\frac{\mathrm{d}w_1}{w_2+w_1} + \frac{\mathrm{d}w_1}{w_2-w_1} \right) \times \\
&\qquad \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x}) \left(\frac{2}{r^2} - \mathcal{F}_\kappa\left[(|\mathbf{w}\cdot\mathbf{y}|-|\kappa|)^+\right](r) \right) \Bigg) \Bigg](k).
\end{aligned}$$

Substituting $\int_{\mathbb{R}^2} \mathrm{d}\mathbf{y} h_n(\mathbf{y}-\mathbf{x}) = 1$, and taking Fourier transforms with respect to κ out of the $\mathrm{d}\mathbf{y}$ integrals yields the required result. ∎

The delicate nature of the explicit distributions identified in this section shows that the solution formula (3.3) for N^F is nontrivial and requires case-by-case analysis. If the target payoff F is not smooth, the formula is unlikely to be numerically tractable, and alternative spanning methods must be developed. The following section investigates the use of neural networks for discrete spanning, which may be more practical in terms of applications.

5 Benefits of Unrestricted Neural Network Spanning Versus Other Discrete Spanning Strategies

Continuum spanning, such as done in previous sections, shows that perfect replication can in principle be achieved with an infinite-width ReLU network. In practice, only a finite number of vanilla basket calls may be traded, and discrete spanning may be viewed as more relevant for industry applications. The use of neural networks is natural for this purpose since the discrete spanning error (1.2) corresponds to the error function of a one-hidden-layer feedforward neural network. In this section, the strike variable k of the target payoff $F(\mathbf{x}, k)$ is viewed as a fixed parameter, and we simply write $F(\mathbf{x})$ for ease of notation.

For two positive integers d, l let $\mathcal{NN}_{d,l}$ denote the family of functions that take a vector $\mathbf{x} \in \mathbb{R}^d$ as input and return a value in $\mathbb{R}$ through the sequential mapping

$$\mathbb{R}^d \ni \mathbf{x} \overset{\tilde{F}}{\longmapsto} \alpha + \boldsymbol{\mu} \cdot \mathbf{x} + \sum_{i=1}^{l} \nu_i \left(\eta_i (\mathbf{w}^{(i)} \cdot \mathbf{x} - k_i) \right)^+ \in \mathbb{R}\,, \tag{5.1}$$

where $\boldsymbol{\mu}, \mathbf{w}^{(i)} = [w_1^{(i)}, \ldots w_d^{(i)}]^T$ are vector versions of the quantities introduced in (1.2), while the sign of each parameter η_i determines whether a basket call or put is used. The sign of each parameter ν_i determines whether a long or short position is taken in basket option i with strike $k_i > 0$, in quantity $|\nu_i \eta_i|$.

Remark 5.1. When spanning with basket calls only, i.e. $\eta_i > 0$, approximating $F(\mathbf{x})$ by $\tilde{F}(\mathbf{x}) \in \mathcal{NN}_{d,l}$ is equivalent to a discretization of (2.8) for fixed $k > 0$. Original: that would only be required to hold for a single value of $k > 0$ (in other words, ν in (2.8) would be allowed to depend on k). This can be seen by rewriting (5.1) as

$$\tilde{F}(\mathbf{x}) = \alpha + \boldsymbol{\mu} \cdot \mathbf{x} + \sum_{i=1}^{l} \underbrace{\nu_i \eta_i \frac{k_i}{k}}_{=:\nu_i'} \Big(\underbrace{\frac{k}{k_i} \mathbf{w}^{(i)}}_{=:\mathbf{w}'^{(i)}} \cdot \mathbf{x} - k \Big)^+$$

which is a discretization of (2.8) with explicit affine terms to the right hand side.

The family of functions (5.1) corresponds to an unrestricted one-hidden-layer residual neural network [5] with ReLU activation function architecture characterized by the set of trainable parameters

$$\theta = (\mathbf{W}, \boldsymbol{\mu}, \mathbf{k}, \alpha, \boldsymbol{\nu}, \boldsymbol{\eta}),$$

where $\mathbf{W} = [\mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(l)}]^T \in \mathbb{R}^{l \times d}$ stores the $w_j^{(i)}$ in matrix form, while $\mathbf{k} \in (\mathbb{R}_+^*)^l, \boldsymbol{\nu} \in \mathbb{R}^l$, and $\boldsymbol{\eta} \in \mathbb{R}^l$ store the k_i, ν_i and η_i in vector form. We seek an approximation

$$\hat{F} \in \underset{\tilde{F} \in \mathcal{NN}_{d,l}}{\arg\min}\, \widehat{\mathbb{E}} \left[(F(\mathbf{x}) - \tilde{F}(\mathbf{x}))^2 \right], \tag{5.2}$$

where $\widehat{\mathbb{E}}[.] = \frac{[.]_1 + \cdots + [.]_n}{n}$ denotes the sample mean over n observations drawn from values of $\mathbf{x}$ that may be deterministically or randomly sampled.

This section summarizes our main practical results compared to two other spanning strategies that may more naturally come to mind but turn out to perform poorly:

- Spanning with single-asset payoffs: This "marginal" approach is attractive to practitioners because single-asset vanilla options are more liquid and can often be traded

[5] At the ImageNet 2015 competition, deep residual neural networks outperformed other methods such as gradient boosting and nonresidual deep neural network architectures.

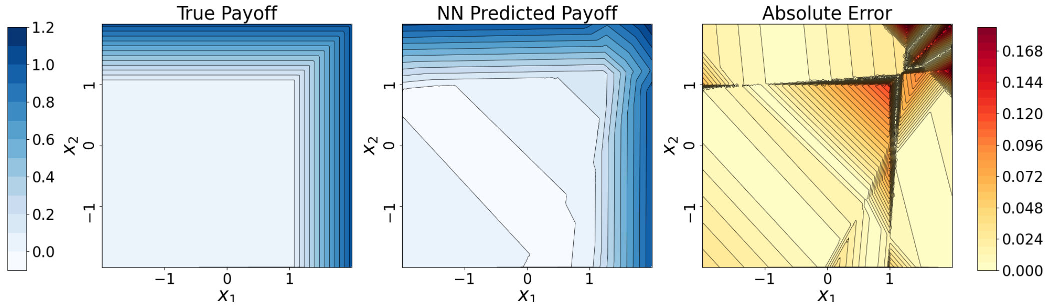
on exchanges;

- Spanning with predetermined basket payoffs, i.e. with fixed basket weights $w_1^{(i)}, \ldots, w_d^{(i)}$ and strikes k_i: This level of control may be beneficial to practitioners in order to define a tractable universe of spanning instruments. Another benefit of this approach is that it can be solved using classical linear regression techniques.
- Spanning with long-only basket payoffs, i.e. the components of the basket weights $\mathbf{w}^{(i)}$ of each basket payoff $1 \leqslant i \leqslant l$ are positive.

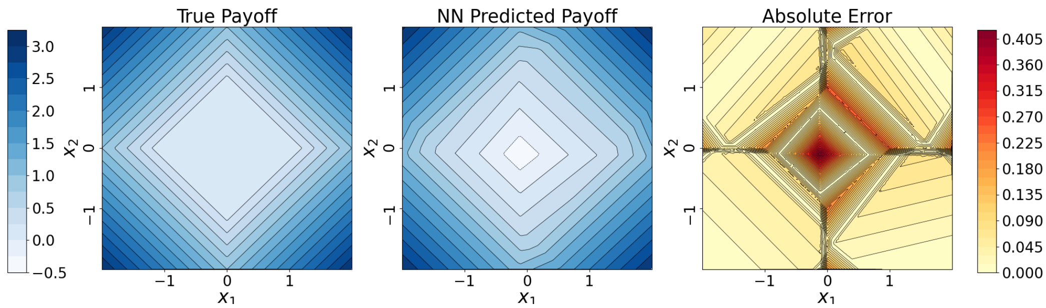
5.1 Spanning Metrics

We use two metrics to assess the distance between the target payoff F and a predictor $\tilde{F} \in \mathcal{NN}_{d,l}$: mean squared error $\text{MSE} = \widehat{\mathbb{E}}\big[(F(\mathbf{x}) - \tilde{F}(\mathbf{x}))^2\big]$ for the loss function, and mean absolute error $\text{MAE} = \widehat{\mathbb{E}}\big|F(\mathbf{x}) - \tilde{F}(\mathbf{x})\big|$ for reporting. The choice of MAE for reporting is motivated by its ease of financial interpretation as average absolute dollar mismatch between the target payoff and the spanning portfolio. However, MAE is not differentiable at certain points and as such it is not a good choice of loss function for gradient calculation, which is why we opted for MSE loss.

In Figure 2 we use contour plots to report the target payoff F, its approximation $\hat{F} \in \mathcal{NN}_{2,l}$ and the pointwise absolute spanning error $|F(x_1, x_2) - \hat{F}(x_1, x_2)|$ for best-of call and dispersion call payoffs. We can see that the predicted surface provided by the unrestricted neural network spanning strategy (NN) fits the target payoff well. However, higher spanning errors are observed around areas where the target payoff is nondifferentiable.



(a) Best-of call $F(x_1, x_2) = \big(\max(x_1, x_2) - 1\big)^+$. MAE $= 0.02$.



(b) Dispersion call $F(x_1, x_2) = \big(|x_1| + |x_2| - 1\big)^+$. MAE $= 0.051$.

Figure 2: Contour plots of the target payoff (*left*), NN prediction (*center*) and absolute spanning error (*right*) for the (a) best-of call, (b) dispersion call payoffs.

5.2 Limitations of Spanning with Single-Asset Payoffs

When spanning the target payoff $F(\mathbf{x})$ with single-asset vanilla payoffs only, equations (5.1)-(5.2) become

$$\min_{\alpha,\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\eta},\mathbf{k},\mathbf{E}} \widehat{\mathbb{E}}\left[\left(F(\mathbf{x}) - \alpha - \boldsymbol{\mu}\cdot\mathbf{x} - \sum_{i=1}^{l} \nu_i \left(\eta_i(\mathbf{e}_i \cdot \mathbf{x} - k_i)\right)^+\right)^2\right], \tag{5.3}$$

where $\mathbf{e}_i \in \{0,1\}^d$ is a "one-hot" vector with all coefficients equal to 0, except the one corresponding to the selected asset, which is 1 (the index of which remains free for optimization), and $\mathbf{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_l]^T \in \{0,1\}^{l\times d}$ is the corresponding matrix. We solve this optimization problem numerically by Adam stochastic gradient descent (Kingma & Ba 2015) with a restricted neural network architecture, for the best-of call and worst-of put payoffs on 2 to 5 underlying assets (see Table 1). Single-asset spanning does provide some risk reduction compared to the unhedged case for which all parameters are zero and the MAE corresponds to the payoff average absolute value. However, Figure 3 shows that the spanning error of this strategy is substantially higher than that of our core unrestricted NN approach explained in Eqn. (5.1). Mathematically, this is hardly surprising given that single-asset option spanning attempts to reproduce a "joint distribution" (the best-of or worst-of payoff) with "marginal distributions" only (the single-asset payoffs).

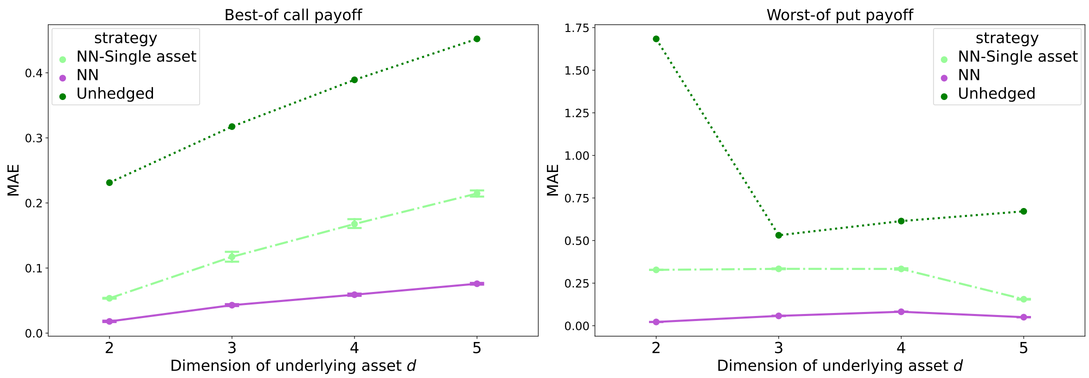


Figure 3: MAE comparison of the single-asset and unrestricted NN spanning strategies for $d = 2, \ldots, 5$ assets: best-of call (*left*) and worst-of put (*right*). Error bars are 95% confidence intervals.

5.3 Limitations of Spanning with Predetermined Basket Payoffs

When spanning the target payoff $F(\mathbf{x})$ with predetermined basket option payoffs, the strikes k_i, call/put selectors η_i, and basket weights $\mathbf{w}^{(i)}$ of each basket payoff $1 \leqslant i \leqslant l$ are fixed and the spanning problem takes the simpler form

$$\min_{\alpha,\boldsymbol{\mu},\boldsymbol{\nu}} \widehat{\mathbb{E}}\left[\left(F(\mathbf{x}) - \alpha - \boldsymbol{\mu}\cdot\mathbf{x} - \sum_{i=1}^{l} \nu_i \left(\eta_i(\mathbf{w}^{(i)}\cdot\mathbf{x} - k_i)\right)^+\right)^2\right], \tag{5.4}$$

which is a classic linear least-squares regression problem. Due to parameter redundancy we choose unit strikes $k_i = 1$, and we also set selectors $\eta_i = 1$. In this case the explicit solution of (5.4) is given by the regression coefficients $\widehat{\boldsymbol{\beta}} = \left[\boldsymbol{\mu}^\top, \boldsymbol{\nu}^\top\right] \in \mathbb{R}^{d+l}$ and constant $\hat{\alpha} \in \mathbb{R}$ with

$$\widehat{\boldsymbol{\beta}} = \left(\widehat{\mathrm{Var}}\,(\mathbf{z})\right)^{-1} \widehat{\mathrm{Cov}}\,(\mathbf{z}, F(\mathbf{x}))\,, \qquad \hat{\alpha} = \widehat{\mathbb{E}}\,[F(\mathbf{x})] - \widehat{\boldsymbol{\beta}}^\top \widehat{\mathbb{E}}\,[\mathbf{z}]\,,$$

where $\mathbf{z} = \left[x_1, \ldots, x_d, (\mathbf{w}^{(1)}\cdot\mathbf{x} - 1)^+, \ldots, (\mathbf{w}^{(l)}\cdot\mathbf{x} - 1)^+\right]^\top \in \mathbb{R}^{d+l}$ is the vector of explanatory variables (underlying assets and basket payoffs), $\widehat{\mathrm{Cov}}$ is the sample covariance operator, and $\widehat{\mathrm{Var}}\,(\mathbf{z})$ is the sample covariance matrix which must be nonsingular. In practice, we found this calculation to be numerically unstable due to conditioning issues: basket payoffs tend to overlap, making the columns of $\mathbf{z}$ loosely dependent, particularly for large l. This issue is further compounded when the sampling of $\mathbf{x}$ (and thus $\mathbf{z}$) is sparse. To circumvent this practical difficulty, we used singular value decomposition (SVD) (Golub & Van Loan 2013, Theorems 2.5.2 and 5.5.1).

Figure 4 reports the spanning error for the best-of call on 2 to 5 underlying assets, together with 95% confidence intervals over 30 different runs[6] that were obtained using the following methods:

[6] A *run* is defined as a new training routine with the same fixed weights $\mathbf{w}$ for methods (a) and (c), and with new random weights $\mathbf{w}$ for methods (b) and (d).

(a) SVD with regular grid sampling of basket weights $\mathbf{w}$ (Regular+SVD);
(b) SVD with i.i.d. uniform random sampling of $\mathbf{w}$ (Uniform+SVD);
(c) Stochastic gradient descent with regular grid sampling of $\mathbf{w}$ (LS-GD);
(d) Unrestricted neural network approach with free $\mathbf{w}$ initialized with i.i.d. uniform random sampling (NN).

We can see that all fixed-weights methods (a) to (c) resulted in substantial spanning errors compared to (d) unrestricted NN, except perhaps in dimension $d = 2$ where the error magnitude is smaller. Remarkably enough, increasing the number l of basket payoffs does not materially reduce the spanning error for any method (a) to (d), which suggests that only a limited number of basket payoffs may be needed to obtain a satisfactory hedge of the best-of call. Finally, SVD methods (a) and (b) often perform poorly: when sampling $\mathbf{w}$ along a regular grid, the spanning error (a) even "exploded" in dimensions $d = 3, 4, 5$ due to ill-conditioning of the design matrix generated by observations of $\mathbf{z}$, as shown in inner panels.

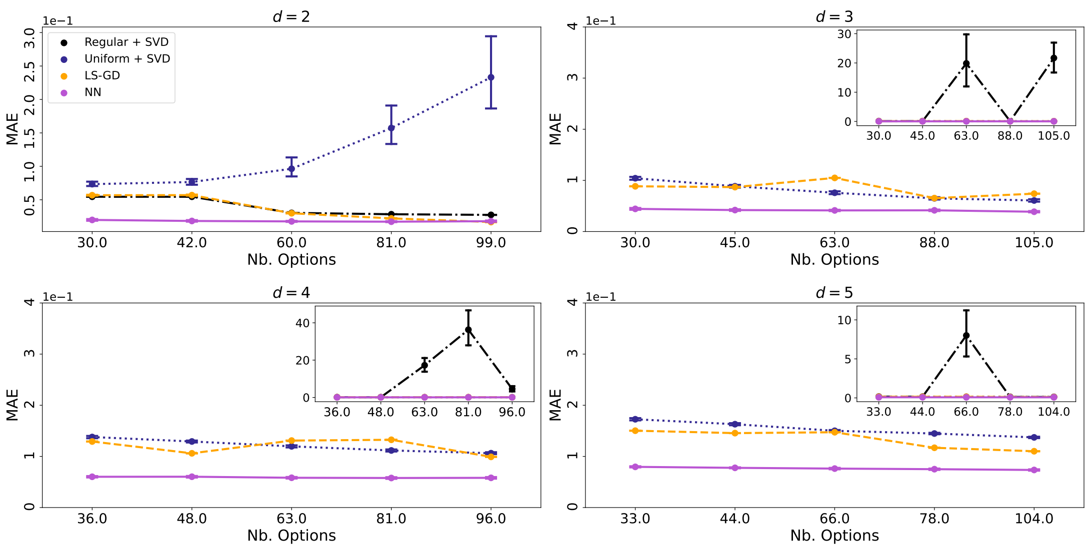


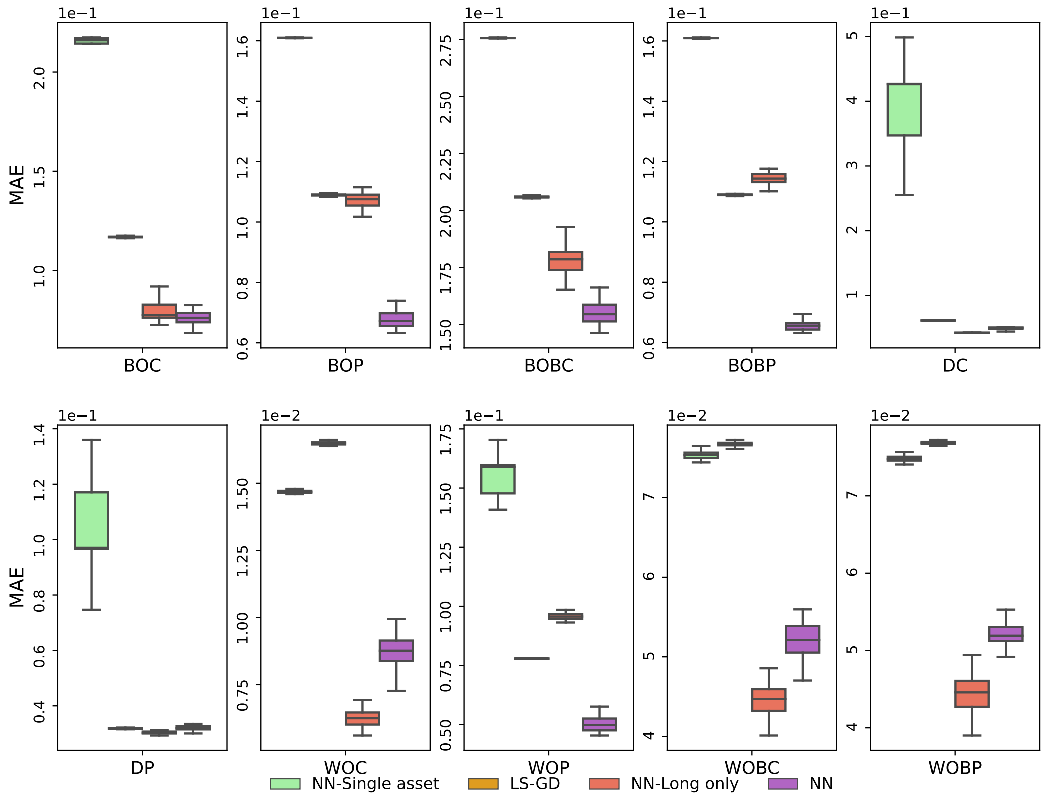
Figure 4: MAE with 95% confidence intervals for the best-of call on $d = 2$ to 5 underlying assets for methods (a) to (d), as a function of the number of basket options l. Large errors obtained with (a) are reported in inside panels for $d = 3, 4, 5$ for readability.

It is worth noting that, from a neural network architecture perspective, the spanning problem (5.4) corresponds to a particular category of extreme learning machines or ELMs (Huang, Zhu & Siew 2004, 2006). ELMs are known to be universal approximators, but they typically require a very large number l of hidden units to achieve satisfactory performance. In the context of payoff spanning, this suggests that only a large number l of predetermined basket payoffs in methods (a) to (c) would be able to approach the performance of the unrestricted NN method (d), which would be impractical for hedge execution.

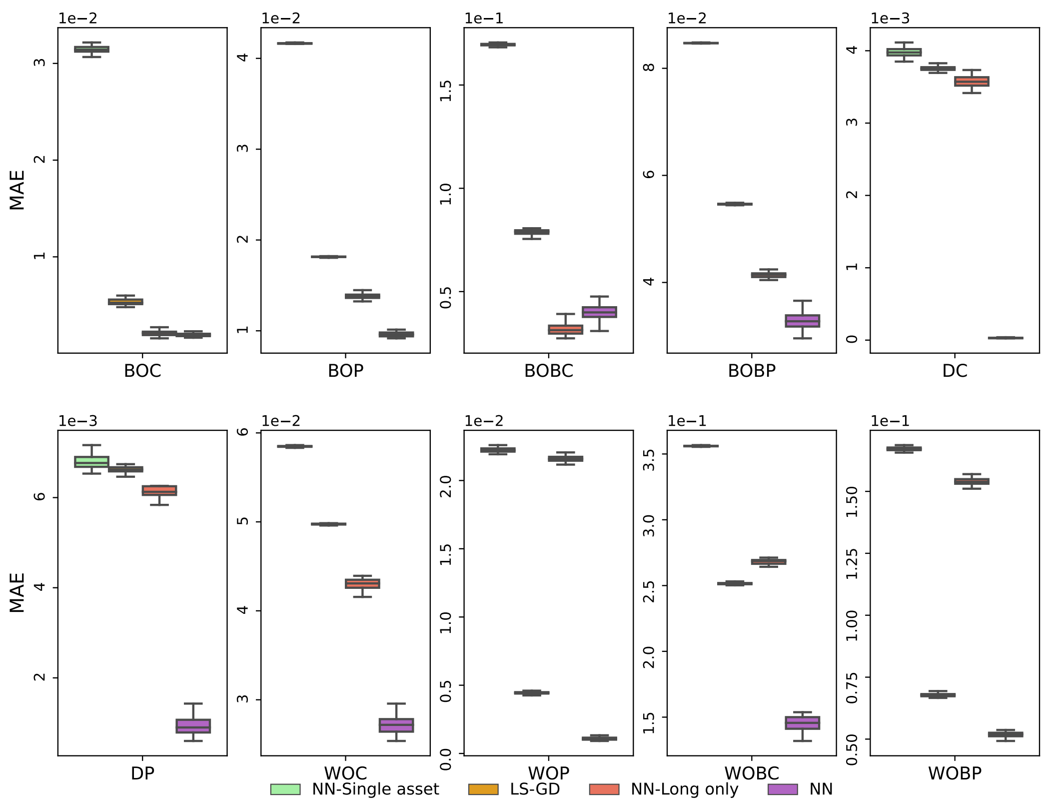
5.4 Summary of Restricted and Unrestricted Spanning Results

Figure 5 reports the performance of four spanning strategies in dimension $d = 5, 20$ and 50 using 50 different network initialization: (i) Adam training of single asset portfolio (NN-Single asset) as per Section 5.2, (ii) least-squares approach with gradient descent (LS-GD) which is further as per Section 5.3, (iii) Adam training of long only basket portfolio (NN-Long only), and (iv) unrestricted neural network with Adam training (NN).

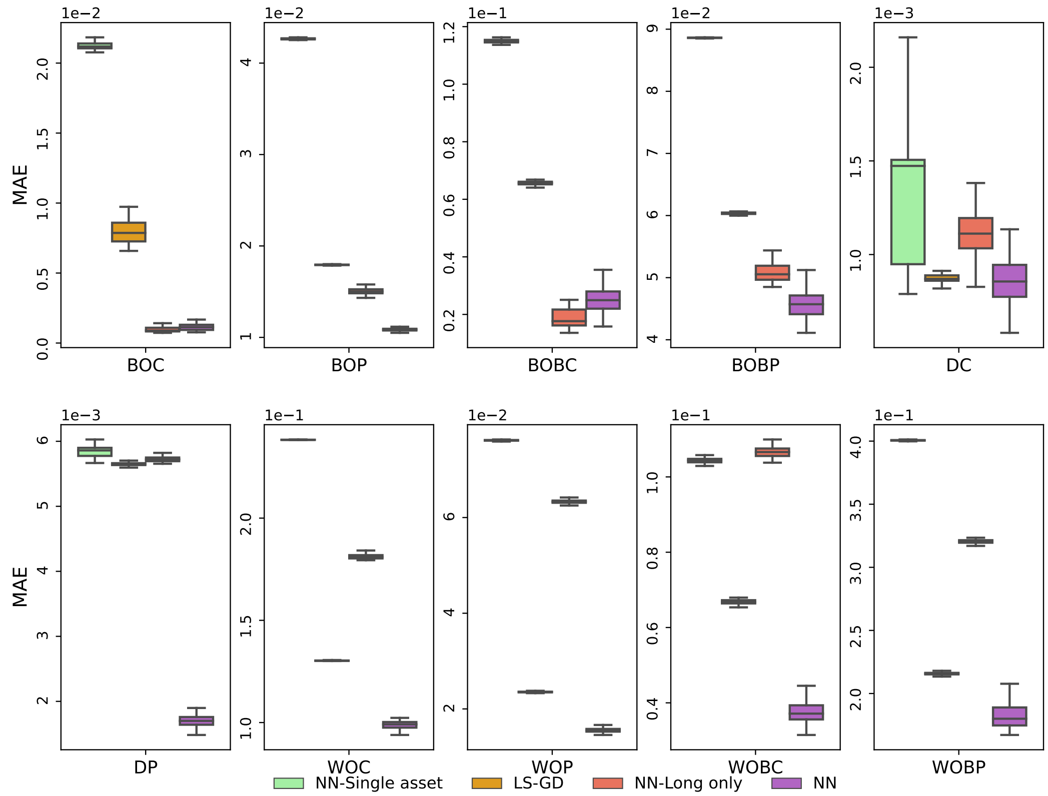
We can see that the spanning portfolios suggested by the neural network give satisfying MAE results in most cases. Unrestricted NN outperforms all other strategies in terms of MAE and standard deviation: MAE increases with the underlying asset dimension d but remains fairly low. The low standard deviation figures also signals that unrestricted NN is the most stable among the four strategies.



(a) $d = 5$ and $l = 78$.



(b) $d = 20$ and $l = 410$.



(c) $d = 50$ and $l = 808$.

Figure 5: Average MAE and 95% error bars over 50 runs by spanning strategy and target payoff in dimension $d = 5, 20$ and 50.

5.5 Stability Issues

In linear regression models, the loss function is convex and thus easier to optimize. In contrast, it is nonconvex in neural network parameterizations such as (5.1)-(5.2): optimization is more difficult, algorithms such as Adam method typically converge to different local minima for different initializations, yet loss values usually remain small (Choromanska, Henaff, Mathieu, Arous & LeCun 2015). We observed this phenomenon in our study: the optimal neural network parameters vary with each training, but we obtained persistently small and stable errors. Financially, this means that spanning performance remains strong but the particular optimal hedge identified based on a particular training set does not have a stable interpretative meaning with respect to the target option.

Figures 6 and 7 show how the optimizer solution for the dispersion call approaches the theoretical solution that we derived in Proposition 4.3 in dimensions $d = 1, 2$. Our results are remarkably consistent with the accumulation of basket call quantities ν_i predicted by theory. In dimension $d = 1$ our numerical results perfectly match the exact solution (4.12) (see also Remark 4.2) after we process the learning with an ℓ^2 regularization (Figure 6, right panel.) In dimension $d = 2$ our numerical results after

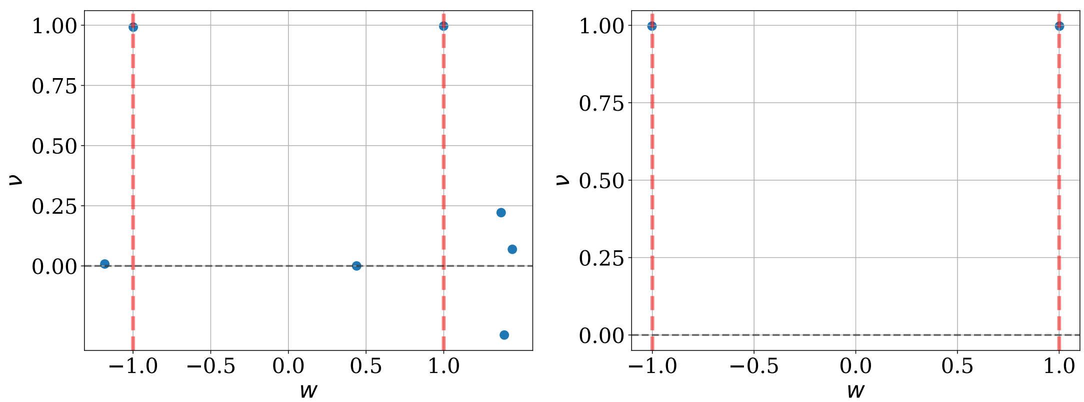
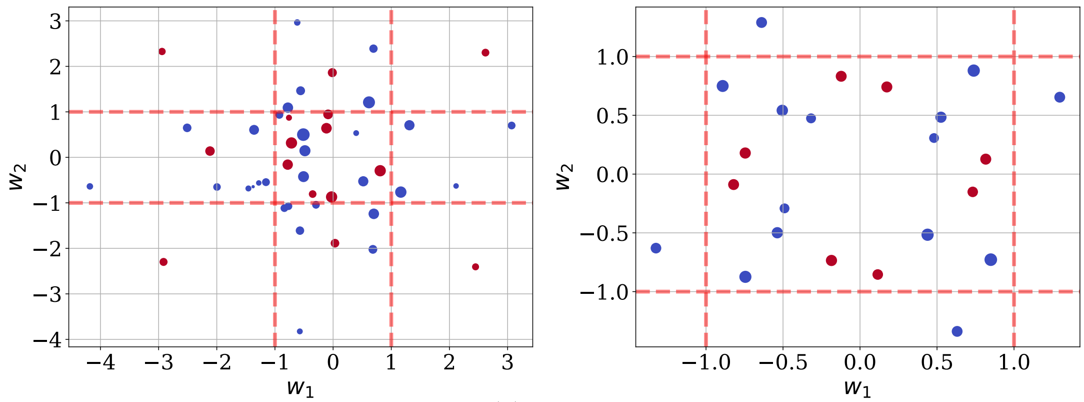
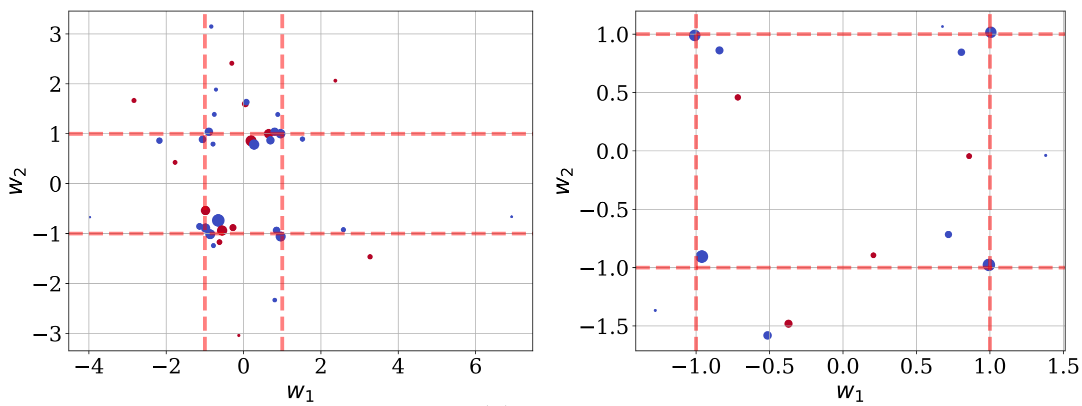


Figure 6: Scatter plots of the optimal basket call quantity ν_i against the optimal basket weight $w^{(i)}$ predicted by (5.1)-(5.2) for the single-asset dispersion call payoff $F(x) = (|x| - 1)^+$: (*left*) without regularization; (*right*) with regularization. The training asset price x is sampled in $[-2, 2]$. The basket call strikes are set to $k_i = 1$.

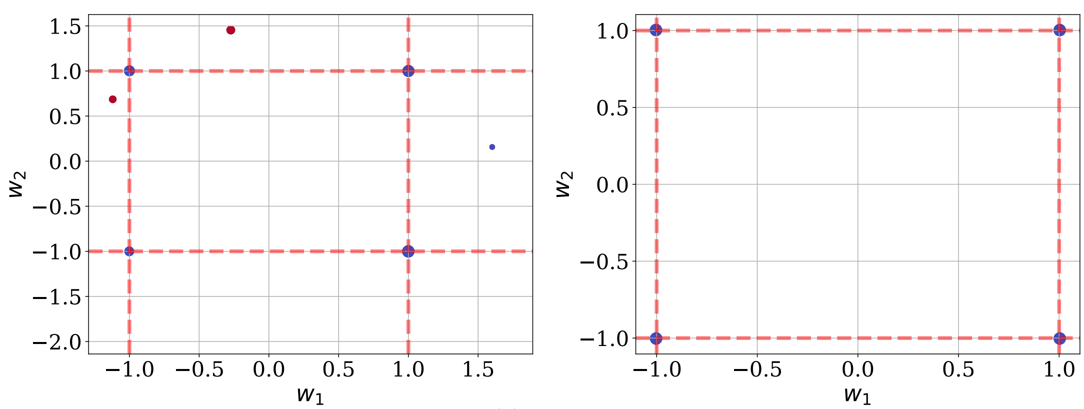
regularization are consistent with the accumulation of basket calls around discontinuity points $w_1 = \pm 1$ and $w_2 = \pm 1$ predicted by the theoretical formula (4.5). (Figure 7, right panel).



(a) $k = 1$



(b) $k = 1.5$



(c) $k = 2$

Figure 7: Scatter plots of the optimal basket call weights $(w_1^{(i)}, w_2^{(i)})$ predicted by (5.1)-(5.2) for the two-asset dispersion call with payoff $F(x_1, x_2) = (|x_1| + |x_2| - k)^+$ for $k = 1, 1.5, 2$: (*left*) without regularization; (*right*) with regularization.
The training asset prices $\mathbf{x}$ are sampled in $[-2, 2]^2$. The blue and red points represent respectively long ($\nu_i > 0$) and short ($\nu_i < 0$) positions while point sizes reflect absolute quantities $|\nu_i|$.

6 Conclusion and Perspectives

Identifying static hedges for exotic option payoffs is important for both theory and practice. Theorem 3.3 expands on existing continuum replication theory of European option payoffs with vanilla options to formulate a general and rigorous solution of the continuum spanning problem. As an application, Proposition 4.3 derives the continuum solution replicating the industry ℓ^1 dispersion call. In addition to the derivation of explicit solutions for other absolutely homogeneous payoffs such as best-of and worst-of, other formulations of the continuum spanning problem are open for future research, for example by letting ν in (1.1) depend on k and/or integrating the spanning portfolio over k, which may allow solutions to exist for non-homogeneous payoffs.

Leveraging the parallel between vanilla basket calls and ReLU functions, we also examined how neural networks can be used to numerically solve the corresponding discrete spanning problem and identify finite static hedges, in comparison to other restricted spanning strategies and optimization schemes such as least-squares SVD. Our empirical study suggests that our unrestricted NN approach yields superior results in terms of static hedging error for any dimension 2 to 50. We expect this approach to be of great practical interest for the exotic derivatives industry, particularly if combined with delta-hedging of the residual payoff mismatch, which could be investigated in future research.

A Correspondence between Carr-Madan spanning and basket call spanning in dimension $d = 1$

Proposition A.1. *In dimension $d = 1$ and for $k > 0$, the Carr-Madan spanning formula* (2.7) *of a twice differentiable payoff $F(x) \equiv F(x, k)$ such that $F(\lambda x, \lambda k) = \lambda F(x, k)$ for all $\lambda > 0$, and*[7] *$\partial_x F(0, k) = \partial^2_{x^2} F(0, k) = 0$, can be rewritten as* (2.8) *(for $x \geqslant 0$), with*

$$\nu(\mathrm{d}w) = \frac{1}{w^3}(\partial^2_{x^2} F)\left(\frac{1}{w}, 1\right) \mathrm{d}w. \tag{A.1}$$

Proof. By change of variable $K \mapsto w = k/K$ in (2.7),

$$\begin{aligned} F(x) &= \int_0^\infty \frac{k}{w^2}\left(x - \frac{k}{w}\right)^+ (\partial^2_{x^2} F)\left(\frac{k}{w}, k\right) \mathrm{d}w = \int_0^\infty \frac{k}{w^3}\,(wx-k)^+\,(\partial^2_{x^2} F)\left(\frac{k}{w}, k\right) \mathrm{d}w \\ &= \int_{-\infty}^\infty \frac{k}{w^3}\,(wx-k)^+\,(\partial^2_{x^2} F)\left(\frac{k}{w}, k\right) \mathrm{d}w,\ x \in \mathbb{R}_+, \end{aligned} \tag{A.2}$$

where we used in the last step that $(wx - k)^+ = 0$ for $w < 0 < \frac{k}{x}$. If $F(\lambda x, \lambda k) = \lambda F(x, k)$ holds for $\lambda > 0$, then $\partial^2_{x^2} F(\lambda x, \lambda k) = \lambda^{-1} \partial^2_{x^2} F(x, k)$, and

$$\frac{k}{w^3}(\partial^2_{x^2} F)\left(\frac{k}{w}, k\right) = \frac{1}{w^3}(\partial^2_{x^2} F)\left(\frac{1}{w}, 1\right),\ k > 0. \tag{A.3}$$

[7] The affine term in (2.7) is thereby removed for consistency with (2.8).

Hence for $x \geqslant 0$ (2.8) is satisfied with $\nu(dw)$ as per (A.1). ∎

Example A.1. Consider the one-dimensional payoff $F(x,k) = G_1(x,k) = \sqrt{x^2}e^{-\frac{k^2}{x^2}}, k > 0, x \in \mathbb{R}_+$ (see Propositions 4.1-4.2). We have

$$\partial_x G_1(x,k) = \mathbf{1}_{x \neq 0} \frac{e^{-\frac{k^2}{x^2}}(2k^2 + x^2)}{x\sqrt{x^2}}, \; \partial^2_{x^2} G_1(x,k) = \mathbf{1}_{x \neq 0} \frac{2k^2 e^{-\frac{k^2}{x^2}}(2k^2 - x^2)}{x^4\sqrt{x^2}}$$

(both understood as 0 for $x = 0$). Thus $G_1(0,k) = \partial_x G_1(0,k) = 0$. The Carr-Madan spanning formula for G_1 then reads

$$G_1(x,k) = \int_0^\infty (x - K) \frac{2k^2 e^{-\frac{k^2}{K^2}}(2k^2 - K^2)}{K^4\sqrt{K^2}} \, \mathrm{d}K.$$

Besides (A.1) yields, for $k > 0, x \in \mathbb{R}_+$,

$$G_1(x,k) = \int_{-\infty}^{\infty} (wx - k)^+ 2e^{-w^2}(2w^2 - 1) \, \mathrm{d}w$$

(as $(wx - k)^+ = 0$ for $w < 0$), which is a representation of the form (2.8).

B Cauchy Principal Values

The Cauchy principal value is a method to assign a rational value to certain improper integrals that would otherwise be undefined in the Lebesgue-Stieltjes sense (Estrada & Kanwal 2000, Section 1.5), (King 2009, Section 2.4).

Definition B.1. For a function $f \in L^1(\mathbb{R})$ and a constant $c \in \mathbb{R}$, the singular integral

$$\int_{-\infty}^{+\infty} \frac{f(x)}{c - x} \, \mathrm{d}x = \lim_{\alpha \to 0+} \int_{-\infty}^{c-\alpha} \frac{f(x)}{c - x} \, \mathrm{d}x + \lim_{\beta \to 0+} \int_{c+\beta}^{+\infty} \frac{f(x)}{c - x} \, \mathrm{d}x$$

may not exist when α and β tend to 0+ independently. However, the integral exists restricted to the diagonal $\alpha = \beta$. The corresponding limit

$$⨍_{-\infty}^{+\infty} \frac{f(x)}{c - x} \, \mathrm{d}x := \lim_{\epsilon \to 0+} \left(\int_{-\infty}^{c-\epsilon} \frac{f(x)}{c - x} \, \mathrm{d}x + \int_{c+\epsilon}^{+\infty} \frac{f(x)}{c - x} \, \mathrm{d}x \right) = \lim_{\epsilon \to 0+} \int_{\epsilon < |x - c|} \frac{f(x)}{c - x} \, \mathrm{d}x < \infty \quad \text{(B.1)}$$

is called the Cauchy principal value or Cauchy principal integral of $\frac{f(x)}{c-x}$ against $\mathrm{d}x$; $\frac{1}{\pi} ⨍_{-\infty}^{+\infty} \frac{f(x)}{c-x} \, \mathrm{d}x$ is the Hilbert transform of $f \in L^1(\mathbb{R})$ at point $c \in \mathbb{R}$.

Definition B.2. For a constant $c \in \mathbb{R}$, $\mathrm{p.v.}_x \frac{1}{x-c}$ denotes the distribution on $\mathcal{S}(\mathbb{R})$ (Reed & Simon 1980, Proposition 6 page 136) acting on functions $\varphi \in \mathcal{S}(\mathbb{R})$ via the Cauchy principal value integral as

$$\left\langle \mathrm{p.v.}_x \frac{1}{x - c}, \varphi(x) \right\rangle_x := ⨍_{-\infty}^{\infty} \frac{\varphi(x)}{x - c} \, \mathrm{d}x. \quad \text{(B.2)}$$

Definition B.3. Following Kanwal (2004, Lemma page 185 in Section 7.2), the linear form

$$\mathcal{S}(\mathbb{R}^2) \ni \varphi \mapsto \left\langle \text{p.v.}_{x_1} \frac{1}{x_1 - c_1}, \left\langle \text{p.v.}_{x_2} \frac{1}{x_2 - c_2}, \varphi(x_1, x_2) \right\rangle_{x_2} \right\rangle_{x_1}$$

is a well defined distributions on $\mathcal{S}(\mathbb{R}^2)$, called the direct product of the $\mathcal{S}(\mathbb{R})$ distributions $\text{p.v.}_{x_1} \frac{1}{x_1-c_1}$ and $\text{p.v.}_{x_2} \frac{1}{x_2-c_2}$ and denoted by $\text{p.v.}_{x_1} \frac{1}{x_1-c_1} \text{p.v.}_{x_2} \frac{1}{x_2-c_2}$. This construction can be iterated to define the direct product

$$\prod_{j=1}^{d} \text{p.v.}_{x_j} \frac{1}{x_j - c_j} \text{ as a distribution on } \mathcal{S}(\mathbb{R}^d), \tag{B.3}$$

for any positive integer $d \geqslant 1$.

Lemma B.1. *For any $\varphi(\mathbf{x}) = \prod_{j=1}^{d} \varphi_j(x_j)$ with $\varphi_j \in \mathcal{S}(\mathbb{R})$, for $j = 1, \ldots, d, d \geqslant 2$, there exists a constant $C > 0$ such that,*

$$\underbrace{\left| \int_{\epsilon<|x_2-x_1|} \frac{\mathrm{d}x_2}{x_2 - x_1} \cdots \int_{\epsilon<|x_d-x_1|} \frac{\mathrm{d}x_d}{x_d - x_1} \varphi(\mathbf{x}) \right|}_{d-1 \text{ integrals}} \leqslant C|\varphi_1(x_1)|, \qquad x_1 \in \mathbb{R}, 0 < \epsilon {\color{red}<} 1,$$

where the constant C may depend on φ but not on x_1 or ϵ.

Proof. Factoring $\varphi_1(x_1)$ out and separating integrals,

$$\int_{\epsilon<|x_2-x_1|} \frac{\mathrm{d}x_2}{x_2 - x_1} \cdots \int_{\epsilon<|x_d-x_1|} \frac{\mathrm{d}x_d}{x_d - x_1} \varphi(\mathbf{x}) = \varphi_1(x_1) \prod_{j=2}^{d} \int_{\epsilon<|x_j-x_1|} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j - x_1}. \tag{B.4}$$

For $j = 2, \ldots, d$,

$$\begin{aligned} \int_{\epsilon<|x_j-x_1|} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j - x_1} &= \int_{\epsilon<|x_j-x_1|<1} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j - x_1} + \int_{|x_j-x_1|\geqslant 1} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j - x_1} \\ &= \int_{\epsilon<|x_j-x_1|<1} \mathrm{d}x_j \frac{\varphi_j(x_j) - \varphi_j(x_1)}{x_j - x_1} + \int_{|x_j-x_1|\geqslant 1} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j - x_1}, \end{aligned}$$

where we used $\varphi_j(x_1) \int_{\epsilon<|x_j-x_1|<1} \frac{\mathrm{d}x_j}{x_j-x_1} = 0$ in the last step (as integral of an odd function over a symmetric domain). Since $\varphi_j \in \mathcal{S}(\mathbb{R})$, φ_j' is bounded and by the mean value inequality, $|\varphi_j(x_j) - \varphi_j(x_1)| \leqslant |x_j - x_1| \sup \left|\varphi_j'\right|$, whence

$$\left| \int_{\epsilon<|x_j-x_1|<1} \mathrm{d}x_j \frac{\varphi_j(x_j) - \varphi_j(x_1)}{x_j - x_1} \right| \leqslant 2 \sup \left|\varphi_j'\right|. \tag{B.5}$$

Turning our attention to the second integral,

$$\left|\int_{|x_j-x_1|\geqslant 1} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j-x_1}\right| \leqslant \left|\int_{\mathbb{R}} \mathrm{d}x_j |\varphi_j(x_j)|\right| = \|\varphi_j\|_{L^1(\mathbb{R})} < +\infty, \tag{B.6}$$

since $\varphi_j \in \mathcal{S}(\mathbb{R})$. Combining (B.5) and (B.6) yields $\left|\int_{\epsilon<|x_j-x_1|} \mathrm{d}x_j \frac{\varphi_j(x_j)}{x_j-x_1}\right| < +\infty,\ j = 2,\ldots,d$. In view of (B.4) the lemma is thus proven. ∎

Proposition B.2. *For any $c \in \mathbb{R}$ and multiplicatively separable function $\mathcal{S}(\mathbb{R}^d) \ni \varphi(\mathbf{x}) = \prod_{j=1}^d \varphi_j(x_j)$ with $\varphi_j \in \mathcal{S}(\mathbb{R})$ for $j = 1,\ldots,d$, the composite integral*

$$⨍_{-\infty}^{\infty} \frac{\varphi_1(x_1)\,\mathrm{d}x_1}{x_1-c} \prod_{j=2}^{d} ⨍_{-\infty}^{\infty} \frac{\varphi_j(x_j)\,\mathrm{d}x_j}{x_j-x_1} \tag{B.7}$$

is well defined.

Proof. By Definitions B.1, B.2 and B.3,

$$\varphi_1(x_1)\prod_{j=2}^{d} ⨍_{-\infty}^{\infty} \frac{\varphi_j(x_j)\,\mathrm{d}x_j}{x_j-x_1} = \varphi_1(x_1)\prod_{j=2}^{d} \lim_{\epsilon\to 0+} \int_{\epsilon<|x_j-x_1|} \mathrm{d}x_2 \frac{\varphi_j(x_j)}{x_j-x_1},$$

which in view of Lemma B.1 is integrable with respect to x_1. Hence (B.7) is well defined in accordance with (B.1). ∎

C Fourier Transform

Definition C.1. For $f \in L^1(\mathbb{R}^q)$, the Fourier and inverse Fourier transforms $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ of f are the following functions on $\mathbb{R}^q$ (King 2009, Section 2.6 and 15.6):

$$\mathcal{F}f(\mathbf{z}) = \int_{\mathbb{R}^q} f(\mathbf{s})e^{-i\mathbf{z}\cdot\mathbf{s}}\,\mathrm{d}\mathbf{s}, \qquad \mathcal{F}^{-1}f(\mathbf{s}) = \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q} f(\mathbf{z})e^{i\mathbf{s}\cdot\mathbf{z}}\,\mathrm{d}\mathbf{z}. \tag{C.1}$$

When f depends on several variables, we write $\mathcal{F}_x[f(x,y)](t)$ or $\mathcal{F}[f(\cdot,y)](t)$ to indicate that the transform is taken with respect to specific variables only.

Remark C.1. The Fourier transform is injective on $L^1(\mathbb{R}^q)$ (Knapp 2007, Corollary 8.5). If $f \in L^1(\mathbb{R}^q)$, then $g = \mathcal{F}f$ is bounded, uniformly continuous and vanishes at infinity (Knapp 2007, Proposition 8.1 and Theorem 8.3). If, in addition, $g \in L^1(\mathbb{R}^q)$, then $\mathcal{F}^{-1}g = f$ (Knapp 2007, Theorem 8.4).

Remark C.2. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous automorphisms of $\mathcal{S}(\mathbb{R}^q)$ (Kanwal 2004, Theorem 2 page 143), hence $\varphi = \mathcal{F}\mathcal{F}^{-1}\varphi = \mathcal{F}^{-1}\mathcal{F}\varphi$, for $\varphi \in \mathcal{S}(\mathbb{R}^q)$.

Example C.3. For any $c \in \mathbb{R}$, we compute

$$\mathcal{F}_k\left[\left(|c|-|k|\right)^+\right](r) = \int_{\mathbb{R}} \mathrm{d}k e^{-irk}\big(|c|-|k|\big)^+ = \int_{-|c|}^{|c|} \mathrm{d}k e^{-irk}\big(|c|-|k|\big)$$

$$= \int_0^{|c|} \mathrm{d}k \left(e^{-irk}+e^{irk}\right)\left(|c|-k\right) = \frac{1}{r^2}(2-2\cos(rc)) = \frac{1}{r^2}(2-e^{irc}-e^{-irc}), \quad \text{and}$$

$$\int_{\mathbb{R}} \frac{\mathrm{d}r}{r^2}\big(2-2\cos(rc)\big) = 2\pi|c|.$$

Hence

$$\mathcal{F}_k\left[\left(|c|-|k|\right)^+\right](r) = \frac{2-2\cos(rc)}{r^2} \in L^1(\mathbb{R}). \tag{C.2}$$

Definition C.2. For a distribution T on $\mathcal{S}(\mathbb{R}^q)$, its Fourier and inverse Fourier transforms $\mathcal{F}T$ and $\mathcal{F}^{-1}T$ are the distributions on $\mathcal{S}(\mathbb{R}^q)$ given as (Kanwal 2004, Theorem 3 page 147), (King 2009, Section 10.4 and 10.10):

$$\begin{cases} \big\langle (\mathcal{F}T)_{\mathbf{s}}, \varphi(\mathbf{s})\big\rangle_{\mathbf{s}} = \langle T_{\mathbf{z}}, \mathcal{F}\varphi(\mathbf{z})\rangle_{\mathbf{z}}\,, & \varphi \in \mathcal{S}(\mathbb{R}^q), \\ \big\langle (\mathcal{F}^{-1}T)_{\mathbf{z}}, \varphi(\mathbf{z})\big\rangle_{\mathbf{z}} = \big\langle T_{\mathbf{s}}, \mathcal{F}^{-1}\varphi(\mathbf{s})\big\rangle_{\mathbf{s}}\,, & \varphi \in \mathcal{S}(\mathbb{R}^q). \end{cases} \tag{C.3}$$

Example C.4. The following are two well-known examples of distributional Fourier transforms (King 2009, page 489), (Kammler 2008, page 415):

$$\langle \mathcal{F}_x[\operatorname{sgn} x \ \mathrm{d}x]_\lambda, \varphi(\lambda)\rangle_\lambda = \int_{\mathbb{R}} \mathrm{d}x \operatorname{sgn} x \int_{\mathbb{R}} \mathrm{d}\lambda e^{-i\lambda x}\varphi(\lambda) = \left\langle \frac{2}{i}\,\mathrm{p.v.}_\lambda \frac{1}{\lambda}, \varphi(\lambda)\right\rangle_\lambda \tag{C.4}$$

$$\langle \mathcal{F}_x[\cos x \ \mathrm{d}x](\mathrm{d}\lambda), \varphi(\lambda)\rangle_\lambda = \int_{\mathbb{R}} \mathrm{d}x \cos x \int_{\mathbb{R}} \mathrm{d}\lambda e^{-i\lambda x}\varphi(\lambda) = \pi\varphi(1) + \pi\varphi(-1) \tag{C.5}$$

D Proof of (4.10)

For any vector $\mathbf{z} \in \mathbb{R}^q$ and any index $j = 1, \ldots, q$, we denote in this appendix $\mathbf{z}_{<j} \in \mathbb{R}^{j-1}$ the subvector of the first $j-1$ coefficients of $\mathbf{z}$ (with $\mathbf{x}_{<1} = \emptyset$), and $\mathbf{z}_{\neq j} \in \mathbb{R}^{q-1}$ the subvector without the j^{th} coefficient.

Lemma D.1. *The measure* $\mathrm{d}\mathbf{x}\delta_{\|\mathbf{x}\|_1}(\mathrm{d}k) + \mathrm{d}\mathbf{x}\delta_{-\|\mathbf{x}\|_1}(\mathrm{d}k)$ *is equal to*

$$\begin{aligned} &\mathrm{d}k \mathbf{1}_{|x_1|<|k|}(\mathrm{d}x_1)\mathbf{1}_{|x_2|<|k|-|x_1|}(\mathrm{d}x_2)\ldots\mathbf{1}_{|x_{d-1}|<|k|-\|\mathbf{x}_{<d-1}\|_1}(\mathrm{d}x_{d-1})\times \\ &\qquad \big(\delta_{|k|-\|\mathbf{x}_{<d}\|_1}(\mathrm{d}x_d) + \delta_{-|k|+\|\mathbf{x}_{<d}\|_1}(\mathrm{d}x_d)\big) =: \delta_D(\mathrm{d}(\mathbf{x},k)). \end{aligned} \tag{D.1}$$

Proof. Let $f : \mathbb{R}^{d+1} \to \mathbb{C}$ be any δ_D-integrable function. By sifting property of the Dirac measures $\delta_{\|\mathbf{x}\|_1}(\mathrm{d}k)$ and $\delta_{-\|\mathbf{x}\|_1}(\mathrm{d}k)$,

$$\int_{\mathbb{R}^{d+1}} \big(\mathrm{d}\mathbf{x}\delta_{\|\mathbf{x}\|_1}(\mathrm{d}k) + \mathrm{d}\mathbf{x}\delta_{-\|\mathbf{x}\|_1}(\mathrm{d}k)\big) f(\mathbf{x},k) = \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} f(\mathbf{x}, \|\mathbf{x}\|_1) + \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} f(\mathbf{x}, -\|\mathbf{x}\|_1). \tag{D.2}$$

Splitting the first right-hand side integral above,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} f(\mathbf{x}, \|\mathbf{x}\|_1) = &\int_{\mathbb{R}^{d-1}} \mathrm{d}\mathbf{x}_{<d} \int_0^\infty \mathrm{d}x_d f(\mathbf{x}, \|\mathbf{x}_{<d}\|_1 + x_d) \\ &+ \int_{\mathbb{R}^{d-1}} \mathrm{d}\mathbf{x}_{<d} \int_{-\infty}^0 \mathrm{d}x_d f(\mathbf{x}, -\|\mathbf{x}_{<d}\|_1 - x_d).\end{aligned} \tag{D.3}$$

By change of variables $(x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1}, k = \|\mathbf{x}\|_1 = \|\mathbf{x}_{<d}\|_1 + x_d)$ in the first right-hand side integral above, and change of variables $(x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1}, k = \|\mathbf{x}\|_1 = \|\mathbf{x}_{<d}\|_1 - x_d)$ in the second integral, where both Jacobians are 1, and adapting integral regions,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} f(\mathbf{x}, \|\mathbf{x}\|_1) = &\int_0^\infty \mathrm{d}k \int_{-k}^{k} \mathrm{d}x_1 \dots \int_{-k+\|\mathbf{x}_{<d-1}\|_1}^{k-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, k - \|\mathbf{x}_{<d}\|_1, k) \\ &+ \int_0^\infty \mathrm{d}k \int_{-k}^{k} \mathrm{d}x_1 \dots \int_{-k+\|\mathbf{x}_{<d-1}\|_1}^{k-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, \|\mathbf{x}_{<d}\|_1 - k, k).\end{aligned} \tag{D.4}$$

Following similar steps, the second right-hand side integral in (D.2) may be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} f(\mathbf{x}, -\|\mathbf{x}\|_1) = &\int_{-\infty}^0 \mathrm{d}k \int_{k}^{-k} \mathrm{d}x_1 \dots \int_{k+\|\mathbf{x}_{<d-1}\|_1}^{-k-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, -\|\mathbf{x}_{<d}\|_1 - k, k) \\ &+ \int_{-\infty}^0 \mathrm{d}k \int_{k}^{-k} \mathrm{d}x_1 \dots \int_{k+\|\mathbf{x}_{<d-1}\|_1}^{-k-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, k + \|\mathbf{x}_{<d}\|_1, k).\end{aligned} \tag{D.5}$$

Combining (D.4) and (D.5) and piecing $\mathrm{d}k$ integrals together,

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}} \left(\mathrm{d}\mathbf{x}\delta_{\|\mathbf{x}\|_1}(\mathrm{d}k) + \mathrm{d}\mathbf{x}\delta_{-\|\mathbf{x}\|_1}\right) f(\mathbf{x}, k) \\ &= \int_{-\infty}^{\infty} \mathrm{d}k \int_{-|k|}^{|k|} \mathrm{d}x_1 \dots \int_{-|k|+\|\mathbf{x}_{<d-1}\|_1}^{|k|-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, |k| - \|\mathbf{x}_{<d}\|_1, k) \\ &\qquad + \int_{-\infty}^{\infty} \mathrm{d}k \int_{-|k|}^{|k|} \mathrm{d}x_1 \dots \int_{-|k|+\|\mathbf{x}_{<d-1}\|_1}^{|k|-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} f(\mathbf{x}_{<d}, \|\mathbf{x}_{<d}\|_1 - |k|, k) \\ &= \int_{\mathbb{R}} \mathrm{d}k \int_{-|k|}^{|k|} \mathrm{d}x_1 \dots \int_{-|k|+\|\mathbf{x}_{<d-1}\|_1}^{|k|-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} \int_{\mathbb{R}} \mathrm{d}x_d \delta_{|k|-\|\mathbf{x}_{<d}\|_1}(\mathrm{d}x_d) f(\mathbf{x}_{<d}, x_d, k) \\ &\qquad + \int_{\mathbb{R}} \mathrm{d}k \int_{-|k|}^{|k|} \mathrm{d}x_1 \dots \int_{-|k|+\|\mathbf{x}_{<d-1}\|_1}^{|k|-\|\mathbf{x}_{<d-1}\|_1} \mathrm{d}x_{d-1} \int_{\mathbb{R}} \mathrm{d}x_d \delta_{-|k|+\|\mathbf{x}_{<d}\|_1}(\mathrm{d}x_d) f(\mathbf{x}_{<d}, x_d, k) \\ &= \int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x}, k)) f(\mathbf{x}, k),\end{aligned} \tag{D.6}$$

where we used the sifting property again in the second equality. ∎

By sifting property of the Dirac mass distribution ?measure, the left-hand side in

(4.10)

$$\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} e^{-i\|\mathbf{x}\|_1} \mathcal{F}^{-1}\phi_e(\mathbf{x}) + \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} e^{i\|\mathbf{x}\|_1} \mathcal{F}^{-1}\phi_e(\mathbf{x})$$
$$= \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \delta_{\|\mathbf{x}\|_1}(\mathrm{d}k) e^{-ik} \mathcal{F}^{-1}\phi_e(\mathbf{x}) + \int_{\mathbb{R}^d} \mathrm{d}\mathbf{x} \int_{\mathbb{R}} \delta_{-\|\mathbf{x}\|_1}(\mathrm{d}k) e^{-ik} \mathcal{F}^{-1}\phi_e(\mathbf{x})$$
$$= \frac{1}{2} \int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1}\phi_e(\mathbf{x}),$$

by Lemma D.1. Hence proving (4.10) is reduced to showing that

$$\frac{1}{2} \int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1}\phi_e(\mathbf{x}) = \left\langle T^d_{\mathbf{w}}, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}}. \tag{D.7}$$

D.1 Dimension $d = 1$

By definition of δ_D in (D.1),

$$\int_{\mathbb{R}^2} \delta_D(\mathrm{d}(x, k)) e^{-ik} \mathcal{F}^{-1}\phi_e(x) = \int_{\mathbb{R}} \mathrm{d}k e^{-ik} \int_{\mathbb{R}} \big(\delta_k(\mathrm{d}x) + \delta_{-k}(\mathrm{d}x)\big) \mathcal{F}^{-1}\phi_e(x)$$
$$= \mathcal{F}_k[\mathcal{F}^{-1}\phi_e(k) + \mathcal{F}^{-1}\phi_e(-k)](1) = \phi_e(1) + \phi_e(-1)$$
$$= \langle \delta_1(\mathrm{d}w) + \delta_{-1}(\mathrm{d}w), \phi_e(w) \rangle_w = 2 \left\langle T^1_w, \phi_e(w) \right\rangle_w,$$

which is (D.7) for $d = 1$.

D.2 Dimension $d = 2$

For ease of reading, we include this proof written for $d = 2$ which otherwise relies on the arguments used in Appendix D.3 for the general case $d \geqslant 2$.

Lemma D.2. *Let, for $d = 2$,*

$$\Phi_\pm(k) = \left\langle \mathbf{1}_{|x_1|<|k|}(\mathrm{d}x_1) \delta_{\pm|k|\mp|x_1|}(\mathrm{d}x_2), \mathcal{F}^{-1}\varphi(\mathbf{x}) \right\rangle_{x_1, x_2}, \ \varphi \in \mathcal{S}(\mathbb{R}^2).$$

If $\varphi = \phi_e \in S_e$, then

$$\Phi_+(k) + \Phi_-(k) = \frac{1}{2\pi^2} \left\langle \sin(w_2|k|)\, \mathrm{d}w_2, ⨍_{\mathbb{R}} \mathrm{d}w_1 \left(\frac{1}{w_2 - w_1} + \frac{1}{w_1 + w_2}\right) \phi_e(w_1, w_2) \right\rangle_{w_2}$$
$$+ \frac{1}{2\pi^2} \left\langle \sin(w_1|k|)\, \mathrm{d}w_1, ⨍_{\mathbb{R}} \mathrm{d}w_2 \left(\frac{1}{w_1 - w_2} + \frac{1}{w_1 + w_2}\right) \phi_e(w_1, w_2) \right\rangle_{w_1}. \tag{D.8}$$

Proof. For $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\begin{aligned}\Phi_+(k) &= \Big\langle \mathbf{1}_{|x_1|<|k|}(\mathrm{d}x_1), \big\langle \delta_{|k|-|x_1|}(\mathrm{d}x_2), \mathcal{F}^{-1}\varphi(\mathbf{x})\big\rangle_{x_2}\Big\rangle_{x_1} \\ &= \int_{\mathbb{R}} \mathrm{d}x_1 \mathbf{1}_{|x_1|<|k|}\mathcal{F}^{-1}\varphi(x_1, |k|-|x_1|) \\ &= \frac{1}{(2\pi)^2}\int_{|x_1|<|k|} \mathrm{d}x_1 \int_{\mathbb{R}^2} \mathrm{d}w_1\,\mathrm{d}w_2 e^{iw_1x_1+iw_2(|k|-|x_1|)}\varphi(w_1,w_2)\end{aligned}$$

Splitting the $\mathrm{d}x_1$ integral at the origin, then rewriting the domain of $\mathrm{d}w_1\,\mathrm{d}w_2$ integration as a limit,

$$\begin{aligned}\Phi_+(k) &= \frac{1}{(2\pi)^2}\int_0^{|k|}\mathrm{d}x_1\int_{\mathbb{R}^2}\mathrm{d}w_1\,\mathrm{d}w_2 e^{ix_1(w_1-w_2)}e^{iw_2|k|}\varphi(w_1,w_2) \\ &\quad + \frac{1}{(2\pi)^2}\int_{-|k|}^{0}\mathrm{d}x_1\int_{\mathbb{R}^2}\mathrm{d}w_1\,\mathrm{d}w_2 e^{ix_1(w_1+w_2)}e^{iw_2|k|}\varphi(w_1,w_2) \\ &= \frac{1}{(2\pi)^2}\int_0^{|k|}\mathrm{d}x_1\Big(\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2 e^{ix_1(w_1-w_2)}e^{iw_2|k|}\varphi(w_1,w_2)\Big) \\ &\quad + \frac{1}{(2\pi)^2}\int_{-|k|}^{0}\mathrm{d}x_1\Big(\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2 e^{ix_1(w_1+w_2)}e^{iw_2|k|}\varphi(w_1,w_2)\Big).\end{aligned} \tag{D.9}$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^2)$, for any $\epsilon \geqslant 0$, there exists $M > 0$ such that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)e^{ix_1(w_1\pm w_2)}e^{iw_2|k|}\Big| \\ &\qquad\leqslant \int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\big|\varphi(w_1,w_2)e^{ix_1(w_1\pm w_2)}e^{iw_2|k|}\big| \\ &\qquad\leqslant \int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\big|\varphi(w_1,w_2)\big| < M < \infty,\end{aligned}$$

and the integral $\int_0^{\pm|k|} M\,\mathrm{d}x_1$ is finite for any $|k| < \infty$. Hence, by dominated convergence and Fubini's theorem,

$$\begin{aligned}\Phi_+(k) &= \frac{1}{(2\pi)^2}\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\int_0^{|k|}\mathrm{d}x_1 e^{ix_1(w_1-w_2)}e^{iw_2|k|} \\ &\quad + \frac{1}{(2\pi)^2}\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1+w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\int_{-|k|}^{0}\mathrm{d}x_1 e^{ix_1(w_1+w_2)}e^{iw_2|k|}.\end{aligned}$$

Solving integrals with respect to x_1,

$$\begin{aligned}\Phi_+(k) &= \frac{1}{(2\pi)^2}\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1-w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\frac{-ie^{i|k|w_1}+ie^{iw_2|k|}}{w_1-w_2} \\ &\quad + \frac{1}{(2\pi)^2}\lim_{\epsilon\to0+}\int_{\mathbb{R}^2\backslash\{|w_1+w_2|\leqslant\epsilon\}}\mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\frac{ie^{-i|k|w_1}-ie^{iw_2|k|}}{w_1+w_2}.\end{aligned}$$

Similar steps would show that

$$\Phi_-(k) = \frac{1}{(2\pi)^2}\lim_{\epsilon\to 0+}\int_{\mathbb{R}^2\setminus\{|w_1-w_2|\leqslant\epsilon\}} \mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\,\mathrm{d}x_1\frac{ie^{-i|k|w_1}-ie^{-iw_2|k|}}{w_1-w_2}$$
$$+\frac{1}{(2\pi)^2}\lim_{\epsilon\to 0+}\int_{\mathbb{R}^2\setminus\{|w_1+w_2|\leqslant\epsilon\}} \mathrm{d}w_1\,\mathrm{d}w_2\varphi(w_1,w_2)\frac{-ie^{i|k|w_1}+ie^{-iw_2|k|}}{w_1+w_2}.$$

Plugging Euler's trigonometric formulas into the above expressions and combining,

$$\begin{aligned}&\Phi_+(k)+\Phi_-(k)\\ =&\frac{1}{2\pi^2}\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\mathrm{d}w_1\sin(w_1)\Big(\int_{\epsilon<|w_1-w_2|}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1-w_2}+\int_{\epsilon<|w_1+w_2|}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1+w_2}\Big)\\ &+\frac{1}{2\pi^2}\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\mathrm{d}w_2\sin(w_2)\Big(\int_{\epsilon<|w_1-w_2|}\mathrm{d}w_1\frac{\varphi(w_1,w_2)}{w_2-w_1}+\int_{\epsilon<|w_1+w_2|}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1+w_2}\Big).\end{aligned} \tag{D.10}$$

For $\varphi(w_1,w_2)=\varphi_1(w_1)\varphi_2(w_2)$ with $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R})$, in view of Lemma B.1, we have by dominated convergence

$$\begin{aligned}&\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\mathrm{d}w_1\sin(w_1)\int_{\epsilon<|w_1-w_2|}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1-w_2}\\ &\qquad=\int_{\mathbb{R}}\mathrm{d}w_1\sin(w_1)\lim_{\epsilon\to 0+}\int_{\epsilon<|w_1-w_2|}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1-w_2}=\int_{\mathbb{R}}\mathrm{d}w_1\sin(w_1)⨍_{\mathbb{R}}\mathrm{d}w_2\frac{\varphi(w_1,w_2)}{w_1-w_2}.\end{aligned}$$

Following similar steps for the remaining terms in (D.10), we recover (D.8) for $\varphi(w_1,w_2)=\varphi_1(w_1)\varphi_2(w_2)$ with $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R})$, from which the result for $\varphi=\phi_e\in S_e$ follows by linearity. ∎

By definition of δ_D in (D.1),

$$\delta_D(\mathrm{d}(\mathbf{x},k))=\mathrm{d}k\mathbf{1}_{|x_1|<|k|}(\mathrm{d}x_1)(\delta_{|k|-|x_1|}+\delta_{-|k|+|x_1|})(\mathrm{d}x_2).$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^3}\delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x})\\ &\qquad=\int_{\mathbb{R}}\mathrm{d}ke^{-ik}\underbrace{\big\langle\mathbf{1}_{|x_1|<|k|}(\mathrm{d}x_1)(\delta_{|k|-|x_1|}+\delta_{-|k|+|x_1|})(\mathrm{d}x_2),\mathcal{F}^{-1}\phi_e(\mathbf{x})\big\rangle_{x_1,x_2}}_{\Phi_+(k)+\Phi_-(k)}.\end{aligned} \tag{D.11}$$

By Lemma D.2,

$$\begin{aligned}&\int_{\mathbb{R}^3}\delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x})\\ =&\frac{1}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}ke^{-ik}\int_{\mathbb{R}}\mathrm{d}w_2\sin(w_2|k|)⨍_{\mathbb{R}}\mathrm{d}w_1\Big(\frac{1}{w_2-w_1}+\frac{1}{w_1+w_2}\Big)\phi_e(w_1,w_2)\\ &+\frac{1}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}ke^{-ik}\int_{\mathbb{R}}\mathrm{d}w_1\sin(w_1|k|)⨍_{\mathbb{R}}\mathrm{d}w_2\Big(\frac{1}{w_1-w_2}+\frac{1}{w_1+w_2}\Big)\phi_e(w_1,w_2).\end{aligned} \tag{D.12}$$

Let $f(w_2) := \fint_{\mathbb{R}} \mathrm{d}w_1(\frac{1}{w_2-w_1} + \frac{1}{w_1+w_2})\phi_e(w_1,w_2)$. Substituting Euler's sine formula, then splitting the integrand while applying the reflective change of variable $k \mapsto -k$ to the second resulting integral, the first term in (D.12) may be rewritten as

$$
\begin{aligned}
&\frac{i}{4\pi^2}\int_{\mathbb{R}} \mathrm{d}k e^{-ik}\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_2(e^{-iw_2k} - e^{iw_2k})f(w_2)\\
=&\frac{i}{4\pi^2}\left(\int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_2 e^{-ik(w_2+1)}f(w_2) + \int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_2 e^{-ik(w_2-1)}f(w_2)\right)\\
=&\frac{1}{2\pi^2}\left\langle \mathrm{p.v.}_{w_2}\left(\frac{1}{w_2+1} + \frac{1}{w_2-1}\right), f(w_2)\right\rangle_{w_2},
\end{aligned}
$$

where we substituted (C.4) in the last step. In addition,

$$
\begin{aligned}
&\frac{1}{2\pi^2}\left\langle \mathrm{p.v.}_{w_2}\left(\frac{1}{w_2+1} + \frac{1}{w_2-1}\right), f(w_2)\right\rangle_{w_2}\\
=&\frac{1}{2\pi^2}\left\langle \mathrm{p.v.}_{w_2}\left(\frac{1}{w_2+1} + \frac{1}{w_2-1}\right), \fint_{\mathbb{R}} \mathrm{d}w_1(\frac{1}{w_2-w_1} + \frac{1}{w_1+w_2})\phi_e(w_1,w_2)\right\rangle_{w_2}\\
=&\frac{1}{2\pi^2}\left\langle \mathrm{p.v.}_{w_2}\left(\left(\frac{1}{w_2+1} + \frac{1}{w_2-1}\right)\mathrm{p.v.}_{w_1}\left(\frac{1}{w_2-w_1} + \frac{1}{w_2+w_1}\right)\right), \phi_e(w_1,w_2)\right\rangle_{w_1,w_2}.
\end{aligned}
\tag{D.13}
$$

ok for me. Following similar steps for the remaining terms to the right-hand side of (D.12), we may conclude that

$$
\frac{1}{2}\int_{\mathbb{R}^3} \delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x}) = \left\langle T^2_{\mathbf{w}}, \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}},
$$

which is (D.7) for $d = 2$.

D.3 General Dimension $d \geqslant 2$

Let $\phi_e \in S_e$ and $\Phi_1(k) \equiv \Phi_1(\mathbf{x}_{<1},k)$ result from the following backward recurrence:

$$
\begin{aligned}
&\Phi_d(\mathbf{x}_{<d},k) = \left\langle (\delta_{|k|-\|\mathbf{x}_{<d}\|_1} + \delta_{-|k|+\|\mathbf{x}_{<d}\|_1})(\mathrm{d}x_d), \mathcal{F}^{-1}\phi_e(\mathbf{x})\right\rangle_{x_d} \text{ and, for } j = d-1,\dots,1,\\
&\Phi_j(\mathbf{x}_{<j},k) = \left\langle \mathbf{1}_{|x_j|<|k|-\|\mathbf{x}_{<j}\|_1}(\mathrm{d}x_j), \Phi_{j+1}(\mathbf{x}_{<j+1},k)\right\rangle_{x_j}.
\end{aligned}
\tag{D.14}
$$

Lemma D.3. *For any $m = d-1,\dots,1$,*

$$
\begin{aligned}
&\Phi_m(\mathbf{x}_{<m},k) = \frac{2}{(2\pi)^{d-m+1}}\times\\
&\left\langle \sum_{j=m}^{d}\left[\mathrm{sc}_m\big(w_j(|k| - \|\mathbf{x}_{<m}\|_1)\big)\prod_{j'\neq j,j'=m}^{d}\mathrm{p.v.}_{w_{j'}}(\frac{1}{w_j-w_{j'}} + \frac{1}{w_j+w_{j'}})\right]\mathrm{d}w_j, \Xi_m\right\rangle_{w_m,\dots,w_d},
\end{aligned}
\tag{D.15}
$$

where we denote

$$\mathrm{sc}_m(\cdot) = \begin{cases} (-1)^{\frac{d-m-1}{2}} \sin(\cdot), & \textit{if } (d-m) \textit{ odd} \\ (-1)^{\frac{d-m}{2}} \cos(\cdot), & \textit{if } (d-m) \textit{ even} \end{cases}, \textit{ and } \Xi_m = \mathcal{F}^{-1}_{\mathbf{w}_{<m}}[\phi_e(\mathbf{w})](\mathbf{x}_{<m}, w_m, \dots, w_d).$$

Proof. We proceed by backward induction. Following the same steps as in Lemma D.2 proves (D.15) for $m = d-1$, i.e.

$$\begin{aligned}\Phi_{d-1}(\mathbf{x}_{<d-1}, k) = \Big\langle \frac{1}{2\pi^2} \sum_{j=d-1}^{d} \Big(& \sin(w_j(|k| - \|\mathbf{x}_{<d-1}\|_1)) \times \\ & \prod_{j' \neq j, j'=d-1}^{d} \mathrm{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\big)\Big) \, \mathrm{d}w_j, \Xi_{d-1} \Big\rangle_{w_{d-1}, w_d}.\end{aligned} \tag{D.16}$$

Moreover, if (D.15) is satisfied for some index $m = d-1, \dots, 2$, then it must also be satisfied at index $m-1$ as proven underneath, that is,

$$\begin{aligned}\Phi_{m-1}(\mathbf{x}_{<m-1}, k) = \frac{2}{(2\pi)^{d-m+2}} \Big\langle \sum_{j=m-1}^{d} \Big(& \mathrm{sc}_{m-1}(w_j(|k| - \|\mathbf{x}_{<m}\|_1)) \times \\ & \prod_{j' \neq j, j'=m-1}^{d} \mathrm{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\big)\Big) \, \mathrm{d}w_j, \Xi_{d-1} \Big\rangle_{w_{m-1} \dots w_d}.\end{aligned} \tag{D.17}$$

- Suppose $d-m$ is odd; then $\mathrm{sc}_m(\cdot) = (-1)^{\frac{d-m-1}{2}} \sin(\cdot)$ and

$$\begin{aligned}\Phi_{m-1}(\mathbf{x}_{<m-1}, k) = \Big\langle \mathbf{1}_{|x_{m-1}| < |k| - \|\mathbf{x}_{<m-1}\|_1}(\mathrm{d}x_{m-1}), \Phi_m(k) \Big\rangle_{x_{m-1}} = \frac{(-1)^{\frac{d-m-1}{2}} 2}{(2\pi)^{d-m+1}} \times \\ \Bigg[\int_0^{|k| - \|\mathbf{x}_{<m-1}\|_1} \mathrm{d}x_{m-1} \Big\langle \sum_{j=m}^{d} \Big(& \sin(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1 - x_{m-1})) \times \\ & \prod_{j' \neq j, j'=m}^{d} \mathrm{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\big)\Big) \, \mathrm{d}w_j, \Xi_m \Big\rangle_{w_m, \dots, w_d} \\ + \int_{-|k| + \|\mathbf{x}_{<m-1}\|_1}^{0} \mathrm{d}x_{m-1} \Big\langle \sum_{j=m}^{d} \Big(& \sin(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1 + x_{m-1})) \times \\ & \prod_{j' \neq j, j'=m}^{d} \mathrm{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\big)\Big) \, \mathrm{d}w_j, \Xi_m \Big\rangle_{w_m, \dots, w_d} \Bigg].\end{aligned} \tag{D.18}$$

Substituting Euler's sine formula into the above together with

$$\Xi_m = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}w_{m-1} e^{-ix_{m-1}w_{m-1}} \Xi_{m-1} = \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j \pm w_{m-1}|} \mathrm{d}w_{m-1} e^{-ix_{m-1}w_{m-1}} \Xi_{m-1}$$

which stems from slicing the Fourier transform Ξ_m along x_{m-1} and then rewriting the integral domain as a limit, the first term inside the square bracket in (D.18) may be rewritten as

$$
\begin{aligned}
&\frac{i}{4\pi}\int_0^{|k|-\|\mathbf{x}_{<m-1}\|_1} \mathrm{d}x_{m-1}\Big\langle \sum_{j=m}^{d}\Big(\prod_{j'\neq j, j'=m}^{d} \text{p.v.}_{w_{j'}}\big(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\big)\Big)\,\mathrm{d}w_j,\\
&\qquad \lim_{\epsilon\to 0+}\int_{\epsilon<|w_j-w_{m-1}|}\mathrm{d}w_{m-1}e^{-ix_{m-1}w_{m-1}}e^{-iw_j(|k|-\|\mathbf{x}_{<m-1}\|_1-x_{m-1})}\Xi_{m-1}\Big\rangle_{w_m\ldots,w_d}\\
-&\frac{i}{4\pi}\int_0^{|k|-\|\mathbf{x}_{<m-1}\|_1} \mathrm{d}x_{m-1}\Big\langle \sum_{j=m}^{d}\Big(\prod_{j'\neq j, j'=m}^{d} \text{p.v.}_{w_{j'}}\big(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\big)\Big)\,\mathrm{d}w_j,\\
&\qquad \lim_{\epsilon\to 0+}\int_{\epsilon<|w_j+w_{m-1}|}\mathrm{d}w_{m-1}e^{-ix_{m-1}w_{m-1}}e^{iw_j(|k|-\|\mathbf{x}_{<m-1}\|_1-x_{m-1})}\Xi_{m-1}\Big\rangle_{w_m\ldots,w_d}.
\end{aligned}
\tag{D.19}
$$

By a combination of Definition B.1, dominated convergence and Fubini's theorem ?(see proof of Lemma D.2) No more need ref here we may bring the $\mathrm{d}x_{m-1}$ integrals into the second argument of the inner product and then exchange $\mathrm{d}x_{m-1}$ with $\lim_{\epsilon\to 0+}$ and $\mathrm{d}w_j$. As such, the first term in (D.19) is equal to

$$
\begin{aligned}
&\frac{i}{4\pi}\Big\langle \sum_{j=m}^{d}\Big(\prod_{j'\neq j, j'=m}^{d} \text{p.v.}_{w_{j'}}\big(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\big)\Big)\,\mathrm{d}w_j, \lim_{\epsilon\to 0+}\int_{\epsilon<|w_j+w_{m-1}|}\mathrm{d}w_{m-1}\times\\
&\qquad \int_0^{|k|-\|\mathbf{x}_{<m-1}\|_1}\mathrm{d}x_{m-1}e^{-ix_{m-1}(w_{m-1}-w_j)-iw_j(|k|-\|\mathbf{x}_{<m-1}\|_1)}\Xi_{m-1}\Big\rangle_{w_m\ldots,w_d}\\
=&\frac{1}{4\pi}\Big\langle \sum_{j=m}^{d}\Big(\prod_{j'\neq j, j'=m}^{d} \text{p.v.}_{w_{j'}}\big(\frac{1}{w_j-w_{j'}}+\frac{1}{w_j+w_{j'}}\big)\Big)\,\mathrm{d}w_j,\\
&\qquad \lim_{\epsilon\to 0+}\int_{\epsilon<|w_j-w_{m-1}|}\mathrm{d}w_{m-1}\frac{e^{iw_{m-1}(|k|-\|\mathbf{x}_{<m-1}\|_1)}-e^{iw_j(|k|-\|\mathbf{x}_{<m-1}\|_1)}}{w_j-w_{m-1}}\Xi_{m-1}\Big\rangle_{w_m\ldots,w_d}
\end{aligned}
$$

where we solved the $\mathrm{d}x_{m-1}$ integral in the last step. Following similar steps for the second term in (D.19), and then the second term within square brackets in (D.18), we obtain

$$\Phi_{m-1}(\mathbf{x}_{<m-1}, k) = \frac{(-1)^{\frac{d-m-1}{2}}}{(2\pi)^{d-m+2}} \Big\langle \sum_{j=m}^{d} \Big(\prod_{j' \neq j, j'=m}^{d} \text{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}} \big) \Big) \, \mathrm{d}w_j,$$

$$\lim_{\epsilon \to 0+} \int_{\epsilon < |w_j - w_{m-1}|} \mathrm{d}w_{m-1} \frac{e^{iw_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1)} - e^{iw_j(|k| - \|\mathbf{x}_{<m-1}\|_1)}}{w_j - w_{m-1}} \Xi_{m-1}$$

$$+ \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j + w_{m-1}|} \mathrm{d}w_{m-1} \frac{e^{iw_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1)} - e^{-iw_j(|k| - \|\mathbf{x}_{<m-1}\|_1)}}{w_j + w_{m-1}} \Xi_{m-1}$$

$$+ \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j - w_{m-1}|} \mathrm{d}w_{m-1} \frac{e^{-iw_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1)} - e^{-iw_j(|k| - \|\mathbf{x}_{<m-1}\|_1)}}{w_j - w_{m-1}} \Xi_{m-1}$$

$$+ \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j + w_{m-1}|} \mathrm{d}w_{m-1} \frac{e^{-iw_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1)} - e^{iw_j(|k| - \|\mathbf{x}_{<m-1}\|_1)}}{w_j + w_{m-1}} \Xi_{m-1} \Big\rangle_{w_m, \ldots, w_d}$$

Splitting and rearranging integrands, and substituting Euler's cosine formula,

$$\begin{aligned}
\Phi_{m-1}(\mathbf{x}_{<m-1}, k) &= \frac{(-1)^{\frac{d-m-1}{2}} 2}{(2\pi)^{d-m+2}} \Big\langle \sum_{j=m}^{d} \Big(\prod_{j' \neq j, j'=m}^{d} \text{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}} \big) \Big) \, \mathrm{d}w_j, \\
& \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j + w_{m-1}|} \frac{\mathrm{d}w_{m-1} \cos(w_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1))}{w_j + w_{m-1}} \Xi_{m-1} \\
& + \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j + w_{m-1}|} \frac{\mathrm{d}w_{m-1} \cos(w_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1))}{w_j - w_{m-1}} \Xi_{m-1} \\
& - \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j + w_{m-1}|} \frac{\mathrm{d}w_{m-1} \cos(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1))}{w_j + w_{m-1}} \Xi_{m-1} \\
& - \lim_{\epsilon \to 0+} \int_{\epsilon < |w_j - w_{m-1}|} \frac{\mathrm{d}w_{m-1} \cos(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1))}{w_j - w_{m-1}} \Xi_{m-1} \Big\rangle_{w_m, \ldots, w_d}.
\end{aligned} \tag{D.20}$$

By Definition B.1, dominated convergence and Fubini's theorem ?the limits above correspond to Cauchy integrals. Rearranging integrands, the first term in (D.20) may thus be rewritten as

$$\begin{aligned}
&\frac{(-1)^{\frac{d-m-1}{2}}}{(2\pi)^{d-m+2}} \Big(\Big\langle \sum_{j=m}^{d} \cos(w_{m-1}(|k| - \|\mathbf{x}_{<m-1}\|_1)), \Big\langle \text{p.v.}_{w_j} \Big(\big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}} \big) \times \\
&\qquad \prod_{j' \neq j, j'=m}^{d} \text{p.v.}_{w_{j'}} \big(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}} \big) \Big), \Xi_{m-1} \Big\rangle_{w_m, \ldots, w_d} \Big\rangle_{w_{m-1}}.
\end{aligned} \tag{D.21}$$

By applying the reflective change of variables $(w_j, w_{m-1}) \mapsto (-w_j, -w_{m-1})$ and recognizing Cauchy principal integrals, the last two integrals of (D.20) is equivalent to

$$\begin{aligned}
&\frac{(-1)^{\frac{d-m-1}{2}}}{(2\pi)^{d-m+2}}\Bigg\langle \sum_{j=m}^{d} \Big(\cos(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1)) \prod_{j'\neq j, j'=m}^{d} \text{p.v.}_{w_{j'}}(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}})\Big)\, \mathrm{d}w_j, \\
&\qquad \Big\langle \text{p.v.}_{w_{m-1}}(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}), \Xi_{m-1}\Big\rangle_{w_{m-1}} \Bigg\rangle_{w_m,\dots,w_d} \\
&= \frac{(-1)^{\frac{d-m-1}{2}}}{(2\pi)^{d-m+2}}\Bigg\langle \sum_{j=m}^{d} \Big(\cos(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1))\times \\
&\qquad \prod_{j'\neq j, j'=m-1}^{d} \text{p.v.}_{w_{j'}}(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}})\Big)\, \mathrm{d}w_j, \Xi_{m-1} \Bigg\rangle_{w_{m-1},\dots,w_d}.
\end{aligned} \tag{D.22}$$

Substituting (D.21) and (D.22) into (D.20) yields

$$\begin{aligned}
\Phi_{m-1}(\mathbf{x}_{<m-1}, k) = &\frac{(-1)^{\frac{d-m-1}{2}}}{(2\pi)^{d-m+2}}\Bigg\langle \sum_{j=m-1}^{d} \Big(\cos(w_j(|k| - \|\mathbf{x}_{<m-1}\|_1))\times \\
&\prod_{j'\neq j, j'=m-1}^{d} \text{p.v.}_{w_{j'}}(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}})\Big)\, \mathrm{d}w_j, \Xi_{m-1} \Bigg\rangle_{w_{m-1},\dots,w_d},
\end{aligned}$$

as required for $d-m$ odd.

• Suppose $d-m$ is even; then $\mathrm{sc}_m(\cdot) = (-1)^{\frac{d-m}{2}}\cos(\cdot)$ and following similar steps would prove (D.17), thereby establishing the identity for any $m = d-1, \dots, 1$. ∎

By definition of δ_D in (D.1),

$$\int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1}\phi_e(\mathbf{x}) = \int_{\mathbb{R}} \mathrm{d}k e^{-ik} \Phi_1(k), \tag{D.23}$$

where $\Phi_1(k) \equiv \Phi_1(\mathbf{x}_{<1}, k)$ as defined by the backward recurrence (D.14). By Lemma D.3,

$$\begin{aligned}
\Phi_1(k) = \frac{2}{(2\pi)^d} \sum_{j=1}^{d} \Bigg\langle &\mathrm{sc}_1(w_j|k|)\, \mathrm{d}w_j, \\
&\Big\langle \prod_{j'\neq j, j'=1}^{d} \text{p.v.}_{w_{j'}}(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}), \phi_e(\mathbf{w}) \Big\rangle_{\mathbf{w}_{\neq j}} \Bigg\rangle_{w_j}.
\end{aligned} \tag{D.24}$$

Substituting (D.24) into (D.23),

$$\int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x}) = \frac{2}{(2\pi)^d}\sum_{j=1}^{d}\int_{\mathbb{R}} e^{-ik}\times$$
$$\int_{\mathbb{R}} \mathrm{d}w_j \mathrm{sc}_1(w_j|k|)\left\langle \prod_{j'\neq j, j'=1}^{d} \mathrm{p.v.}_{w_{j'}}\left(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\right), \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}_{\neq j}}. \tag{D.25}$$

In odd dimension $d \geqslant 3$, substituting (C.5) yields (D.7) as required. In even dimension, replacing $\mathrm{sc}_1(\cdot) = (-1)^{\frac{d}{2}-1}\sin(\cdot)$ into (D.25) yields

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x}) &= \frac{(-1)^{\frac{d}{2}-1}2}{(2\pi)^d}\sum_{j=1}^{d}\int_{\mathbb{R}} \mathrm{d}k e^{-ik}\int_{\mathbb{R}} \mathrm{d}w_j \sin(w_j|k|)\times \\ &\qquad \underbrace{\left\langle \prod_{j'\neq j, j'=1}^{d} \mathrm{p.v.}_{w_{j'}}\left(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\right), \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}_{\neq j}}}_{=: f_j(w_j)} \\ &= \frac{(-1)^{\frac{d}{2}-1}2}{(2\pi)^d}\sum_{j=1}^{d}\int_{\mathbb{R}} \mathrm{d}k e^{-ik}\int_{\mathbb{R}} \mathrm{d}w_j \sin(w_j|k|) f_j(w_j),\end{aligned}$$

by which we recognize the same integrals as in (D.12). Substituting Euler's sine formula, then splitting the integrand while applying the reflective change of variable $k \mapsto -k$ to the second resulting integral,

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x}) \\ =&\frac{i(-1)^{\frac{d}{2}-1}}{(2\pi)^d}\sum_{j=1}^{d}\int_{\mathbb{R}} \mathrm{d}k e^{-ik}\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_j (e^{-iw_jk} - e^{iw_jk}) f_j(w_j) \\ =&\frac{i(-1)^{\frac{d}{2}-1}}{(2\pi)^d}\sum_{j=1}^{d}\left(\int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_j e^{-ik(w_j+1)} f_j(w_j) + \int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{sgn}(k)\int_{\mathbb{R}} \mathrm{d}w_j e^{-ik(w_j-1)} f_j(w_j)\right) \\ =&\frac{(-1)^{\frac{d}{2}-1}2}{(2\pi)^d}\sum_{j=1}^{d}\left\langle \mathrm{p.v.}_{w_j}\left(\frac{1}{w_j+1} + \frac{1}{w_j-1}\right), f_j(w_j)\right\rangle_{w_j},\end{aligned}$$

where we substituted (C.4) in the last step. Substituting $f_j(w_j)$ into the above equation, we recover

$$\int_{\mathbb{R}^{d+1}} \delta_D(\mathrm{d}(\mathbf{x},k))e^{-ik}\mathcal{F}^{-1}\phi_e(\mathbf{x}) = \frac{(-1)^{\frac{d}{2}-1}2}{(2\pi)^d}\times$$
$$\sum_{j=1}^{d}\left\langle \mathrm{p.v.}_{w_j}\left(\frac{1}{w_j+1} + \frac{1}{w_j-1}\right), \left\langle \prod_{j'\neq j, j'=1}^{d} \mathrm{p.v.}_{w_{j'}}\left(\frac{1}{w_j - w_{j'}} + \frac{1}{w_j + w_{j'}}\right), \phi_e(\mathbf{w})\right\rangle_{\mathbf{w}_{\neq j}}\right\rangle_{w_j},$$

which is (D.7). ok for me

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