Global existence for quasilinear wave equations

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Table of contents

Motivation and Examples

Motivation

Let (M, g) be an asymptotically flat $d + 1$ dimensional Lorentzian manifold, and consider the equation

$$\begin{cases} g^{\alpha\beta}\partial^2_{\alpha\beta}\phi = \mathcal{N}(\partial\phi, \partial\phi) \\ \phi(t = 0) = f \\ \partial_t\phi(t = 0) = g \end{cases}$$

where $\alpha, \beta = 0, 1, 2, \ldots, d$ range over Cartesian coordinates and $\mathcal{N}$ is a quadratic nonlinearity.
What can we say about solutions to the equation?

Motivation

Let (M, g) be an asymptotically flat $d + 1$ dimensional Lorentzian manifold, and consider the equation

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where $\alpha, \beta = 0, 1, 2, \ldots, d$ range over Cartesian coordinates and $\mathcal{N}$ is a quadratic nonlinearity.

What can we say about solutions to the equation?

Our primary motivations will be from the fields of **general relativity** and **compressible fluids**, though similar equations show up in e.g. electromagnetism or gauge theory.

Primary regime of interest is small data and $d = 3$, for reasons that will be expanded on later.

Examples

Einstein vacuum

$$\mathrm{Ric}_{\mu\nu}(g) = 0.$$

In appropriate coordinate system, takes the form

$$\begin{aligned}
-\frac{1}{2}(g^{-1})^{\alpha\beta}\partial_{\alpha\beta}g_{\mu\nu} &+ \frac{1}{2}(g^{-1})^{\alpha\sigma}(g^{-1})^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\beta g_{\alpha\nu} \\
&+ \frac{1}{2}(g^{-1})^{\alpha\sigma}(g^{-1})^{\beta\rho}\partial_\nu g_{\sigma\rho}\partial_\alpha g_{\beta\mu} \\
&- \frac{1}{2}(g^{-1})^{\alpha\sigma}(g^{-1})^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\nu g_{\alpha\beta} \\
&+ F_{\mu\nu}(g, \partial g) = 0
\end{aligned}$$

with flat solution given by $g = \mathrm{diag}(-1, 1, 1, 1)$ (Minkowski).

Examples II

Compressible Euler

$$\begin{cases} D_t\rho = -\rho\nabla\cdot u \\ D_t u = -\frac{\nabla p}{\rho} \\ D_t s = 0 \end{cases}$$

where ρ is the density, $p = p(\rho)$ is the pressure, u is the velocity, s is the entropy, and $D_t := \partial_t + u\cdot\nabla$ is the material derivative. (Can be rewritten as a wave equation for the velocity and logarithmic density).

The linear wave equation

$$\Box\phi := (-\partial_t^2 + \Delta)\phi = 0$$

Previous Work

1. Stability of Minkowski:
   - Christodulu, Klainerman '93 [1]
   - Lindblad, Rodnianski '10 [2]
   - Keir '18 [3]
   - Shen '23 [4]
   - Dafermos, Rodnianski '10 [5]
   - Hintz, Vasy '20 [6]
   - ...
2. Shock formation in Euler:
   - Christodoulou '07 [7].
   - Speck, Holzegel, Luk, Wong '16 [8]
   - ...
3. Wave equations: Lindblad '08 [9], John '81 [10], Yu '24 [11], ...

Review of the linear wave equation

Key properties of the linear wave equation:

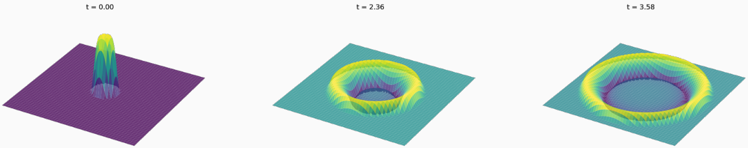
1. Conservation of energy: $\|\partial\phi(t,\cdot)\|_{L^2} = \|\partial\phi(0,\cdot)\|$.
2. **Finite speed of propagation**: if initial data is $\equiv 0$ in "backward light cone", then solution is 0.
3. **Dispersive decay:** Near the "wave zone" $\{r \approx t\}, |\partial\phi| \sim t^{-(d-1)/2}$

Can be read off from solutions using the fundamental solution, but for quasilinear problems, need robust methods of e.g. proving decay.

$d = 3$ corresponds to critical rate of t^{-1} decay.

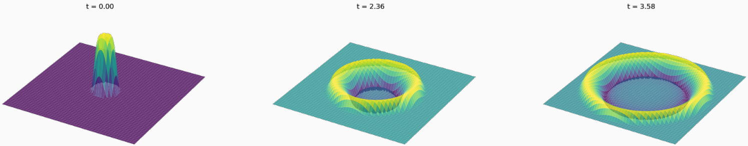
What does a free wave look like?

Why $t^{-(d-1)/2}$ decay?
Heuristic picture: solution begins life with support $\approx B(0,1)$, and is propagated along "forward light cone."



After time t, solution is supported on annulus of radius $\approx t$, so $|\operatorname{supp} \phi(t, \cdot)| \approx t^{d-1}$.

Why $(d-1)/2$ decay?



After time t, solution is supported on annulus of radius $\approx t$, so $|\operatorname{supp}\phi(t,\cdot)| \approx t^{d-1}$. On the other hand, $\|\partial\phi(t,\cdot)\|_{L^2}$ is conserved, which is consistent with decay rate above.
Can get improved decay rate for derivatives in direction of propagation, which will be critical in analyzing the nonlinearity.

The semilinear problem

General strategy to prove global existence

General method for obtaining global existence for initial data of size $\varepsilon^{3/2}$:

1. Identify suitable set of **weighted** vector fields $\Gamma \subseteq TM$ and commute to obtain equations for $\Gamma^k \phi$.
2. **Bootstrap:** assume energy estimates bounds of the form $\left\| \Gamma^k \partial \phi \right\|_{L^2} \leq \mathcal{O}(\varepsilon)$ hold up to some time $T_* < \infty$.
3. Use weighted vector fields to obtain pointwise estimates **with improved decay**: e.g. $|\partial \phi| \lesssim \varepsilon / t^{1+\delta}$.
4. Use this to show $\|\mathcal{N}\|_{L^1 L^2} = \mathcal{O}(\varepsilon^2)$, hence **can improve energy estimates:**

$$\left\| \Gamma^k \partial \phi \right\|_{L^2} \leq \mathcal{O}(\varepsilon^{3/2}) + \mathcal{O}(\varepsilon^2) \ll \mathcal{O}(\varepsilon).$$

5. Conclude $T_* = \infty$.

Obstructions

Consider the equation

$$\Box\phi = (\partial_t \phi)^2.$$

In [10], John showed that there are *no nontrivial global solutions* when $d = 3$.

What goes wrong?

Closing (or not) the bootstrap

Functional estimate: if $\Box\phi = F$, then

$$\|\partial\phi(T,\cdot)\|_{L^2} \leq \|\partial\phi(0,\cdot)\|_{L^2} + \|F\|_{L^1([0,T];L^2)}\,.$$

Suppose $\|\partial\phi\|_{L^2} \lesssim \varepsilon \implies |\partial\phi| \leq \varepsilon t^{-1}$. Then

$$\left\|(\partial_t\phi)^2\right\|_{L^1L^2} \leq \|\partial\phi\|_{L^1L^\infty}\,\|\partial\phi\|_{L^\infty L^2} \lesssim \varepsilon^2 \log T \neq \mathcal{O}(\varepsilon).$$

Still can show "almost global" existence, e.g. $T_* \approx e^{\mathcal{O}(1/\varepsilon)}$.
Actually sharp; John's proof shows $T_* \leq e^{\mathcal{O}(1/\varepsilon)}$.

Null structure

Consider now Nirenberg's example: the equation

$$\Box\phi = (\partial_t \phi)^2 - |\nabla\phi|^2 .$$

All small data solutions are now global (via change of variable $\psi = e^{\phi} - 1$), but why?

A return to the linear wave equation

Consider choosing coordinates $u := t - |x|, v := t + |x|, \theta \in \mathbb{S}^2$ on $\mathbb{R}^{3+1}$.
The vector fields $\partial_v, \partial_\theta$ are tangential to the *forward light cone* $\{u = u_0\}$.
In this frame, the nonlinearity above *factors* as

$$4\partial_u\phi\partial_v\phi - |\nabla\!\!\!/\phi|^2$$

which **decays at rate** $t^{-3/2}$.

Slower decaying data

Can extend this method to prove small data global existence for any equation satisfying **null condition** and some equations satisfying "**weak null condition**" say for data in $C_c^\infty(B(0,1))$.

Weakest possible assumption is *energy class*; that is, solutions for which $\left\|\Gamma^k \partial\phi\right\|_{L^2} < \infty$.

Next question: what's the weakest type of decay for which global existence still holds?

Actually necessary for some applications: **positive mass theorem** says that any compactly supported perturbation of Minkowski is Minkowski.

(Usual working assumption is Schwarzschild or $\mathcal{O}(1/t)$ tails).

Target decay class

We consider data satisfying the decay condition

$$|\partial^\alpha \phi(0, x)| \leq |x|^{-\delta-|\alpha|} \tag{1}$$

for any $\delta > 0$. Roughly speaking: decay like $r^{-\delta}$, and every derivative gains you a power. Now

$$\|\partial\phi\|_{L^2(B(0,R))} \approx \varepsilon R^{1/2-\delta},$$

which is far from bounded.

Bootstrap issues

Now have to worry about following issues:

1. How do you do energy estimates? Local well posedness?
2. Weaker pointwise estimates: $|\partial\phi| \lesssim t^{-1/2-\delta}$, $\left|\bar{\partial}\phi\right| \lesssim t^{-1-\delta}$ decay, which makes the nonlinearity much worse than L^1L^2.

Global existence for semilinear problems

Theorem

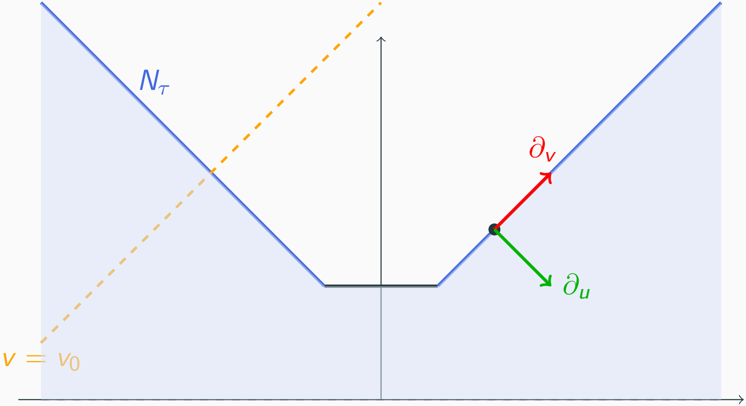
Let $d = 3, N \geq 15, \delta > 0$, and fix an equation satisfying the null or weak null conditions. There exists $\varepsilon_0 > 0$ such that if initial data satisfies the target decay condition with $\varepsilon < \varepsilon_0$ for all $|\alpha| \leq N$, then the solution exists globally in time. Furthermore, we have the decay rates

1. $|\partial\phi| \lesssim \varepsilon t^{-1}$,
2. $\left|\bar{\partial}\phi\right| \lesssim \varepsilon t^{-1-\delta/2}$.

Remark: there is a much simpler Picard iteration-type proof for this result, but it requires $N = \mathcal{O}(1/\delta)$ derivatives, and uses techniques very different than what we will be using for the quasilinear problem.

Energy estimates

Geometric setup: recall we've chosen coordinates $u := t - |x|, v := t + |x|, \theta \in \mathbb{S}^2$ on $\mathbb{R}^{3+1}$.
Consider foliating $\mathbb{R}^{3+1}$ by surfaces of the form $N_\tau := \{u = \tau\} \cap \{|x| > 1\}$.



The energy identity

Theorem

Suppose ϕ is a solution to $\Box\phi = F$. Then

$$\int_{Bulk} \partial_t \phi F = \int_{\partial} T_\phi(\partial_t, \nu).$$

Here ν is the unit normal, and T_ϕ is a 2-tensor given by

$$T_\phi(X, Y) = X\phi Y\phi - \langle X, Y\rangle \langle \nabla\phi, \nabla\phi\rangle /2.$$

Key fact: when boundary component is $\{t = t_0\}$, RHS is

$$(\partial_t\phi)^2 + \sum_i (\partial_i\phi)^2,$$

and when boundary is N_τ, RHS is

$$(\partial_v\phi)^2 + |\nabla\!\!\!/\phi|^2 .$$

“The energy identity is the only method known to man that does not lose derivatives.” - J. Zhao.

Pointwise estimates

How do you actually obtain pointwise estimates?
Consider the following set of vector fields, which generate (conformal) isometries of Minkowski:

1. Translations: ∂_t, ∂_i.
2. **Scaling:** $S := t\partial_t + x^i\partial_i = u\partial_u + v\partial_v$.
3. Rotations: $\Omega_{ij} := x_i\partial_j - x_j\partial_i$.
4. Lorentz boosts: $\Omega_{0j} := t\partial_i + x_i\partial_t$.

Fundamental theorem of the vector field method

Theorem (Klainerman-Sobolev)

Let Γ be the full set of commuting vector fields. For any sufficiently smooth ϕ, we have the pointwise bounds

$$|\partial\phi(u,v,\theta)| \lesssim u^{-1/2}v^{-1} \sum_{|\alpha|\leq 4} \|\Gamma^\alpha \partial\phi\|_{L^2} .$$

Furthermore,

$$\left|\bar{\partial}\phi(u,v,\theta)\right| \lesssim v^{-3/2} \sum_{|\alpha|\leq 4} \|\Gamma^\alpha \partial\phi\|_{L^2} .$$

Elliptic estimates

We actually prefer to use an elliptic estimate that requires commuting with fewer vector fields.

Theorem (Luk, Oh '23)

Set $\Gamma := \{S, \partial_t, \Omega_{ij}\}$. *Fixing* $U \lesssim R$, *set* $A := \{u \sim U, r \sim R\}$ *and set* B *to be a enlarged copy of* A. *Also let* $s := |\alpha| + |\beta|$. *Then we have the estimate*

$$\left\|(u\partial_u)^\alpha (r\bar{\partial})^\beta \phi\right\|_{L^2(A)} \lesssim \left\|\Gamma^{\leq s}\phi\right\|_{L^2(B)} + UR\left\|(u\partial_u)^{\leq s}(r\bar{\partial})^{\leq s}\Box\phi\right\|_{L^2(B)}.$$

Ideas for the elliptic estimates

We combine the elliptic estimates above with the following rescaled Sobolev inequality:

Theorem

$$\|\phi\|_{L^\infty(B_R)} \lesssim R^{-d/2} \sum_{|\alpha|\leq (d+1)/2} \|(R\partial)^\alpha \phi\|_{L^2(B_{2R})}$$

in all odd space dimensions d.

This implies the estimates

$$\|\phi\|_{L^\infty(\{u\sim U, r\sim R\})} \lesssim R^{-3/2} U^{-1/2} \sum_{|\alpha|+|\beta|+|\gamma|\leq 5} \left\|(u\partial_u)^\alpha (r\partial_r)^\beta \Omega^\gamma \phi\right\|_{L^2(\{u\sim U, r\sim R\})}.$$

Finishing the pointwise estimates

Suppose we are in the region $r \gg t$. We estimate derivatives in the following way.

- **Good derivatives.** By conservation of energy, the integral of $\left|\Gamma^k \bar{\partial}\phi\right|$ over any N_τ is of size comparable to data. By the elliptic estimate and the rescaled Klainerman-Sobolev estimate on N_τ, this implies $\left|\bar{\partial}\phi\right| \lesssim r^{-3/2} \|\partial\phi\| \lesssim r^{-1-\delta}$.
- **Bad derivatives.** Using the elliptic estimate over a $u \sim U, r \sim R$ estimate gives $|\partial\phi| \lesssim r^{-1}u^{-1/2} \lesssim r^{-1/2-\delta}$ since we lose a factor of r integrating over spacetime instead of just a surface.

Similar but more complicated elliptic estimates for region $t \gg r$.

Pointwise estimates for bad derivatives

We still need to recover t^{-1} decay for $\partial_u \phi$. Although the estimates above only yield a decay rate of $t^{-1/2+\delta}$, the key thing to note is that ∂_u satisfies a transport equation which allows us to improve this decay a posteriori. In particular, we have that

$$\partial_v(r\partial_u\phi) = r\not\Delta\phi + \partial_v\phi + r\mathcal{N}(\partial\phi, \partial\phi) = \mathcal{O}(r^{-1-\delta})$$

which is integrable.

Tools for the quasilinear problem

Tools for the quasilinear problem

We move now to the realm of quasilinear equations.

Model equation

$$-(1+\phi)\partial_t^2\phi + \Delta\phi = 0.$$

(Corresponds to $g = \mathrm{diag}(-1/(1+\phi), 1, 1, 1)$).
New issues: derivative loss, additional term $\phi\partial_t^2\phi$.

Quasilinear equations

First attempt: just put the quasilinear term $\phi\partial_t^2\phi$ into error. Ignoring loss of derivatives, we have

$$\left\|\phi\partial_t^2\phi\right\|_{L^1L^2} \lesssim \|\phi\|_{L^1L^\infty}\left\|\partial_t^2\phi\right\|_{L^\infty L^2} \text{ or } \|\phi\|_{L^\infty L^2}\left\|\partial_t^2\phi\right\|_{L^1L^\infty}$$

but we only have $|\phi| \lesssim v^{-\delta}$, and $\|\partial_t^2\phi\|_{L^2}$ is at best bounded. Similarly, the best pointwise decay for $\partial_t^2\phi$ we can get is t^{-1}, and ϕ isn't bounded in energy.

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but we only have $|\phi| \lesssim v^{-\delta}$, and $\|\partial_t^2\phi\|_{L^2}$ is at best bounded.
Similarly, the best pointwise decay for $\partial_t^2\phi$ we can get is t^{-1}, and ϕ isn't bounded in energy.
Need derivatives on both terms in order to gain decay.
Solution: construct specialized vector fields Γ depending on the metric g that commute better with the equation.

Global existence

Theorem

Let $d = 3, N \geq 20$, and $\delta > 0$. There exists $\varepsilon_0 > 0$ such that if the initial data satisfies the target decay condition with $\varepsilon < \varepsilon_0$ for all $|\alpha| \leq N$, then the solution to the model equation exists globally in time (with similar decay rates to the semilinear case).

Theorem

Let $d = 3, N \geq 20$, and $\delta > 0$. There exists $\varepsilon_0 > 0$ such that if "initial data" to the Einstein vacuum equations satisfies the target decay condition with $\varepsilon < \varepsilon_0$ for all $|\alpha| \leq N$, then the solution to the EVE exists globally in time, is future geodesically complete, and decays asymptotically back to the Minkowski solution.

Null geometry adapted to g

How do we construct vector fields that "see" the geometry of g?

- Construct an analogue of Minkowski $t - r$ by solving eikonal equation $\langle \nabla u, \nabla u \rangle = 0$ with initial data on appropriately chosen hypersurface.
- Let $\mu^{-1} := \langle \nabla u, \nabla r \rangle$ (inverse foliation density) and define $L := \mu \nabla u$ so that $L(r) = 1$ (analogue of "good derivative" ∂_v; note $\mu \equiv 1$ in Minkowski).
- Use (u, r, θ) coordinates, where θ are angular coordinates propagated by L.

- The frame $\{L, \partial_\theta\}$ spans the tangent space to constant u hypersurfaces.
- Complete the frame with vector field T, chosen to satisfy normalization/orthogonality conditions. (Analogue of Minkowski ∂_t).
- Define "scaling vector field" $S := rL + u\mu T$, and commute with $\Gamma := \{S, T, r^2 \nabla\!\!\!/\}$.
- In these coordinates,

$$\tilde{\Box}_g \phi = -L(-L + 2T)\phi + \Delta\!\!\!/\,\phi - \mathrm{tr}_{g\!\!\!/}\,\chi T\phi - \mathrm{tr}_{g\!\!\!/}\,\kappa L\phi - \zeta^\sharp \phi$$

which allows us to easily compute commutators.

Geometric estimates

Additional geometric quantities that appear in the estimates:

1. Second fundamental forms χ, κ associated to constant u, r "spheres."
2. Torsion ζ associated to null hypersurfaces of constant u (0 on Minkowski).
3. Gauss curvature K used for elliptic estimates.
4. **Ricci curvature** $\mathrm{Ric}_{\mu\nu}$ appears when differentiating χ, κ terms.

Variety of technical difficulties associated to losing derivatives, but eventually can get estimates to close.

Thanks!

Thanks!
Questions?

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Morawetz estimates

It is also useful to have estimates for bulk energy terms. We state this estimate below.

Theorem (Morawetz)

$$\sup_{k\in\mathbb{N}} \int_{r\sim 2^k} \left(r^{-1}\phi^2 + r\left|\partial\phi\right|^2\right) r^{-2} dx$$
$$\lesssim \|\partial\psi\|^2_{L^2(Future)} + \|\partial\psi\|^2_{L^2(Past)} + \int_{Bulk} \left|\partial_r \psi r F\right|.$$

Takeaway: integral over all of time but a compact set in space scales like $r^{1/2}$.

Proof of Morawetz

We begin again with writing the wave equation as

$$-\partial_u \partial_v \psi + \not{\Delta} \psi = -\partial_t^2 \psi + \partial_r^2 \psi + \not{\Delta} \psi = rF$$

where $\psi := r\phi$. Let $h(r)$ be a (bounded, smooth) function to be determined later. Multiplying by $h(r)\partial_r \psi$ and commuting, we deduce that

$$h(r)\partial_r \psi(-\partial_t^2 \psi + \partial_r^2 \psi + r^{-1} \mathring{\not{\Delta}} \phi) = h(r)\partial_r \psi rF \qquad (2)$$

(here we use the fact that $\mathring{\not{\Delta}} = r^2 \not{\Delta}$ and commuted the multiplication of ϕ by r with the angular derivatives).

We would like to write the left hand side as

$$-\partial_t \left(h(r)\partial_r\psi\partial_t\psi\right) + \frac{1}{2}\partial_r \left(h(r)\left[(\partial_t\psi)^2 + (\partial_r\psi)^2\right]\right)$$
$$+\mathring{\not{d}}\mathrm{iv}(h(r)\partial_r\psi r^{-1}\mathring{\not\nabla}\phi) + \mathrm{Error}.$$

Denote the terms in (2) by A, B, C and the terms in the above equation by I, II, III. We now calculate the error terms. First, we have

$$A = I + h(r)\partial_t\partial_r\psi\partial_t\psi.$$

(Note all t derivatives of h vanish identically by construction). Now we compute that

$$II = B + h(r)\partial_r\partial_t\psi\partial_t\psi + \frac{1}{2}\left(h'(r)\left[(\partial_t\psi)^2 + (\partial_r\psi)^2\right]\right)$$

so

$$B = II - h(r)\partial_r\partial_t\psi\partial_t\psi - \frac{1}{2}\left(h'(r)\left[(\partial_t\psi)^2 + (\partial_r\psi)^2\right]\right)$$

Finally,

$$III = C + \frac{h(r)}{r}\left\langle \mathring{\nabla\!\!\!/}\partial_r\psi, \mathring{\nabla\!\!\!/}\phi \right\rangle \tag{3}$$
$$= C + h(r)\left\langle \mathring{\nabla\!\!\!/}\partial_r\phi, \mathring{\nabla\!\!\!/}\phi \right\rangle + \frac{h(r)}{r}\left\langle \mathring{\nabla\!\!\!/}\phi, \mathring{\nabla\!\!\!/}\phi \right\rangle \tag{4}$$
$$= C + \frac{1}{2}\left(\partial_r\left(h(r)\left|\mathring{\nabla\!\!\!/}\phi\right|^2\right) - h'(r)\left|\mathring{\nabla\!\!\!/}\phi\right|^2\right) + \frac{h(r)}{r}\left\langle \mathring{\nabla\!\!\!/}\phi, \mathring{\nabla\!\!\!/}\phi \right\rangle \tag{5}$$

Thus, the overall error term is equal to

$$-\left[\frac{1}{2}\left(h'(r)\left[(\partial_t\psi)^2 + (\partial_r\psi)^2\right]\right) + \frac{1}{2}\left(\partial_r\left(h(r)\left|\mathring{\nabla\!\!\!/}\phi\right|^2\right) - h'(r)\left|\mathring{\nabla\!\!\!/}\phi\right|^2\right) + \frac{h(r)}{r}\left|\mathring{\nabla\!\!\!/}\phi\right|^2\right]$$

(note that we used the equality $\partial_r\partial_t = \partial_t\partial_r$ to cancel the first error term).

Integrating by parts, we obtain the equality

$$\int_{Bulk} \frac{1}{2}\left(h'(r)\left[(\partial_t\psi)^2+(\partial_r\psi)^2\right]\right)+\left(\frac{h(r)}{r}-\frac{h'(r)}{2}\right)\left|\mathring{\nabla}\phi\right|^2 \tag{6}$$

$$=-\left(\int_{Future} h(r)\partial_r\psi\partial_t\psi-\int_{Past} h(r)\partial_r\psi\partial_t\psi+\int_{Bulk} h(r)\partial_r\psi rF\right). \tag{7}$$

Now set $h(r)=\frac{r}{2^k+r}$. Note that $h(r)\le 1$, and we have the equality

$$h'(r)=\frac{2^k}{(2^k+r)^2} \implies \frac{h(r)}{r}-\frac{h'(r)}{2}=\frac{2^k+2r}{2(2^k+r)^2}.$$

We obtain the estimate

$$\int_{Bulk} \frac{1}{2}\left(\frac{2^k}{(2^k+r)^2}\left[(\partial_t\psi)^2+(\partial_r\psi)^2\right]\right)+\left(\frac{2^k+2r}{2(2^k+r)^2}\right)\left|\mathring{\nabla}\phi\right|^2 \tag{8}$$

$$\leq -\left(\int_{Future} h(r)\partial_r\psi\partial_t\psi - \int_{Past} h(r)\partial_r\psi\partial_t\psi + \int_{Bulk} h(r)\partial_r\psi rF\right). \tag{9}$$

Hardy

We now quickly prove a functional estimate that will be used above. Observe that for any $\alpha > 1$ we have

$$\int_{r_1}^{r_2} r^{-\alpha} f^2 dr = \frac{1}{\alpha - 1} \left(\int_{r_1}^{r_2} 2(r^{-\alpha+1}) ff' dr - r^{-\alpha+1} f^2 \,|_{r_1}^{r_2} \right)$$

Applying Young's inequality gives

$$\begin{aligned} \int_{r_1}^{r_2} r^{-\alpha} f^2 dr \leq \frac{1}{2} \int_{r_1}^{r_2} r^{-\alpha} f^2 dr \\ + \frac{1}{\alpha - 1} \left(\left(\int_{r_1}^{r_2} (r^{-\alpha+2}) f'^2 dr \right) - r^{-\alpha+1} f^2 \,|_{r_1}^{r_2} \right) \end{aligned}$$

and hence

$$\begin{aligned} \int_{r_1}^{r_2} r^{-\alpha} f^2 dr + \left[r^{-\alpha+1} f^2 \right] (r_2) \\ \lesssim_\alpha \left(\int_{r_1}^{r_2} (r^{-\alpha+2}) f'^2 dr \right) + \left[r^{-\alpha+1} f^2 \right] (r_1). \end{aligned}$$

Now note that when $r \sim 2^k$ the coefficients in (8) control r^{-1}. Applying the inequality above with $f = \int r\phi = \int \psi, \alpha = 3$, we deduce that

$$\sup_{k\in\mathbb{N}} \int_{r\sim 2^k} r^{-1}\phi^2 \lesssim \sup_{k\in\mathbb{N}} \int_{r\sim 2^k} r^{-1} |\partial\psi| \leq \sup_k (8).$$

Using the fact that $\partial\psi = \phi + r\partial\phi$ and the triangle inequality, we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} \int_{r\sim 2^k} r^{-1}\phi^2 + r |\partial\phi|^2 &\lesssim \sup_k (8) \\ &\leq \|\partial\psi\|^2_{L^2(Future)} + \|\partial\psi\|^2_{L^2(Past)} + \int_{Bulk} |\partial_r \psi r F| .\end{aligned}$$

Pointwise estimates towards timelike infinity

So far, we've only discussed how to do pointwise estimates in the region $u \lesssim r$. It remains to prove decay in the regime $u \gg r$, or, equivalently, $t \gg r$. Since the weights attached to good derivatives only grow as $r \to \infty$, we will need to use the equation to exchange good derivatives for derivatives with weights in t. To do so, in this regime, we combine a variant of the spacetime elliptic estimate from above with an elliptic estimate involving Δ to improve u decay in this region. The eventual goal is to show that

$$|\partial\phi(U, R)|^2 \lesssim U^{-3} R^{\varepsilon}$$

We will need the following functional estimate together with the obvious observation $\Delta = \Box + \partial_t^2$.

Theorem (Luk, Oh '23)

For any $\gamma \in (-3/2, -1/2)$, *we have the functional estimate*

$$\sum_{|\alpha| \leq 2} \|(\langle r \rangle \partial)^\alpha \phi\|_{L^{2,\gamma}} \leq C_\gamma \|\Delta \phi\|_{L^{2,\gamma+2}} . \tag{10}$$

where

$$\|f\|_{L^{2,\gamma}} = \|f\|_{L^2(\mathbb{R}^3, r^{2\gamma} dx)}$$

For any fixed U, R, we have

$$\begin{aligned}
|\partial\phi(U,R)|^2 &\lesssim U^{-1}R^{-\varepsilon}\int_{u\sim U}\left\|(u\partial_u)^\alpha(r\partial_r)^\beta\Omega^\gamma\partial\phi\right\|^2_{L^{2,-3/2+\varepsilon/2}} \\
&\lesssim U^{-1}R^{-\varepsilon}\int_{u\sim U}\left\|\Gamma^{\leq 3}\partial\phi\right\|^2_{L^{2,-3/2+\varepsilon/2}} \\
&\lesssim U^{-1}R^{-\varepsilon}\int_{u\sim U}\left\|\Delta\Gamma^{\leq 3}\partial\phi\right\|^2_{L^{2,1/2+\varepsilon/2}} \\
&\lesssim U^{-1}R^{-\varepsilon}\int_{u\sim U}\left\|\partial^2_{tt}\Gamma^{\leq 3}\partial\phi\right\|^2_{L^{2,1/2+\varepsilon/2}} + \mathit{better}
\end{aligned}$$

Use Morawetz to conclude this is enough decay to close.

Geometric quantities that need to be controlled

Lemma

We have the equality

$$\mathrm{Ric}_{LL} = L(\cdots) + L^\mu L^\nu \tilde{\Box}_g g_{\mu\nu} + \bar{\partial} g \partial g$$

Proof sketch We recall that the Ricci tensor is given by

$$\mathrm{Ric}_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$$

and hence

$$\mathrm{Ric}_{LL} = L^\mu L^\nu \partial_\alpha \Gamma^\alpha_{\mu\nu} - L^\nu L(\Gamma^\alpha_{\nu\alpha}) + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{LL} - \Gamma^\alpha_{L\beta}\Gamma^\beta_{\alpha L}.$$

We first note that

$$\Gamma^\alpha_{\alpha\beta} = g^{\alpha\alpha'}(\partial_\alpha g_{\beta\alpha'} + \partial_\beta g_{\alpha'\alpha} - \partial_{\alpha'} g_{\alpha\beta}) = g^{\alpha\alpha'}(\partial_\beta g_{\alpha'\alpha})$$

since the first and third terms are antisymmetric with respect to switching α and α'. We can thus expand the third term as

$$L^\mu L^\nu g^{\alpha\alpha'}(\partial_\beta g_{\alpha'\alpha}) g^{\beta\beta'}(\partial_{\beta'} g_{\mu\nu})$$

Now we expand the first term as

$$
\begin{aligned}
&L^\mu L^\nu \partial_\alpha (g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}))/2 \\
&\quad = L^\mu L^\nu (\partial_\alpha g^{\alpha\beta})(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})/2 \\
&\qquad + L^\mu L^\nu g^{\alpha\beta} \partial_\alpha (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})/2 \\
&\quad = L^\nu (\partial_\alpha g^{\alpha\beta})(L g_{\nu\beta}) + L^\mu L^\nu g^{\alpha\alpha'} \partial_\alpha g_{\alpha'\beta'} g^{\beta\beta'} (\partial_\beta g_{\mu\nu})/2 \\
&\qquad + L^\nu g^{\alpha\beta} L(\partial_\alpha g_{\nu\beta}) - L^\mu L^\nu g^{\alpha\beta} (\partial^2_{\alpha\beta} g_{\mu\nu})/2 \\
&\quad = L^\nu (\partial_\alpha g^{\alpha\beta})(L g_{\nu\beta}) + L^\mu L^\nu g^{\alpha\alpha'} \partial_\alpha g_{\alpha'\beta'} g^{\beta\beta'} (\partial_\beta g_{\mu\nu})/2 \\
&\qquad + L^\nu g^{\alpha\beta} L(\partial_\alpha g_{\nu\beta}) - L^\mu L^\nu \tilde{\Box}_g g_{\mu\nu}/2
\end{aligned}
$$

and the fourth term as

$$
\begin{aligned}
-(&L^\mu L^\nu g^{\alpha\alpha'} (\partial_\mu g_{\beta\alpha'} + \partial_\beta g_{\mu\alpha'} - \partial_{\alpha'} g_{\beta\mu}) \\
&\quad g^{\beta\beta'} (\partial_\nu g_{\alpha\beta'} + \partial_\alpha g_{\beta'\nu} - \partial_{\beta'} g_{\nu\alpha}))/4.
\end{aligned}
$$

All of the terms above involving a contraction of the form $L^\mu \partial_\mu$ are already admissible quadratic terms. Similarly, all the products of the form $g^{\alpha\alpha'} \partial_\alpha(\cdots)\partial_{\alpha'}(\cdots)$ are admissible quadratic terms, as

$$g^{\alpha\alpha'} = -\frac{1}{2} L^\alpha \underline{L}^{\alpha'} - \frac{1}{2} \underline{L}^{\alpha'} L^\alpha + \not{g}^{AB} (X_A)^\alpha (X_B)^{\alpha'}$$

so there are no terms of the form $\underline{L}(\cdots)\underline{L}(\cdots)$ above. The primary term lacking null structure is

$$-(L^\mu L^\nu g^{\alpha\alpha'} g^{\beta\beta'} (\partial_\beta g_{\mu\alpha'})(\partial_\alpha g_{\beta'\nu}))/4$$

but this exactly cancels one of the terms above.

Appendix: strong non-global existence

The goal of this appendix is to show that all global solutions to the equation

$$\begin{cases} \Box u = (\partial_t u)^2 \\ u(t=0) = u_0 \\ \partial_t u(t=0) = u_1 \end{cases} \tag{11}$$

with u_i smooth and compactly supported are trivial, implying that all nontrivial solutions blowup in finite time. Following the Keir/Luk notes, we will deduce this via a reduction to spherical means and an ODE blowup type result. We begin with the Darboux equation. For $h \in C^\infty(\mathbb{R}^n)$, define

$$M_h(x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)} h(y)dy = \int_{\mathbb{S}^1} h(x+rz)dz.$$

We claim the following:

Theorem

With M_h defined as above, we have

$$\Delta_x M_h(x, r) = \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) M_h(x, r).$$

Proof.

By definition, we have

$$|B(0,1)| \int_0^R r^{n-1} M_h(x,r) dr = \int_{|y| \leq R} h(x+y) dy.$$

Taking Δ_x on both sides and integrating by parts, we deduce that

$$\begin{aligned} |B(0,1)| \int_0^R r^{n-1} \Delta_x M_h(x,r) dr &= \int_{|y| \leq R} \Delta_x h(x+y) dy \\ &= \int_{|y| \leq R} \partial^i \partial_i h(x+y) dy \\ &= \int_{|y| = R} \frac{y^i}{R} \partial_i h(x+y) dy. \end{aligned}$$

$\square$

Proof.

Changing variables to $z = y/R$, this is further equal to

$$R^{n-1} \int_{\mathbb{S}^1} z^i \partial_i h(x + rz) dy = |B(0,1)| \, R^{n-1} \partial_r M_h(x,r).$$

Now taking derivatives with respect to r, we deduce that

$$R^{n-1} \Delta_x M_h(x,r) = (n-1) R^{n-2} \partial_r M_h(x,r) + R^{n-1} \partial_r^2 M_h(x,r)$$

as desired. □

We will also need the following calculation, where all functions are now living in $\mathbb{R}^{n+1}$:

Lemma

If $\Box u = F$*, then*

$$M_F(0,r) = -\partial_t^2 M_u(0,r) + \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) M_u(0,r).$$

where now M_F *implicitly also may depend on time.*

Proof.

For any fixed r, we have

$$(-\partial_t^2 + \Delta_x) M_u(x,r) = \Box_x M_u(x,r) = M_{\Box u}(x,r)$$

so using the previous equation and plugging in $x = 0$ yields the result. $\square$

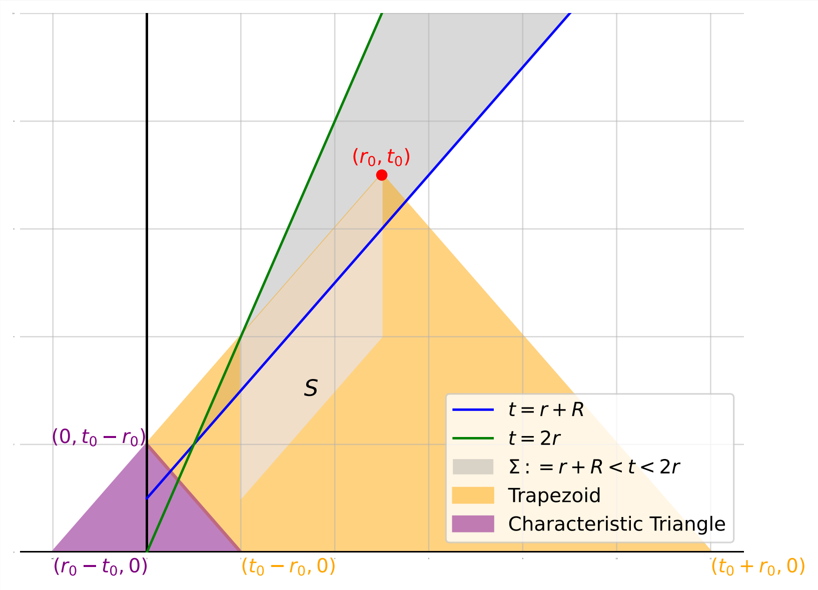
Finally, we will need the following explicit formula for solutions to the wave equation in $1+1$ dimensions.

Theorem

The solution to the equation $\Box v = F$ *with initial data* $v(t=0) = v_0$ *and* $\partial_t v(t=0) = v_1$ *is given by*

$$v(t,r) = \frac{1}{2}\left[v_0(t-r) + v_0(t+r) + \int_{|r-r'|\leq t} v_1(r')dr' \right.$$
$$\left. + \int_{T(t,r)} F(t',r')dt'r' \right]$$

where $T(t,r) := \{(t',r') \mid t' \leq t, |r-r'| \leq t-t'\}$ *is the backward light cone from* (t,r)*.*



(r_0, t_0)
S
$(0, t_0 - r_0)$
$(r_0 - t_0, 0)$
$(t_0 - r_0, 0)$
$(t_0 + r_0, 0)$
$t = r + R$
$t = 2r$
$\Sigma := r + R < t < 2r$
Trapezoid
Characteristic Triangle

Now suppose we have a global C^2 solution of (11), and take R to be such that the initial data is supported inside of $B(x, R)$. Define $v(t, r) := M_u(0, r)$ and $u := t - r$. Note that $\partial_r^2(rv) = r(\partial_r^2 + \frac{2}{r}\partial_r)v$, so using 12, we know that rv satisfies the $1 + 1$ dimensional wave equation

$$\partial_t^2(rv) - \partial_r^2(rv) = rF =: rM_{(\partial_t u)^2}$$

In particular, using 13 and dividing by r, we have that

$$v(t_0, r_0) = \frac{1}{2r_0}\left(\tilde{V} + \int_{T(r_0,t_0)} rFdrdt\right)$$

where $\tilde{V}$ is a solution to $\Box\tilde{V} = 0$ with the correct data.

For $(t_0, r_0) \in \Sigma := \{r + R < t < 2r\}$, the contribution from the homogeneous solution vanishes, and hence

$$v(t_0, r_0) = \frac{1}{2r_0}\left(\int_{T(r_0,t_0)}^{t_0+r_0} rFdrdt\right) \tag{12}$$

$$= \frac{1}{2r_0}\left(\int_{T(r_0,t_0)-T(0,u_0)} rFdrdt\right) \tag{13}$$

$$\geq \frac{1}{2r_0}\left(\int_{T^*(r_0,t_0)} r(\partial_t v)^2 drdt\right) \tag{14}$$

where the last inequality follows from Jensen's inequality.

By positivity, we can further restrict the area of integration on the right hand side to the set

$$\{u_0 < r < r_0, -R < u < u_0\}$$

to replace the right hand side by

$$\frac{1}{2r_0} \int_{u_0}^{r_0} r dr \int_{r-R}^{r+u_0} (\partial_t v)^2 dt.$$

Now note that

$$|v(r, r+u_0)| = \left| \int_{r-R}^{r+u_0} \partial_t v(r,t) dt \right| \leq (u_0 + R)^{1/2} \left| \int (\partial_t v)^2 \right|^{1/2}$$

so plugging this into the previous equation yields

$$\begin{aligned} v(t_0, r_0) &\geq \frac{1}{2r_0} \int_{u_0}^{r_0} rdr \int_{r-R}^{r+u_0} (\partial_t v)^2 dt \\ &\geq \frac{1}{2r_0(u_0+R)} \int_{u_0}^{r_0} rv(r, r+u_0) dr \end{aligned}$$

Now define

$$\beta(r_0) := \int_{u_0}^{r_0} rv(r, r+u_0)^2 dr$$

and note

$$\beta'(r_0) = r_0 v(r_0, r+u_0)^2 \geq \frac{1}{4(R+u_0)r_0} \beta^2$$

by the equation above.

Integrating this functional inequality implies that, if $\beta(r_0) \neq 0$, then

$$\frac{1}{\beta\left(r_0\right)} \geq \frac{1}{\beta\left(r_0\right)} - \frac{1}{\beta(r)} \geq \frac{1}{4}\frac{1}{(R+u_0)^2}\log\frac{r}{r_0}$$

for all r, which is impossible. We conclude that $\beta = 0$ in Σ, hence $v = 0$ in Σ. Now using 12, we deduce that $v \equiv 0$ on a full slice, which concludes.