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Discrete Mathematics

Workshop 5 — Revised Proof

1. Prove: $P(x)\, R\, P(x)$ (Reflexive)

Let $P(x)$ be an arbitrary polynomial with arbitrary integer coefficients.

The relation R states that $P(x)\, R\, Q(x)$ if and only if $n \mid (P(x) - Q(x))$ for an arbitrary integer n.

We know that $P(x) - P(x) = 0$. Similarly, we also know that $n \mid 0$ for any integer n.

Because of this, it follows that $P(x)\, R\, P(x)$.

Therefore, R is reflexive.

2. Prove: $P(x)\, R\, Q(x) \implies Q(x)\, R\, P(x)$ (Symmetric)

Let $P(x)$ and $Q(x)$ be arbitrary polynomials with arbitrary integer coefficients.

By the definition of R, we will assume $n \mid (P(x) - Q(x))$. This means $P(x) - Q(x) = nK(x)$ for some polynomial $K(x)$, due to the divides relationship.

Rearranging,

$$Q(x) - P(x) = -(P(x) - Q(x)) = -nK(x) = n(-K(x)).$$

Since $-K(x)$ is a polynomial, it follows that $n \mid (Q(x) - P(x))$, so $Q(x)\, R\, P(x)$.

Therefore, R is symmetric.

3. Prove: $P(x)\, R\, Q(x) \wedge Q(x)\, R\, Z(x) \implies P(x)\, R\, Z(x)$ (Transitive)

Let $P(x)$, $Q(x)$, and $Z(x)$ be arbitrary polynomials with arbitrary integer coefficients.

We will assume $P(x)\, R\, Q(x)$ and $Q(x)\, R\, Z(x)$. By the definition of R:

$$P(x) - Q(x) = nK(x), \quad Q(x) - Z(x) = nJ(x),$$

for arbitrary polynomials $K(x)$ and $J(x)$.

Adding these equations:

$$P(x) - Z(x) = (P(x) - Q(x)) + (Q(x) - Z(x)) = nK(x) + nJ(x) = n(K(x) + J(x)).$$

Let $F(x) = K(x) + J(x)$. Since $F(x)$ is a polynomial, it follows that $n \mid (P(x) - Z(x))$.

Finally, it then follows that $P(x)\, R\, Z(x)$, and R is transitive.

4. Conclusion

Since R is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$