Please submit your solutions (both tex and PDF files) to WyoCourses; and hand in a printed version in class. If you are a graduate student, please also finish Problem 5.

The following problems are mainly selected from Ross's book.

1. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

   Solution:

   $$\text{All possible assignments} = 20! \approx 2.4329 \times 10^{18}$$

2. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

   Solution:

   (a) All area codes:
   Possible outcomes for 1st digit: 8
   Possible outcomes for 2nd digit: 2
   Possible outcomes for 3rd digit: 9

   $$\text{All possible area codes} = 8 \times 2 \times 9 = 144$$

   (b) Area codes starting with a 4:
   Possible outcomes for 1st digit: 1
   Possible outcomes for 2nd digit: 2
   Possible outcomes for 3rd digit: 9

   $$\text{All possible area codes starting with } 4 = 1 \times 2 \times 9 = 18$$

3. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

   Solution:
   Out of 10 children, 7 of them will get 1 gift each and 3 will get no gift. Of the 7 children that receive gifts, there are 7! possible assignments of the gifts.

   $$\text{Possible distinct results} = \binom{10}{7} 7! = 120 \times 5040 = 604800$$

4. Prove that

   $$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0};$$

   Solution:

   (a) The left hand side is the number of ways to choose r things from $n+m$ things

   (b) $n+m$ can be divided into n number of things and m number of things

   (c) Now with one group of n things and one group of m things, choosing r can be done by choosing 0 from the group of n things and r from m things,

   (d) Or 1 thing from n and $r-1$ thing from m

   (e) Or 2 things from n and $r-2$ things from m
   ...and so on

   (f) The sum of the ways that r things can be chosen from a group of n things and a group of m things is the same as the number of ways to choose r from $n+m$ things (the left hand side)

   (g) The left hand side and the right hand side are equal and then prove

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

Solution: Using the previous proof, let $m = n = r$

$$\binom{n+m}{r} = \binom{n+n}{n} = \binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Using the fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k!)} = \binom{n}{n-k}$

Then therefore $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$