Please submit your solutions (both tex and PDF files) to WyoCourses; and hand in a printed version in class. If you are a graduate student, please also finish Problem 6.

The following problems are mainly selected from Ross's book.

1. Let X be the winnings of a gambler. Let $p(i) = P(X = i)$ and suppose that

$$p(0) = 1/3; p(1) = p(-1) = 13/55; p(2) = p(-2) = 1/11; p(3) = p(-3) = 1/165.$$

Compute the conditional probability that the gambler wins i, $i = 1, 2, 3$, given that he wins a positive amount.

Solution: Let E be the event that the gambler wins a positive amount of money.

$$\begin{aligned} P(E) &= P(1) + P(2) + P(3) \\ &= 13/55 + 1/11 + 1/165 \\ &= 55/165 \\ &= 1/3 \\ P(X = i|E) &= \frac{p(i)}{P(E)} \\ &= \frac{p(i)}{\frac{1}{3}} \\ &= 3p(i) \end{aligned}$$

2. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, $40, 33, 25$, and 50 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

   (a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

   (b) Compute $E[X]$ and $E[Y]$.

Solution:

   (a) $E[X] > E[Y]$, because there are 148 students to choose from, but only 4 bus drivers.

   (b)

$$\begin{aligned} E[X] &= 40 \times 40/148 + 33 \times 33/148 + 25 \times 25/148 + 50 \times 50/148 \\ &= 5814/148 \\ &\approx 39.28 \\ E[Y] &= 40 \times 1/4 + 33 \times 1/4 + 25 \times 1/4 + 50 \times 1/4 \\ &= 148/4 \\ &= 37 \end{aligned}$$

3. Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately 0:135, increasing to approximately $0 : 268$ if the man does have cancer. If, on the basis of other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that

   (a) the test indicated an elevated PSA level?

(b) the test did not indicate an elevated PSA level?

Denote by C the event that the patient has cancer and by E the event that the test indicates and elevated PSA level.

Solution: From the information given, we know $P(E|C^c) = .135$, $P(E|C) = .268$, $P(C) = .7$, and $P(C^c) = 1 - .7 = .3$. The problem is asking (a)$P(C|E) =$? and (b)$P(C|E^c) =$?

(a)

$$\begin{aligned} P(C|E) &= \frac{P(CE)}{P(E)} \\ &= \frac{P(E^c|C)P(C)}{P(E|C)P(C) + P(E|C^c)P(C^c)} \\ &= \frac{(0.268)(0.7)}{(0.268)(0.7) + (0.135)(0.3)} \\ &= \frac{0.1876}{0.2281} \\ &\approx 0.8224 \end{aligned}$$

(b)

$$\begin{aligned} P(C|E^c) &= \frac{P(CE^c)}{P(E^c)} \\ &= \frac{P(E^c|C)P(C)}{P(E^c|C)P(C) + P(E^c|C^c)P(C^c)} \\ &= \frac{(1-0.268)(0.7)}{(1-0.268)(0.7) + (0.135)(0.3)} \\ &= \frac{0.5124}{0.7719} \\ &\approx 0.6638 \end{aligned}$$

4. Approximately $80,000$ marriages took place in the state of New York last year. Estimate the probability that for at least one of these couples,

   (a) both partners were born on April 30;

   (b) both partners celebrated their birthday on the same day of the year.

   State your assumptions.

   Solution:

   (a) We are assuming Poisson distribution to approximate since the probability that both partners were born on April 30, $p = (1/365)(1/365) \approx 7.506 \times 10^{-6}$, is very small and $n = 80,000$.

   $$\lambda = np = (80000)(7.506 \times 10^{-6}) \approx 0.6$$

   Let X be a random variable of the number of couples with both partners born on April 30

   $$\begin{aligned} P\{X \geq 1\} &= 1 - P\{X = 0\} \\ &= 1 - e^{-\lambda}\frac{\lambda^0}{0!} \\ &= 1 - e^{-0.6} \\ &\approx 0.4512 \end{aligned}$$

(b) We are assuming Poisson distribution to approximate since the probability that both partners have the same birthday, $p = (1/365) \approx 0.0027$, is small and $n = 80,000$.

$$\lambda = np = (80000)(0.0027) \approx 219.178$$

Let X be a random variable of the number of couples with both partners born on April 30

$$\begin{aligned} P\{X \geq 1\} &= 1 - P\{X = 0\} \\ &= 1 - e^{-\lambda}\frac{\lambda^0}{0!} \\ &= 1 - e^{-219.178} \\ &\approx 1 \end{aligned}$$

5. Consider n independent trials, each of which results in one of the outcomes $1, \cdots, k$ with respective probabilities $p_1, \cdots, p_k$, $\sum_{i=1}^{k} p_k = 1$. Show that if all the p_i are small, then the probability that no trial outcome occurs more than once is approximately equal to $\exp(-n(n-1)\sum_i p_i^2/2)$.

Solution: The probability, p, that 2 of the trial outcomes will be the same is

$$p = \sum_{i=1}^{k} p_i^2$$

Of n trials, the possibility that 2 of the outcomes are the same is

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Using Poisson distribution,

$$\lambda = np = \frac{n(n-1)}{2}\sum_{i=1}^{k} p_i^2$$

Let X be the random variable of the number of 2 trials with the same outcome. The probability that no 2 outcomes are the same is

$$\begin{aligned} P\{X = 0\} &= e^{-\lambda}\frac{\lambda^0}{0!} \\ &= e^{-\lambda} \\ &= exp\left(\frac{n(n-1)}{2}\sum_{i=1}^{k} p_i^2\right) \end{aligned}$$