Chapter One

Gaussian matrix ensembles

The Gaussian ensembles are introduced as Hermitian matrices with independent elements distributed as Gaussians, and joint distribution of all independent elements invariant under conjugation by appropriate unitary matrices. The Hermitian matrices are divided into classes according to the elements being real, complex or real quaternion, and their invariance under conjugation by orthogonal, unitary, and unitary symplectic matrices, respectively. These invariances are intimately related to time reversal symmetry in quantum physics, and this in turn leads to the eigenvalues of the Gaussian ensembles being good models of the highly excited spectra of certain quantum systems. Calculation of the eigenvalue p.d.f.'s is essentially an exercise in change of variables, and to calculate the corresponding Jacobians both wedge products and metric forms are used. The p.d.f.'s coincide with the Boltzmann factor for a log-gas system at three special values of the inverse temperature $\beta = 1, 2$ and 4. Thus the eigenvalues behave as charged particles, all of like sign, which are in equilibrium. The Coulomb gas analogy, through the study of various integral equations, allows for the prediction of the leading asymptotic form of the eigenvalue density. After scaling, this leading asymptotic form is referred to as the Wigner semicircle law. The Wigner semicircle law is applied to the study of the statistics of critical points for a model of high-dimensional energy landscapes, and to relating matrix integrals to some combinatorial problems on the enumeration of maps. Conversely, the latter considerations also lead to the proof of the Wigner semicircle law in the case of the GUE. The shifted mean Gaussian ensembles are introduced, and it is shown how the Wigner semicircle law can be used to predict the condition for the separation of the largest eigenvalue. In the last section a family of random tridiagonal matrices, referred to as the Gaussian β-ensemble, are presented. These interpolate continuously between the eigenvalue p.d.f.'s of the Gaussian ensembles studied previously.

1.1 RANDOM REAL SYMMETRIC MATRICES

Quantum mechanics singles out three classes of random Hermitian matrices. We will begin our study by specifying one of these—Hermitian matrices with all entries real, or equivalently real symmetric matrices. The independent elements are taken to be distributed as independent Gaussians, but with the variance different for the diagonal and off-diagonal entries.

DEFINITION 1.1.1 *A random real symmetric $N \times N$ matrix $\mathbf{X}$ is said to belong to the Gaussian orthogonal ensemble (GOE) if the diagonal and upper triangular elements are independently chosen with p.d.f.'s*

$$\frac{1}{\sqrt{2\pi}} e^{-x_{jj}^2/2} \quad \text{and} \quad \frac{1}{\sqrt{\pi}} e^{-x_{jk}^2},$$

respectively.

The p.d.f.'s of Definition 1.1.1 are examples of the normal (or Gaussian) distribution

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

denoted $\mathrm{N}[\mu, \sigma]$. With this notation, note that an equivalent construction of GOE matrices is to let $\mathbf{Y}$ be an $N \times N$ random matrix of independent standard Gaussians $\mathrm{N}[0, 1]$ and to form $\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T)$.

The joint p.d.f. of all the independent elements is

$$P(\mathbf{X}) := \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-x_{jj}^2/2} \prod_{1\le j<k\le N} \frac{1}{\sqrt{\pi}} e^{-x_{jk}^2} = A_N \prod_{j,k=1}^{N} e^{-x_{jk}^2/2}$$
$$= A_N e^{-\sum_{j,k=1}^{N} x_{jk}^2/2} = A_N e^{-(1/2)\mathrm{Tr}\mathbf{X}^2}, \tag{1.1}$$

where A_N is the normalization and Tr denotes the trace. This structure is behind the choices of the independent Gaussians in Definition 1.1.1. It provides the starting point to identify features of the GOE which make it relevant to quantum physics [447].

Proposition 1.1.2 *Let $\mathbf{X}$ be a member of the GOE and let $\mathbf{R}$ be an $N \times N$ real orthogonal matrix. One has $P(\mathbf{R}^T\mathbf{X}\mathbf{R}) = P(\mathbf{X})$. Furthermore, the most general p.d.f. satisfying this equation which has the factorization property $P(\mathbf{X}) = \prod_{1\le j\le k\le N} f_{jk}(x_{jk})$ for f_{jk} differentiable is*

$$P(\mathbf{X}) = Ae^{-a\sum_{j,k=1}^{N}(x_{jk})^2 - b\sum_{j=1}^{N} x_{jj}} = Ae^{-a\mathrm{Tr}(\mathbf{X}^2) - b\mathrm{Tr}\mathbf{X}}.$$

Proof. See Exercises 1.1 q.1. □

Proposition 1.1.3 *Define the entropy S of the joint p.d.f. P of the independent elements of $\mathbf{X}$ by $S[P] := -\int P\log P\,\mu(d\mathbf{X}) =: -\langle \log P\rangle_P$ where $\mu(d\mathbf{X}) := \prod_{1\le j\le k\le N} dx_{jk}$. Then P as given by (1.1) maximizes S subject to the constraint $\langle \mathrm{Tr}\mathbf{X}^2\rangle_P = N^2$.*

Proof. Because of the constraint on the second moment, and the normalization constraint, we can write

$$S[P] = -\langle \log P\rangle_P - \lambda\Big(\langle \mathrm{Tr}\,\mathbf{X}^2\rangle_P - N^2\Big) + (\log A + 1)\Big(\langle 1\rangle_P - 1\Big),$$

where λ and $-(\log A + 1)$ are Lagrange multipliers. The condition for a maximum is $\delta S = 0$, where the variation is made with respect to P. This gives

$$-\log P - \lambda \mathrm{Tr}\mathbf{X}^2 + \log A = 0$$

and thus $P = Ae^{-\lambda \mathrm{Tr}\mathbf{X}^2}$. The value of λ is determined to be $\frac{1}{2}$ from the given constraint. □

From these properties an understanding of the applicability of the GOE in the study of quantum energy spectra can be obtained. However as a further prerequisite some theory from quantum mechanics is required [401], [284].

1.1.1 Time reversal in quantum systems

First it is necessary to understand the relevance of an $N \times N$ matrix to quantum energy spectra. A basic axiom of quantum mechanics says the energy spectrum of a quantum system is given by the eigenvalues of its (Hermitian) Hamiltonian operator H, the latter being in general infinite dimensional. Now, to model the discrete portion of the spectrum of a complicated quantum system, a reasonable approximation is to replace H by a finite-dimensional $N \times N$ Hermitian matrix, which has a discrete spectrum only.

Next we need to understand the significance of real symmetric matrices in quantum mechanics. This is related to the fact that in general the structure of a matrix modeling H is constrained by the symmetries of H.

Definition 1.1.4 *A quantum Hamiltonian H is said to have a symmetry A if*

$$[H, A] = 0,$$

where $[\cdot,\cdot]$ denotes the commutator.

One basic symmetry of most quantum systems is time reversal.

DEFINITION 1.1.5 *A general time reversal operator T is any antiunitary operator, which means $T = UK$ where U is unitary and K is the complex conjugation operator.*

Hence we say a quantum system has a time reversal symmetry if the Hamiltonian commutes with an antiunitary operator.

Study of time reversal operators in the context of physical systems further restricts their form. For systems with an even number or no spin $\frac{1}{2}$ particles, it is required that

$$T^2 = 1,$$

while for a finite-dimensional system with an odd number of spin $\frac{1}{2}$ particles

$$T^2 = -1 \qquad \text{and} \qquad T = \mathbf{Z}_{2N} K,$$

where $\mathbf{Z}_{2N}$ is a $2N \times 2N$ block diagonal matrix with each 2×2 diagonal block given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1.2}$$

(a tensor product formula for $\mathbf{Z}_{2N}$ is given in Exercises 1.1 q.2) which has the effect of reversing the spins. Real symmetric matrices arise in the former situation.

PROPOSITION 1.1.6 *Let H be a quantum Hamiltonian which is invariant with respect to a time reversal symmetry T, where T has the additional property $T^2 = 1$. Then H can always be given a T-invariant orthogonal basis, and with respect to this basis the (in general infinite) matrix representation of H is real.*

Proof. See Exercises 1.1 q.3. □

The above result tells us that a matrix chosen to model the discrete energy spectra of a quantum system with a time reversal symmetry T such that $T^2 = 1$ must be real symmetric. A further general property in quantum mechanics is that two operators related by a similarity transformation of unitary operators are equally valid descriptions of the operator, in that all observables are the same for both operators. A requirement of (1.1) is therefore that any two real symmetric matrices related by a similarity transformation of unitary matrices must have the same p.d.f. for the elements. For the two real symmetric matrices to be so related the unitary matrix must be real orthogonal (or i times a real orthogonal matrix; see Exercises 1.1 q.4). Thus this requirement is guaranteed by Proposition 1.1.2.

We are assuming no information on the Hamiltonian other than the time reversal symmetry. Proposition 1.1.3 says that the p.d.f. (1.1) is the most random subject to the given constraint, in that it maximizes the entropy.

These considerations thus show the applicability of the GOE in the study of quantum spectra. Explicitly, it is hypothesized that the statistical properties of the highly excited states of a complex quantum system with a time reversal symmetry $T^2 = 1$ coincide with the statistical properties of the bulk eigenvalues from large GOE matrices (see Section 7.1.1 for the notion of bulk eigenvalues). Here it is assumed that both spectra have been scaled (technically referred to as *unfolded*) so that the mean spacing is unity. The meaning of a complex quantum system requires further explanation. Wigner first made this hypothesis for the spectra of heavy nuclei in the 1950's. In 1984 Bohigas, Giannoni and Schmit made the same hypothesis for a single particle quantum billiard system, provided the underlying classical mechanics is chaotic and the system has a time reversal symmetry $T^2 = 1$. It is of interest to note that a GOE hypothesis also applies to eigenmodes of microwave cavities (this is not surprising as the Helmholtz equation is formally equivalent to the stationary Schrödinger equation), and also to the eigenmodes of systems governed by classical wave equations — vibrations of irregular shaped metal plates, electromechanical eigenmodes of aluminium and quartz blocks, among other examples. (For references to the original literature, and an extended discussion of GOE hypotheses, see [276].)

EXERCISES 1.1

1. The objective of this exercise is to prove Proposition 1.1.2.

(i) Note that the invariance $P(\mathbf{R}^T\mathbf{X}\mathbf{R}) = P(\mathbf{X})$ with $\mathbf{R}$ a permutation matrix requires that the distribution of all elements on the diagonal be equal, $f_{jj} = f$, and similarly the distribution of all elements on the off diagonal be equal, $f_{jk} = g$ $(j < k)$, for some f and g.

(ii) Choose

$$\mathbf{R} = \begin{bmatrix} 1 & \epsilon & 0 & \dots & 0 \\ -\epsilon & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

where $|\epsilon| \ll 1$. Ignoring terms $\mathrm{O}(\epsilon^2)$, show that

$$\mathbf{R}^{-1}\mathbf{X}\mathbf{R} = \begin{bmatrix} x_{11} - 2\epsilon x_{12} & x_{12} + \epsilon(x_{11} - x_{22}) & x_{13} - \epsilon x_{23} & \dots & x_{1N} - \epsilon x_{2N} \\ * & x_{22} + 2\epsilon x_{12} & x_{23} + \epsilon x_{13} & \dots & x_{2N} + \epsilon x_{1N} \\ * & * & x_{33} & \dots & x_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & x_{NN} \end{bmatrix},$$

where the elements $*$ are such that the matrix is symmetric.

(iii) Use the result of (ii) to show that at first order in ϵ the requirement

$$\prod_{j=1}^N f(x_{jj}) \prod_{1 \le j < k \le N} g(x_{jk}) = \prod_{j=1}^N f(\tilde{x}_{jj}) \prod_{1 \le j < k \le N} g(\tilde{x}_{jk}),$$

where $\tilde{x}_{jk} := [\mathbf{R}^{-1}\mathbf{X}\mathbf{R}]_{jk}$ implies

$$\frac{(x_{11} - x_{22})g'(x_{12})}{g(x_{12})} - 2\frac{x_{12}f'(x_{11})}{f(x_{11})} + 2\frac{x_{12}f'(x_{22})}{f(x_{22})} - \sum_{j=3}^N \left(\frac{x_{2j}g'(x_{1j})}{g(x_{1j})} - \frac{x_{1j}g'(x_{2j})}{g(x_{2j})} \right) = 0,$$

which in turn, by separation of variables, implies

$$-\frac{f'(x_{11})}{f(x_{11})} + \frac{f'(x_{12})}{f(x_{12})} + \frac{(x_{11} - x_{22})g'(x_{12})}{2x_{12}g(x_{12})} = \gamma$$

for some constant γ.

(iv) By a further separation of variables in the last equation conclude

$$\frac{g'(x_{12})}{x_{12}g(x_{12})} = -b$$

for some constant b. Solve this differential equation.

(v) Note that the invariance $P(\mathbf{R}^T\mathbf{X}\mathbf{R}) = P(\mathbf{X})$ requires that P be a symmetric function of the eigenvalues, and thus a function of $\mathrm{Tr}(\mathbf{X}^k)$ $k = 1, 2, \dots$. Now combine the results of (i) and (iv) to deduce the result.

2. Let $\mathbf{A} = [a_{ij}]$ be a $p \times q$ matrix and $\mathbf{B} = [b_{i'j'}]$ be an $r \times s$ matrix. The tensor product, denoted $\mathbf{A} \otimes \mathbf{B}$, is the $pr \times qs$ matrix with elements

$$(\mathbf{A} \otimes \mathbf{B})_{ii',jj'} = a_{i,j} b_{i',j'},$$

and thus

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}.$$

With $\mathbf{Z}_{2N}$ defined as above (1.2), show that

$$\mathbf{Z}_{2N} = \mathbf{1}_N \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.3}$$

3. [401] Let $\vec{\psi}_1 = \alpha_1 \vec{\phi}_1 + T(\alpha_1 \vec{\phi}_1)$, where α_1 is a scalar, $\vec{\psi}_1$ and $\vec{\phi}_1$ are vectors, T is anti-unitary and $T^2 = 1$. Note that $T\vec{\psi}_1 = \vec{\psi}_1$. Here Proposition 1.1.6 will be established.
   (i) From the antiunitarity property it follows that in general $\langle \vec{u} | T\vec{v} \rangle = \overline{\langle T\vec{u} | \vec{v} \rangle}$, where $\langle \cdot | \cdot \rangle$ denotes the inner product. Use this to show that $\langle \vec{u} | \vec{v} \rangle = \overline{\langle T\vec{u} | T\vec{v} \rangle}$.
   (ii) Suppose $\vec{\phi}_2$ is orthogonal to $\vec{\psi}_1$. Use (i) to show that $\vec{\psi}_2 := \alpha_2 \vec{\phi}_2 + T(\alpha_2 \vec{\phi}_2)$ is orthogonal to $\vec{\psi}_1$, and note how this construction can be used to create an orthogonal basis of vectors with the T-invariance property $T\vec{\psi}_n = \vec{\psi}_n$.
   (iii) Consider a Hamiltonian H which has symmetry T. Use the above properties of T to show that with respect to the basis $\{\vec{\psi}_n\}$ the matrix elements $\langle \vec{\psi}_m | H \vec{\psi}_n \rangle$ are real.

4. Let $\mathbf{X}$ be an arbitrary real symmetric $N \times N$ matrix and suppose $\mathbf{X}' = \mathbf{U}^{-1}\mathbf{X}\mathbf{U}$, where $\mathbf{U}$ is unitary and $\mathbf{X}'$ is real symmetric. Assume that the only symmetry of $\mathbf{X}$ and $\mathbf{X}'$ in general (other than some constant times the identity) is the time reversal operator T with $T^2 = 1$.
   (i) Deduce that $T\mathbf{U}T^{-1}\mathbf{U}^{-1}$ commutes with $\mathbf{X}$.
   (ii) Use (i) to show $T\mathbf{U} = c\mathbf{U}T$ and take the inverse of this equation to conclude $c = \pm 1$.
   (iii) Use (ii) and q.3(i) to show that with respect to the T invariant basis $\{\vec{\psi}_n\}$, $\langle \vec{\psi}_n | \mathbf{U} \vec{\psi}_m \rangle = c \overline{\langle \vec{\psi}_n | \mathbf{U} \vec{\psi}_m \rangle}$. Hence conclude that $\mathbf{U}$ has either real elements ($c = 1$) or pure imaginary elements ($c = -1$) and is thus either a real orthogonal matrix or i times a real orthogonal matrix.

1.2 THE EIGENVALUE P.D.F. FOR THE GOE

The p.d.f. for the elements of the matrices in the GOE is given by (1.1). We want to calculate the corresponding eigenvalue p.d.f. This was first accomplished as long ago as 1939 [299]. We will follow a more recent treatment [410].

The new variables and the final expression

The p.d.f. (1.1) has $N(N+1)/2$ independent variables, whereas there are only N eigenvalues, say, $\lambda_1 < \cdots < \lambda_N$. The remaining variables are linear combinations of the independent elements of the eigenvectors, denoted $p_1, \ldots, p_{N(N-1)/2}$ say. Our task is to change variables

$$\exp\Big(-\frac{1}{2}\mathrm{Tr}(\mathbf{X}^2)\Big) \prod_{1 \le j \le k \le N} dx_{jk} = \exp\Big(-\frac{1}{2}\sum_{l=1}^{N} \lambda_l^2\Big)|J| \prod_{j=1}^{N} d\lambda_j \prod_{j=1}^{N(N-1)/2} dp_j,$$

where the Jacobian is given by

$$J := \det \begin{bmatrix} \frac{\partial x_{11}}{\partial \lambda_1} & \frac{\partial x_{12}}{\partial \lambda_1} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_1} \\ \frac{\partial x_{11}}{\partial \lambda_2} & \frac{\partial x_{12}}{\partial \lambda_2} & \cdots & \frac{\partial x_{NN}}{\partial \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{11}}{\partial p_{N(N-1)/2}} & \frac{\partial x_{12}}{\partial p_{N(N-1)/2}} & \cdots & \frac{\partial x_{NN}}{\partial p_{N(N-1)/2}} \end{bmatrix}.$$

Thus we must evaluate the Jacobian and then integrate over the variables $p_1, \ldots, p_{N(N-1)/2}$ to obtain the eigenvalue p.d.f.

Below we will show that J factorizes,

$$J = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)\, f(p_1, \dots, p_{N(N-1)/2})$$

so the integration over the variables $p_1, \dots, p_{N(N-1)/2}$ only alters the normalization constant. Hence the final expression for the eigenvalue p.d.f. of the GOE is

$$\frac{1}{C_N} \exp\Big(-\frac{1}{2} \sum_{j=1}^{N} \lambda_j^2 \Big) \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|, \tag{1.4}$$

where C_N is the normalization constant.

From the viewpoint of application to quantum mechanics, the important feature is the product of differences due to the Jacobian. It can be proved that the correlations are determined entirely by the product of differences, in the sense that the same so-called bulk correlations (see Section 7.1) result if the one body terms $e^{-\lambda^2/2}$ are replaced by some different functional forms $e^{-V(\lambda)/2}$, provided the local density is constant [125]. This feature is referred to as *universality* and gives rise to the notion [53] that the essential feature of a random matrix hypothesis applying to a quantum system is that the spectral correlations are geometrical, meaning that they are due to this Jacobian.

1.2.1 Wedge products

In the theory of multivariable calculus (see, e.g., [485]) the wedge product operation, which is linear and antisymmetric, is defined to give a signed volume element in the tangent space at a point in the manifold. However, for our purpose the latter concept plays no explicit role, and we can make do with the following definition.

DEFINITION 1.2.1 *With* $du_i(j) := \delta_{i,j} du_i$ *define*

$$du_1 \wedge \cdots \wedge du_N =: \bigwedge_{j=1}^{N} du_j := \det[du_i(j)]_{i,j=1,\dots,N}. \tag{1.5}$$

Note that it follows from (1.5) that

$$\int_\Omega f(u_1, \dots, u_N) du_1 \wedge \cdots \wedge du_N = \int_\Omega f(u_1, \dots, u_N) du_1 \cdots du_N,$$

since only the diagonal entries in the determinant are nonzero.

When changing variables from $\{u_1, \dots, u_N\}$ to $\{v_1, \dots, v_N\}$ the fundamental formula

$$du_i = \sum_{l=1}^{N} \frac{\partial u_i}{\partial v_l} dv_l$$

applies. Substituting this in (1.5), and noting the factorization

$$\Big[\sum_{l=1}^{N} \frac{\partial u_i}{\partial v_l} dv_l(j) \Big]_{i,j=1,\dots,N} = \Big[\frac{\partial u_i}{\partial v_j} \Big]_{i,j=1,\dots,N} \Big[dv_i(j) \Big]_{i,j=1,\dots,N}$$

shows

$$\bigwedge_{j=1}^{N} du_j = \det \Big[\frac{\partial u_i}{\partial v_j} \Big]_{i,j=1,\dots,N} \bigwedge_{j=1}^{N} dv_j. \tag{1.6}$$

The determinant in (1.6) is precisely the Jacobian for the change of variables. The practical use of calculating

Jacobians from this formula relies on an alternative way of calculating the l.h.s. of (1.6) in terms of $\{v_j\}$. For the problem at hand, this in turn is done by using the special feature that all the variables are connected by matrix relations. The following definitions are helpful.

DEFINITION 1.2.2 *For any $N \times N$ matrix $\mathbf{X} = [x_{jk}]$, the matrix of differentials is defined as*

$$d\mathbf{X} = \begin{bmatrix} dx_{11} & dx_{12} & \dots & dx_{1N} \\ dx_{21} & dx_{22} & \dots & dx_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ dx_{N1} & dx_{N2} & \dots & dx_{NN} \end{bmatrix}.$$

With this definition the usual product rule for differentiation holds,

$$d(\mathbf{XY}) = d\mathbf{X}\,\mathbf{Y} + \mathbf{X}\,d\mathbf{Y}.$$

DEFINITION 1.2.3 *The symbol $(d\mathbf{X})$ denotes the wedge product of the independent elements of $d\mathbf{X}$. In particular, if $\mathbf{X}$ is a real symmetric matrix,*

$$(d\mathbf{X}) = \bigwedge_{1 \le j \le k \le N} dx_{jk},$$

while if $\mathbf{X} = [x_{jk} + iy_{jk}]_{j,k=1,\dots,N}$ is Hermitian ($x_{jk} = x_{kj}$, $y_{jk} = -y_{kj}$)

$$(d\mathbf{X}) = \bigwedge_{j=1}^{N} dx_{jj} \bigwedge_{1 \le j < k \le N} dx_{jk} dy_{jk}.$$

In integration formulas only the absolute value of the Jacobian occurring in the change of variables formula (1.6) is required, so consequently there is no need to strictly adhere to the ordering of wedge products specified in Definition 1.2.3 (according to the definition, reversing the order of two differentials changes the sign of the wedge product). Because of this, any overall factor of -1 will be ignored in subsequent formulas involving $(d\mathbf{X})$. With this convention $(d\mathbf{X})$ will be referred to as a *volume form*, or *volume measure*.

In preparation for the calculation of J, we note a result for the wedge product $(\mathbf{A}^T d\mathbf{MA})$, where $\mathbf{A}$ is a real $N \times N$ matrix and $\mathbf{M}$ is a real symmetric $N \times N$ matrix [410].

PROPOSITION 1.2.4 *Let $\mathbf{A}$ and $\mathbf{M}$ be real $N \times N$ matrices, and suppose furthermore that $\mathbf{M}$ is symmetric. We have*

$$(\mathbf{A}^T d\mathbf{MA}) = (\det \mathbf{A})^{N+1} (d\mathbf{M}).$$

Proof. We note from Definition 1.2.3 that

$$(\mathbf{A}^T d\mathbf{MA}) = p(\mathbf{A})(d\mathbf{M}), \tag{1.7}$$

where p is a polynomial in the elements of $\mathbf{A}$. Furthermore, if $\mathbf{B}$ is also an $N \times N$ matrix, then

$$(\mathbf{B}^T \mathbf{A}^T d\mathbf{MAB}) = p(\mathbf{B})(\mathbf{A}^T d\mathbf{MA}) = p(\mathbf{B})p(\mathbf{A})(d\mathbf{M}),$$

so we must have $p(\mathbf{AB}) = p(\mathbf{A})p(\mathbf{B})$, for arbitrary $\mathbf{A}$ and $\mathbf{B}$. But it is known [377] that the only polynomial in the matrix elements satisfying such a factorization is

$$p(\mathbf{A}) = (\det \mathbf{A})^k, \quad k \in \mathbb{Z}_{\ge 0}.$$

The value of k can be determined by making the special choice $\mathbf{A} = \mathrm{diag}(a, 1, \dots, 1)$ in (1.7).

For an alternative proof of this result, see Exercises 1.3 q.2. □

1.2.2 Calculation of the Jacobian

From Definition 1.2.3 and (1.6) we see

$$J \bigwedge_{i=1}^{N} d\lambda_i \bigwedge_{j=1}^{N(N-1)/2} dp_j = (d\mathbf{X}).$$

To calculate $(d\mathbf{X})$ in terms of the eigenvalues and eigenvectors we use the fact that all symmetric matrices are orthogonally diagonalizable [8] (see Exercises 1.9 q.3) to write

$$\mathbf{X} = \mathbf{R}\mathbf{L}\mathbf{R}^T. \tag{1.8}$$

Here $\mathbf{L}$ is a diagonal matrix consisting of the N eigenvalues of $\mathbf{X}$ and the columns of the real orthogonal matrix $\mathbf{R}$ consist of the corresponding normalized eigenvectors. Using the notation of Definition 1.2.2, the product rule for differentiation gives

$$d\mathbf{X} = d\mathbf{R}\,\mathbf{L}\mathbf{R}^T + \mathbf{R}d\mathbf{L}\,\mathbf{R}^T + \mathbf{R}\mathbf{L}d\mathbf{R}^T.$$

Rather than take the wedge product of both sides of this equation, it is simpler to premultiply by $\mathbf{R}^T$ and postmultiply by $\mathbf{R}$ to obtain

$$\begin{aligned}\mathbf{R}^T d\mathbf{X}\mathbf{R} &= \mathbf{R}^T d\mathbf{R}\,\mathbf{L} + \mathbf{L}d\mathbf{R}^T\mathbf{R} + d\mathbf{L}\\ &= \mathbf{R}^T d\mathbf{R}\,\mathbf{L} - \mathbf{L}\mathbf{R}^T d\mathbf{R} + d\mathbf{L},\end{aligned} \tag{1.9}$$

where to obtain the last line the formula $d\mathbf{R}^T\mathbf{R} = -\mathbf{R}^T d\mathbf{R}$ has been used (this follows from $\mathbf{R}\mathbf{R}^T = \mathbf{1}$).

According to Proposition 1.2.4

$$(\mathbf{R}^T d\mathbf{X}\mathbf{R}) = (\det \mathbf{R})^{N+1}(d\mathbf{X}). \tag{1.10}$$

But $\mathbf{R}$ is an orthogonal matrix and so $\det \mathbf{R} = \pm 1$. As already noted, since only the modulus of J occurs in the change of variables formula, this sign factor can be ignored.

The wedge product of the r.h.s. of (1.9) can be taken with the aid of the following result.

Proposition 1.2.5 *With the notation $\vec{r}_k = (r_{1k}, r_{2k}, \dots, r_{Nk})^T$ for the kth column of $\mathbf{R}$, we have*

$$\mathbf{R}^T d\mathbf{R}\mathbf{L} - \mathbf{L}\mathbf{R}^T d\mathbf{R} + d\mathbf{L}$$
$$= \begin{bmatrix} d\lambda_1 & (\lambda_2-\lambda_1)\vec{r_1}^{\,T}\cdot d\vec{r_2} & \dots & (\lambda_N-\lambda_1)\vec{r_1}^{\,T}\cdot d\vec{r_N} \\ (\lambda_2-\lambda_1)\vec{r_1}^{\,T}\cdot d\vec{r_2} & d\lambda_2 & \dots & (\lambda_N-\lambda_2)\vec{r_2}^{\,T}\cdot d\vec{r_N} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N-\lambda_1)\vec{r_1}^{\,T}\cdot d\vec{r_N} & (\lambda_N-\lambda_2)\vec{r_2}^{\,T}\cdot d\vec{r_N} & \dots & d\lambda_N \end{bmatrix}.$$

Proof. This is obtained by explicitly forming the matrix products, and simplifying the resulting expression by noting from $d\mathbf{R}^T d\mathbf{R} = -\mathbf{R}^T d\mathbf{R}$ that $\mathbf{R}^T d\mathbf{R}$ is antisymmetric. □

From Proposition 1.2.5 and Definition 1.2.3, the wedge product of the r.h.s. of (1.9) can be written down (note in particular that the matrix in Proposition 1.2.5 is symmetric), whereas (1.10) gives the wedge product of the l.h.s. of (1.9). Equating these expressions gives

$$(d\mathbf{X}) = \prod_{1\le j<k\le N} (\lambda_k - \lambda_j) \bigwedge_{j=1}^{N} d\lambda_j \, (\mathbf{R}^T d\mathbf{R}). \tag{1.11}$$

The factorization property of the Jacobian between the eigenvalues and the variables involving the eigenvectors is evident and the expression (1.4) for the eigenvalue p.d.f. of the GOE follows. The p.d.f. for the components of the eigenvectors is calculated in Exercises 1.2 q.2.

1.2.3 Scaling of the Jacobian

Here we will show how the eigenvalue factor in the Jacobian can be deduced by considering a simple scaling property of the wedge product. Since there are $N(N+1)/2$ independent elements in $\mathbf{X}$, $(d\mathbf{X})$ consists of the product of $N(N+1)/2$ independent differentials. Thus if we multiply $\mathbf{X}$ by a scalar a, we have that $(d\,a\mathbf{X}) = a^{N(N+1)/2}(d\mathbf{X})$. On the other hand, with $\mathbf{X} = \mathbf{RLR}^T$, we know that $(d\mathbf{X})$ is a polynomial in $\lambda_1, \dots, \lambda_N$. Since $a\mathbf{X} = \mathbf{R}a\mathbf{LR}^T$, the scaling property of $(d\,a\mathbf{X})$ gives that in fact $(d\mathbf{X})$ is a homogeneous polynomial of degree $N(N-1)/2$ (here we have subtracted N from $N(N+1)/2$ to account for the scaling of the measure $d\lambda_1 \cdots d\lambda_N$). Furthermore, analysis of the 2×2 case reveals that the Jacobian must vanish linearly for $\lambda_j \to \lambda_k$ (see Exercises 1.2 q.3). Hence the polynomial factor is necessarily proportional to $\prod_{j<k}(\lambda_k - \lambda_j)$, in agreement with the above calculation.

1.2.4 Metric forms

Another approach to deriving (1.11) is through the use of a metric form defined on the space of symmetric matrices [300]. For an $N \times N$ real symmetric matrix $\mathbf{X}$, the metric form of the line element ds is specified by

$$(ds)^2 = \mathrm{Tr}(d\mathbf{X}d\mathbf{X}^T) = \sum_{j=1}^{N}(dx_{jj})^2 + 2\sum_{j<k}(dx_{jk})^2 \tag{1.12}$$

(of course $d\mathbf{X}^T = d\mathbf{X}$, but it is convenient to write as presented), and the volume measure is

$$(d\mathbf{X}) = \bigwedge_{j\le k} dx_{jk}.$$

If one now makes a change of variables, expressing the elements x_{jk} in terms of some new variables y_{jk} such that

$$(ds)^2 = \sum_{j=1}^{N}(h_{jj}dy_{jj})^2 + 2\sum_{j<k}(h_{jk}dy_{jk})^2, \tag{1.13}$$

where the h_{jk} typically depend on $\{y_{jk}\}$, the corresponding volume measure is

$$(d\mathbf{X}) = \Big(\bigwedge_{j\le k} h_{jk}\Big)(d\mathbf{Y}), \tag{1.14}$$

thus giving a change of variable formula for the volume measure.

More generally the metric forms method gives that if $(ds)^2$ is a symmetric quadratic form in some independent infinitesimals $\{dy_\mu\}$, so that

$$(ds)^2 = \sum_{\mu,\nu} g_{\mu,\nu}dy_\mu dy_\nu, \qquad g_{\mu,\nu} = g_{\nu,\mu}, \tag{1.15}$$

then the corresponding volume measure is

$$\Big(\det[g_{\mu,\nu}]\Big)^{1/2} \bigwedge_{\mu} dy_\mu. \tag{1.16}$$

Comparing (1.15) with (1.13), we see that there are only diagonal terms present in the formula for the line element. The determinant is then the product of the diagonal terms, which is consistent with (1.14).

We can apply this formalism by noting from (1.9) and Proposition 1.2.5 that

$$\mathrm{Tr}(d\mathbf{X}d\mathbf{X}^T) = 2\sum_{j<k}(\lambda_k - \lambda_j)^2(\vec{r}_j \cdot d\vec{r}_k)^2 + \sum_{j=1}^{N}(d\lambda_j)^2.$$

Application of (1.14) then reclaims (1.11).

EXERCISES 1.2 1. (i) Let $\mathbf{R}$ be a $N \times N$ real orthogonal matrix. Show that in general $\mathbf{R}$ has $N^2 - N(N-1)/2 - N$ independent elements.

(ii) Use (i) to show that the number of independent elements on the two sides of the equation $\mathbf{X} = \mathbf{R}\mathbf{L}\mathbf{R}^T$, where $\mathbf{X}$ is real symmetric and $\mathbf{L}$ is diagonal, are equal.

(iii) For $\vec{a}, \vec{b}$, $N \times 1$ real column vectors related by $\vec{a} = \mathbf{A}\vec{b}$ for some $N \times N$ matrix $\mathbf{A}$, show that

$$(d\vec{a}) = |\det \mathbf{A}|(d\vec{b}). \tag{1.17}$$

2. [284] Here the distribution of the components of the eigenvectors in the GOE is calculated.

(i) Note that for matrices in the GOE every eigenvector can be transformed by an arbitrary real orthogonal matrix, and still remain an eigenvector of a matrix in the GOE. Conclude from this that the only invariant of the eigenvectors is their norm, and so the joint distribution of the components $(u_1, \dots, u_N)$ is given by

$$\frac{1}{C}\delta\Big(1 - \sum_{p=1}^{N} u_p^2\Big),$$

where $C = 2\pi^{N/2}/\Gamma(N/2)$ represents the surface area of the unit $(N-1)$-sphere.

(ii) Show that the marginal joint distribution $p(u_1, \dots, u_n)$, obtained by integrating out the variables $u_{n+1}, \dots, u_N$, is given by

$$p(u_1, \dots, u_n) = \pi^{-n/2}\frac{\Gamma(N/2)}{\Gamma((N-n)/2)}\Big(1 - \sum_{p=1}^{n} u_p^2\Big)^{(N-n-2)/2}.$$

For this purpose write the delta function in (i) as a Fourier integral.

(iii) From (ii) show that for large N,

$$\frac{1}{N^{n/2}}p\Big(\frac{u_1}{\sqrt{N}}, \dots, \frac{u_n}{\sqrt{N}}\Big) \sim \Big(\frac{2}{\pi}\Big)^{n/2} e^{-\frac{1}{2}\sum_{p=1}^{n} u_p^2}. \tag{1.18}$$

(iv) Show that forming a vector $(u_1, \dots, u_N)$ in which each component has distribution $x_j/(x_1^2 + \cdots + x_N^2)^{1/2}$, with the x_js standard normal random variables, implies that the vector is uniformly distributed on the unit $(N-1)$-sphere and thus has joint density as in (i). Use this fact to rederive (1.18).

3. (i) For a general 2×2 real symmetric matrix

$$\mathbf{A} := \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

show that the (unordered) eigenvalues are given by

$$\lambda_\pm = \frac{1}{2}(a+c) \pm \frac{1}{2}\Big((a-c)^2 + 4b^2\Big)^{1/2}.$$

Note that the condition for a degenerate eigenvalue is $b = 0$ and $a = c$, and thus has codimension 2 in the space of matrix entries.

(ii) For the matrix in (i) parametrize the matrix of eigenvectors as

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and from the diagonalization equation $\mathbf{A} = \mathbf{R}\,\mathrm{diag}[\lambda_+, \lambda_1]\mathbf{R}^T$, read off that

$$a = \lambda_+ \cos^2\theta + \lambda_- \sin^2\theta, \quad b = (\lambda_+ - \lambda_-)\cos\theta\sin\theta, \quad c = \lambda_+ \sin^2\theta + \lambda_- \cos^2\theta.$$

(iii) Deduce from (ii) that

$$J := \begin{vmatrix} \frac{\partial a}{\partial \lambda_+} & \frac{\partial b}{\partial \lambda_+} & \frac{\partial c}{\partial \lambda_+} \\ \frac{\partial a}{\partial \lambda_-} & \frac{\partial b}{\partial \lambda_-} & \frac{\partial c}{\partial \lambda_-} \\ \frac{\partial a}{\partial \theta} & \frac{\partial b}{\partial \theta} & \frac{\partial c}{\partial \theta} \end{vmatrix} = (\lambda_+ - \lambda_-).$$

1.3 RANDOM COMPLEX HERMITIAN AND QUATERNION REAL HERMITIAN MATRICES

Since most physical systems possess a time reversal symmetry, the GOE correctly models statistical properties of the spectra of many quantum systems (recall the discussion at the end of Section 1.1). Nonetheless the considerations of time reversal symmetry of Section 1.1.1 indicate two further random matrix ensembles [149].

1.3.1 The Gaussian unitary ensemble

For a quantum system without time reversal symmetry the only constraint on the complex Hermitian matrix used to model the discrete portion of the energy spectrum is that two matrices related by a similarity transformation of unitary operators have the same joint p.d.f. for the elements. This requirement is satisfied by the following choice of matrix ensemble.

DEFINITION 1.3.1 *A random Hermitian $N \times N$ matrix $\mathbf{X}$ is said to belong to the Gaussian unitary ensemble (GUE) if the diagonal elements (which must be real) and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ are independently chosen with p.d.f.'s*

$$\frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \quad \text{and} \quad \frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} = \frac{2}{\pi} e^{-2|x_{jk}|^2},$$

respectively. Equivalently, the diagonal entries have distribution $\mathrm{N}[0, 1/\sqrt{2}]$*, while the upper triangular elements have distribution* $\mathrm{N}[0, \frac{1}{2}] + i\mathrm{N}[0, \frac{1}{2}]$*, and* $\mathbf{X}$ *can be specified in terms of the complex random matrix* $\mathbf{Y}$ *with entries independently chosen from* $\mathrm{N}[0, 1/\sqrt{2}] + i\mathrm{N}[0, 1/\sqrt{2}]$*, according to* $\mathbf{X} = (\mathbf{Y} + \mathbf{Y}^\dagger)/2$.

From this definition the joint p.d.f. of all the independent elements is

$$P(\mathbf{X}) := \prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \prod_{1 \le j < k \le N} \frac{2}{\pi} e^{-2|x_{jk}^2|} = A_N \prod_{j,k=1}^{N} e^{-|x_{jk}|^2} = A_N e^{-\mathrm{Tr}(\mathbf{X}^2)},$$

where A_N is the normalization. The invariance $P(\mathbf{U}^{-1}\mathbf{X}\mathbf{U}) = P(\mathbf{X})$ for any unitary matrix $\mathbf{U}$ follows immediately.

1.3.2 The Gaussian symplectic ensemble

In Section 1.1.1 it was remarked that in quantum systems with a time reversal symmetry T, either $T^2 = 1$ or $T^2 = -1$ with $T = \mathbf{Z}_{2N}K$. Consideration of the former case leads to real symmetric matrices. Here the latter possibility will be discussed.

Now, since T commutes with the $2N \times 2N$ matrix $\mathbf{X}$ modeling the Hamiltonian, $\mathbf{X}$ must in addition to being Hermitian have the property

$$\mathbf{X} = T\mathbf{X}T^{-1} = \mathbf{Z}_{2N}K\mathbf{X}K^{-1}\mathbf{Z}_{2N}^{-1} = \mathbf{Z}_{2N}K\mathbf{X}K\mathbf{Z}_{2N}^{-1} = \mathbf{Z}_{2N}\bar{\mathbf{X}}\mathbf{Z}_{2N}^{-1}. \tag{1.19}$$

Since $\mathbf{Z}_{2N}$ is block diagonal, with blocks (1.2), a $2N \times 2N$ matrix $\mathbf{X}$ with this property can be viewed as an $N \times N$ matrix with elements consisting of 2×2 blocks of the form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \tag{1.20}$$

where z and w are complex numbers. A 2×2 matrix of this form is said to be *real quaternion*. From an abstract perspective the quaternions are an algebra with elements of the form

$$a_0 + a_1 i + a_2 j + a_3 k, \qquad i^2 = j^2 = k^2 = -1, \ \ ijk = -1, \tag{1.21}$$

where $a_0, \dots, a_3$ are scalars. The basis elements $1, i, j, k$ can be realized as 2×2 matrices with complex elements given by

$$\mathbf{1} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{e}_1 := i\sigma_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \mathbf{e}_2 := i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{e}_3 := i\sigma_x = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \tag{1.22}$$

respectively. Forming a general linear combination, consisting of real scalar multiples of these basis elements, gives the structure (1.20).

For future reference we note that with a real quaternion $\mathbf{q}$ written in the form $\mathbf{q} = c_0\mathbf{1} + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ the dual, denoted $\bar{\mathbf{q}}$ or $\mathbf{q}^D$, is defined as

$$\bar{\mathbf{q}} = \mathbf{q}^D = c_0\mathbf{1} - c_1\mathbf{e}_1 - c_2\mathbf{e}_2 - c_3\mathbf{e}_3. \tag{1.23}$$

With this definition the dual of (1.20) is

$$\begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix}. \tag{1.24}$$

Furthermore, with $|\mathbf{q}|^2 := \bar{\mathbf{q}}\mathbf{q} = \mathbf{q}\bar{\mathbf{q}}$, we have $|\mathbf{q}|^2 = c_0^2 + c_1^2 + c_2^2 + c_3^2$, the relation $|\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1||\mathbf{q}_2|$ holds, and each nonzero $\mathbf{q}$ has a unique inverse, $\mathbf{q}^{-1} = \bar{\mathbf{q}}/|\mathbf{q}|^2$.

An $N \times N$ matrix with real quaternion elements is said to be *quaternion real*. This structure underlies the definition of the third and final ensemble of Gaussian random matrices as motivated by quantum physics.

Definition 1.3.2 *A random Hermitian $N \times N$ matrix $\mathbf{X}$ with real quaternion elements is said to belong to the Gaussian symplectic ensemble (GSE) if the elements z_{jj} of each diagonal real quaternion (which must be real) are independently chosen with p.d.f.*

$$\sqrt{\frac{2}{\pi}} e^{-2z_{jj}^2}$$

(or equivalently have distribution $\mathrm{N}[0, 1/2]$*) while the upper triangular off-diagonal elements $z_{jk} = u_{jk} + iv_{jk}$ and $w_{jk} = u'_{jk} + iv'_{jk}$ are independently chosen with p.d.f.*

$$\frac{4}{\pi} e^{-4|z_{jk}|^2} \qquad \textit{and} \qquad \frac{4}{\pi} e^{-4|w_{jk}|^2}$$

(or equivalently have distribution $\mathrm{N}[0, 1/2\sqrt{2}] + i\mathrm{N}[0, 1/2\sqrt{2}]$*). Thus $\mathbf{X} = (\mathbf{Y} + \mathbf{Y}^\dagger)/2$, where $\mathbf{Y}$ is an $N \times N$ random matrix of independent real quaternions with z and w in (1.20) having distribution $N[0, \frac{1}{2}] + iN[0, \frac{1}{2}]$.*

A fundamental property of quaternion real Hermitian matrices, which follows from the first equation in (1.19), is that their spectrum is doubly degenerate (see Exercises 1.3 q.1).

It follows from Definition 1.3.2 that the joint p.d.f. of all the independent elements of the GSE is given by

$$P(\mathbf{X}) = A_N e^{-2\mathrm{Tr}(\mathbf{X}^2)},$$

where A_N denotes the normalization and Tr denotes the trace with $\mathbf{X}^2$ regarded as a quaternion real matix (i.e., Tr($\mathbf{X}^2$) equals the sum of the scalar multiples of $\mathbf{1}_2$ on the diagonal of $\mathbf{X}^2$). This satisfies the general requirement of being invariant with respect to similarity transformations of appropriate unitary matrices. In fact the appropriate unitary matrices are those which under a similarity transformation map a quaternion real Hermitian matrix into another quaternion real Hermitian matrix. This subgroup of unitary matrices is specified by the following result.

PROPOSITION 1.3.3
(a) Let $\mathbf{X}$ be an arbitrary $N \times N$ Hermitian matrix with real quaternion elements, so that in general the only symmetry of $\mathbf{X}$ (other than some multiple of the identity) is the operator $T = \mathbf{Z}_{2N}K$. Then any unitary matrix $\mathbf{U}$ which under a similarity transformation maps $\mathbf{X}$ into another Hermitian matrix with real quaternion elements must commute or anticommute with T.
(b) A unitary matrix $\mathbf{U}$ which commutes with $T = \mathbf{Z}_{2N}K$ has the property

$$\mathbf{U}\mathbf{Z}_{2N}\mathbf{U}^T = \mathbf{Z}_{2N}, \tag{1.25}$$

which implies $\mathbf{U}$ is equivalent to a symplectic matrix, while a unitary matrix $\mathbf{U}$ which anticommutes with T has the property $-\mathbf{U}\mathbf{Z}_{2N}\mathbf{U}^T = \mathbf{Z}_{2N}$.

Proof. (a) Let $\mathbf{X}'$ be such that $\mathbf{U}^{-1}\mathbf{X}\mathbf{U} = \mathbf{X}'$. Since both $\mathbf{X}$ and $\mathbf{X}'$ are quaternion real, T commutes with both of these matrices. This implies $\mathbf{X}T\mathbf{U}T^{-1} = T\mathbf{U}T^{-1}\mathbf{X}'$. Comparing these two equations gives that $T\mathbf{U}T^{-1}\mathbf{U}^{-1}$ commutes with $\mathbf{X}$. But the only operators which commute with $\mathbf{X}$ are T and some multiple of the identity, so the above combination of operators must equal one of these operators. We see that the first choice leads to $T = 1$, which is a contradiction, while the second gives $T\mathbf{U} = \pm\mathbf{U}T$ (regarding the signs, recall Exercises 1.1 q.4(ii)) as required.
(b) At the beginning of this subsection it was noted that any matrix, in this case $\mathbf{U}$, which commutes with T has the property $\mathbf{U} = \mathbf{Z}_{2N}\bar{\mathbf{U}}\mathbf{Z}_{2N}^{-1}$. Equation (1.25) follows after noting $\bar{\mathbf{U}} = (\mathbf{U}^{-1})^T$ and rearranging. To continue, recall that by definition a matrix is symplectic if

$$\mathbf{X}^T\mathbf{J}_{2N}\mathbf{X} = \mathbf{J}_{2N}, \qquad \mathbf{J}_{2N} := \begin{bmatrix} \mathbf{0}_N & \mathbf{1}_N \\ -\mathbf{1}_N & \mathbf{0}_N \end{bmatrix}. \tag{1.26}$$

If $\mathbf{X}$ is also unitary, this implies $\mathbf{X}$ has the block structure

$$\mathbf{X} = \begin{bmatrix} \mathbf{Z} & \mathbf{W} \\ -\bar{\mathbf{W}} & \bar{\mathbf{Z}} \end{bmatrix}$$

(cf. (1.20)). Now, the matrix $\mathbf{J}_{2N}$ is related to $\mathbf{Z}_{2N}$ by a similarity transformation $\mathbf{J}_{2N} = \mathbf{Q}^{-1}\mathbf{Z}_{2N}\mathbf{Q}$, where $\mathbf{Q}$ is a unitary matrix with elements ± 1 (there must therefore be exactly one nonzero element in each row/column). We thus conclude from (1.25) that $\mathbf{Q}^{-1}\mathbf{U}\mathbf{Q}$ is symplectic. The only difference in the anticommuting case is a minus sign, which gives the second result. □

1.3.3 The eigenvalue p.d.f.'s

The calculation of the eigenvalue p.d.f.'s from the joint p.d.f.'s for the elements of the GUE and GSE can be done in a similar way to that presented in Section 1.2 for the GOE. The required working is sketched in Exercises 1.3 q.3 and q.4, and the final results are summarized in the following, which for completeness also contains the eigenvalue p.d.f. for the GOE.

PROPOSITION 1.3.4 *Let $\mathbf{H}$ be a Hermitian matrix with real ($\beta = 1$), complex ($\beta = 2$), or real quaternion ($\beta = 4$) elements, and let $\mathbf{H}$ be decomposed in terms of its eigenvalues and eigenvectors via the formula $\mathbf{H} = \mathbf{U}\mathbf{L}\mathbf{U}^\dagger$, where $\mathbf{L}$ is a diagonal matrix consisting of the eigenvalues of $\mathbf{H}$, and $\mathbf{U}$ is a unitary matrix with real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) elements consisting of the corresponding eigenvectors. We have*

$$(d\mathbf{H}) = \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta \bigwedge_{j=1}^{N} d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}), \tag{1.27}$$

and hence for an appropriate choice of the normalization $G_{\beta,N}$, which is given explicitly by (1.163),

$$\frac{1}{G_{\beta,N}} \exp\Big(-\frac{\beta}{2}\sum_{j=1}^{N}\lambda_j^2\Big) \prod_{1\le j<k\le N} |\lambda_k-\lambda_j|^\beta, \tag{1.28}$$

with $\beta = 1,2$ and 4 is the eigenvalue p.d.f. for the GOE, GUE and GSE, respectively.

We remark that for the decomposition $\mathbf{H} = \mathbf{U}\mathbf{L}\mathbf{U}^\dagger$ to be unique the eigenvalues must be ordered and the first component of the eigenvectors must be real and positive. Because (1.28) is a symmetric function of the eigenvalues, the ordering constraint can conveniently be removed, and the normalization appropriately adjusted. In particular, $G_{\beta,N}$ is the normalization without the ordering constraint.

1.3.4 Relationship to Lie algebras

The sets of matrices

$$\begin{aligned} gl(N,\mathbb{R}) &:= \{\text{all } N\times N \text{ real matrices}\}, \\ gl(N,\mathbb{C}) &:= \{\text{all } N\times N \text{ complex matrices}\}, \\ u^*(2N) &:= \{\text{all } N\times N \text{ real quaternion matrices}\} \end{aligned}$$

are each closed under commutation and so form matrix Lie algebras.

Now, in general a matrix can be decomposed as the sum of a Hermitian and an anti-Hermitian matrix. We see that the Hermitian component of the above Lie algebras corresponds to Hermitian matrices with real, complex and real quaternion elements, respectively. This is significant for a number of reasons. One is from a classification perspective. One can identify ten infinite families of matrix Lie algebras, in correspondence with the ten infinite families of symmetric spaces, as catalogued by Cartan [295]. We will see that each of the remaining seven cases also occurs in a basic quantum mechanics problem constrained by a global symmetry. Furthermore, the identification with matrix Lie algebras implies a one-to-one correspondence between the ten families of Hermitian matrices and the ten families of unitary matrices. This comes about because of the relationship between matrix Lie algebras and symmetric spaces. To each matrix Lie algebra there corresponds a noncompact and compact symmetric space, with the former being isomorphic to a certain set of Hermitian matrices, and the latter isomorphic to a certain set of unitary matrices. Some more details are given in Section 2.1.2. The isomorphism with symmetric spaces has the consequence that the eigenvalue-dependent portion of the Jacobian in $(d\mathbf{H})$ can be written in the form

$$\prod_{\vec{\alpha}\in R_+} |\langle\vec{\alpha},\vec{\lambda}\rangle|^{m_\alpha},$$

where $\vec{\alpha}$, an N component Euclidean vector, is a so-called root of the root system corresponding to the symmetric space, $\langle\cdot,\cdot\rangle$ is the dot product, R_+ is the set of positive roots, and m_α the multiplicity of $\vec{\alpha}$. This structure, in the case of the symmetric spaces corresponding to the classical groups, appears in the so called *Weyl integration formula* [540]. For the symmetric spaces corresponding to the Gaussian ensembles the positive roots are $\vec{e}_j - \vec{e}_k$ $(j<k)$ (root system of type A — see Section 4.7.2) with multiplicities $m_\alpha = \beta$, and this reclaims the eigenvalue-dependent portion of (1.27). However ,we will not pursue the derivation of these facts, which can be found in [295].

1.3.5 Octonions and the $N = 2$, $\beta = 8$ Gaussian ensemble

The p.d.f. (1.28) for $N = 2$, $\beta = 8$ can be realized as the eigenvalues of a random Hermitian matrix with real octonion elements. To see this, we must first revise aspects of the theory of real octonions [513]. The real octonions can be constructed out of the real quaternions. Let p_1, p_2, q_1, q_2 be abstract real quaternions, and thus linear combinations with real coefficients of $\{1, i, j, k\}$ as specified by (1.21). Let $\bar{q}$ denote the quaternionic dual defined by (1.23), and let l denote a quantity algebraically distinct from the real quaternions. The real octonion algebra then consists of elements of the form $a = p_1 + p_2 l$, $b = q_1 + q_2 l$, with addition and multiplication defined by

$$a + b = (p_1 + q_1) + (p_2 + q_2)l, \qquad ab = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)l, \tag{1.29}$$

respectively. It follows that the real octonions are an eight-dimensional algebra with basis

$$1,\ e_1 := i,\ e_2 := j,\ e_3 := k,\ e_4 := l,\ e_5 := il,\ e_6 := jl,\ e_7 := kl.$$

In general

$$a(bc) \neq (ab)c$$

(for example with $a = e_5$, $b = e_6$, $c = e_7$ we have $a(bc) = -e_4$ and $(ab)c = e_4$), so unlike the real quaternions the real octonions are not associative. On the other hand, with $\bar{a} := \bar{p}_1 - p_2 l$, $\bar{p}_1$ denoting the quaternionic dual (1.23), we have $\overline{ab} = \bar{b}\bar{a}$ and thus with $|a| := \sqrt{a\bar{a}} = \sqrt{\bar{a}a}$,

$$|ab| = |a||b|. \tag{1.30}$$

Furthermore, with a general real octonion written as $a = a_0 + \sum_{j=1}^{7} a_j e_j$, we have

$$|a| = \sqrt{a_0^2 + a_1^2 + \cdots + a_7^2}, \tag{1.31}$$

and it is also true that each $a \neq 0$ has a unique inverse specified by

$$a^{-1} = \bar{a}/(\bar{a}a). \tag{1.32}$$

The properties (1.30)–(1.32) say that the real octonions are a normed division algebra. In fact a theorem of Hurwitz [301] says that up to isomorphisms, the only normed division algebras over the reals, with a unit element, are the reals, complex numbers, real quaternions and real octonions.

Because the real octonions are not associative, they cannot be represented as a matrix algebra. Nonetheless, the actions of right and left multiplication by a given real octonion a on a general real octonion x can be represented as a matrix. To specify these matrices, we first require the corresponding result for the real quaternions, which follows immediately from the explicit form of the multiplication rule.

PROPOSITION 1.3.5 *Let $x = x_0 + x_1 i + x_2 j + x_3 k$ be a real quaternion and let $\vec{x} = (x_0, x_1, x_2, x_3)^T$ denote the column vector formed from the coefficients. Then for $\vec{a}$ a real quaternion*

$$\vec{ax} = \phi(a)\vec{x}, \qquad \vec{xa} = \tau(a)\vec{x}$$

where

$$\phi(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, \quad \tau(a) = \mathbf{K}\phi^T(a)\mathbf{K} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix},$$

with $\mathbf{K} = \mathrm{diag}[1, -1, -1, -1]$.

Using Proposition 1.3.5, the corresponding result for the real octonions follows from the multiplication rule (1.29).

PROPOSITION 1.3.6 *Let $x = x_0 + \sum_{j=1}^{7} x_j e_j$ be a real octonion, and let $\vec{x} = (x_0, x_1, \ldots, x_7)^T$ denote the column vector formed from the coefficients. Then with $a = a^{(1)} + a^{(2)} l$ a real octonion, and thus $a^{(1)}, a^{(2)}$*

real quaternions, and $\tilde{\mathbf{K}} := \mathrm{diag}[\mathbf{K}, \mathbf{1}_4]$ *we have*

$$a\vec{x} = \omega(a)\vec{x}, \qquad \vec{x}a = \nu(a)\vec{x},$$

where

$$\omega(a) = \begin{bmatrix} \phi(a^{(1)}) & -\tau(a^{(2)})\mathbf{K} \\ \phi(a_{(2)})\mathbf{K} & \tau(a^{(1)}) \end{bmatrix} = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix},$$

$$\nu(a) = \tilde{\mathbf{K}}\omega^T(a)\tilde{\mathbf{K}}.$$

Consider now the 2×2 Hermitian matrix with real octonion entries

$$\mathbf{A} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}.$$

For $\mathbf{A}$ to be Hermitian, the elements a and c must in fact be real, and thus

$$\omega(\mathbf{A}) = \begin{bmatrix} a\mathbf{1}_8 & \omega(b) \\ \omega^T(b) & c\mathbf{1}_8 \end{bmatrix}. \tag{1.33}$$

Adding together appropriate (octonion) multiples of rows and columns shows that this matrix is similar to the matrix

$$\begin{bmatrix} a\mathbf{1}_8 & \mathbf{1}_4 \otimes \begin{bmatrix} b & 0 \\ 0 & \bar{b} \end{bmatrix} \\ \mathbf{1}_4 \otimes \begin{bmatrix} \bar{b} & 0 \\ 0 & b \end{bmatrix} & c\mathbf{1}_8 \end{bmatrix}$$

and thus the characteristic polynomial is given by

$$\det(\omega(\mathbf{A}) - \lambda\mathbf{1}_{16}) = ((a-\lambda)(c-\lambda) - b\bar{b})^8.$$

This shows that each eigenvalue is eightfold degenerate.

Regarding the eigenvectors, as the number of independent real elements in (1.33) is ten, and there are two distinct eigenvalues, there are a total of eight independent components. This implies that in the analogue of Proposition 1.2.5, exactly eight components are to be multiplied together in any one term, and consequently

$$(d\mathbf{A}) = (\lambda_1 - \lambda_2)^8 d\lambda_1 d\lambda_2 (\mathbf{U}^\dagger d\mathbf{U}). \tag{1.34}$$

Furthermore, choosing the elements a, c and the components b_j of $\omega(b)$ in (1.33) to have the Gaussian distributions

$$\frac{2}{\sqrt{\pi}} e^{-4a^2}, \quad \frac{2}{\sqrt{\pi}} e^{-4c^2}, \quad \sqrt{\frac{8}{\pi}} e^{-8b_j^2},$$

respectively, we have that the joint distribution of the independent elements is proportional to

$$e^{-\mathrm{Tr}((\omega(\mathbf{A}))^2)/2}.$$

This together with (1.34) implies that the eigenvalue p.d.f. is given by (1.28) with $N = 2, \beta = 8$.

EXERCISES 1.3 1. The aim of this exercise is to show that if a $2N \times 2N$ Hermitian matrix $\mathbf{X}$ commutes with the time reversal operator $T = \mathbf{Z}_{2N}K$, then the eigenvalues of $\mathbf{X}$ are doubly degenerate (this is known as Kramer's degeneracy).

(i) Suppose $\vec{\phi}$ is an eigenvector of $\mathbf{X}$ with eigenvalue λ. State why $T\vec{\phi}$ is also an eigenvector with eigenvalue λ.

(ii) Use the facts that T satisfies the formula of Exercises 1.1 q.3(i) and $T^2 = -1$ to show that $\langle \vec{\phi} | T\vec{\phi} \rangle = 0$ and hence deduce the desired result.

2. [389, p. 32] Let $\mathbf{A}$ and $\mathbf{M}$ be $N \times N$ matrices, where $\mathbf{A}$ is nonsingular. In this exercise it will be shown that for $\mathbf{A}$ real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$), and $\mathbf{M}$ real symmetric ($\beta = 1$), Hermitian ($\beta = 2$) and quaternion real Hermitian ($\beta = 4$),

$$(\mathbf{A}^\dagger d\mathbf{M}\mathbf{A}) = \Big(\det(\mathbf{A}^\dagger \mathbf{A})\Big)^{\beta(N-1)/2+1} (d\mathbf{M}), \tag{1.35}$$

up to a $\pm$ sign. In the case $\beta = 1$, this is the statement of Proposition 1.2.4. The idea is to decompose $\mathbf{A}$ in terms of elementary matrices $\mathbf{A} = \mathbf{E}_p \mathbf{E}_{p-1} \cdots \mathbf{E}_1$. Each elementary matrix is either a permutation matrix $\mathbf{E}^{(j \leftrightarrow k)}$ (the identity matrix with rows j and k interchanged), a matrix $\mathbf{E}^{(j \mapsto \alpha j)}$ which multiplies row j by the constant α with α real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$) (the identity matrix with row j multiplied by α), or the matrix $\mathbf{E}^{(j \mapsto j+k)}$ which adds together two rows (the identity matrix with row k added to row j).

(i) Show by explicit calculation that for any matrix $\mathbf{X}$ of the same type as $\mathbf{M}$

$$(\mathbf{E}^{(j \leftrightarrow k)} d\mathbf{X} \mathbf{E}^{(j \leftrightarrow k)\dagger}) = (d\mathbf{X}), \qquad (\mathbf{E}^{(j \mapsto j+k)} d\mathbf{X} \mathbf{E}^{(j \mapsto j+k)\dagger}) = (d\mathbf{X}),$$

while, up to a $\pm$ sign,

$$(\mathbf{E}^{(j \mapsto \alpha j)} d\mathbf{X} \mathbf{E}^{(j \mapsto \alpha j)\dagger}) = |\alpha|^{\beta(N-1)+2} (d\mathbf{X}) = |\det \mathbf{E}^{(j \mapsto \alpha j)}|^{\beta(N-1)+2} (d\mathbf{X}).$$

(ii) Use the result of (i) to deduce the stated result.

For printing purposes, the symbol α^* rather than $\bar{\alpha}$ is used in the exercises below to denote the complex conjugate of α.

3. The aim of this exercise is to calculate the change of variables from the independent elements of a Hermitian matrix $\mathbf{X}$ to the eigenvalues $\lambda_1, \dots, \lambda_N$ and other independent variables.

(i) From the diagonalization formula $\mathbf{X} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$, where $\mathbf{L} := \mathrm{diag}[\lambda_1, \dots, \lambda_N]$ and $\mathbf{U}$ is a unitary matrix with columns given by the eigenvectors of $\mathbf{X}$, show that

$$\mathbf{U}^{-1} d\mathbf{X} \mathbf{U} = \mathbf{U}^{-1} d\mathbf{U} \mathbf{L} - \mathbf{L} \mathbf{U}^{-1} d\mathbf{U} + d\mathbf{L}$$

and write down a formula for the Jacobian in terms of $(d\mathbf{X})$. Use the result of q.2 to show that the wedge product of the independent elements on the l.h.s. is equal to $(d\mathbf{X})$.

(ii) Show that $\mathbf{U}^{-1} d\mathbf{U} \mathbf{L} - \mathbf{L} \mathbf{U}^{-1} d\mathbf{U} + d\mathbf{L}$ equals

$$\begin{bmatrix} d\lambda_1 & (\lambda_2 - \lambda_1) \vec{u}_1^\dagger \cdot d\vec{u}_2 & \dots & (\lambda_N - \lambda_1) \vec{u}_1^\dagger \cdot d\vec{u}_N \\ (\lambda_2 - \lambda_1)(\vec{u}_1^\dagger \cdot d\vec{u}_2)^* & d\lambda_2 & \dots & (\lambda_N - \lambda_2) \vec{u}_2^\dagger \cdot d\vec{u}_N \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N - \lambda_1)(\vec{u}_1^\dagger \cdot d\vec{u}_N)^* & (\lambda_N - \lambda_2)(\vec{u}_2^\dagger \cdot d\vec{u}_N)^* & \dots & d\lambda_N \end{bmatrix}.$$

(iii) Use the facts that $\vec{u}_j^\dagger \cdot d\vec{u}_k$ has independent real and imaginary parts and that only the elements on and above the diagonal are independent to conclude that the wedge product of the independent elements of the matrix in (ii) equals

$$\prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2 \bigwedge_{j=1}^N d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}).$$

(iv) Show that the factor dependent on the λ_j's is consistent with the form required by the scaling $\mathbf{X} \mapsto a\mathbf{X}$ (recall Section 1.2.3).

4. Here the objective is the same as in q.3 above, except $\mathbf{X}$ is now an $N \times N$ Hermitian matrix with real quaternion elements.

   (i) From the diagonalization formula $\mathbf{X} = \mathbf{ULU}^{-1}$ where $\mathbf{L} = \mathrm{diag}[\lambda_1 \mathbf{1}_2, \dots, \lambda_N \mathbf{1}_2]$ and $\mathbf{U}$ is an $N \times N$ unitary matrix with real quaternion elements, write down the formulas analogous to those in q.3(i) and q.3(ii) above. To write down the analogue of (ii) use a matrix notation for the quaternion elements,

$$\vec{\mathbf{u}}_k = (\mathbf{u}_{1k}, \dots, \mathbf{u}_{Nk})^T, \quad \vec{\mathbf{u}}_j^\dagger \cdot \vec{\mathbf{u}}_k = \sum_{p=1}^N \mathbf{u}_{pj}^\dagger \mathbf{u}_{pk}.$$

   (ii) Use the facts that $\vec{\mathbf{u}}_j^\dagger \cdot d\vec{\mathbf{u}}_k$ has four independent terms, corresponding to the real and imaginary parts of the two independent terms in each real quaternion element, to deduce the formula analogous to q.3(iii) above. Also repeat the scaling analysis of q.3(iv) above.

5. A Hermitian matrix with zero real part is antisymmetric.

   (i) Show that the nonzero eigenvalues of antisymmetric Hermitian matrices come in $\pm$ pairs, λ_j and $-\lambda_j$, say, with corresponding eigenvectors $\vec{\phi}_j$ and $\vec{\phi}_j^*$, and that for N odd $\lambda = 0$ is an eigenvalue.

   (ii) Use (i) to deduce that the equation of q.3(ii) holds with $\vec{u}_j = \vec{\phi}_j$, $\lambda_{N/2+j} = -\lambda_j$, $\vec{u}_{N/2+j} = \vec{u}_j^*$ $(j = 1, \dots, N/2)$ N even, and $\vec{u}_N^* = \vec{u}_N = \phi_0$, $\lambda_N = 0$, $\vec{u}_j = \vec{\phi}_j$, $\lambda_{(N-1)/2+j} = -\lambda_j$ $(j = 1, \dots, (N-1)/2)$ N odd. Use the fact that eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal to deduce that $\vec{\phi}_0{}^\dagger \cdot d\vec{\phi}_j^* = 0$ $(j \neq 0)$, and note too that $\vec{\phi}_0^\dagger \cdot d\vec{\phi}_k$ and $\vec{\phi}_0^{*\dagger} \cdot d\vec{\phi}_j^*$ are not independent.

   (iii) Use (ii) to show that for an antisymmetric Hermitian $N \times N$ matrix $\mathbf{H}^i$, diagonalized by $\mathbf{H}^i = \mathbf{ULU}^{-1}$,

$$(d\mathbf{H}^i) = \prod_{1 \le j < k \le N/2} (\lambda_j^2 - \lambda_k^2)^2 \bigwedge_{j=1}^{N/2} d\lambda_j \, (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ even},$$

$$(d\mathbf{H}^i) = \prod_{j=1}^{(N-1)/2} \lambda_j^2 \prod_{1 \le j < k \le (N-1)/2} (\lambda_j^2 - \lambda_k^2)^2 \bigwedge_{j=1}^{(N-1)/2} d\lambda_j \, (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ odd}.$$

   (iv) Conclude from the result of (iii) that for a random antisymmetric Hermitian $N \times N$ matrix with upper triangular elements ix_{jk} chosen with p.d.f. $\sqrt{1/\pi}e^{-x_{jk}^2}$, the eigenvalue p.d.f. of the positive eigenvalues is equal to

$$\frac{1}{C_N} \prod_{j=1}^{N/2} e^{-\lambda_j^2} \prod_{1 \le j < k \le N/2} (\lambda_j^2 - \lambda_k^2)^2, \quad N \text{ even},$$

$$\frac{1}{C_N} \prod_{j=1}^{(N-1)/2} \lambda_j^2 e^{-\lambda_j^2} \prod_{1 \le j < k \le (N-1)/2} (\lambda_j^2 - \lambda_k^2)^2, \quad N \text{ odd},$$

   where the normalizations C_N are given explicitly in (4.157) below.

6. (i) Let $\mathbf{Q}^\mathrm{r}$ be a quaternion real Hermitian matrix in which all entries of each real quaternion are real, and let $\mathbf{H}$ be the Hermitian matrix formed by replacing each quaternion element (1.20) by the scalar $z + iw$. Show that $\mathbf{Q}^\mathrm{r}$ and $\mathbf{H}$ have the same distinct eigenvalues, and that the eigenvalues of $\mathbf{Q}^\mathrm{r}$ are doubly degenerate with eigenvectors of the form $\vec{\psi}^{(1)} = \begin{bmatrix} \phi_k^\mathrm{r} \\ \phi_k^\mathrm{i} \end{bmatrix}_{k=1,\dots,N}$ and $\vec{\psi}^{(2)} = \mathbf{Z}_{2N} \vec{\psi}^{(1)}$ where $\vec{\phi} = [\phi_k^\mathrm{r} + i\phi_k^\mathrm{i}]_{k=1,\dots,N}$ is an eigenvector of $\mathbf{H}$. Hence write down the eigenvalue p.d.f. of $\mathbf{Q}^\mathrm{r}$.

(ii) Let $\mathbf{Q}$ be an $N \times N$ real quaternion Hermitian matrix. Proceed in a converse fashion to (i) to write down a $4N \times 4N$ real symmetric matrix $\mathbf{R}$ such that $\mathbf{Q}$ and $\mathbf{R}$ have the same distinct eigenvalues, and thus the same eigenvalue p.d.f. Also, relate the corresponding eigenvectors. Similarly, for $\mathbf{H}$ an $N \times N$ complex Hermitian matrix, replace each entry $x + iy$ by its 2×2 real matrix representation

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \tag{1.36}$$

to obtain a doubly degenerate $2N \times 2N$ matrix for which the distinct eigenvalues coincide with those of $\mathbf{H}$.

7. Consider a quaternion real Hermitian matrix $\mathbf{Q}^{\mathrm{i}}$ in which all entries of each real quaternion are pure imaginary so that $\mathbf{Q}^{\mathrm{i}}$ is antisymmetric.

(i) With the pair of eigenvectors corresponding to the doubly degenerate eigenvalues λ_j denoted by $\vec{\mathbf{u}}_j$ as in q.4, note from the theory of q.5(i) that $\vec{\mathbf{u}}_j^*$ is equal to the pair of eigenvectors corresponding to the doubly degenerate eigenvalue $-\lambda_j$.

(ii) Use the fact that eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal to deduce that

$$\vec{\mathbf{u}}_j^\dagger \cdot \vec{\mathbf{u}}_j^* := \sum_{p=1}^N \mathbf{u}_{pj}^\dagger \mathbf{u}_{pj}^* = \sum_{p=1}^N \begin{bmatrix} 0 & -2\mathrm{Im}\,(z_{pj} w_{pj}) \\ -2\mathrm{Im}\,(z_{pj} w_{pj}) & 0 \end{bmatrix}, \quad \mathbf{u}_{jp} := \begin{bmatrix} z_{pj} & w_{pj} \\ -w_{pj}^* & z_{pj}^* \end{bmatrix},$$

and conclude from this that $\vec{\mathbf{u}}_j^\dagger \cdot d\vec{\mathbf{u}}_j^*$ has only one independent component. Note too that $\vec{\mathbf{u}}_j^\dagger \cdot d\vec{\mathbf{u}}_k$ and $\vec{\mathbf{u}}_j^{*\dagger} \cdot d\vec{\mathbf{u}}_k^*$ are not independent.

(iii) With the analogue of the equation of q.3(ii) in the quaternion case modified as in the first sentence of q.5(ii) (N even case), show from (i) that for an antisymmetric $N \times N$ quaternion real Hermitian matrix $\mathbf{Q}^{\mathrm{i}}$ diagonalized by $\mathbf{Q}^{\mathrm{i}} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$,

$$(d\mathbf{Q}^{\mathrm{i}}) = \prod_{j=1}^{N/2} \lambda_j \prod_{1 \le j < k \le N/2} (\lambda_j^2 - \lambda_k^2)^4 \bigwedge_{j=1}^{N/2} d\lambda_j \, (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ even},$$

$$(d\mathbf{Q}^{\mathrm{i}}) = \prod_{j=1}^{(N-1)/2} \lambda_j^5 \prod_{1 \le j < k \le (N-1)/2} (\lambda_j^2 - \lambda_k^2)^4 \bigwedge_{j=1}^{(N-1)/2} d\lambda_j \, (\mathbf{U}^\dagger d\mathbf{U}), \quad N \text{ odd}.$$

8. [146] Let $\mathbf{Q}$ be a quaternion real matrix with the property that $i\mathbf{Q}$ is Hermitian.

(i) Note that $\mathbf{Q}$ must anticommute with the time reversal operator $T = \mathbf{Z}_{2N} K$, and use this to show that if $|\phi\rangle$ is an eigenvector of $\mathbf{Q}$ with eigenvalue λ, then $T\vec{\phi}$ is an eigenvector with eigenvalue $-\lambda$.

(ii) Proceed as in q.4 to show that with $i\mathbf{Q}$ diagonalized by $i\mathbf{Q} = \mathbf{U} i \mathbf{L} \mathbf{U}^\dagger$, where

$$\mathbf{L} = \mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_N, -\lambda_N),$$

and $\mathbf{U}$ is unitary with real quaternion elements in which all elements are real,

$$(d\mathbf{Q}) = \prod_{j=1}^N (2\lambda_j)^2 \prod_{1 \le j < k \le N} (\lambda_k^2 - \lambda_j^2)^2 \bigwedge_{j=1}^N d\lambda_j (\mathbf{U}^\dagger d\mathbf{U}). \tag{1.37}$$

9. Let $G_{\beta,N}$ be the normalization in (1.28), which has the evaluation (1.163) below, and let

$$A_{\beta,N} = \Big(\frac{\beta}{2\pi}\Big)^{N/2} \Big(\frac{\beta}{\pi}\Big)^{\beta N(N-1)/4}$$

so that

$$A_{\beta,N} \int e^{-(\beta/2)\mathrm{Tr}(\mathbf{H}^2)} \, (d\mathbf{H}) = 1.$$

Deduce from this last equation and (1.27) that for $\beta = 1, 2$

$$\int (\mathbf{U}^\dagger d\mathbf{U}) = \frac{N!}{A_{\beta,N} G_{\beta,N}}, \tag{1.38}$$

where the first entry of each column of $\mathbf{U}$ is chosen to be real and positive.

10. Define the matrix

$$\tilde{\mathbf{H}} = \Big(\frac{c}{\operatorname{Tr} \mathbf{H}^2} \Big)^{1/2} \mathbf{H},$$

where $\mathbf{H}$ is a member of one of the Gaussian ensembles and $c > 0$ is a constant. By changing variables from the elements of $\mathbf{H}$ to the elements of $\tilde{\mathbf{H}}$, and $\operatorname{Tr} \mathbf{H}^2$, show that the eigenvalues $\{\tilde{\lambda}_j\}$ of $\tilde{\mathbf{H}}$ have distribution proportional to

$$\delta\Big(c - \sum_{j=1}^N \tilde{\lambda}_j^2\Big) \prod_{1 \le j < k \le N} |\tilde{\lambda}_k - \tilde{\lambda}_j|^\beta.$$

1.4 COULOMB GAS ANALOGY

The eigenvalue p.d.f. (1.28) can be identified with the Boltzmann factor of a particular log-gas, an observation which goes back to Dyson [146]. To appreciate this, we must revise some basic theory from statistical mechanics [390], and show how the Boltzmann factor of a log-gas, or more generally a one-component Coulomb system, is computed.

1.4.1 Boltzmann factors

The canonical formalism of statistical mechanics applies to any mechanical system of N particles free to move in a fixed domain Ω, in equilibrium at absolute temperature T. A fundamental postulate gives the p.d.f. for the event that the particles are at positions $\vec{r}_1, \dots, \vec{r}_N$ as

$$\frac{1}{\hat{Z}_N} e^{-\beta U(\vec{r}_1,\dots,\vec{r}_N)}.$$

Here $U(\vec{r}_1, \dots, \vec{r}_N)$ denotes the total potential energy of the system, $\beta := 1/(k_B T)$ (k_B is Boltzmann's constant), and the normalization $\hat{Z}_N$ is given by

$$\hat{Z}_N = \int_\Omega d\vec{r}_1 \cdots \int_\Omega d\vec{r}_N e^{-\beta U(\vec{r}_1,\dots,\vec{r}_N)}. \tag{1.39}$$

The term $e^{-\beta U(\vec{r}_1,\dots,\vec{r}_N)}$ is referred to as the *Boltzmann factor* and $\hat{Z}_N/N! =: Z_N$ is called the *(canonical) partition function.*

For log-potential Coulomb systems the potential energy U is calculated according to the laws of two-dimensional electrostatics, and Ω must be one- or two-dimensional. The particles can be thought of as infinitely long parallel charged lines, which are perpendicular to the confining domain. In a vacuum the electrostatic potential Φ at a point $\vec{r} = (x, y)$ due to a two-dimensional unit charge at $\vec{r}\,' = (x', y')$ is given by the solution of the *Poisson equation*

$$\nabla_{\vec{r}}^2 \Phi(\vec{r}, \vec{r}\,') = -2\pi\delta(\vec{r} - \vec{r}\,'), \tag{1.40}$$

where

$$\nabla_{\vec{r}}^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is straightforward to verify that the solution of the Poisson equation is (see Exercises 1.4 q.1)

$$\Phi(\vec{r},\vec{r}\,') = -\log(|\vec{r}-\vec{r}\,'|/l), \tag{1.41}$$

where l is some arbitrary length scale which will henceforth be set to unity.

A Coulomb system is said to consist of *one component* if all N particles are of like charge, q, say. To stop the particles from all repelling to the boundary, a neutralizing background charge density $-q\rho_b(\vec{r})$ is imposed, with the electroneutrality condition $\int_\Omega \rho_b(\vec{r})\,d\vec{r} = N$. The total potential energy U therefore consists of the sum of the electrostatic energy of the particle-particle interaction

$$U_1 := -q^2 \sum_{1\le j<k\le N} \log|\vec{r}_k - \vec{r}_j|,$$

the particle-background interaction

$$U_2 := q^2 \sum_{j=1}^N V(\vec{r}_j) \qquad \text{where} \qquad V(\vec{r}_j) := \int_\Omega \log|\vec{r}-\vec{r}_j|\,\rho_b(\vec{r})\,d\vec{r}, \tag{1.42}$$

and the background-background interaction

$$U_3 := -\frac{q^2}{2}\int_\Omega d\vec{r}\,'\,\rho_b(\vec{r}\,')\int_\Omega d\vec{r}\,\rho_b(\vec{r})\log|\vec{r}\,'-\vec{r}| = -\frac{q^2}{2}\int_\Omega \rho_b(\vec{r}\,')V(\vec{r}\,')\,d\vec{r}\,'. \tag{1.43}$$

The factor of $\frac{1}{2}$ in U_3 is included to compensate for the double counting of the potential energy implicit in the double integration.

From this expression for U we conclude that the Boltzmann factor of a one-component log-potential Coulomb system (log-gas) is of the form

$$e^{-\beta U_3}\prod_{l=1}^N e^{-\Gamma V(\vec{r}_l)} \prod_{1\le j<k\le N} |\vec{r}_k - \vec{r}_j|^\Gamma, \tag{1.44}$$

where $\Gamma := q^2/k_BT$. Furthermore, for a given geometry and background density the potentials $V(\vec{r})$ and U_3 can readily be evaluated. As an illustration, we have the following result.

PROPOSITION 1.4.1 *The Boltzmann factor of a one-component log-potential Coulomb system of N particles of charge $q = 1$, confined to a circle of radius R with a uniform neutralizing background, is given by*

$$R^{-N\beta/2}\prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^\beta,$$

where the position of each particle has been specified in polar coordinates.

Proof. It is generally true that for two points $\vec{r}$ and $\vec{r}\,'$ in the plane $|\vec{r}-\vec{r}\,'| = |z-z'|$, where z and z' are the corresponding points in the complex plane. Hence, if $\vec{r}$ and $\vec{r}\,'$ are both on a circle of radius R with positions specified using polar coordinates, then $|\vec{r}-\vec{r}\,'| = R|e^{i\theta} - e^{i\theta'}|$. Use of this formula gives the required expression for the product over pairs in (1.44). It also allows the potential $V(\vec{r})$ to be written as

$$V(\vec{r}) = \frac{N}{2\pi R}\int_0^{2\pi} \log|Re^{i\theta'} - Re^{i\theta}|\,R d\theta' = N\log R + \frac{N}{2\pi}\int_0^{2\pi}\log|e^{i\theta'}-1|\,d\theta'.$$

But it is straightforward to show that the last integral vanishes (see Exercises 1.4 q.2), and so $V(\vec{r}) = N\log R$. Use of this result gives $U_3 = -\frac{q^2}{2}N^2\log R$. Substituting these evaluations in (1.44) and noting that since $q=1$, $\Gamma = \beta$ gives the desired expression for the Boltzmann factor. □

The Boltzmann factor, being proportional to the p.d.f. for the location of the particles, occurs in the definition of all statistical quantities associated with the equilibrium state. In particular the *canonical average* of any function $f(\vec{r}_1, \dots, \vec{r}_N)$ is given by

$$\langle f \rangle := \frac{1}{\hat{Z}_N} \int_\Omega d\vec{r}_1 \cdots \int_\Omega d\vec{r}_N \, f(\vec{r}_1, \dots, \vec{r}_N) e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}. \tag{1.45}$$

With $f = \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)$ the canonical average is called the one-point correlation function, or particle density

$$\rho_{(1)}(\vec{r}) := \Big\langle \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \Big\rangle = \frac{N}{\hat{Z}_N} \int_\Omega d\vec{r}_2 \cdots \int_\Omega d\vec{r}_N \, e^{-\beta U(\vec{r}, \vec{r}_2, \dots, \vec{r}_N)}, \tag{1.46}$$

where the equality is valid for a system of identical particles, and thus when the Boltzmann factor is a symmetric function of the particle coordinates.

1.4.2 The potential and calculation of $\rho_b(y)$

Comparison of (1.28) with (1.44) shows immediately that the eigenvalue p.d.f. is identical to the Boltzmann factor of a one-component log-potential Coulomb system confined to a line, with the position of the charged particles corresponding identically to the location of the eigenvalues. Furthermore, the background charge density $-q\rho_b(y)$ is such that

$$\frac{x^2}{2} + C = \int_{-\infty}^{\infty} \rho_b(y) \log |x - y| \, dy, \tag{1.47}$$

where C is a constant.

Note that it is not possible to satisfy (1.47) for $|x| \to \infty$, since in this limit the r.h.s. is to leading order $N \log |x|$ and is thus a different order from the l.h.s. Instead we seek to solve the integral equation for $\rho_b(y)$ with support on the finite interval $(-a, a)$ say, and x confined to the same interval. Then (1.47) reads

$$\frac{x^2}{2} + C = \int_{-a}^{a} \rho_b(y) \log |x - y| \, dy, \quad x \in (-a, a). \tag{1.48}$$

The solution of the equation can be computed exactly by the method of eigenfunction expansions (see, e.g., [448]).

Proposition 1.4.2 *Suppose all the eigenvalues $\{\lambda_n\}_{n=0,1,\dots}$ and corresponding normalized eigenfunctions $\{\phi_n\}_{n=0,1,\dots}$ of a linear operator A are known, all the eigenvalues are nonzero, and the eigenfunctions form a complete set. Then the operator equation $g = Af$, where g is given, has the solution*

$$f = \sum_{n=0}^{\infty} \frac{\langle g | \phi_n \rangle}{\lambda_n} \phi_n,$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product.

Proof. Since the eigenfunctions form a complete set, $g = \sum_{n=0}^\infty \langle g|\phi_n\rangle \phi_n$ Also $f = \sum_{n=0}^\infty \langle f|\phi_n\rangle \phi_n$ and so $Af = \sum_{n=0}^\infty \langle f|\phi_n\rangle \lambda_n \phi_n$. The result follows by equating the coefficients of ϕ_n in the operator equation. □

To make use of this method, it is necessary to make the further change of variables

$$y = a \cos\theta, \qquad x = a\cos\sigma, \qquad \sin\theta \rho_b(a\cos\theta) =: a\phi(\theta)$$

so that (1.48) reads

$$-\frac{1}{4}\cos 2\sigma - \left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{C}{a^2}\right) = -\int_0^\pi \log|\cos\theta - \cos\sigma| \, \phi(\theta) \, d\theta. \tag{1.49}$$

Note that $\cos\theta - \cos\sigma = 2\sin(\sigma-\theta)/2\, \sin(\sigma+\theta)/2$. Since $2|\sin(\sigma-\theta)/2|$ gives the chord length for two points on the unit circle with angles σ and θ, the r.h.s. of (1.49) can be interpreted as giving (up to an additive constant) the electrostatic potential at the angle σ due to a charge density $\phi(\theta)$ and $\phi(2\pi-\theta)$ between 0 and π and π and 2π, respectively, on the unit circle.

The eigenvalues and eigenfunctions of the integral operator

$$A[\phi](\sigma) := -\int_0^\pi \log|\cos\theta - \cos\sigma|\phi(\theta)\, d\theta \tag{1.50}$$

are known (see Exercises 1.4 q.4). They are

$$\lambda_0 = \pi\log 2, \quad \phi_0(\theta) = \frac{1}{\pi^{1/2}}, \quad \lambda_n = \frac{\pi}{n}, \quad \phi_n(\theta) = \left(\frac{2}{\pi}\right)^{1/2}\cos n\theta \quad (n=1,2,\dots).$$

In terms of these eigenfunctions

$$-\frac{1}{4}\cos 2\sigma - \left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{C}{a^2}\right) = -\frac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\phi_2(\sigma) - \left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{C}{a^2}\right)\pi^{\frac{1}{2}}\phi_0(\sigma),$$

and so from the general formula of Proposition 1.9 the solution of the transformed integral equation is

$$\phi(\theta) = -\frac{1}{2\pi}(\cos 2\theta - 1) - \frac{1}{\pi\log 2}\left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{C}{a^2} + \frac{1}{2}\log 2\right),$$

where $1/(2\pi)$ has been added and subtracted for later convenience. Reverting back to the original variables we obtain the following result.

PROPOSITION 1.4.3 *The solution of the integral equation*

$$\frac{x^2}{2} + C = \int_{-a}^{a} \rho_b(y)\log|x-y|\, dy, \quad -a \le x \le a,$$

is

$$\rho_b(y) = \frac{a}{\pi}\sqrt{1-\left(\frac{y}{a}\right)^2} - \frac{1}{\pi\log 2}\left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{C}{a^2} + \frac{1}{2}\log 2\right)\frac{a}{\sqrt{1-(y/a)^2}}.$$

We see that there are two drastically different classes of solution depending on the value of C. Unless we choose

$$C = N\log a - \frac{a^2}{4} - \frac{a^2}{2}\log 2 \tag{1.51}$$

the density profile $\rho_b(y)$ has an inverse square root singularity at $y = \pm a$. However, with C according to (1.51) the term proportional to $(1-(y/a)^2)^{-1/2}$ vanishes and a physically sensible result is obtained. Making this choice of C and fixing a by the neutrality condition $\int_{-a}^{a}\rho_b(y)\, dy = N$ gives the desired analogy between the Boltzmann factor of a one-component log-potential Coulomb system and the eigenvalue p.d.f.'s of Proposition 1.3.4.

PROPOSITION 1.4.4 *The Boltzmann factor of the one-component log-potential Coulomb system with particles of charge $q=1$ at $x_1,\dots,x_N$, confined to the interval $[-\sqrt{2N}, \sqrt{2N}]$, with a neutralizing background charge density*

$$-\rho_b(y) = -\frac{\sqrt{2N}}{\pi}\sqrt{1-\frac{y^2}{2N}},$$

is

$$A \exp\Big(-\frac{\beta}{2}\sum_{j=1}^{N} x_j^2\Big) \prod_{1\le j<k\le N} |x_k - x_j|^\beta, \quad A = \exp\left(-\frac{\beta N^2}{4}\log(N/2) + \frac{3\beta N^2}{8}\right).$$

Proof. Apply the general formula (1.44) for the Boltzmann factor of a one-component log-potential Coulomb system with $\vec{r}_k = x_k$. From Proposition 1.4.3 and (1.51) with $a = \sqrt{2N}$

$$V(x) = \frac{x^2}{2} + 4N\left(\frac{1}{4}\log\sqrt{N/2} - \frac{1}{8}\right),$$

$$U_3 = -q^2\left[\frac{N^2}{\pi}\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx + 2N^2\left(\frac{1}{4}\log\sqrt{N/2} - \frac{1}{8}\right)\right].$$

A simple change of variables $x = \cos\theta$ shows that the integral in the above equals $\pi/8$. The stated formula for the Boltzmann factor follows. □

Proposition 1.4.4 can be used to predict the eigenvalue density profile for Gaussian β-ensembles with eigenvalue p.d.f. (1.28). Physically, we expect that to leading order in N Coulomb systems are locally charge neutral, which for a one-component system implies that to leading order the particle density will equal the background density. For the log-potential system of Proposition 1.4.4 this gives the particle density as

$$\rho_{(1)}(y) = \frac{\sqrt{2N}}{\pi}\sqrt{1 - \frac{y^2}{2N}}. \tag{1.52}$$

But the statistical properties of the log-potential system in Proposition 1.4.4 are identical to those of the eigenvalues of Gaussian random matrices, so we expect that the eigenvalue density profile will to leading order in N be given by this formula. Consequently we expect the so-called global density

$$\tilde{\rho}_{(1)}(Y) := \lim_{N\to\infty}\sqrt{2/N}\,\rho_{(1)}(\sqrt{2N}Y) \tag{1.53}$$

to obey the limit formula

$$\tilde{\rho}_{(1)}(x) = \begin{cases} \frac{2}{\pi}(1-x^2)^{1/2}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \tag{1.54}$$

known as the *Wigner semicircle law*. The validity of this statement is known rigorously from [328]. Wigner's derivation, which is applicable to GUE matrices, is given in Exercises 1.6 q.1.

In Figure 1.1 we have plotted the empirical eigenvalue density for 1000 10×10 matrices from the GUE, using the variable $Y = y/\sqrt{2N}$. The accuracy of the Wigner semicircle law is evident.

1.4.3 The complex electric field and calculation of $\rho_b(y)$

The integral equation (1.48) is the special case $V(x) = x^2/2$ of the integral equation

$$V(x) + C = \int_{-a}^{a} \rho_b(y)\log|x-y|\,dy, \qquad x \in (-a, a). \tag{1.55}$$

For the log-gas at $\beta = 2$ a rigorous derivation of this integral equation for the particle density is given in Exercises 14.4 q.4 below. We seek the solution such that $\rho_b(y)$ is bounded at $y = \pm a$ and normalized so that

$$\int_{-a}^{a} \rho_b(y)\,dy = N. \tag{1.56}$$

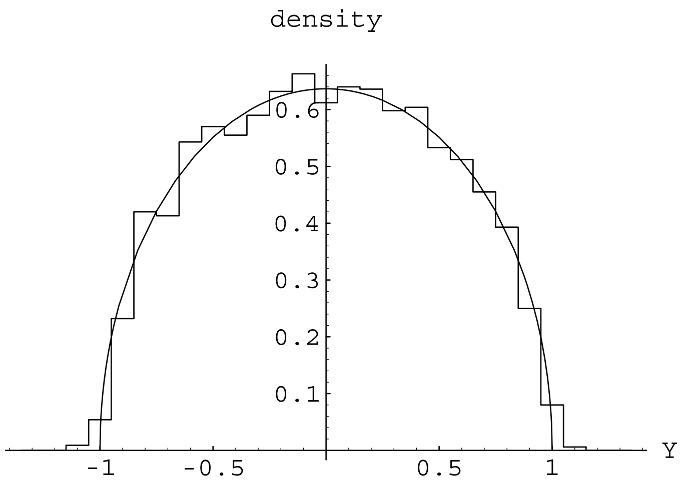

Figure 1.1 Empirical demonstration of the Wigner semicircle law for 10×10 matrices from the GUE.

If the primary concern is the calculation of $\rho_b(y)$ and not C, an alternative to the method of eigenfunctions used above is to introduce the complex electric field

$$E(z) := -\int_{-a}^{a} \frac{\rho_b(y)}{z-y}\, dy. \tag{1.57}$$

Note that for $z \notin [-a,a]$, $E(z)$ is analytic and has the asymptotic behavior

$$E(z) \underset{|z|\to\infty}{\sim} -\frac{N}{z}, \tag{1.58}$$

while for $z \sim \pm a$, by the assumption that $\rho_b(y)$ is bounded,

$$E(z) \underset{z\to\pm a}{\sim} \mathrm{O}(\log(z \mp a)). \tag{1.59}$$

Furthermore, if $\rho_b(y)$ can be analytically continued to a neighborhood of the interval $y \in (-a,a)$, it follows by deforming the path of integration in the neighborhood of $y = x$ and Cauchy's theorem that

$$E^+(x) - E^-(x) = 2\pi i \rho_b(x), \qquad x \in (-a,a), \tag{1.60}$$

where

$$E^{\pm}(x) = \lim_{\epsilon\to 0^+} E(x \pm i\epsilon).$$

Differentiating (1.55) shows

$$\mathrm{Re}\, E(x) = \frac{1}{2}(E^+(x) + E^-(x)) = -V'(x), \qquad \text{for} \quad x \in (-a,a). \tag{1.61}$$

The properties (1.58), (1.59) and (1.61) can be used to characterize $E(z)$, with $\rho_b(x)$ then computed from (1.60). For the quantity $W(z) := e^{E(z)}$ the properties (1.58), (1.59), (1.60), where $\rho_b(x)$ is given, specify a scalar *Riemann-Hilbert problem.*

Restricting attention to potentials $V(x)$ even in x, one can check that the function

$$E(z) = -\frac{1}{\pi}\sqrt{z^2 - a^2} \int_{-a}^{a} \frac{V'(t)}{(z-t)\sqrt{a^2 - t^2}}\, dt, \tag{1.62}$$

with a such that

$$\frac{1}{\pi}\int_{-a}^{a}\frac{tV'(t)}{\sqrt{a^2-t^2}}\,dt = N \tag{1.63}$$

has the properties (1.58), (1.59) and (1.61) and is thus the sought complex electric field. In particular, to verify (1.61), we note that (1.62) gives

$$E^{\pm}(x) = \mp\frac{i}{\pi}\sqrt{a^2-x^2}\lim_{\epsilon\to 0^+}\int_{-a}^{a}\frac{V'(t)}{(x\pm i\epsilon-t)\sqrt{a^2-t^2}}\,dt \tag{1.64}$$

and then make use of Cauchy's theorem. Using this formula in (1.60) gives an explicit formula for $\rho_b(x)$ in terms of the potential V [412].

PROPOSITION 1.4.5 *In the case $V(x)$ even, the solution of the integral equation (1.55) with $\rho_b(y)$ bounded at $y=\pm a$ and normalized as in (1.56) is*

$$\rho_b(y) = \frac{1}{\pi^2}\sqrt{a^2-y^2}\int_{-a}^{a}\frac{V'(y)-V'(t)}{y-t}\frac{1}{\sqrt{a^2-t^2}}\,dt, \tag{1.65}$$

where a is specified by (1.63).

Proof. After substituting (1.64) in (1.60), we subtract an appropriate multiple of the identity

$$\lim_{\epsilon\to 0}\Big(\int_{-a}^{a}\frac{1}{(x+i\epsilon-t)\sqrt{a^2-t^2}}\,dt - \int_{-a}^{a}\frac{1}{(x-i\epsilon-t)\sqrt{a^2-t^2}}\,dt\Big) = 0 \tag{1.66}$$

from both sides to obtain

$$2\pi i\rho_b(x) = -\frac{2i}{\pi}\sqrt{a^2-x^2}\mathrm{Re}\lim_{\epsilon\to 0^+}\int_{-a}^{a}\frac{V'(t)-V'(x)}{(x+i\epsilon-t)\sqrt{a^2-t^2}}\,dt.$$

The limit can be taken inside the integrand because the numerator vanishes for $x=t$, giving (1.65). □

In the special case $V(y)=y^2/2$, (1.65) gives

$$\rho_b(y) = \frac{a}{\pi}\sqrt{1-(y/a)^2},$$

while it follows from (1.63) that $a=\sqrt{2N}$, in agreement with (1.52). In this same special case the explicit form of the complex electric field can also be computed; this is done in Exercises 1.6 q.2. The generalization of (1.65) for $V(x)$ not necessarily even is

$$\rho_b(y) = \frac{1}{\pi^2}\sqrt{(y-a)(b-y)}\int_{a}^{b}\frac{V'(y)-V'(t)}{y-t}\frac{1}{\sqrt{(t-a)(b-t)}}\,dt, \tag{1.67}$$

where a and b are such that

$$\int_{a}^{b}\frac{V'(t)}{\sqrt{(t-a)(b-t)}}\,dt = 0, \qquad \frac{1}{\pi}\int_{a}^{b}\frac{tV'(t)}{\sqrt{(t-a)(b-t)}}\,dt = N, \tag{1.68}$$

as can be checked from a similar analysis.

It can happen that for certain $V(x)$ the solution (1.65) or (1.67) of (1.55) does not in fact correspond to the background density because it becomes negative within the interval $(-a,a)$. An example is the potential $V(x)=-cx^2+gx^4$ for c large enough. Formula (1.65) gives

$$\rho_b(y) = \frac{1}{\pi}(-2c+2ga^2+4gy^2)\sqrt{a^2-y^2}, \tag{1.69}$$

where, according to (1.63),

$$-ca^2 + \frac{3ga^4}{2} = N. \tag{1.70}$$

The solution (1.69) will take on negative values for some y whenever $c > ga^2$. According to (1.70), for this to happen it is sufficient that $c^2 > 2gN$. In such a circumstance, the original assumption that the support is on a single interval breaks down, and one must seek a solution supported on a double interval $(-a_2, -a_1) \cup (a_1, a_2)$.

EXERCISES 1.4 1. (i) By explicit differentiation show that $\Phi(\vec{r}, \vec{r}\,') = -\log\Big(|\vec{r} - \vec{r}\,'|/l\Big)$, satisfies the two-dimensional *Laplace equation* $\nabla^2_{\vec{r}}\Phi(\vec{r}, \vec{r}\,') = 0$ for $\vec{r} \neq \vec{r}\,'$.

(ii) Use the divergence theorem in the plane

$$\int_{\mathcal{D}} \nabla^2 V(\vec{r})\, d\vec{r} = \int_{\mathcal{C}} \vec{n} \cdot \nabla V(\vec{r})\, d\vec{r}$$

with $V(\vec{r}) = \Phi(\vec{r}, \vec{r}\,')$, $\mathcal{D}$ a disk centered on $\vec{r}\,'$ and $\mathcal{C}$ the circle which is the boundary of the disk, to conclude

$$\int_{\mathcal{D}} \nabla^2_{\vec{r}}\Phi(\vec{r}, \vec{r}\,')\, d\vec{r} = -2\pi.$$

Relate this result to the Poisson equation (1.40).

2. Use the power series expansion of $\log(1-z)$ for $|z| < 1$ to show that for all $|\mu| < 1$,

$$\int_0^{2\pi} \log|1 - \mu e^{i\theta}|\, d\theta = 0.$$

Show that this integral is equal to $2\pi \log|\mu|$ for $|\mu| > 1$ by using the result for $|\mu| < 1$, and use the continuity of the integral as a function of μ to deduce its value for $|\mu| = 1$.

3. Suppose there are N mobile particles of charge q in a disk filled with a uniform neutralizing background $\rho_b = N/\pi R^2$. This specifies the two-dimensional one-component plasma confined to a disk.

(i) With the position of the particles specified in polar coordinates, use the integral evaluations of q.2 and the definition of $V(r)$ (1.42) to show

$$V(r) = \pi\rho_b(r^2/2 + R^2 \log R - R^2/2). \tag{1.71}$$

Write down the Poisson equation satisfied by $V(r)$.

(ii) Use this expression for $V(r)$ to calculate U_3 and thus show that the Boltzmann factor is equal to

$$e^{-\Gamma N^2((1/2)\log R - 3/8)} e^{-\pi\Gamma\rho_b \sum_{j=1}^N |\vec{r}_j|^2/2} \prod_{1 \le j < k \le N} |\vec{r}_k - \vec{r}_j|^\Gamma, \qquad \Gamma := q^2\beta. \tag{1.72}$$

4. (i) Assuming the validity of the formula

$$\log|1 - ae^{ix}| = -\sum_{n=1}^\infty \frac{a^n \cos nx}{n}, \qquad 0 \le a < 1, \quad x \in \mathbb{R},$$

for $a = 1$ provided $x \neq 0 \bmod (2\pi)$, deduce that

$$\log|2\sin(x-t)/2| = -\sum_{n=1}^\infty \frac{\cos n(x-t)}{n}$$

for $x - t \neq 0 \bmod (2\pi)$, and write down a similar formula for $\log|2\sin(x+t)/2|$. Hence derive the cosine expansion

$$\log(2|\cos x - \cos t|) = -\sum_{n=1}^{\infty} \frac{2}{n} \cos nx \cos nt. \tag{1.73}$$

(ii) Use the above cosine expansion to verify that the eigenvalues and normalized eigenfunctions of the integral operator

$$A[\hat{\phi}](\sigma) := -\int_0^{\pi} \log|\cos\theta - \cos\sigma| \hat{\phi}(\theta)\, d\theta$$

are as specified below (1.50).

5. The objective of this exercise is to compute the background density and the Boltzmann factor for the one-component log-potential system confined to the interval $(-a, a)$ with unit charges and one-body potential

$$V(x) = \frac{x^2}{2} + g\frac{x^4}{N} + C. \tag{1.74}$$

This calculation is of interest in the graphical expansion of matrix integrals [98], [555], and will be used in this context in the next section.

(i) With $Y = \cos\theta$ verify that

$$\frac{\cos 4\theta - 1}{\sin\theta} = -8Y^2(1 - Y^2)^{1/2}.$$

(ii) Use the eigenfunction expansion method, the result of (i) and Proposition 1.4.3 to show that the solution of the integral equation

$$V(x) = \int_{-a}^{a} \rho_b(y) \log|x - y|\, dy, \qquad -a \le x \le a,$$

which is bounded at $y = \pm a$ is

$$\rho_b(y) = \frac{a}{\pi}\Big(1 + \frac{2ga^2}{N} + \frac{4g}{N}y^2\Big)\sqrt{1 - (y/a)^2},$$

provided

$$C = -a^2\left(\frac{1}{4} - \frac{N}{a^2}\log a + \frac{1}{2}\log 2 + \frac{3ga^2}{8N} + \frac{3ga^2}{2N}\log 2\right).$$

(iii) Use the neutrality condition to show

$$\frac{a^2}{2} + \frac{3ga^4}{2N} = N, \tag{1.75}$$

and use this in the formula for C to obtain the simplification

$$C = -\frac{a^2}{8} + N\log\frac{a}{2} - \frac{N}{4}.$$

(iv) Use the trigonometric Euler integral in Exercises 4.1 q.1(i) below and the neutrality condition to show

$$U_3 = -\frac{CN}{2} + \frac{a^4}{192} - \frac{a^2 N}{24} - \frac{N^2}{16},$$

and thus

$$\begin{aligned}(U_2 + U_3) &- (U_2 + U_3)|_{g=0} \\ &= \frac{g}{N}\sum_{j=1}^{N} x_j^4 - \frac{N^2}{2}\left[\frac{1}{24}\big((a/\sqrt{2N})^2 - 1\big)\big(9 - (a/\sqrt{2N})^2\big) - \log(a/\sqrt{2N})\right],\end{aligned} \tag{1.76}$$

where $(U_2 + U_3)|_{g=0}$ is as implicit in Proposition 1.4.4.

6. (i) For a general potential $u(x)$, use the eigenfunction expansion method to show that the solution $\rho_b(y)$ of the

integral equation

$$u(x) + C = \int_{-a}^{a} \rho_b(y) \log |x-y|\, dy, \quad x \in [-a, a], \tag{1.77}$$

which is bounded at $y = \pm a$ can be written

$$\rho_b(a\cos\theta) = -\frac{2}{a\pi^2 \sin\theta} \sum_{p=1}^{\infty} p\Big(\int_0^{\pi} u(a\cos\sigma)\cos p\sigma\, d\sigma\Big)\Big(\cos p\theta - 1\Big).$$

(ii) For $u(x) = x^{2n}$, $n \in \mathbb{Z}^+$, use the integration formula

$$\int_0^{\pi} \cos^{2n}\sigma \cos 2p\sigma\, d\sigma = \frac{\pi}{2^{2n}}\binom{2n}{n+p},$$

verified using complex exponentials, and the transformation identity

$$\frac{1}{2n}\sum_{p=1}^{n} p\binom{2n}{n+p}\frac{1-\cos 2p\theta}{1-\cos^2\theta} = \sum_{l=1}^{n}\binom{2(n-l)}{n-l}(2\cos\theta)^{2(l-1)}$$

to show that [99]

$$\rho_b(x) = \frac{4n}{\pi}\Big(\frac{a}{2}\Big)^{2n-1}\Big(\sum_{l=1}^{n}\binom{2(n-l)}{n-l}\Big(\frac{2x}{a}\Big)^{2(l-1)}\Big)\sqrt{1-\Big(\frac{x}{a}\Big)^2}.$$

Check that this is consistent with $\rho_b(y)$ in q.2(ii).

7. [124] The task of this exercise is to solve the integral equation

$$\frac{x^2}{2} + C = \int_{-b}^{a} \rho_{\rm b}(y)\log|x-y|\, dy, \tag{1.78}$$

with $a = \sqrt{2N}s$, $s < 1$, subject to the neutrality constraint

$$\int_{-b}^{a} \rho_{\rm b}(y)\, dy = N, \tag{1.79}$$

and to the constraint that $\rho_{\rm b}(y)$ be bounded at $y = -b$.

(i) Change variables according to

$$y = Y + \frac{a-b}{2}, \qquad x = X + \frac{a-b}{2}$$

and then according to

$$Y = \frac{a+b}{2}\cos\theta, \quad Y = \frac{a+b}{2}\cos\sigma, \quad \sin\theta\rho_{\rm b}\Big(\frac{a+b}{2}\cos\theta + \frac{a-b}{2}\Big) = \frac{a+b}{2}\phi(\theta),$$

to rewrite (1.78) to read

$$\begin{aligned}&-\Big(\frac{2}{a+b}\Big)^2\Big(\frac{(a+b)^2}{16}\cos 2\sigma - \frac{a^2-b^2}{2}\cos\sigma + \frac{(a-b)^2}{8} + \frac{(a+b)^2}{16} + C - N\log\frac{a+b}{2}\Big)\\ &= -\int_0^{\pi}\phi(\theta)\log|\cos\theta - \cos\phi|\, d\phi.\end{aligned}$$

(ii) Use the method of derivation of Proposition 1.4.4 to show that only for the value

$$C = N\log\frac{l\sqrt{N}}{\sqrt{2}} - \frac{9Nl^2}{8} + 2Nls - Ns^2, \tag{1.80}$$

where $a+b=\sqrt{N}l$, does (1.78) permit a solution bounded at $y=-b$, and furthermore show that the latter has the explicit form

$$\rho_{\rm b}(y) = \frac{\sqrt{2N}}{\pi}(s-y/\sqrt{2N})^{1/2}(l-s+y/\sqrt{2N})^{1/2} + \sqrt{N}\frac{l-2s}{\sqrt{2}\pi}\Big(\frac{l-s+y/\sqrt{2N}}{s-y/\sqrt{2N}}\Big)^{1/2}. \tag{1.81}$$

(iii) By making use of (4.2) below, show that the neutrality condition (1.79) gives

$$l = \frac{2}{3}(s+\sqrt{s^2+3}). \tag{1.82}$$

8. In this exercise the location of the minimum of the function

$$H(x_1,\dots,x_N) := \frac{1}{2}\sum_{j=1}^N x_j^2 - \sum_{1\le j<k\le N}\log|x_k-x_j|$$

will be determined by following a calculation of Stieltjes [503]. This gives the equilibrium points of the system of Proposition 1.4.4.

(i) Show that H is convex by establishing that for $t_j \ne 0$ $(j=1,\dots,N)$, $\sum_{j,k=1}^N t_j t_k \frac{\partial^2 H}{\partial x_j x_k} > 0$, and conclude that H has a unique minimum.

(ii) Let $g(x)=\prod_{l=1}^N (x-x_l^{(0)})$. Show that the equations for the minimum $\partial H/\partial x_j = 0$ $(j=1,\dots,N)$ can be written

$$g''(x_j) - 2x_j g'(x_j) = 0 \quad (j=1,\dots,N).$$

(iii) Observe that the l.h.s. of the above equation is a polynomial of degree N which vanishes at the zeros of $g(x)$, and so must be proportional to $g(x)$, to deduce the d.e.

$$g''(x) - 2xg'(x) + 2Ng(x) = 0.$$

Hence show that the minimum of $H(x_1,\dots,x_N)$ occurs at the zeros of the Hermite polynomial $H_N(x)$.

1.5 HIGH-DIMENSIONAL RANDOM ENERGY LANDSCAPES

This and the next three sections all relate to the Wigner semicircle law for the eigenvalue density in the Gaussian ensembles. In this section it is the Wigner semicircle law in the case of the GOE which arises.

As we have seen, the GOE was formulated as a model of the eigenvalues of classically chaotic quantum Hamiltonians with a time reversal symmetry. Later, the GOE received prominence for its relevance to the study of the so-called replica trick in the theory of disordered systems [161]. As applied to random matrix theory, the replica trick corresponds to the identity

$$\Big\langle {\rm Tr}\,(\epsilon-\mathbf{H})^{-1}\Big\rangle_{\mathbf{H}\in{\rm GOE}} = \lim_{n\to 0}\frac{1}{n}\frac{\partial Z_n(\epsilon)}{\partial\epsilon}, \qquad Z_n(\epsilon) := \langle {\det}^{\,n}(\epsilon-\mathbf{H})\rangle_{\mathbf{H}\in{\rm GOE}}. \tag{1.83}$$

In practice the difficulty with the implementation of (1.83) is that orthogonal polynomial methods to evaluate $Z_n(\epsilon)$ (see Chapter 5) require n to be a positive integer, and one is then faced with the problem of analytic continuation off the positive integers in order to take the limit.

More recently the GOE has been shown to be of relevance to another problem relating to the theory of disordered systems [248]. The problem is the computation of the distribution of the critical points for certain high-dimensional Gaussian random potentials (often referred to as landscapes). Specifically, consider the energy

$$\mathcal{H} := \frac{\mu}{2}\sum_{j=1}^N x_j^2 + V(x_1,\dots,x_N), \tag{1.84}$$

where $\mu > 0$ and V is Gaussian distributed with zero mean and covariance

$$\langle V(\vec{x}_1)V(\vec{x}_2)\rangle = Nf\Big(\frac{1}{2N}(\vec{x}_1-\vec{x}_2)^2\Big). \tag{1.85}$$

A critical point of $\mathcal{H}$ is characterized by the simultaneous stationarity conditions $\partial\mathcal{H}/\partial x_j = 0$ ($j = 1,\dots,N$). Let $\rho_{(1)}(\vec{x})$ be the density of critical points, so that $\mathcal{N}(D)$ — the expected number of critical points in the region D — is given by $\mathcal{N}(D) = \int_D \rho_{(1)}(\vec{x})\,d\vec{x}$. With $\{\vec{x}_k\}_{k=1,\dots,N^*}$ denoting the critical points, one has the change of variables type formula

$$\sum_{k=1}^{N^*}\delta(\vec{x}-\vec{x}_k) = \prod_{i=1}^{N}\delta\Big(\frac{\partial\mathcal{H}}{\partial x_i}\Big)\Big|\det\Big[\frac{\partial^2\mathcal{H}}{\partial x_i\partial x_j}\Big]_{i,j=1,\dots,N}\Big|, \tag{1.86}$$

and hence $\rho_{(1)}(\vec{x})$ can be computed as the ensemble average of the r.h.s. This form of $\rho_{(1)}(\vec{x})$ is referred to as the *generalized Kac-Rice formula* (see also (15.56) below).

PROPOSITION 1.5.1 *Let GOE$^{\#}$ refer to the GOE with matrices $\mathbf{X}\mapsto\sqrt{N/2f''(0)}\mathbf{X}$. We have*

$$\mathcal{N}(\mathbb{R}^N) = \mu^{-N}\sqrt{\frac{N}{2\pi}}\int_{-\infty}^{\infty}e^{-Nt^2/2}\Big\langle\Big|\det\Big((\mu+\sqrt{f''(0)}t)\mathbf{1}_N-\mathbf{X}\Big)\Big|\Big\rangle_{\mathbf{X}\in\mathrm{GOE}^{\#}}dt. \tag{1.87}$$

Proof. We begin with the formula implied by the sentence including (1.86). Recalling (1.84) this gives

$$\begin{aligned}\rho_{(1)}(\vec{x}) &= \Big\langle\prod_{i=1}^{N}\delta(\mu x_i+\partial_iV)|\det[\mu\delta_{j,k}+\partial_j\partial_kV]_{j,k=1,\dots,N}|\Big\rangle\\ &= \Big\langle\prod_{i=1}^{N}\delta(\mu x_i+\partial_iV)\Big\rangle\Big\langle|\det[\mu\delta_{j,k}+\partial_j\partial_kV]_{j,k=1,\dots,N}|\Big\rangle,\end{aligned} \tag{1.88}$$

where $\partial_i := \partial/\partial x_i$ and the second equality follows by noting that ∂_iV and $\partial_j\partial_kV$ are statistically independent.

The Gaussian field formed by ∂_iV has, according to (1.85), covariance $\langle\partial_jV\partial_kV\rangle = a^2\delta_{j,k}$, $a^2 := -f'(0)$. Hence, after making use of the Fourier integral representation of the delta function, one has

$$\Big\langle\prod_{i=1}^{N}\delta(\mu x_i+\partial_iV)\Big\rangle = \frac{1}{(\sqrt{2\pi a^2})^N}e^{-\mu^2\sum_{j=1}^N x_j^2/2a^2}$$

(see (1.93) below). The second average in (1.88) is independent of x_k, and so we can integrate over $D=\mathbb{R}^N$ to obtain

$$\mathcal{N}(\mathbb{R}^N) = \mu^{-N}\Big\langle\Big|\det[\mu\delta_{j,k}+\partial_j\partial_kV]_{j,k=1,\dots,N}\Big|\Big\rangle.$$

Set $H_{jk} := \partial_j\partial_kV$. It follows from (1.85) that

$$\langle H_{il}H_{jm}\rangle = \frac{f''(0)}{N}\Big(\delta_{ij}\delta_{lm}+\delta_{im}\delta_{lj}+\delta_{il}\delta_{jm}\Big). \tag{1.89}$$

Now let the diagonal elements H_{ii} and upper triangular elements H_{jk} ($j<k$) collectively be indexed H_μ, and form the vector $\vec{H} = (H_\mu)$. Being Gaussian variables, for some matrix $\mathbf{A}$ of appropriate size they have distribution proportional to $\exp(-\frac{1}{2}\vec{H}\mathbf{A}\vec{H})$. Furthermore $\mathbf{A}$ is completely determined by $\langle H_\mu H_\nu\rangle$ (see (1.95) below) with the task being to compute the inverse of the matrix of these averages. The final result can be written in a structured form, showing that $\mathbf{H} := [H_{jk}]$ is a real symmetric Gaussian random matrix with p.d.f. proportional to

$$\exp\Big(-\frac{N}{4f''(0)}\Big(\mathrm{Tr}\,\mathbf{H}^2-\frac{1}{N+2}(\mathrm{Tr}\,\mathbf{H})^2\Big)\Big). \tag{1.90}$$

By completing the square in t we see that

$$\int_{-\infty}^{\infty} e^{-Nt^2/2} \exp\Big(- \frac{N}{4f''(0)} \Big(\mathrm{Tr} \Big(\mathbf{H} - \sqrt{f''(0)} t \mathbf{1}_N \Big)^2 \Big) \Big) \, dt,$$

is proportional to (1.90) and (1.87) follows. (The N-dependent proportionality follows by requiring that $\mathcal{N}(\mathbb{R}^N) \to 1$ for $\mu \to \infty$.) □

With $J := \sqrt{f''(0)}$, changing variables $\mathbf{X} \mapsto J\sqrt{2/N}\mathbf{X}$ in the average of (1.87) gives

$$\begin{aligned} &(J\sqrt{2/N})^N \Big\langle \Big| \det \Big[\sqrt{N/2}((\mu/J) + t)\mathbf{1}_N - \mathbf{X} \Big] \Big| \Big\rangle_{\mathbf{X} \in \mathrm{GOE}} \\ &\quad = (J\sqrt{2/N})^N e^{N((\mu/J)+t)^2/4} \frac{G_{1,N+1}}{(N+1)G_{1,N}} \rho_{(1),N+1}(\sqrt{N/2}((\mu/J)+t)) \end{aligned} \tag{1.91}$$

where $\rho_{(1),N+1}$ refers to the density in the GOE with $N+1$ eigenvalues, and $G_{1,N}$ is given by (1.163). The equality in (1.91) follows by writing the determinant as a product of eigenvalues, writing the average in terms of eigenvalues using (1.27), and recalling the formula for the density (one-particle correlation) (1.46). Substituting (1.91) in (1.87) shows

$$\mathcal{N}(\mathbb{R}^N) = \Big(\frac{J\sqrt{2}}{\mu\sqrt{N}} \Big)^N \Gamma\Big(\frac{N+1}{2} \Big) e^{N(\mu/J)^2/2} \sqrt{\frac{N}{\pi}} \int_{-\infty}^{\infty} e^{-N(t-(\mu/J))^2/4} \rho_{(1),N+1}(\sqrt{N/2}((\mu/J)+t)) \, dt.$$

For large N, after making use of Stirling's formula

$$\Gamma(x+1) \sim (2\pi x)^{1/2} e^{x \log x - x} \qquad \text{as} \quad x \to \infty, \ \mathrm{Re}(x) > 0, \tag{1.92}$$

and noting the delta function type behavior of the integral, we see that for the argument of $\rho_{(1),N+1}$ inside its support, and thus $\mu < J$,

$$\mathcal{N}(\mathbb{R}^N) \sim 2(2\pi)^{1/2} (J/\mu)^N e^{N(\mu/J)^2/2} e^{-N/2} \rho_{(1),N+1}(\sqrt{2N}(\mu/J)).$$

Making use now of (1.52), one obtains that for $\mu < J$

$$\Sigma(\mu) := \lim_{N\to\infty} \frac{1}{N} \log \mathcal{N}(\mathbb{R}^N) = \frac{1}{2}\Big(\frac{\mu^2}{J^2} - 1 \Big) - \log(\mu/J).$$

Note that $\Sigma(J) = 0$. In fact analysis of $\rho_{(1),N+1}(\sqrt{2N}X)$ for $|X| > 1$ undertaken in Exercises 14.4 q.5 below can be used to show that $\Sigma(\mu) = 0$ for $\mu > J$, and so the number of critical points undergoes a phase transition at $\mu = J$.

EXERCISES 1.5 1. (i) Let $\mathbf{A}$ be an $n \times n$ positive definite matrix. By changing variables $\vec{y} = \mathbf{A}^{1/2}\vec{x}$ and completing the square show

$$\begin{aligned} I_n[\mathbf{A}, \vec{b}] &:= \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \exp\Big(-\frac{1}{2} \vec{x}^T \mathbf{A} \vec{x} + \vec{b} \cdot \vec{x} \Big) \\ &= (2\pi)^{n/2} (\det \mathbf{A})^{-1/2} \exp\Big(\frac{1}{2} \vec{b}^T \mathbf{A}^{-1} \vec{b} \Big). \end{aligned} \tag{1.93}$$

(ii) Let

$$\langle f \rangle_{\mathbf{A}} = \frac{1}{I_n[\mathbf{A}, \vec{0}]} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, f \exp\Big(-\frac{1}{2} \vec{x}^T \mathbf{A} \vec{x} \Big).$$

Use (1.93) and the method of derivation of (1.99) below to show that for l even

$$\langle x_{k_1} x_{k_2} \cdots x_{k_l} \rangle_{\mathbf{A}} = \sum_{\substack{\text{all possible} \\ \text{pairings of } (k_1 \cdots k_l)}} \mathbf{A}^{-1}_{k_{p_1} k_{p_2}} \cdots \mathbf{A}^{-1}_{k_{p_{l-1}} k_{p_l}}, \tag{1.94}$$

while for l odd this average vanishes.

(iii) By choosing $l = 2$, deduce from (1.94) that

$$\mathbf{A}^{-1} = [\langle x_j x_k \rangle_{\mathbf{A}}]_{j,k=1,\dots,n},$$

which in words says that the covariance matrix associated with the average $\langle \cdot \rangle_{\mathbf{A}}$ is given by $\mathbf{A}^{-1}$.

(iv) Replace $\vec{b}$ by $i\vec{b}$ in (1.93), and integrate over $\vec{b}_{k+1}, \dots, \vec{b}_n$ $(k \le n)$ to deduce that

$$(2\pi)^{k/2} (\det \mathbf{A})^{-1/2} \int_{-\infty}^{\infty} db_{k+1} \cdots \int_{-\infty}^{\infty} db_n \exp\Big(- \frac{1}{2} \vec{b}^T \tilde{\mathbf{A}}^{-1} \vec{b} \Big)$$
$$= (2\pi)^{k/2} (\det \tilde{\mathbf{A}})^{-1/2} \exp\Big(- \frac{1}{2} \vec{b}^T \mathbf{A}^{-1} \vec{b} \Big)\Big|_{b_{k+1}=\cdots=b_n=0},$$

where $\tilde{\mathbf{A}}$ is the $k \times k$ submatrix of $\mathbf{A}$ formed from the first k rows and columns.

(v) With $\vec{b}$ replaced by $i\vec{b}$, regard the r.h.s. of (1.93) as a p.d.f. for $\vec{b}$ (up to normalization), so that the covariance matrix is now

$$\mathbf{A} = [\langle b_j b_k \rangle_{\mathbf{A}^{-1}}]_{j,k=1,\dots,n}.$$

Show that under the linear change of variables $\vec{b} = \mathbf{L}\vec{c}$, the vector $\vec{c}$ has a Gaussian distribution

$$(2\pi)^{n/2} (\det \mathbf{B})^{-1/2} \exp\Big(- \frac{1}{2} \vec{c}^T \mathbf{B}^{-1} \vec{c} \Big), \quad \mathbf{B} = \langle c_j c_k \rangle_{\mathbf{A}^{-1}}. \tag{1.95}$$

1.6 MATRIX INTEGRALS AND COMBINATORICS

1.6.1 Combinatorics of $\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*}$

In this section we put our knowledge of the asymptotic density for the GUE to use in the solution of a combinatorial problem. It has long been known [98] that the matrix integrals

$$\int f(\mathbf{X}) e^{-\mathrm{Tr}(\mathbf{X}^2)/2} (d\mathbf{X}),$$

for $\mathbf{X}$ a particular class of random matrices and suitable $f(\mathbf{X})$, have combinatorial significance in that they count certain diagrams embedded on surfaces according to their genus. Here, following [557], [239], we will detail such a combinatorial interpretation of the matrix integral

$$\frac{1}{C} \int \mathrm{Tr}(\mathbf{X}^{2k}) e^{-\mathrm{Tr}(\mathbf{X}^2)/2} (d\mathbf{X}) =: \langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*}, \tag{1.96}$$

where C is the normalization, and GUE^* is identical to the GUE except that $\mathbf{X} \mapsto \mathbf{X}/\sqrt{2}$. By changing variables $\mathbf{X} = \mathbf{U}\mathbf{L}\mathbf{U}^{-1}$ for the eigenvalues and eigenvectors we see from the result of Exercises 1.3 q.3 that

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = \frac{1}{C} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N \prod_{l=1}^{N} e^{-\lambda_l^2/2} \Big(\sum_{j=1}^{N} \lambda_j^{2k} \Big) \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^2. \tag{1.97}$$

Changing variables $\lambda_l \mapsto \sqrt{2}\lambda_l$ we see from the definition (1.46) that in terms of the density $\rho_{(1)}(\lambda)$ for the GUE we have

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = 2^k \int_{-\infty}^{\infty} \lambda^{2k} \rho_{(1)}(\lambda)\, d\lambda. \tag{1.98}$$

However, it is not from (1.98) that the combinatorics arise; this comes from the evaluation of (1.96) as a Gaussian integral over the independent elements of the matrix $\mathbf{X}$. The latter task can be achieved by using a particular matrix version of *Wick's theorem*.

PROPOSITION 1.6.1 *Let* $\mathbf{X} = [z_{jk}]_{j,k=1,\dots,N}$, $z_{jk} = x_{jk} + iy_{jk}$ *be Hermitian so that*

$$e^{-\mathrm{Tr}(\mathbf{X}^2)/2}(d\mathbf{X}) = e^{-\mathrm{Tr}(\mathbf{X}^2)/2} \prod_{j=1}^N dx_{jj} \prod_{1 \le j < k \le N} dx_{jk} dy_{jk}.$$

Let I *be a finite ordered set of pairs of indices* (j,k), $1 \le j,k \le N$, *and let* P *denote a matching of the elements of* I *in pairs. Then we have*

$$\Big\langle \prod_{(i,j) \in I} z_{ij} \Big\rangle_{\mathrm{GUE}^*} = \sum_{\substack{\text{pairings} \\ P \text{ of } I}} \prod_{(i,j),\,(k,l)} \langle z_{ij} z_{kl} \rangle_{\mathrm{GUE}^*}. \tag{1.99}$$

Proof. Introducing the Hermitian matrix $\mathbf{Y} = [w_{jk}]_{j,k=1,\dots,N}$ we observe that

$$\Big\langle \prod_{(i,j) \in I} z_{ij} \Big\rangle_{\mathrm{GUE}^*} = \Big(\prod_{(i,j) \in I} \frac{\partial}{\partial w_{ji}} \Big) \Big\langle e^{\mathrm{Tr}(\mathbf{YX})} \Big\rangle_{\mathrm{GUE}^*} \Big|_{\mathbf{Y}=\mathbf{0}}. \tag{1.100}$$

In the integrand, writing

$$\mathrm{Tr}\,\mathbf{X}^2 - 2\mathrm{Tr}\,(\mathbf{YX}) = \mathrm{Tr}((\mathbf{X}-\mathbf{Y})^2) - \mathrm{Tr}\,\mathbf{Y}^2,$$

we see from the change of variables $\mathbf{X} \mapsto \mathbf{X} + \mathbf{Y}$ that

$$\Big\langle e^{\mathrm{Tr}(\mathbf{YX})} \Big\rangle_{\mathrm{GUE}^*} = e^{\mathrm{Tr}(\mathbf{Y}^2)/2} = \prod_{j=1}^N e^{w_{jj}^2/2} \prod_{1 \le j < k \le N} e^{w_{jk} w_{kj}}.$$

Thus (1.100) gives

$$\Big\langle \prod_{(i,j) \in I} z_{ij} \Big\rangle_{\mathrm{GUE}^*} = \sum_{\substack{\text{pairings} \\ P \text{ of } I}} \prod_{(i,j),\,(k,l)} \delta_{i,l}\delta_{j,k},$$

which reduces to (1.99) after noting

$$\langle z_{ij} z_{kl} \rangle_{\mathrm{GUE}^*} = \delta_{i,l}\delta_{j,k}. \tag{1.101}$$

□

Our task is to compute

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} := \Big\langle \sum_{i_1,\dots,i_{2k}=1}^N z_{i_1 i_2} z_{i_2 i_3} \cdots z_{i_{2k-1} i_{2k}} z_{i_{2k} i_1} \Big\rangle_{\mathrm{GUE}^*}. \tag{1.102}$$

According to (1.99) we have

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = \sum_{i_1,\dots,i_{2k}=1}^N \sum_{\substack{\text{pairings } P \text{ of} \\ \{(i_1,i_2),(i_2,i_3),\dots,(i_{2k},i_1)\}}} \prod_{(j,j'),\,(l,l')} \langle z_{i_j i_{j'}} z_{i_l i_{l'}} \rangle_{\mathrm{GUE}^*}, \tag{1.103}$$

and (1.101) shows that various labels must coincide for a given term in this expression to be nonzero. For example, with $k = 4$ consider the particular term in (1.103)

$$\langle z_{i_1 i_2} z_{i_3 i_4} \rangle \langle z_{i_2 i_3} z_{i_8 i_1} \rangle \langle z_{i_4 i_5} z_{i_6 i_7} \rangle \langle z_{i_5 i_6} z_{i_7 i_8} \rangle = (\delta_{i_1,i_4}\delta_{i_2,i_3})(\delta_{i_2,i_1}\delta_{i_3,i_8})(\delta_{i_4,i_7}\delta_{i_5,i_6})(\delta_{i_5,i_8}\delta_{i_6,i_7}). \tag{1.104}$$

For this to be nonzero we must have $i_1 = i_2 = \cdots = i_8$, giving only one independent label. As another example, consider the term

$$\langle z_{i_1 i_2} z_{i_4 i_5} \rangle \langle z_{i_2 i_3} z_{i_3 i_4} \rangle \langle z_{i_5 i_6} z_{i_8 i_1} \rangle \langle z_{i_6 i_7} z_{i_7 i_8} \rangle = (\delta_{i_1,i_5}\delta_{i_2,i_4})(\delta_{i_2,i_4})(\delta_{i_5,i_1}\delta_{i_6,i_8})(\delta_{i_6,i_8}), \tag{1.105}$$

which is nonzero for $i_1 = i_5$, $i_2 = i_4$, $i_6 = i_8$, giving five independent labels, i_1, i_2, i_3, i_6, i_7, say.

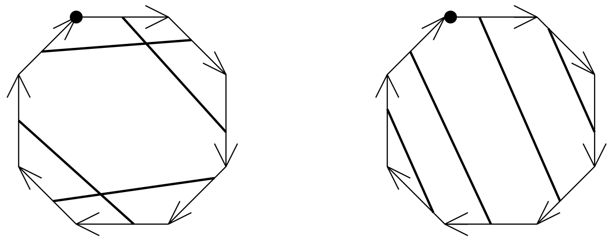

Figure 1.2 Graphical representation of the contributions (1.104) and (1.105). The heavy lines identify edges and the dot marks the location of the vertex labeled i_1, with the other vertices labeled clockwise.

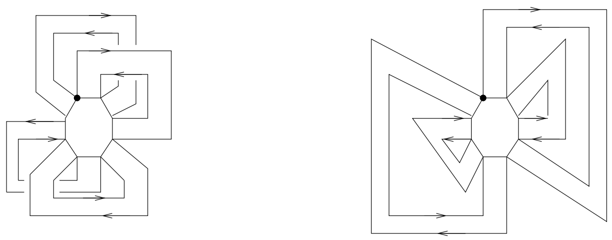

Figure 1.3 The dual graphical representation of Figure 1.2 for the contributions of (1.104) and (1.105). The dot marks the location of the vertex labeled i_1, with the other vertices labeled clockwise.

In general the nonzero terms in (1.103) can be represented graphically in two related ways, both of which involve a regular $2k$-gon, with the vertices labeled $i_1, \dots, i_{2k}$, and the edges oriented clockwise. One method to carry out the pairing between consecutive vertices (i_j, i_k) and consecutive vertices (i_l, i_m) is to join the corresponding edges on the $2k$-gon according to the rule that edges must be joined in opposite directions (see Figure 1.2 for this representation of (1.104) and (1.105)).

Another approach to carrying out the pairing is to draw a straight line segment perpendicular to and outward from the ends of each edge of the $2k$-gon. These segments are to be given the directions of out and in alternately around the $2k$-gon. Then each nonzero contribution to (1.103) can be represented by joining different pairs (i_j, i_k) and (i_l, i_m) of parallel straight line segments to form roadways (see Figure 1.3 for this representation of (1.104) and (1.105)). Note that the joining is such that edges of the roadways have definite directions.

The diagrams of Figure 1.2 can be catalogued according to the number ν of independent vertices after pairing. This of course is just the number of independent summation labels in (1.103) so we can write

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = \sum_{\nu=1}^{k+1} a_\nu(k) N^\nu, \tag{1.106}$$

where $a_\nu(k)$ denotes the number of different pairings which have ν vertices. On the other hand, the diagrams of Figure 1.3 are topological duals of Figure 1.2 with the ν independent vertices now ν independent faces. These can be determined by following the edge of a roadway and its continuation according to its direction, until arriving back at the starting point. The formal meaning of the faces is obtained by shrinking the width of the roadways in Figure 1.3 to single lines, at the same time as shrinking the $2k$-gon to a single vertex so the lines become loops, and then embedding the diagram on a closed surface as a *map*.

DEFINITION 1.6.2 *A map is a graph (a collection of vertices and edges) drawn on a closed surface such that the edges do not intersect and, if we cut the surface along the edges, a disjoint union of sets topologically*

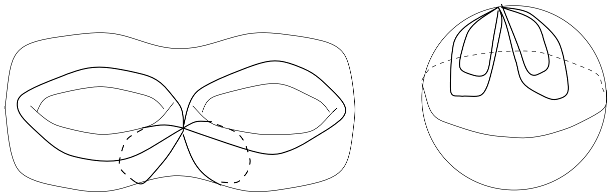

Figure 1.4 Embedding of the graphs of Figure 1.3 (after shrinking the k-gon to a single vertex, and the roadways to single lines) onto a closed surface to form a map with a single vertex.

equivalent to an open disk results. The number of such disks is by definition the number of faces of the map.

It is similarly the case that the number of independent vertices in the diagrams of Figure 1.2 can be specified in terms of the corresponding map.

The index ν in (1.106) determines the genus g (number of holes) of the closed surface. This follows from *Euler's relation*

$$2 - 2g = V - E + F, \tag{1.107}$$

where V denotes the number of vertices, E the number of edges and F the number of faces. In the diagrams of Figure 1.2 $V = \nu$, $F = 1$ and $E = k$, while in the diagrams of Figure 1.3 the roles of V and F are interchanged so that $V = 1$, $F = \nu$ and $E = k$. Either way (1.107) gives

$$\nu = k + 1 - 2g. \tag{1.108}$$

As shown in Figure 1.4, the diagrams of Figure 1.3 can be directly embedded on a surface of particular genus, thereby illustrating (1.108) (thus in the first case $k = 4$, $\nu = 1$, $g = 2$ while in the second case $k = 4$, $\nu = 5$, $g = 0$).

Using (1.108) in (1.106) gives

$$\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = N^{k+1} \sum_{g=0}^{[k/2]} a_{k+1-2g}(k) N^{-2g}. \tag{1.109}$$

In particular

$$\lim_{N\to\infty} N^{-k-1} \langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*} = a_{k+1}(k) \tag{1.110}$$

where $a_{k+1}(k)$ denotes the number of matchings of the $2k$-gon which are planar (i.e. can be embedded on the surface of a sphere, which has $g = 0$). Substituting for $\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GUE}^*}$ using (1.97), and then substituting for $\rho_{(1)}(x)$ using (1.52), evaluating the integral using (4.3) below and simplifying the resulting gamma functions using the duplication formula

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \pi^{1/2}\Gamma(2z), \tag{1.111}$$

one finds

$$a_{k+1}(k) = \frac{1}{k+1}\binom{2k}{k}. \tag{1.112}$$

This number is familiar in combinatorics and is called the kth *Catalan number*. In fact (1.112) can easily be derived without using (1.98), which has the significance of providing an alternative derivation of the Wigner semicircle law (1.52) for Hermitian matrices (see Exercises 1.6 q.1), one which applies to establishing the Wigner semicircle for a large class of symmetric random matrices with independent entries (see, e.g., [551]).

However, this is not the case for the coefficients $a_{k-1}(k), a_{k-3}(k), \dots$ for which the use of (1.98) is the most efficient. We will return to the evaluation of these numbers in Chapter 5 when the exact value of $\rho_{(1)}(\lambda)$ is available.

We remark that $\langle \mathrm{Tr}(\mathbf{X}^{2k}) \rangle_{\mathrm{GOE}}$ allows for a similar combinatorial description in terms of maps on surfaces, although now the surfaces may be nonorientable (corresponding to graphs with twisted ribbons) [348].

1.6.2 Combinatorics of the $\beta = 2$ partition function with a general power series potential

Closely related to the combinatorial interpretation of (1.96) is the combinatorial interpretation of

$$Z_N(\{g_j\}) := \Big\langle \prod_{l=1}^{N} e^{\sum_{j=1}^{\infty} g_j x_l^j / j N^{j/2-1}} \Big\rangle_{\mathrm{GUE}^*} \tag{1.113}$$

when expanded in a power series in $\{g_i\}$. For the latter, expanding the exponentials gives

$$Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j! N^{n_j(j/2-1)}} \Big\langle \prod_{j=1}^{\infty} \Big(\sum_{l=1}^{N} x_l^j \Big)^{n_j} \Big\rangle_{\mathrm{GUE}^*}, \tag{1.114}$$

while

$$\Big\langle \prod_{j=1}^{\infty} \Big(\sum_{l=1}^{N} x_l^j \Big)^{n_j} \Big\rangle_{\mathrm{GUE}^*} = \Big\langle \prod_{j=1}^{\infty} \Big(\mathrm{Tr}\,\mathbf{X}^j \Big)^{n_j} \Big\rangle_{\mathrm{GUE}^*}. \tag{1.115}$$

From the discussion of the previous subsection we know how to give a combinatorial interpretation of (1.115) in the special case $n_j = 1$ $(j = k)$, $n_j = 0$ $(j \ne k)$. A natural generalization of this interpretation extends to the general case [557], [239].

Each factor of $\mathrm{Tr}\,\mathbf{X}^j$ is represented as a j-gon with vertices labeled $i_1, i_2, \dots, i_j$ clockwise, starting at a marked vertex. These labels on vertices extend to labels on pairs of oppositely directed roadway edges coming into and out of each vertex. Whereas in the case of a single factor of $\mathrm{Tr}\,\mathbf{X}^j$ the combinatorial interpretation of computing (1.115) via Wick's theorem involved connecting roadways within the single j-gon, the graphical representation of contributions to (1.115) is to connect roadways among or within any of the n_j j-gons $(j = 1, 2, \dots)$. The resulting structure, referred to as a *labeled fatgraph*, has weight N^{ν}, where ν is the number of faces (which in turn is equal to the number of unpaired labels). The number of edges is equal to $\sum_{j=1}^{\infty} j n_j / 2$, which is required to be an integer, while the number of vertices — defined as the number of j-gons — is equal to $\sum_{j=1}^{\infty} n_j$. Recalling Euler's relation (1.107) we see that (1.114) can thus be written

$$Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \Big(\prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j!} \Big) \sum_g a_g(\{n_j\}) N^{2-2g},$$

where $a_g(\{n_j\})$ is the number of labeled graphs constructed out of n_j j-gons $(j = 1, 2, \dots)$ which can be embedded on a surface of genus g.

The various j-gons in the labeled fatgraph will not in general be connected. However, as $Z_N(\{g_j\})$ is an exponential generating function for these quantities, it is a well-known fact that taking the logarithm restricts to connected components. Thus, denoting this restriction by an asterisk, we have

$$\log Z_N(\{g_j\}) = \sum_{n_1, n_2, \dots = 0}^{\infty} \Big(\prod_{j=1}^{\infty} \frac{g_j^{n_j}}{j^{n_j} n_j!} \Big) \sum_g{}^{*} a_g(\{n_j\}) N^{2-2g}. \tag{1.116}$$

Fatgraphs which are topologically equivalent define a class of maps Γ. For each class the maximum value of $a_g(\{n_j\})$ is $\prod_{j=1}^{\infty} j^{n_j} n_j!$ and furthermore $\prod_{j=1}^{\infty} j^{n_j} n_j! / a_g(\{n_j\})$ is an integer written as $|\mathrm{Aut}\,\Gamma|$. As the notation suggests, $|\mathrm{Aut}\,\Gamma|$ is in fact equal to the order of the group of automorphisms associated with Γ. This

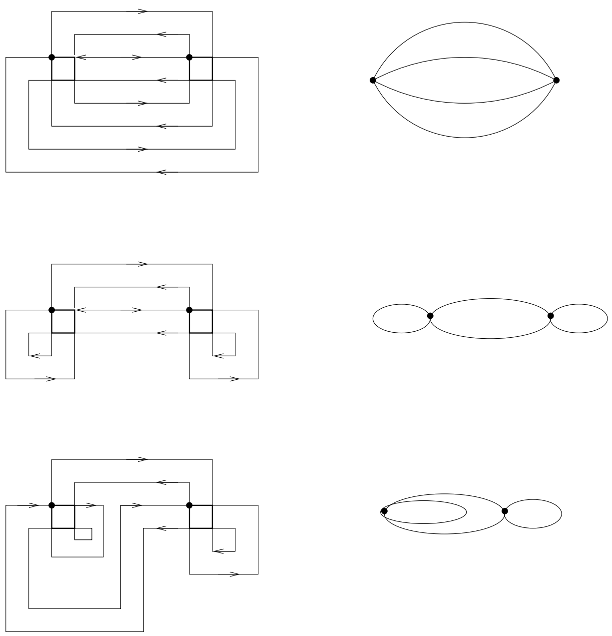

Figure 1.5 Three classes of fatgraphs can be constructed out of two 4-gons. An example from each class, together with the corresponding map is given. For the first class $|\mathrm{Aut}\,\Gamma| = 8$, while for the second and third classes $|\mathrm{Aut}\,\Gamma| = 2$.

can be specified as the number of equivalent labelings of the faces of Γ, which means the number of different labelings in the plane which result from topological transformations of the map on the closed surface.

In terms of $|\mathrm{Aut}\,\Gamma|$ (1.116) reads

$$\log Z_N(\{g_j\}) = \sum_{\text{connected}\,\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} N^{2-2g(\Gamma)} \prod_{j=1}^{\infty} g_j^{V_j(\Gamma)}, \tag{1.117}$$

where n_j in (1.116) has been written $V_j(\Gamma)$ in (1.117) to emphasize that it counts the number of vertices with coordination number j in the corresponding map. In particular

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_N(\{g_j\}) = \sum_{\substack{\text{connected}\,\Gamma \\ g(\Gamma)=0}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \prod_{j=1}^{\infty} g_j^{V_j(\Gamma)}, \tag{1.118}$$

and thus we obtain a generating function for maps weighted according to the coordination number of the vertices. To illustrate (1.118), in Figure 1.5 we display the contributions to the coefficient of g_4^2 in terms of fatgraphs and the corresponding maps.

Suppose $g_j = 0$ for $j \neq 4$. From the definition (1.113) we have

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N^2} \log Z_N\Big(\{g_j\}\Big|_{g_j=0\ (j\neq 4)}\Big) \\ &\quad = \lim_{N\to\infty} \frac{1}{N^2} \log \Big\langle \prod_{l=1}^{N} e^{g_4 x_l^4/4N} \Big\rangle_{\mathrm{GUE}^*} = \lim_{N\to\infty} \frac{1}{N^2} \log \Big\langle \prod_{l=1}^{N} e^{g_4 x_l^4/N} \Big\rangle_{\mathrm{GUE}}. \end{aligned} \tag{1.119}$$

To evaluate this limit we make use of the log-gas interpretation of the average as a ratio of configuration integrals relating to one-component log-potential systems with particular neutralizing background charge densities. The Boltzmann factor for these systems contains constant terms (i.e., terms independent of the particle coordinates) which are not present in (1.119). If these terms, A_N say, were included, its logarithm would then be expected to be proportional to N as the difference between two free energies is being calculated (see (4.160) below). Thus one expects

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_N\Big(\{g_j\}\Big|_{g_j=0\ (j\neq 4)}\Big) = \lim_{N\to\infty} \frac{1}{N^2} \log \frac{1}{A_N}.$$

The value of $1/A_N$ has been calculated in Exercises 1.4 q.5(iv). It is equal to the exponential of the x_j independent terms in (1.76) with g in (1.75) replaced by $-g_4$. This gives

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_N\Big(\{g_j\}\Big|_{g_j=0\ (j\neq 4)}\Big) = -\Big(\frac{1}{24}(u-1)(9-u) - \frac{1}{2}\log u\Big), \tag{1.120}$$

where u is defined in terms of g_4 as the solution of

$$u - 3g_4u^2 = 1, \qquad u \to 1 \text{ as } g_4 \to 0. \tag{1.121}$$

According to the result of Exercises 1.6 q.1(iii),

$$u = -\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}(3g_4)^k. \tag{1.122}$$

After substituting this in the r.h.s. of (1.120), and substituting (1.118) in the l.h.s, the following result is obtained [98].

PROPOSITION 1.6.3 *We have*

$$\sum_{\substack{\text{connected } \Gamma \\ g(\Gamma)=0}} \frac{1}{|\mathrm{Aut}\,\Gamma|} g_4^{V_4(\Gamma)} = \sum_{k=1}^{\infty} \frac{(2k-1)!}{k!(k+2)!}(3g_4)^k. \tag{1.123}$$

Proof. The remaining task is to expand the functions of u on the r.h.s. of (1.120) as power series in g_4. For the quadratic, this is immediate from (1.122) and (1.121). For $\log u$ this follows from the result of Exercises 1.6 q.3. □

Let us denote the coefficient of g_4^k in (1.123) by a_k, which then represents the number of (weighted) planar fatgraphs that can be constructed out of k 4-gons. Making use of Stirling's formula (1.92) shows

$$a_k \sim \frac{12^k}{k^{7/2}\pi}.$$

The particular value of the exponent of the algebraic term $k^{-7/2}$ has meaning in the conformal field theory associated with the graphical expansion [239].

If we cut an edge in any of the planar maps giving rise to (1.123), we obtain a planar fatgraph constructed from 4-gons, but now with two external legs in the same face. The external legs, when distinguished by different labelings, break the symmetry of the maps, so for all classes Γ_2 of such maps $|\mathrm{Aut}\,\Gamma_2| = 1$. Because the legs have been distinguished, and because there are twice as many edges as vertices, one sees [241]

$$\tilde{G} = 1 + 4g_4\frac{\partial}{\partial g_4}G,$$

where $\tilde{G}$ denotes the generating function for the maps with external legs, and G denotes the l.h.s. of (1.123). Substituting the r.h.s. of (1.123) we see that the power series of $\tilde{G}$ has positive integer coefficients, as it must. In particular, the coefficient of g_4^2 is 9. One contribution results from the first map in Figure 1.5, while four result from each of the other two maps therein.

EXERCISES 1.6 1. Here the number $c_k := a_{k+1}(k)$ of diagrams which can be constructed from a $2k$-gon according to the prescription of Figure 1.2, and which contain no intersecting lines, will be computed directly.

(i) Suppose the lines from edge 1 join the lines from edge $2j$ ($j = 1, \dots, k$). Argue that inside these lines there can be c_{j-1} configurations of the allowed type, while there are c_{k-j} configurations of the allowed type joining the edges $2j+1, \dots, 2k$. Hence deduce that

$$c_k = \sum_{j=0}^{k-1} c_j c_{k-1-j}, \qquad c_0 = 1. \tag{1.124}$$

(ii) Verify that the Catalan numbers (1.112) solve this recurrence.

(iii) Introduce the generating function $C(t) = \sum_{k=0}^{\infty} c_k t^k$. Use the recurrence (1.124) to show that $C(t)$ satisfies the quadratic equation

$$C(t) = 1 + t(C(t))^2, \tag{1.125}$$

and consequently has the explicit form

$$tC(t) = \frac{1}{2}(1 - (1-4t)^{1/2}). \tag{1.126}$$

(iv) With the value of $a_{k+1}(k)$ known independently of the average in (1.110) according to the result of (ii), use (1.98) to deduce that the scaled density (1.53) is such that

$$2^{2k} \int_{-\infty}^{\infty} x^{2k} \tilde{\rho}_{(1)}(x)\, dx = \frac{1}{k+1}\binom{2k}{k} \tag{1.127}$$

while the odd moments vanish.

(v) A sufficient condition for a density function to be determined by its moments $\{c_0, c_1, c_2, \dots\}$ is that

$$\sum_{k=0}^{\infty} \frac{c_k t^k}{k!} \tag{1.128}$$

converges for some $t > 0$ (this implies the Fourier transform of the density function is analytic in the neighbourhood of the origin). Verify that this is the case for the moments in (iv). Now use the fact that (1.54) reproduces these values to conclude from (iv) that the Wigner semicircle law is valid.

2. (i) Consider the Gaussian β-ensemble p.d.f. in (1.160) below, scaled so that $\lambda_l \mapsto \sqrt{2\beta N}/J$, $J > 0$. Use the result of Proposition 1.4.4 to show that to leading order the density is then supported on the interval $[-J, J]$ and is given by

$$\frac{2N}{\pi J}\sqrt{1 - (y/J)^2}. \tag{1.129}$$

(ii) Use (1.55) with suitable $V(x)$ and a, to show that for $\rho_b(y)$ given by (1.129) and $z \in (-J, J)$,

$$\int_{-J}^{J} \frac{\rho_b(y)}{z-y}\, dy = \frac{2Nz}{J^2}. \tag{1.130}$$

Also, deduce from (1.127) that

$$\int_{-J}^{J} y^{2k} \rho_b(y)\, dy = \frac{2N}{J}(J/2)^{2k+1} c_k, \tag{1.131}$$

where c_k denotes the kth Catalan number (1.112). From this and the result of q.1(iii) deduce that for $|z| > J$,

$$\int_{-J}^{J} \frac{\rho_b(y)}{z-y}\, dy = \frac{2Nz}{J^2}\Big(1 - (1 - J^2/z^2)^{1/2}\Big). \tag{1.132}$$

3. [541] Let $f(z)$ and $\phi(z)$ be analytic in a neighborhood Ω of $z = a$. According to the Lagrange inversion formula, for t small enough that $|t\phi(z)| < |z-a|$, $z \in \Omega$, the equation $\zeta = a + t\phi(\zeta)$ has one solution in Ω, and furthermore
$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} \Big(f'(a)(\phi(a))^n \Big).$$
Use this formula to show that for x defined as the solution of the equation $x = 1 + yx^p$ with the property $x \to 1$ as $y \to 0$, one has
$$\log x = \sum_{k=1}^{\infty} \frac{(kp-1)!}{k!(kp-k)!} y^k.$$

1.7 CONVERGENCE

Consider for definiteness GUE matrices. As stated the Wigner semicircle law tells us the leading large N form of $\langle \frac{1}{N} \sum_{j=1}^N \sqrt{2N}\delta(\sqrt{2N}y - \lambda_j)\rangle_{\rm GUE}$. As the normalized empirical density integrated over an interval $[a,b]$ is the proportion of eigenvalues in that interval, $\#[a,b]$ say, equivalently the Wigner semicircle law tells us the expected value of this quantity when averaged over GUE matrices. Indeed, this was how Figure 1.1 was produced, with the theoretical means in each bin of the bar graph substituted by their empirical averages.

What if instead one considers $\#[a,b]$ for a sequence of single $n \times n$ matrices, $n = 1, 2, \dots$, each chosen from the GUE and with eigenvalues scaled $\lambda_j \mapsto \lambda_j/\sqrt{2N}$. Does the resulting sequence of values for $\#[a,b]$ converge to that predicted by the Wigner semicircle law? And what is the meaning of convergence in this setting? Regarding the latter point, two possibilities are convergence in probability, and almost sure convergence. Convergence in probability says that for a given $\epsilon > 0$, and sequence of single $n \times n$ GUE matrices $(n = 1, 2, \dots)$, ${\rm Pr}(|\mu_n - \mu| > \epsilon) \to 0$, where μ_n is the empirical value of $\#[a,b]$ for each matrix, and μ is the limiting ensemble average (the value implied by the Wigner semicircle law). Almost sure convergence says that the measure of the sequence of matrices for which $\mu_n \to \mu$ is equal to 1. A well-known consequence of the Borel-Cantelli lemma in probability theory (see, e.g., [66]) is that almost sure convergence is equivalent to the statement that for a given $\epsilon > 0$, $\sum_{n=1}^{\infty} {\rm Pr}(|\mu_n - \mu| > \epsilon) < \infty$. Note that a necessary condition for this is that ${\rm Pr}(|\mu_n - \mu| > \epsilon) \to 0$, and thus almost sure convergence implies convergence in probability. To estimate ${\rm Pr}(|\mu_n - \mu| > \epsilon)$, the *Chebyshev inequality* [66]
$${\rm Pr}(|\mu_n - \mu| > \epsilon) \le \frac{\langle (\mu_n - \mu)^2 \rangle_{\rm GUE}}{\epsilon^2}$$
can be employed. Hence for convergence in probability, it is sufficient that $\langle (\mu_n - \mu)^2\rangle_{\rm GUE} \to 0$ as $n \to \infty$, while for almost sure convergence, it is sufficient that $\sum_{n=1}^{\infty} \langle (\mu_n - \mu)^2 \rangle_{\rm GUE} < \infty$.

In Section 1.6.1 and Exercises 1.6 q.1 the Wigner semicircle law has been studied through its moments. We have shown that $\langle N^{-k-1} {\rm Tr}(\mathbf{X}^{2k})\rangle_{\rm GUE^*} \to m_{2k}$ where m_{2k} is the corresponding moment of the Wigner semicircle law. To study convergence in probability and almost sure convergence one thus must study
$${\rm Var}(N^{-k-1}{\rm Tr}\mathbf{X}^{2k}) := \langle (N^{-k-1}{\rm Tr}\mathbf{X}^{2k})^2\rangle_{\rm GUE^*} - \Big(\langle N^{-k-1}{\rm Tr}\mathbf{X}^{2k}\rangle_{\rm GUE^*}\Big)^2.$$
Now, analogous to (1.102) we have
$$\langle ({\rm Tr}\mathbf{X}^{2k})^2 \rangle_{\rm GUE^*} = \Big\langle \sum_{\substack{i_1,\dots,i_{2k}=1 \\ j_1,\dots,j_{2k}=1}} z_{i_1 i_2} \cdots z_{i_{2k} i_1} z_{j_1 j_2} \cdots z_{j_{2k} j_1} \Big\rangle.$$

Regarding $i_1, \dots, i_{2k}$ as fixed, and taking into consideration (1.103), one sees [274]

$$\begin{aligned}\langle (\mathrm{Tr}\mathbf{X}^{2k})^2\rangle_{\mathrm{GUE}^*} &= \Big\langle \sum_{i_1,\dots,i_{2k}=1} z_{i_1 i_2}\cdots z_{i_{2k} i_1}\Big\rangle_{\mathrm{GUE}^*} \Big\langle \sum_{j_1,\dots,j_{2k}=1} z_{j_1 j_2}\cdots z_{j_{2k} j_1}\Big\rangle_{\mathrm{GUE}^*} \Big(1+\mathrm{O}\Big(\frac{1}{N^2}\Big)\Big)\\ &= \langle \mathrm{Tr}\mathbf{X}^{2k}\rangle^2_{\mathrm{GUE}^*}\Big(1+\mathrm{O}\Big(\frac{1}{N^2}\Big)\Big).\end{aligned}$$

It follows from this that $\sum_{N=1}^{\infty} \mathrm{Var}(N^{-k-1}\mathrm{Tr}\mathbf{X}^{2k}) < \infty$, so we can conclude that almost sure convergence holds and so the Wigner semicircle law is the limiting density of all sequences of GUE matrices, up to a set of measure zero.

1.8 THE SHIFTED MEAN GAUSSIAN ENSEMBLES

The Gaussian orthogonal, unitary and symplectic ensembles have joint p.d.f. for the elements proportional to $\exp(-(\beta/2)\mathrm{Tr}\,\mathbf{H}^2)$. We know that this is equivalent to the independent entries in $\mathbf{H}$ having Gaussian distribution with mean zero and particular variance. It follows that for a fixed Hermitian matrix $\mathbf{H}_0$, a p.d.f. proportional to $\exp(-(\beta/2)\mathrm{Tr}\,(\mathbf{H}-\mathbf{H}_0)^2)$ specifies a Gaussian ensemble in which the mean of each element is equal to the corresponding element in $\mathbf{H}_0$. The simplest case is when all elements of $\mathbf{H}_0$ are constant, equal to c say. Then

$$\mathbf{H} = \mathbf{A} + c\vec{x}\vec{x}^T, \tag{1.133}$$

where $\vec{x}$ is a column vector with all entries equal to 1, and $\mathbf{A}$ is a member of the corresponding zero mean Gaussian ensemble. Thus in this case the shifted mean Gaussian ensembles correspond to a rank 1 perturbation of the original ensembles.

Consider for definiteness the GOE. Diagonalizing $\mathbf{A}$, $\mathbf{A} = \mathbf{OLO}^T$, $\mathbf{L} = \mathrm{diag}\,(a_1,\dots,a_N)$, and writing $\mathbf{O}^T\vec{x} =: \vec{y}$ shows that from the viewpoint of the eigenvalues, the r.h.s. of (1.133) can be replaced by $\mathbf{L}+c\vec{y}\vec{y}^T$. We seek the eigenvalues of this matrix.

PROPOSITION 1.8.1 *The eigenvalues of the matrix*

$$\tilde{\mathbf{H}} := \mathrm{diag}(a_1,\dots,a_N) + c\vec{y}\vec{y}^T$$

are given by the solutions of the equation

$$0 = 1 - c\sum_{i=1}^{N} \frac{y_i^2}{\lambda - a_i}. \tag{1.134}$$

Assuming the ordering $a_1 > \cdots > a_N$, and that $c > 0$, a corollary is that the eigenvalues satisfy the interlacing

$$\lambda_1 > a_1 > \lambda_2 > a_2 > \cdots > \lambda_N > a_N. \tag{1.135}$$

Proof. With $\tilde{\mathbf{A}} = \mathrm{diag}(a_1,\dots,a_N)$ we have

$$\det(\mathbf{1}_N\lambda - \tilde{\mathbf{H}}) = \det(\mathbf{1}_N - \tilde{\mathbf{A}})\det(\mathbf{1}_N\lambda - c\vec{y}\vec{y}^T(\mathbf{1}_N\lambda - \tilde{\mathbf{A}})^{-1}). \tag{1.136}$$

But the matrix product in the second determinant has rank 1, and so

$$\det(\mathbf{1}_N - c\vec{y}\vec{y}^T(\mathbf{1}_N\lambda - \tilde{\mathbf{A}})^{-1}) = 1 - c\mathrm{Tr}(\vec{y}\vec{y}^T(\mathbf{1}_N\lambda - \tilde{\mathbf{A}})^{-1}) = 1 - c\sum_{i=1}^{N}\frac{y_i^2}{\lambda - a_i}. \tag{1.137}$$

The characteristic polynomial (1.136) vanishes at the zeros of this determinant, but not at the zeros of $\det(\mathbf{1}_N\lambda - \tilde{\mathbf{A}})$ due to the poles in (1.137), implying the first result. The interlacing condition can be seen by sketching a graph, and noting in the process that $cy_i^2 > 0$. □

The GUE and GSE lead to the same equation (1.134) but with y_i^2 replaced by $|y_i|^2 := \sum_{s=1}^{\beta}(y_i^{(s)})^2$, where the $y_i^{(s)}$ are the independent real parts of the complex and real quaternion entries, respectively. We remark too that in the case of the GSE the eigenvalues of $\mathbf{A}$ are doubly degenerate and the rank 1 perturbation leaves one copy of the eigenvalues unchanged; see the discussion about (4.20) below.

Of particular interest is the position of the largest eigenvalue of $\tilde{\mathbf{H}}$, and thus $\mathbf{H}$, as a function of c and N, which unlike the other eigenvalues is not trapped by the eigenvalues of $\mathbf{A}$. Following [335] this can analyzed by making use of the results of Exercises 1.6 q.2.

PROPOSITION 1.8.2 *Consider the Gaussian ensembles scaled so that the distribution of the elements is proportional to* $\exp(-(\beta N/J^2)\mathrm{Tr}\,(\mathbf{H}-\mathbf{H}_0)^2)$*, with* $\mathbf{H}_0$ *the constant matrix having all elements equal to* c/N*,* $c > 0$*. Suppose* N *is large. Then for* $2c > J$ *a single eigenvalue splits off from the bulk of the eigenvalues, these being supported on* $(-J, J)$*, and is located at*

$$\lambda = c + \frac{J^2}{4c}.$$

Proof. We seek a solution $\lambda > J$ of

$$1 = \frac{c}{N}\Big\langle \sum_{j=1}^{N} \frac{|y_j|^2}{\lambda - \lambda_j} \Big\rangle, \tag{1.138}$$

where each $|y_j|^2$ has mean unity, and $\{\lambda_j\}$ are the eigenvalues of a member of the specified Gaussian ensemble but with $\mathbf{H}_0 = \mathbf{0}$. We know that the density of the $\{\lambda_j\}$ is then given by the semicircle law (1.129). Hence for N large

$$\Big\langle \sum_{j=1}^{N} \frac{|y_j|^2}{\lambda - \lambda_j} \Big\rangle \sim \int_{-J}^{J} \frac{\rho_b(y)}{\lambda - y}\, dy = \frac{2N\lambda}{J^2}\Big(1 - \Big(1 - \frac{J^2}{\lambda^2}\Big)^{1/2}\Big),$$

where the first relation follows from the fact that the eigenvalues and eigenvectors are independently distributed and $\langle |y_j|^2 \rangle = 1$, while the equality, which requires that $\lambda > J$, follows from (1.132). Substituting this in (1.138) and solving for λ gives the stated result. □

1.9 GAUSSIAN β-ENSEMBLE

The p.d.f. (1.28) is realized by the eigenvalues of the GOE, GUE and GSE for the values of β equal to 1, 2 and 4 respectively. In this section a family of random tridiagonal matrices, referred to as the Gaussian β-ensemble, with (1.28) as their eigenvalue p.d.f. for general $\beta > 0$, will be studied. They can be motivated by the reduction of GOE or GUE matrices to tridiagonal form.

1.9.1 Householder transformations

A familiar technique in numerical linear algebra is the similarity transformation of a real symmetric matrix to tridiagonal form using a sequence of reflection matrices, referred to as *Householder transformations*. Explicitly, let $\mathbf{A}$ be a real symmetric matrix $[a_{ij}]_{i,j=1,\dots,N}$. Then one can construct a sequence of symmetric real orthogonal matrices $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(N-2)}$ such that the transformed matrix

$$\mathbf{U}^{(N-2)}\mathbf{U}^{(N-3)} \cdots \mathbf{U}^{(1)}\mathbf{A}\mathbf{U}^{(1)}\mathbf{U}^{(2)} \cdots \mathbf{U}^{(N-2)} =: \mathbf{B}^{(N-2)} \tag{1.139}$$

is a symmetric tridiagonal matrix. These matrices have the structure

$$\mathbf{U}^{(j)} = \mathbf{1}_N - 2\vec{u}^{(j)}\vec{u}^{(j)T} = \begin{bmatrix} \mathbf{1}_j & \mathbf{0}_{j\times N-j} \\ \mathbf{0}_{N-j\times j} & \mathbf{V}_{N-j\times N-j} \end{bmatrix}, \tag{1.140}$$

where $\vec{u}^{(j)T}\vec{u}^{(j)} = 1$ and $\mathbf{V}_{N-j\times N-j}$ is symmetric real orthogonal. Geometrically $\mathbf{U}^{(j)}$ corresponds to a reflection in the hyperplane orthogonal to $\vec{u}^{(j)}$.

Consider first the construction of $\mathbf{U}^{(1)}$. Choosing the components $u_l^{(1)}$ of $\vec{u}^{(1)}$ as

$$u_1^{(1)} = 0, \qquad u_2^{(1)} = \left[\frac{1}{2}\Big(1 - \frac{a_{12}}{\alpha}\Big)\right]^{1/2}, \qquad u_l^{(1)} = -\frac{a_{1l}}{2\alpha u_2^{(1)}} \ (l \ge 3), \tag{1.141}$$

where $\alpha = (a_{12}^2 + \cdots + a_{1N}^2)^{1/2}$, we then have $\vec{u}^{(1)T}[a_{l1}]_{l=1,\dots,N} = (a_{12} - \alpha)/2u_2^{(1)}$. This in turn implies that

$$\mathbf{B}^{(1)} := \mathbf{U}^{(1)}\mathbf{A}\mathbf{U}^{(1)} \tag{1.142}$$

has

$$b_{11} = a_{11}, \qquad b_{12} = b_{21} = \alpha, \qquad b_{1k} = b_{k1} = 0 \ (k \ge 3)$$

and is thus tridiagonal with respect to the first row and column. The matrices $\mathbf{U}^{(j)}$, $j = 2, 3, \dots$ in order are now defined by the formulas (1.141), but with $u_1^{(j)} = u_2^{(j)} = \cdots = u_j^{(j)} = 0$, and the analogue of the entries a_{1l} replaced by the elements in the first row of the bottom right $(N-j+1)\times(N-j+1)$ submatrix of $\mathbf{B}^{(j-1)}$.

A number of works (see [157] and references therein) posed the question as to the form of $\mathbf{B}^{(N-2)}$ when $\mathbf{A}$ is a member of the GOE. It was found that like $\mathbf{A}$ itself, the elements of $\mathbf{B}^{(N-2)}$ are all independent (apart from the requirement that $\mathbf{B}^{(N-2)}$ be symmetric) with a distribution that can be calculated explicitly.

Proposition 1.9.1 *Let* $\mathrm{N}[0,1]$ *refer to the standard normal distribution as defined below Definition 1.1.1, and let* $\tilde{\chi}_k$ *denote the square root of the gamma distribution* $\Gamma[k/2, 1]$*, the latter being specified by the p.d.f.* $(1/\Gamma(k/2))u^{k/2-1}e^{-u}$*,* $u > 0$*, and realized by the sum of the squares of* k *independent Gaussian distributions* $\mathrm{N}[0, 1/\sqrt{2}]$*. (The p.d.f. of* $\tilde{\chi}_k$ *is thus equal to* $(2/\Gamma(k/2))u^{k-1}e^{-u^2}$*,* $u > 0$*.) For* $\mathbf{A}$ *a member of the GOE, the tridiagonal matrix* $\mathbf{B}^{(N-2)}$ *obtained by successive Householder transformations is given by*

$$\begin{bmatrix} \mathrm{N}[0,1] & \tilde{\chi}_{N-1} & & & & \\ \tilde{\chi}_{N-1} & \mathrm{N}[0,1] & \tilde{\chi}_{N-2} & & & \\ & \tilde{\chi}_{N-2} & \mathrm{N}[0,1] & \tilde{\chi}_{N-3} & & \\ & \ddots & \ddots & \ddots & & \\ & & \tilde{\chi}_2 & \mathrm{N}[0,1] & \tilde{\chi}_1 & \\ & & & \tilde{\chi}_1 & \mathrm{N}[0,1] & \end{bmatrix}.$$

Proof. Let GOE_n denote the ensemble of $n \times n$ GOE matrices. From the Householder algorithm, the first row and column of $\mathbf{B}^{(N-2)}$ are he same as those of $\mathbf{B}^{(1)}$ in (1.142), and thus from (1.141) we have

$$b_{11}^{(N-2)} = \mathrm{N}[0,1], \qquad b_{12}^{(N-2)} = \tilde{\chi}_{N-1},$$

where use has been made of the assumption that $\mathbf{A}$ is a member of GOE_N, and the definition of $\tilde{\chi}_{N-1}^2$ as a sum of squares of Gaussians. To proceed further we must compute the distribution of the bottom $N-1\times N-1$ block of $\mathbf{B}^{(1)}$. In general, denoting such a block of the matrix $\mathbf{X}$ by $\mathbf{X}_{N-1}$, it follows from (1.140) that $\mathbf{B}_{N-1}^{(1)} = \mathbf{V}_{N-1}\mathbf{A}_{N-1}\mathbf{V}_{N-1}$. Since the elements of the real orthogonal matrix $\mathbf{V}_{N-1}$ are independent of the elements of $\mathbf{A}_{N-1}$, which itself is a member of GOE_{N-1}, it follows immediately from the general invariance of the GOE under orthogonal transformations that $\mathbf{B}_{N-1}^{(1)}$ is also a member of GOE_{N-1}. Applying the Householder transformation to $\mathbf{B}_{N-1}^{(1)}$, we thus get

$$b_{22}^{(N-2)} = N[0,1], \qquad b_{23}^{(N-2)} = \tilde{\chi}_{N-2}.$$

Continuing inductively gives the stated result. □

1.9.2 Tridiagonal matrices

The result of Proposition 1.9.1 suggests investigating the Jacobian for the change of variables from a general real symmetric tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} a_n & b_{n-1} & & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & & \\ & b_{n-2} & a_{n-2} & b_{n-3} & & \\ & \ddots & \ddots & \ddots & & \\ & & & b_2 & a_2 & b_1 \\ & & & & b_1 & a_1 \end{bmatrix}, \tag{1.143}$$

to its eigenvalues and variables relating to its eigenvectors. First, for each eigenvalue λ_k and corresponding eigenvector $\vec{v}_k$, it is easy to see by direct substitution that once the first component $v_k^{(1)} =: q_k$ of $\vec{v}_k$ is specified, all other components can be expressed in terms of λ_k and the elements of $\mathbf{T}$. To make the eigendecomposition unique we specify that $q_k > 0$, and furthermore note that $\mathbf{T}$, being symmetric, can be orthogonally diagonalized, and so doing this we have

$$\sum_{k=1}^{n} q_k^2 = 1. \tag{1.144}$$

The Jacobian for the change of variables from

$$\vec{a} := (a_n, a_{n-1}, \dots, a_1), \qquad \vec{b} := (b_{n-1}, \dots, b_1), \tag{1.145}$$

to

$$\vec{\lambda} := (\lambda_1, \dots, \lambda_n), \qquad \vec{q} := (q_1, \dots, q_{n-1}) \tag{1.146}$$

can be calculated using the method of wedge products. However, one must first establish some auxiliary results.

PROPOSITION 1.9.2 *Let $(\mathbf{X})_{11}$ denote the top-left hand entry of the matrix $\mathbf{X}$. We have*

$$((\mathbf{T} - \lambda \mathbf{1})^{-1})_{11} = \sum_{j=1}^{n} \frac{q_j^2}{\lambda_j - \lambda}. \tag{1.147}$$

Also

$$\prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 = \frac{\prod_{i=1}^{n-1} b_i^{2i}}{\prod_{i=1}^{n} q_i^2}. \tag{1.148}$$

Proof. Now

$$((\mathbf{T} - \lambda \mathbf{1})^{-1})_{11} = \vec{e}_1 \cdot (\mathbf{T} - \lambda \mathbf{1})^{-1} \vec{e}_1,$$

where $\vec{e}_1 := (1, 0, \dots, 0)^T$. Since $\{\vec{v}_j\}$ is an orthonormal set,

$$\vec{e}_1 = \sum_{j=1}^{n} (\vec{e}_1 \cdot \vec{v}_j) \vec{v}_j = \sum_{j=1}^{n} q_j \vec{v}_j, \tag{1.149}$$

and substituting into the above equation gives (1.147). This derivation makes no explicit use of $\mathbf{T}$ being tridiagonal, rather just that (1.149) holds, for which it is sufficient $\mathbf{T}$ be real symmetric.

To derive (1.148) [140] we begin by recalling that in general for $\mathbf{X}$ an $n \times n$ nonsingular matrix,

$$(\mathbf{X}^{-1})_{11} = \frac{\det \mathbf{X}_{n-1}}{\det \mathbf{X}}, \tag{1.150}$$

where $\mathbf{X}_{n-1}$ denotes the bottom right $n-1 \times n-1$ submatrix of $\mathbf{X}$. Hence we can rewrite (1.147) to read

$$\frac{\prod_{i=1}^{n-1}(\lambda - \lambda_i^{(n-1)})}{\prod_{i=1}^{n}(\lambda - \lambda_i)} = \sum_{j=1}^{n} \frac{q_j^2}{\lambda - \lambda_j}, \tag{1.151}$$

where $\{\lambda_i^{(n-1)}\}$ denotes the eigenvalues of $\mathbf{X}_{n-1}$. It follows from this that

$$q_j^2 = \frac{P_{n-1}(\lambda_j)}{P_n'(\lambda_j)}, \qquad P_k(\lambda) := \prod_{i=1}^{k}(\lambda - \lambda_i^{(k)}), \tag{1.152}$$

where $P_k(\lambda)$ is the characteristic polynomial of the bottom right $k \times k$ submatrix of $\mathbf{T}$, say $\mathbf{T}_k$, and $\{\lambda_i^{(k)}\}$ the corresponding eigenvalues. Hence

$$\prod_{i=1}^{n} q_i^2 = \frac{\prod_{i=1}^{n} |P_{n-1}(\lambda_i)|}{\prod_{1 \le i < j \le n}(\lambda_i - \lambda_j)^2}. \tag{1.153}$$

Next, by expanding along the first row of $\lambda \mathbf{1}_k - \mathbf{T}_k$, one obtains the three-term recurrence

$$P_k(\lambda) = (\lambda - a_k)P_{k-1}(\lambda) - b_{k-1}^2 P_{k-2}(\lambda) \tag{1.154}$$

and it follows from this that

$$\prod_{i=1}^{k-1} |P_k(\lambda_i^{(k-1)})| = b_{k-1}^{2(k-1)} \prod_{i=1}^{k-1} |P_{k-2}(\lambda_i^{(k-1)})|.$$

Since

$$\prod_{i=1}^{k-1} |P_{k-2}(\lambda_i^{(k-1)})| = \prod_{i=1}^{k-1}\prod_{j=1}^{k-2} |\lambda_i^{(k-1)} - \lambda_j^{(k-2)}| = \prod_{j=1}^{k-2} |P_{k-1}(\lambda_j^{(k-2)})|, \tag{1.155}$$

this can be rewritten as

$$\prod_{i=1}^{k-1} |P_k(\lambda_i^{(k-1)})| = b_{k-1}^{2(k-1)} \prod_{j=1}^{k-2} |P_{k-1}(\lambda_j^{(k-2)})|,$$

and iteration shows

$$\prod_{i=1}^{n-1} |P_n(\lambda_i^{(n-1)})| = \prod_{i=1}^{n-1} b_i^{2i}.$$

Use of (1.155) with $k = n+1$ and substitution into (1.153) gives (1.148). □

PROPOSITION 1.9.3 *The Jacobian for the change of variables (1.145) to (1.146) can be written as*

$$\frac{1}{q_n} \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i}. \tag{1.156}$$

Proof. [223] Rewriting (1.147) in the form

$$((\mathbf{1} - \lambda \mathbf{T})^{-1})_{11} = \sum_{j=1}^{n} \frac{q_j^2}{1 - \lambda \lambda_j} \tag{1.157}$$

and equating successive powers of λ on both sides gives

$$\begin{gathered}1 = \sum_{j=1}^n q_j^2, \quad a_n = \sum_{j=1}^n q_j^2 \lambda_j, \quad * + b_{n-1}^2 = \sum_{j=1}^n q_j^2 \lambda_j^2, \\ * + a_{n-1} b_{n-1}^2 = \sum_{j=1}^n q_j^2 \lambda_j^3, \quad * + b_{n-2}^2 b_{n-1}^2 = \sum_{j=1}^n q_j^2 \lambda_j^4, \\ * + a_{n-2} b_{n-2}^2 b_{n-1}^2 = \sum_{j=1}^n q_j^2 \lambda_j^5, \dots, \quad * + a_1 b_1^2 \cdots b_{n-2}^2 b_{n-1}^2 = \sum_{j=1}^n q_j^2 \lambda_j^{2n-1},\end{gathered}$$

where the $*$ denotes terms involving only variables already having appeared on the l.h.s. of preceding equations (thus the variables $a_n, b_{n-1}, a_{n-1}, b_{n-2}, \dots$ occur in a triangular structure). The first of these equations implies

$$q_n dq_n = - \sum_{j=1}^{n-1} q_j dq_j. \tag{1.158}$$

Taking differentials of the remaining equations, substituting for $q_n dq_n$, and then taking wedge products of both sides (making use of the triangular structure on the l.h.s.) shows

$$\prod_{j=1}^{n-1} b_j^{4j-1} d\vec{a} \wedge d\vec{b} = q_n^2 \prod_{j=1}^{n-1} q_j^3 \det \Big[[\lambda_k^j - \lambda_n^j]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \Big] d\vec{\lambda} \wedge d\vec{q},$$

where

$$d\vec{a} := \bigwedge_{j=1}^n da_j, \quad d\vec{b} := \bigwedge_{j=1}^{n-1} db_j, \quad d\vec{\lambda} := \bigwedge_{j=1}^n d\lambda_j, \quad d\vec{q} := \bigwedge_{j=1}^{n-1} dq_j.$$

By definition the Jacobian J is positive and such that $d\vec{a} \wedge d\vec{b} = \pm J d\vec{\lambda} \wedge d\vec{q}$ for some sign $\pm$. Making use of the determinant evaluation (1.175) below we thus read off that

$$J = \frac{1}{q_n} \frac{\prod_{j=1}^{n-1} b_j}{\prod_{j=1}^n q_j} \Big(\frac{\prod_{j=1}^n q_j^2}{\prod_{j=1}^{n-1} b_j^{2j}} \Big)^2 \prod_{1 \le j < k \le n} (\lambda_k - \lambda_j)^4.$$

Recalling (1.148) shows J is equal to (1.156). □

Using Proposition 1.9.3, the fact that the tridiagonal matrix of Proposition 1.9.1 has the same eigenvalue p.d.f. as GOE matrices can be reclaimed. Moreover, one can prescribe a tridiagonal matrix with eigenvalue p.d.f. (1.28) for general $\beta > 0$ [140].

PROPOSITION 1.9.4 *Let $\beta > 0$ be fixed. In the notation of Proposition 1.9.1 define the Gaussian β-ensemble as the set of symmetric tridiagonal matrices*

$$\mathbf{T}_\beta := \begin{bmatrix} \mathrm{N}[0,1] & \tilde{\chi}_{(N-1)\beta} & & & & \\ \tilde{\chi}_{(N-1)\beta} & \mathrm{N}[0,1] & \tilde{\chi}_{(N-2)\beta} & & & \\ & \tilde{\chi}_{(N-2)\beta} & \mathrm{N}[0,1] & \tilde{\chi}_{(N-3)\beta} & & \\ & \ddots & \ddots & \ddots & & \\ & & \tilde{\chi}_{2\beta} & \mathrm{N}[0,1] & \tilde{\chi}_\beta \\ & & & \tilde{\chi}_\beta & \mathrm{N}[0,1] \end{bmatrix}. \tag{1.159}$$

The eigenvalues and first component of the eigenvectors (which form the vector $\vec{q}$) are independent, with the *distribution of the former given by*

$$\frac{1}{\tilde{G}_{\beta,N}} \prod_{l=1}^{N} e^{-\lambda_l^2/2} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^\beta \, d\vec{\lambda}, \qquad \tilde{G}_{\beta,N} = (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)\beta/2)}{\Gamma(1+\beta/2)}, \tag{1.160}$$

and the distribution of the latter given by

$$\frac{1}{c_{\beta,N} q_N} \prod_{i=1}^{N} q_i^{\beta-1} \, d\vec{q}, \qquad q_i > 0, \ \sum_{i=1}^{N} q_i^2 = 1, \quad \text{where} \quad c_{\beta,N} = \frac{\Gamma^N(\beta/2)}{2^{N-1}\Gamma(\beta N/2)}. \tag{1.161}$$

Proof. Denote the joint distribution of $\mathbf{T}_\beta$ by $P(\mathbf{T}_\beta)$. We have

$$\begin{aligned} P(\mathbf{T}_\beta)(d\mathbf{T}_\beta) &= \frac{2^{N-1}}{(2\pi)^{N/2}} \prod_{l=1}^{N-1} \frac{b_l^{\beta l-1} e^{-b_l^2}}{\Gamma(\beta l/2)} \prod_{l=1}^{N} e^{-a_l^2/2} d\vec{a} \wedge d\vec{b} \\ &= \frac{2^{N-1}}{(2\pi)^{N/2}} \prod_{l=1}^{N-1} \frac{1}{\Gamma(\beta l/2)} \frac{1}{q_N} \frac{\prod_{l=1}^{N-1} b_l^{\beta l}}{\prod_{l=1}^{N} q_l} e^{-\mathrm{Tr}(\mathbf{T}_\beta^2)/2} d\vec{\lambda} \wedge d\vec{q}, \end{aligned}$$

where the second equality follows using (1.156). But

$$e^{-\mathrm{Tr}(\mathbf{T}_\beta^2)/2} = e^{-\sum_{j=1}^{N} \lambda_j^2/2}, \qquad \prod_{l=1}^{N-1} b_l^{\beta l} = \prod_{l=1}^{N} q_l^\beta \prod_{1\le i<j\le N} |\lambda_j - \lambda_i|^\beta,$$

where the latter formula follows from (1.148), so indeed the dependence on $\vec{\lambda}$ and $\vec{q}$ factorizes into the functional forms specified in (1.160) and (1.161). The normalization for (1.161) follows from the Dirichlet integral [541]

$$\int_{\sum_{i=1}^{n+1} \rho_i = 1,\, \rho_i > 0} d\rho_1 \cdots d\rho_n \prod_{i=1}^{n+1} \rho_i^{s_i - 1} = \frac{\Gamma(s_1)\cdots\Gamma(s_{n+1})}{\Gamma(s_1 + \cdots + s_{n+1})} \tag{1.162}$$

with $n = N-1$, $s_i = \beta/2$ and the change of variables $\rho_i = q_i^2$. With this normalization specified, the value of $\tilde{G}_{\beta,N}$ follows (an extra factor of $N!$ is included to effectively remove the ordering on $\{\lambda_i\}$ implicit in the above working; recall the remark below Proposition 1.3.4). □

We remark that the evaluation of $\tilde{G}_{\beta,N}$ given in (1.160) implies, after a simple change of variables, that the normalization constant in (1.28) has the evaluation

$$G_{\beta,N} = \beta^{-N/2 - N\beta(N-1)/4} (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)\beta/2)}{\Gamma(1+\beta/2)}. \tag{1.163}$$

Another point of interest is that the recurrence (1.154) with

$$a_k \in \mathrm{N}[0,1], \qquad b_k^2 \in \Gamma[k\beta/2, 1] \tag{1.164}$$

can be used to generate the characteristic polynomial for a member of the Gaussian β-ensemble, so the p.d.f. (1.160) can be sampled by simply computing the zeros of this polynomial.

1.9.3 Sturm sequences

For tridiagonal matrices, the task of computing the cumulative microscopic eigenvalue density $N(\mu)$, that is, the number of eigenvalues less than μ, has a number of special features. This in turn follows from special features of the corresponding *Sturm sequences* [13].

DEFINITION 1.9.5 *Let $\mathbf{A}_n$ be a general $n \times n$ matrix, and let $\mathbf{A}_{n-k}$ $(k = 1, \ldots, n-1)$ denote the matrix* *obtained by deleting the first k rows and columns. Let $d_i := \det \mathbf{A}_i$ $(i = 1, \dots, n)$ and set $d_0 := 1$. The Sturm sequence refers to $(d_0, d_1, \dots, d_n)$.*

PROPOSITION 1.9.6 *Let $\mathbf{A}_n$ be a real symmetric matrix with no repeated eigenvalues and no zero eigenvalues, and similarly $\mathbf{A}_{n-k}$. The number of sign changes in the Sturm sequence (reading from right-to-left, say) is equal to the number of negative eigenvalues of $\mathbf{A}_n$.*

Proof. For a given $k = 2, \dots, n$ it is a fundamental result (see Exercises 4.2 q.2(iii) below) that the eigenvalues $\{a_i\}$ of $\mathbf{A}_k$ interlace the eigenvalues $\{\alpha_i\}$ of $\mathbf{A}_{k-1}$,

$$a_k < \alpha_{k-1} < a_{k-1} < \cdots < \alpha_1 < a_1.$$

We know too that the determinant is equal to the product of eigenvalues. Consequently, the number of negative eigenvalues of $\mathbf{A}_k$ equals the number of negative eigenvalues of $\mathbf{A}_{k-1}$, if d_k/d_{k-1} is positive, while we must add one if d_k/d_{k-1} is negative. Iteratively applying this for $k = n, \dots, 1$ gives the stated result. □

Applying Proposition 1.9.6 to the matrix $\mathbf{A}_n - \mu \mathbf{1}_n$ gives that $N(\mu)$ is equal to the number of sign changes in the Sturm sequence for $\mathbf{A}_n - \mu \mathbf{1}_n$. In the case that $\mathbf{A}_n$ is the tridiagonal matrix (1.143), one has that $d_k = (-1)^k P_k(\mu)$ as specified by (1.154), and using the recurrence (1.154) shows that $r_i := d_i/d_{i-1}$ can be specified by the recursive formula

$$r_i = \begin{cases} a_1 - \mu, & i = 1, \\ (a_i - \mu) - b_{i-1}^2/r_{i-1}, & i = 2, \dots, n. \end{cases} \tag{1.165}$$

As each sign change in the Sturm sequence $\{d_i\}$ corresponds to a negative in the ratio sequence $\{r_i\}$, we see that the number of negative values in $\{r_i\}$ equals $N(\mu)$. This latter result can be related to so-called *shooting eigenvectors*.

DEFINITION 1.9.7 *The vector $\vec{x}$ satisfying all but the first of the n linear equations implied by the matrix equation $(\mathbf{A}_n - \mu \mathbf{1}_n)\vec{x} = \vec{0}$, with $\vec{x} = (x_n, \dots, x_1)^T$ and x_1 given, is referred to as a shooting eigenvector. (Note that the first equation can only be satisfied as well if and only if μ is an eigenvalue.)*

For the tridiagonal matrix (1.143), and with x_{n+1} defined as the first component of $(\mathbf{A}_n - \mu \mathbf{1}_n)\vec{x}$, a recurrence for the ratio $s_i = x_i/x_{i-1}$, $i = 2, \dots, n+1$ is readily obtained, and comparison with (1.165) shows $s_i = -r_{i-1}/b_{i-1}$ (in the case $i = n+1$ this requires setting $b_n := 1$). Thus with each $b_i > 0$, the number of positive values in $\{s_i\}$ equals $N(\mu)$. This can equivalently be stated in terms of $\{x_i\}$.

PROPOSITION 1.9.8 *The number of sign changes in the shooting eigenvector $\vec{x}$ equals $n - N(\mu)$, which is the number of eigenvalues of $\mathbf{A}$ greater than μ.*

1.9.4 Prüfer phases

There is a parametrization, in terms of *Prüfer phases* and amplitudes, of the shooting vectors well suited to analysis of the large n limit of the bulk eigenvalues (see Section 13.6). To introduce the parametrization, first observe that the three-term recurrence satisfied by the shooting vector

$$b_j x_{j+1} + a_j x_j + b_{j-1} x_{j-1} = \mu x_j \qquad (j = 1, \dots, n;\ b_0 := 0, b_n := -1) \tag{1.166}$$

is equivalent to the matrix equation

$$\begin{bmatrix} (\mu - a_j)/b_j & -1/b_j \\ b_j & 0 \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} u_{j+1} \\ v_{j+1} \end{bmatrix} \qquad (j = 1, \dots, n), \tag{1.167}$$

where

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b_{j-1} \end{bmatrix} \begin{bmatrix} x_j \\ x_{j-1} \end{bmatrix} \tag{1.168}$$

(note that the matrix in (1.167) has unit determinant and so as a transformation is volume preserving). Choosing the initial condition $u_1 = 1, v_1 = 0$ we see that

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix} = T_j \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $T_j := V_{j-1} \cdots V_1$ is referred to as a transfer matrix

DEFINITION 1.9.9 *The Prüfer phases θ_j^μ and amplitudes $R_j^\mu > 0$ are such that*

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} R_j^\mu \cos\theta_j^\mu \\ R_j^\mu \sin\theta_j^\mu \end{bmatrix}, \tag{1.169}$$

where $-\pi/2 < \theta_{j+1}^\mu - \theta_j^\mu < 3\pi/2$.

Note that it follows from (1.167) and (1.168) that $\{\theta_j^\mu\}$ satisfies the first order recurrence

$$b_j^2 \cot\theta_{j+1}^\mu = -\tan\theta_j^\mu + (\mu - a_j), \qquad \theta_1^\mu = 0. \tag{1.170}$$

A consequence is an identity which tells us that θ_j^μ is a decreasing function of μ (see also Exercises 1.9 q.5).

PROPOSITION 1.9.10 *We have*

$$(R_j^\mu)^2 \frac{\partial}{\partial\mu}\theta_j^\mu = -\sum_{l=1}^{j-1} u_l^2. \tag{1.171}$$

Proof. Differentiating (1.170) with respect to μ and making use of (1.168) and (1.169) gives the recurrence

$$(R_{j+1}^\mu)^2 \frac{\partial\theta_{j+1}^\mu}{\partial\mu} = (R_j^\mu)^2 \frac{\partial\theta_j^\mu}{\partial\mu} - u_j^2.$$

This together with the initial condition $\partial\theta_1^\mu/\partial\mu = 0$ implies (1.171). □

We are now in a position to relate θ_n^μ to $N(\mu)$ for the tridiagonal matrix (1.143) [326] . First note from the recurrence (1.166) that for $\mu \to \infty$, x_j is positive while $x_{j-1}/x_j \to 0$. Recalling (1.169), this implies $\lim_{\mu\to\infty} \theta_j^\mu = 0$. But it has just been shown that θ_j^μ is a decreasing function of μ. The facts that $x_{n+1} = u_{n+1} = R_{n+1}^\mu \cos\theta_{n+1}^\mu$ and that $x_{n+1} = 0$ if and only if μ is an eigenvalue then imply the kth largest eigenvalue λ_k of $\mathbf{T}$ is such that $\theta_{n+1}^{\lambda_k} = (\pi/2) + \pi(k-1)$, and moreover that θ_{n+1}^μ relates to the number of eigenvalues of $\mathbf{T}$ greater that μ, $n - N(\mu)$, according to

$$\Big| \frac{1}{\pi}\theta_{n+1}^\mu - (n - N(\mu)) \Big| \le \frac{1}{2}. \tag{1.172}$$

EXERCISES 1.9 1. The objective of this exercise is to derive the Vandermonde determinant evaluation

$$\det[x_j^{k-1}]_{j,k=1,\dots,N} := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{vmatrix} = \prod_{1\le j<k\le N} (x_k - x_j). \tag{1.173}$$

(i) Verify that both the determinant and product of differences are antisymmetric polynomials which are homogeneous of degree $\frac{1}{2}N(N-1)$ and hence must be proportional.

(ii) Show that the proportionality constant is unity by comparing the coefficients of the term $x_1^0 x_2^1 \dots x_N^{N-1}$ on both sides.

2. (i) In the Vandermonde determinant identity (1.173), replace N by pN. Subtract row one from row two, divide this row by $x_2 - x_1$ and take the limit $x_2 \to x_1$ by first differentiating the top and bottom lines with respect

to x_2. Next subtract the first and second row from the third, divide this row by $(x_3 - x_1)^2$ and take the limit $x_3 \to x_1$ by differentiating top and bottom lines with respect to x_3 twice. Proceed in this fashion by subtracting rows $1, 2, \dots, j-1$ from row j $(j = 4, \dots, p)$, dividing by $(x_j - x_1)^{j-1}$, and taking the limit $x_j \to x_1$ by differentiating top and bottom lines with respect to x_j $j-1$ times. Repeat this procedure for successive blocks of p variables to deduce the confluent Vandermonde determinant identity

$$\det \begin{bmatrix} x_j^{k-1} \\ \binom{k-1}{1} x_j^{k-2} \\ \vdots \\ \binom{k-p+1}{p-1} x_j^{k-p} \end{bmatrix}_{\substack{j=1,\dots,N \\ k=1,\dots,pN}} = \prod_{1 \le j < k \le N} (x_k - x_j)^{p^2}. \tag{1.174}$$

(ii) Consider the identity (1.174) in the case $p = 2$. Take the transpose of the determinant, and rearrange columns so that it reads

$$(-1)^{N(N-1)/2} \det \left[[\lambda_k^{j-1}]_{\substack{j=1,\dots,2N \\ k=1,\dots,N}} [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2N \\ k=1,\dots,N}} \right] = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^4.$$

Subtract column N from columns $1, \dots, k-1$, then expand by the first row to deduce that

$$(-1)^{(N-1)(N-2)/2} \det \left[[\lambda_k^j - \lambda_N^j]_{\substack{j=1,\dots,2N-1 \\ k=1,\dots,N-1}} [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2N-1 \\ k=1,\dots,N}} \right] = \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^4. \tag{1.175}$$

3. [546], [249] In this exercise the Householder transformation will be used to establish an identity of relevance to the Schur decomposition (15.3) below, and also to establish the diagonalization formula (1.8).

(i) Let $\vec{a}$ and $\vec{b}$ be unit vectors, and form the unit vector $\vec{v} = (\vec{a} + \vec{b})/|\vec{a} + \vec{b}|$. From the fact that $\vec{v}$ bisects the angle of $\vec{a}$ and $\vec{b}$, deduce from the geometrical interpretation of the Householder transformation

$$\mathbf{U}_N = \mathbf{1} - 2\vec{v}\vec{v}^T$$

as a reflection in the hyperplane orthogonal to $\vec{v}$ that $\mathbf{U}_N \vec{a} = -\vec{b}$, $\mathbf{U}_N \vec{b} = -\vec{a}$. Also derive these equations algebraically.

(ii) Let $\mathbf{A}_N$ be an $N \times N$ matrix and let λ be an eigenvalue of $\mathbf{A}_N$ with corresponding normalized eigenvector $\vec{w}$. Let $\vec{e}_1 := (1, 0, \dots, 0)^T$ be an $N \times 1$ elementary vector. In (i) set $\vec{a} = \vec{e}_1$, $\vec{b} = \vec{w}$, and use the formulas therein to deduce that

$$\mathbf{U}_N \mathbf{A}_N \mathbf{U}_N \vec{e}_1 = \lambda \vec{e}_1.$$

Hence conclude

$$\mathbf{U}_N \mathbf{A}_N \mathbf{U}_N = \begin{bmatrix} \lambda & \vec{\alpha}_{N-1}^T \\ \vec{0}_{N-1} & \mathbf{A}_{N-1} \end{bmatrix} \tag{1.176}$$

for some $1 \times (N-1)$ vector $\vec{\alpha}_{N-1}^T$ and $(N-1) \times (N-1)$ matrix $\mathbf{A}_{N-1}$.

(iii) Let $\mathbf{P}_N$ be a real orthogonal diagonal matrix (each diagonal entry ± 1). With $\mathbf{V}_N = \mathbf{P}_N \mathbf{U}_N$ note from (1.176) that

$$\mathbf{V}_N \mathbf{A}_N \mathbf{V}_N^T = \begin{bmatrix} \lambda & \vec{\beta}_{N-1}^T \\ \vec{0}_{N-1} & \tilde{\mathbf{A}}_{N-1} \end{bmatrix} \tag{1.177}$$

for some $\vec{\beta}_{N-1}^T$, $\tilde{\mathbf{A}}_{N-1}$. Now use a Householder transformation of the form

$$\begin{bmatrix} 1 & \vec{0}_{N-1}^T \\ \vec{0}_{N-1} & \mathbf{U}_{N-1} \end{bmatrix}$$

to reduce $\tilde{\mathbf{A}}_{N-1}$ to triangular form and proceed inductively to deduce that there exists a real orthogonal matrix $\mathbf{R}$ such that

$$\mathbf{R}\mathbf{A}_N \mathbf{R}^T = \mathbf{T}, \tag{1.178}$$

where $\mathbf{T}$ is upper triangular with diagonal entries equal to the eigenvalues of $\mathbf{A}_{N-1}$, and note that $\mathbf{R}$ is unique up to an overall sign of each column.

(iv) Show that (1.178) implies the diagonalization formula (1.8).

4. [156] In this exercise the change of variables implied by (1.176) will be used to derive a generalization of (1.11).

(i) From the decomposition (1.176) in the case $\mathbf{A}_N$ is symmetric so that $\vec{\alpha}_{N-1} = \vec{0}_{N-1}$, deduce that

$$\mathbf{U}_N d\mathbf{A}_N \mathbf{U}_N = \mathbf{U}_N d\mathbf{U}_N \begin{bmatrix} \lambda_1 & 0^T_{N-1} \\ \vec{0}_{N-1} & \mathbf{A}_{N-1} \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0^T_{N-1} \\ \vec{0}_{N-1} & \mathbf{A}_{N-1} \end{bmatrix} \mathbf{U}_N d\mathbf{U}_N + \begin{bmatrix} d\lambda_1 & 0^T_{N-1} \\ \vec{0}_{N-1} & d\mathbf{A}_{N-1} \end{bmatrix},$$

where use is made of the fact that $\mathbf{U}_N d\mathbf{U}_N$ is antisymmetric. Make further use of this latter fact to show that it is permissible to write

$$\mathbf{U}_N d\mathbf{U}_N := \begin{bmatrix} 0 & -d\vec{s}^{\,T}_{N-1} \\ d\vec{s}_{N-1} & d\tilde{\mathbf{U}}_{N-1} \end{bmatrix}$$

and so obtain

$$\mathbf{U}_N d\mathbf{A}_N \mathbf{U}_N = \begin{bmatrix} 0 & d\vec{s}^{\,T}_{N-1}(\lambda_1 - \mathbf{A}_{N-1}) \\ (\lambda_1 - \mathbf{A}_{N-1}) d\vec{s}_{N-1} & d\tilde{\mathbf{U}}_{N-1}\mathbf{A}_{N-1} - \mathbf{A}_{N-1} d\tilde{\mathbf{U}}_{N-1} \end{bmatrix} + \begin{bmatrix} d\lambda_1 & 0^T_{N-1} \\ \vec{0}_{N-1} & d\mathbf{A}_{N-1} \end{bmatrix}.$$

(ii) Using (1.10) and (1.17) read off from the final equation in (i) that

$$(d\mathbf{A}_N) = |\det(\lambda_1 - \mathbf{A}_{N-1})| d\lambda_1 (d\vec{s}_{N-1})(d\mathbf{A}_{N-1}), \tag{1.179}$$

where use has been made of the fact that $d\tilde{\mathbf{U}}_{N-1}$ is a function of the components of $d\vec{s}_{N-1}$ and thus $d\vec{s}_{N-1} \wedge d\tilde{\mathbf{U}}_{N-1} = 0$.

(iii) Iterate (1.179) to obtain a result equivalent to (1.11).

5. (i) Use the fact that $r_i = -P_i(\mu)/P_{i-1}(\mu)$, to show that the Prüfer phase for $j = 2, \dots, n$ satisfies

$$\cot \theta^\mu_j = \frac{1}{(b_{j-1})^2} \frac{P_{j-1}(\mu)}{P_{j-2}(\mu)}. \tag{1.180}$$

(ii) Use the fact that $\{P_j(\mu)\}_{j=0,1,\dots}$ are, as a consequence of their obeying the three-term recurrence (1.154), a set of orthogonal polynomials with respect to an inner product defined by its moment (the so called Favard theorem, see, e.g., [384]), together with (5.13) below to show that $P'_i(\mu)P_{i-1}(\mu) - P'_{i-1}(\mu)P_i(\mu) > 0$. After differentiating (1.180) with respect to μ, use this fact to deduce $d\theta^\mu_j/d\mu < 0$.