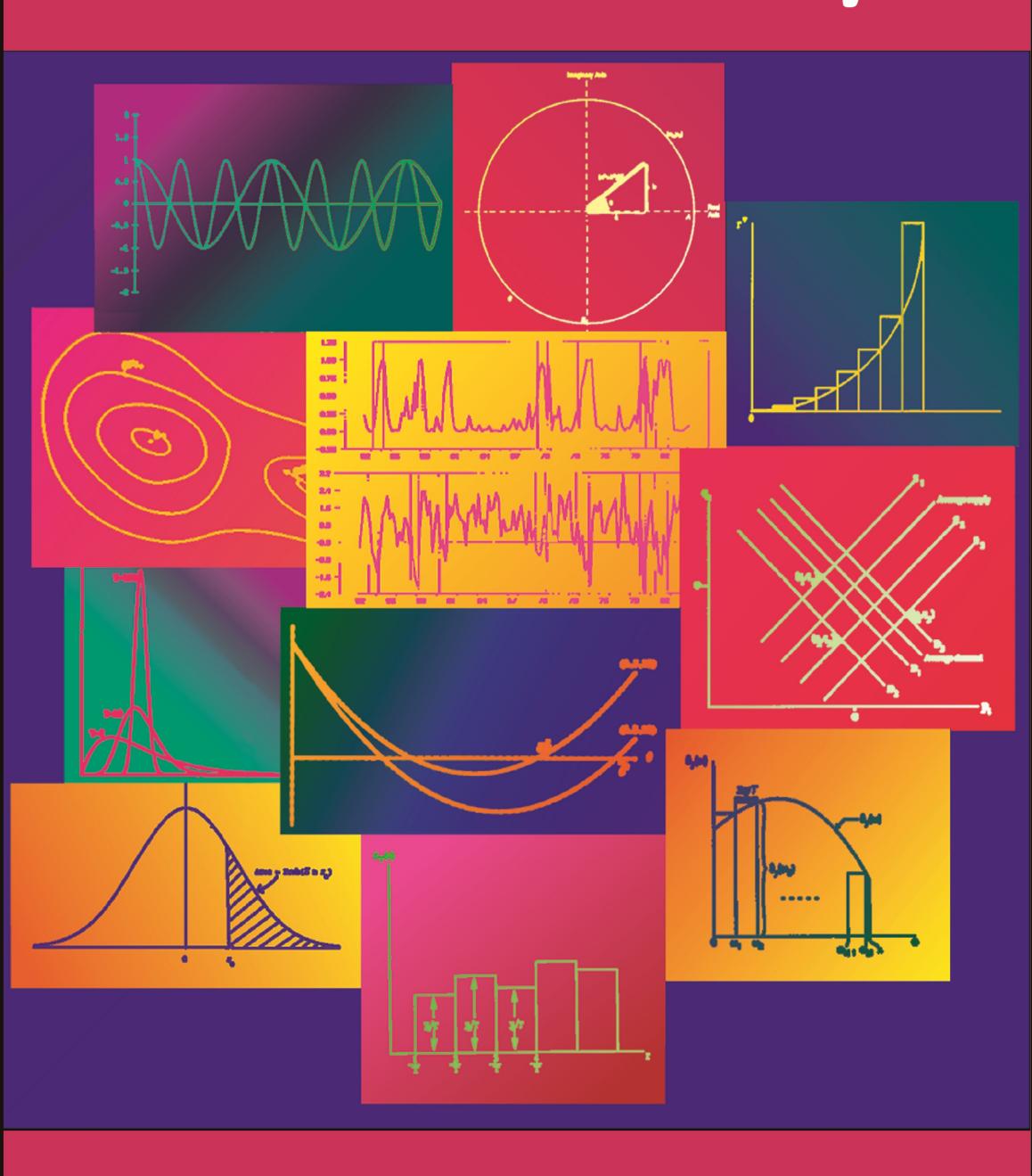


Time Series Analysis



James D. Hamilton

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Preface

Much of economics is concerned with modeling dynamics. There has been an explosion of research in this area in the last decade, as “time series econometrics” has practically come to be synonymous with “empirical macroeconomics.”

Several texts provide good coverage of the advances in the economic analysis of dynamic systems, while others summarize the earlier literature on statistical inference for time series data. There seemed a use for a text that could integrate the theoretical and empirical issues as well as incorporate the many advances of the last decade, such as the analysis of vector autoregressions, estimation by generalized method of moments, and statistical inference for nonstationary data. This is the goal of *Time Series Analysis*.

A principal anticipated use of the book would be as a textbook for a graduate econometrics course in time series analysis. The book aims for maximum flexibility through what might be described as an integrated modular structure. As an example of this, the first three sections of Chapter 13 on the Kalman filter could be covered right after Chapter 4, if desired. Alternatively, Chapter 13 could be skipped altogether without loss of comprehension. Despite this flexibility, state-space ideas are fully integrated into the text beginning with Chapter 1, where a state-space representation is used (without any jargon or formalism) to introduce the key results concerning difference equations. Thus, when the reader encounters the formal development of the state-space framework and the Kalman filter in Chapter 13, the notation and key ideas should already be quite familiar.

Spectral analysis (Chapter 6) is another topic that could be covered at a point of the reader’s choosing or skipped altogether. In this case, the integrated modular structure is achieved by the early introduction and use of autocovariance-generating functions and filters. Wherever possible, results are described in terms of these rather than the spectrum.

Although the book is designed with an econometrics course in time series methods in mind, the book should be useful for several other purposes. It is completely self-contained, starting from basic principles accessible to first-year graduate students and including an extensive math review appendix. Thus the book would be quite suitable for a first-year graduate course in macroeconomics or dynamic methods that has no econometric content. Such a course might use Chapters 1 and 2, Sections 3.1 through 3.5, and Sections 4.1 and 4.2.

Yet another intended use for the book would be in a conventional econometrics course without an explicit time series focus. The popular econometrics texts do not have much discussion of such topics as numerical methods; asymptotic results for serially dependent, heterogeneously distributed observations; estimation of models with distributed lags; autocorrelation- and heteroskedasticity-consistent

standard errors; Bayesian analysis; or generalized method of moments. All of these topics receive extensive treatment in *Time Series Analysis*. Thus, an econometrics course without an explicit focus on time series might make use of Sections 3.1 through 3.5, Chapters 7 through 9, and Chapter 14, and perhaps any of Chapters 5, 11, and 12 as well. Again, the text is self-contained, with a fairly complete discussion of conventional simultaneous equations methods in Chapter 9. Indeed, a very important goal of the text is to develop the parallels between (1) the traditional econometric approach to simultaneous equations and (2) the current popularity of vector autoregressions and generalized method of moments estimation.

Finally, the book attempts to provide a rigorous motivation for the methods and yet still be accessible for researchers with purely applied interests. This is achieved by relegation of many details to mathematical appendixes at the ends of chapters, and by inclusion of numerous examples that illustrate exactly how the theoretical results are used and applied in practice.

The book developed out of my lectures at the University of Virginia. I am grateful first and foremost to my many students over the years whose questions and comments have shaped the course of the manuscript. I also have an enormous debt to numerous colleagues who have kindly offered many useful suggestions, and would like to thank in particular Donald W. K. Andrews, Jushan Bai, Peter Bearse, Stephen R. Blough, John Cochrane, George Davis, Michael Dotsey, John Elder, Robert Engle, T. Wake Epps, Marjorie Flavin, John Geweke, Eric Ghysels, Carlo Giannini, Clive W. J. Granger, Alastair Hall, Bruce E. Hansen, Kevin Hassett, Tomoo Inoue, Ravi Jagannathan, Kenneth F. Kroner, Jaime Marquez, Rocco Mosconi, Edward Nelson, Masao Ogaki, Adrian Pagan, Peter C. B. Phillips, Peter Rappoport, Glenn Rudebusch, Raul Susmel, Mark Watson, Kenneth D. West, Halbert White, and Jeffrey M. Wooldridge. I would also like to thank Pok-sang Lam and John Rogers for graciously sharing their data. Thanks also go to Keith Sill and Christopher Stomberg for assistance with the figures, to Rita Chen for assistance with the statistical tables in Appendix B, and to Richard Mickey for a superb job of copy editing.

James D. Hamilton

Time Series Analysis

Difference Equations

1.1. First-Order Difference Equations

This book is concerned with the dynamic consequences of events over time. Let's say we are studying a variable whose value at date t is denoted y_t . Suppose we are given a dynamic equation relating the value y takes on at date t to another variable w_t and to the value y took on in the previous period:

$$y_t = \phi y_{t-1} + w_t. \quad [1.1.1]$$

Equation [1.1.1] is a *linear first-order difference equation*. A *difference equation* is an expression relating a variable y_t to its previous values. This is a *first-order* difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. Note that it expresses y_t as a linear function of y_{t-1} and w_t .

An example of [1.1.1] is Goldfeld's (1973) estimated money demand function for the United States. Goldfeld's model related the log of the real money holdings of the public (m_t) to the log of aggregate real income (I_t), the log of the interest rate on bank accounts (r_{bt}), and the log of the interest rate on commercial paper (r_{ct}):

$$m_t = 0.27 + 0.72m_{t-1} + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}. \quad [1.1.2]$$

This is a special case of [1.1.1] with $y_t = m_t$, $\phi = 0.72$, and

$$w_t = 0.27 + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}.$$

For purposes of analyzing the dynamics of such a system, it simplifies the algebra a little to summarize the effects of all the input variables (I_t , r_{bt} , and r_{ct}) in terms of a scalar w_t as here.

In Chapter 3 the input variable w_t will be regarded as a random variable, and the implications of [1.1.1] for the statistical properties of the output series y_t will be explored. In preparation for this discussion, it is necessary first to understand the mechanics of difference equations. For the discussion in Chapters 1 and 2, the values for the input variable $\{w_1, w_2, \dots\}$ will simply be regarded as a sequence of deterministic numbers. Our goal is to answer the following question: If a dynamic system is described by [1.1.1], what are the effects on y of changes in the value of w ?

Solving a Difference Equation by Recursive Substitution

The presumption is that the dynamic equation [1.1.1] governs the behavior of y for all dates t . Thus, for each date we have an equation relating the value of

y for that date to its previous value and the current value of w :

<i>Date</i>	<i>Equation</i>	
0	$y_0 = \phi y_{-1} + w_0$	[1.1.3]
1	$y_1 = \phi y_0 + w_1$	[1.1.4]
2	$y_2 = \phi y_1 + w_2$	[1.1.5]
\vdots	\vdots	
t	$y_t = \phi y_{t-1} + w_t$	[1.1.6]

If we know the starting value of y for date $t = -1$ and the value of w for dates $t = 0, 1, 2, \dots$, then it is possible to simulate this dynamic system to find the value of y for any date. For example, if we know the value of y for $t = -1$ and the value of w for $t = 0$, we can calculate the value of y for $t = 0$ directly from [1.1.3]. Given this value of y_0 and the value of w for $t = 1$, we can calculate the value of y for $t = 1$ from [1.1.4]:

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1,$$

or

$$y_1 = \phi^2 y_{-1} + \phi w_0 + w_1.$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$ from [1.1.5]:

$$y_2 = \phi y_1 + w_2 = \phi(\phi^2 y_{-1} + \phi w_0 + w_1) + w_2,$$

or

$$y_2 = \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2.$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value y_{-1} and the history of w between date 0 and date t :

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \cdots + \phi w_{t-1} + w_t. \quad [1.1.7]$$

This procedure is known as solving the difference equation [1.1.1] by *recursive substitution*.

Dynamic Multipliers

Note that [1.1.7] expresses y_t as a linear function of the initial value y_{-1} and the historical values of w . This makes it very easy to calculate the effect of w_0 on y_t . If w_0 were to change with y_{-1} and w_1, w_2, \dots, w_t taken as unaffected, the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t. \quad [1.1.8]$$

Note that the calculations would be exactly the same if the dynamic simulation were started at date t (taking y_{t-1} as given); then y_{t+j} could be described as a

function of y_{t-1} and $w_t, w_{t+1}, \dots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1}y_{t-1} + \phi^jw_t + \phi^{j-1}w_{t+1} + \phi^{j-2}w_{t+2} + \dots + \phi w_{t+j-1} + w_{t+j}. \quad [1.1.9]$$

The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j. \quad [1.1.10]$$

Thus the *dynamic multiplier* [1.1.10] depends only on j , the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}). The multiplier does not depend on t ; that is, it does not depend on the dates of the observations themselves. This is true of any linear difference equation.

As an example of calculating a dynamic multiplier, consider again Goldfeld's money demand specification [1.1.2]. Suppose we want to know what will happen to money demand two quarters from now if current income I_t were to increase by one unit today with future income I_{t+1} and I_{t+2} unaffected:

$$\frac{\partial m_{t+2}}{\partial I_t} = \frac{\partial m_{t+2}}{\partial w_t} \times \frac{\partial w_t}{\partial I_t} = \phi^2 \times \frac{\partial w_t}{\partial I_t}.$$

From [1.1.2], a one-unit increase in I_t will increase w_t by 0.19 units, meaning that $\partial w_t / \partial I_t = 0.19$. Since $\phi = 0.72$, we calculate

$$\frac{\partial m_{t+2}}{\partial I_t} = (0.72)^2(0.19) = 0.098.$$

Because I_t is the log of income, an increase in I_t of 0.01 units corresponds to a 1% increase in income. An increase in m_t of $(0.01) \cdot (0.098) \approx 0.001$ corresponds to a 0.1% increase in money holdings. Thus the public would be expected to increase its money holdings by a little less than 0.1% two quarters following a 1% increase in income.

Different values of ϕ in [1.1.1] can produce a variety of dynamic responses of y to w . If $0 < \phi < 1$, the multiplier $\partial y_{t+j} / \partial w_t$ in [1.1.10] decays geometrically toward zero. Panel (a) of Figure 1.1 plots ϕ^j as a function of j for $\phi = 0.8$. If $-1 < \phi < 0$, the multiplier $\partial y_{t+j} / \partial w_t$ will alternate in sign as in panel (b). In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. Again the absolute value of the effect decays geometrically toward zero. If $\phi > 1$, the dynamic multiplier increases exponentially over time as in panel (c). A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system [1.1.1] exhibits explosive oscillation as in panel (d).

Thus, if $|\phi| < 1$, the system is stable; the consequences of a given change in w_t will eventually die out. If $|\phi| > 1$, the system is explosive. An interesting possibility is the borderline case, $\phi = 1$. In this case, the solution [1.1.9] becomes

$$y_{t+j} = y_{t-1} + w_t + w_{t+1} + w_{t+2} + \dots + w_{t+j-1} + w_{t+j}. \quad [1.1.11]$$

Here the output variable y is the sum of the historical inputs w . A one-unit increase in w will cause a permanent one-unit increase in y :

$$\frac{\partial y_{t+j}}{\partial w_t} = 1 \quad \text{for } j = 0, 1, \dots.$$

We might also be interested in the effect of w on the present value of the stream of future realizations of y . For a given stream of future values $y_t, y_{t+1},$

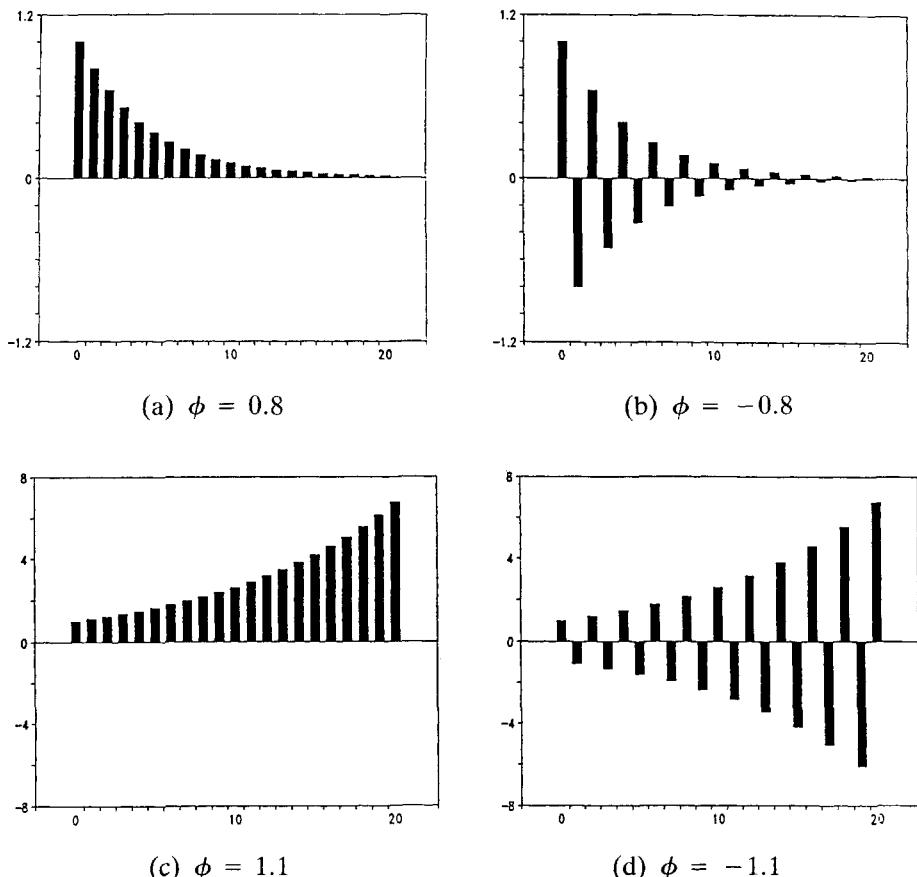


FIGURE 1.1 Dynamic multiplier for first-order difference equation for different values of ϕ (plot of $\partial y_{t+j} / \partial w_t = \phi^j$ as a function of the lag j).

y_{t+2}, \dots and a constant interest rate¹ $r > 0$, the *present value* of the stream at time t is given by

$$y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \dots \quad [1.1.12]$$

Let β denote the discount factor:

$$\beta \equiv 1/(1+r).$$

Note that $0 < \beta < 1$. Then the present value [1.1.12] can be written as

$$\sum_{j=0}^{\infty} \beta^j y_{t+j}. \quad [1.1.13]$$

Consider what would happen if there were a one-unit increase in w_t with w_{t+1}, w_{t+2}, \dots unaffected. The consequences of this change for the present value of y are found by differentiating [1.1.13] with respect to w_t and then using [1.1.10]

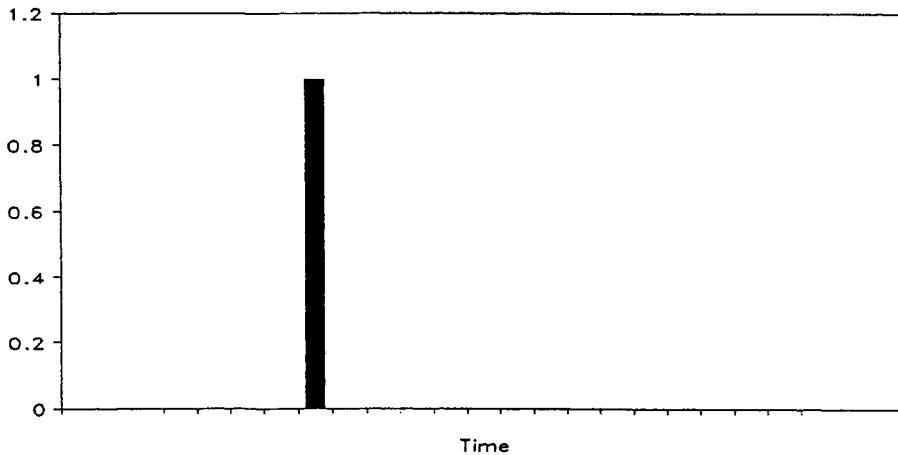
¹The interest rate is measured here as a fraction of 1; thus $r = 0.1$ corresponds to a 10% interest rate.

to evaluate each derivative:

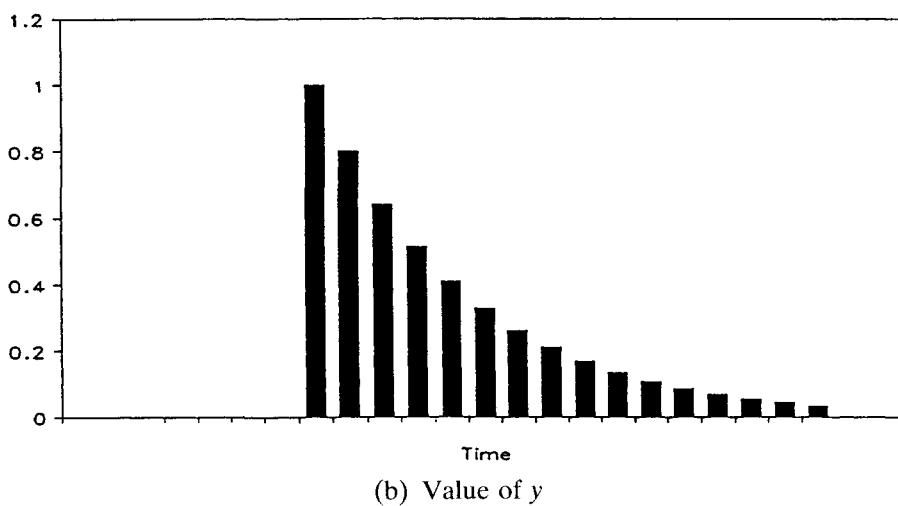
$$\sum_{j=0}^{\infty} \beta^j \frac{\partial y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \beta^j \phi^j = 1/(1 - \beta\phi), \quad [1.1.14]$$

provided that $|\beta\phi| < 1$.

In calculating the dynamic multipliers [1.1.10] or [1.1.14], we were asking what would happen if w_t were to increase by one unit with $w_{t+1}, w_{t+2}, \dots, w_{t+j}$ unaffected. We were thus finding the effect of a purely transitory change in w . Panel (a) of Figure 1.2 shows the time path of w associated with this question, and panel (b) shows the implied path for y . Because the dynamic multiplier [1.1.10] calculates the response of y to a single impulse in w , it is also referred to as the *impulse-response function*.



(a) Value of w



(b) Value of y

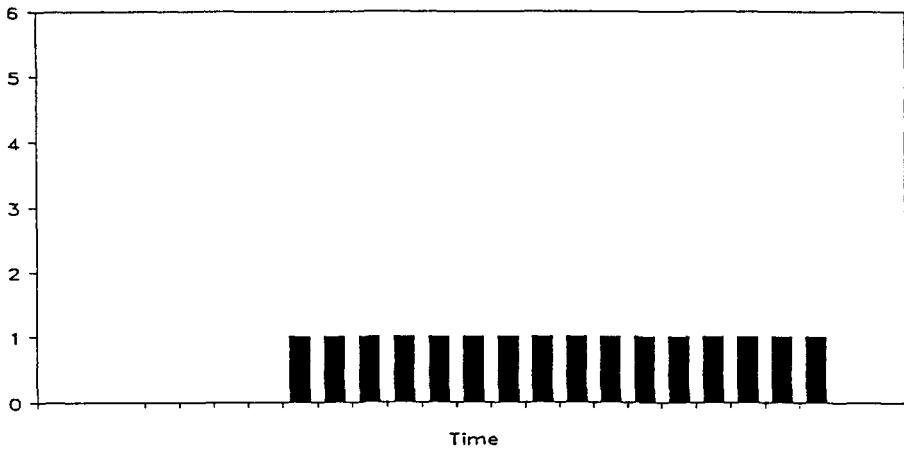
FIGURE 1.2 Paths of input variable (w_t) and output variable (y_t) assumed for dynamic multiplier and present-value calculations.

Sometimes we might instead be interested in the consequences of a permanent change in w . A permanent change in w means that w_t, w_{t+1}, \dots , and w_{t+j} would all increase by one unit, as in Figure 1.3. From formula [1.1.10], the effect on y_{t+j} of a permanent change in w beginning in period t is given by

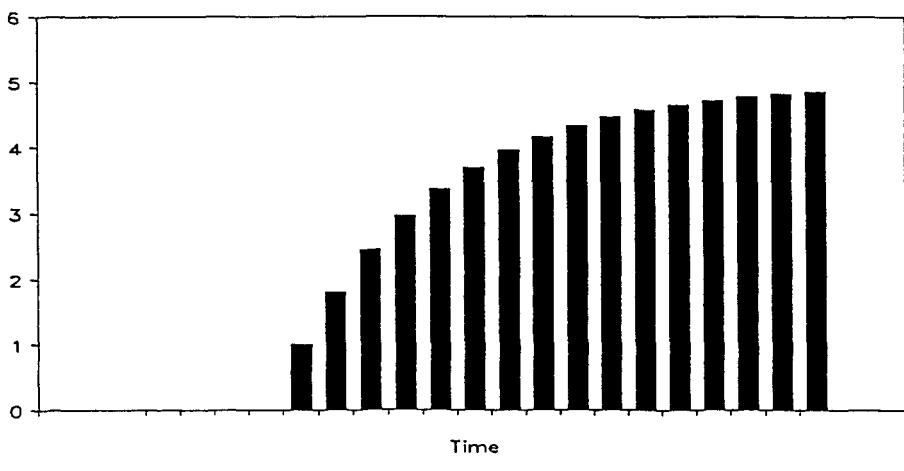
$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + \phi^{j-2} + \cdots + \phi + 1.$$

When $|\phi| < 1$, the limit of this expression as j goes to infinity is sometimes described as the “long-run” effect of w on y :

$$\lim_{j \rightarrow \infty} \left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} \right] = 1 + \phi + \phi^2 + \cdots = 1/(1 - \phi). \quad [1.1.15]$$



(a) Value of w



(b) Value of y

FIGURE 1.3 Paths of input variable (w_t) and output variable (y_t) assumed for long-run effect calculations.

For example, the long-run income elasticity of money demand in the system [1.1.2] is given by

$$\frac{0.19}{1 - 0.72} = 0.68.$$

A permanent 1% increase in income will eventually lead to a 0.68% increase in money demand.

Another related question concerns the cumulative consequences for y of a one-time change in w . Here we consider a transitory disturbance to w as in panel (a) of Figure 1.2, but wish to calculate the sum of the consequences for all future values of y . Another way to think of this is as the effect on the present value of y [1.1.13] with the discount rate $\beta = 1$. Setting $\beta = 1$ in [1.1.14] shows this cumulative effect to be equal to

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = 1/(1 - \phi), \quad [1.1.16]$$

provided that $|\phi| < 1$. Note that the cumulative effect on y of a transitory change in w (expression [1.1.16]) is the same as the long-run effect on y of a permanent change in w (expression [1.1.15]).

1.2. *p*th-Order Difference Equations

Let us now generalize the dynamic system [1.1.1] by allowing the value of y at date t to depend on p of its own lags along with the current value of the input variable w_t :

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t. \quad [1.2.1]$$

Equation [1.2.1] is a linear p th-order difference equation.

It is often convenient to rewrite the p th-order difference equation [1.2.1] in the scalar y_t as a first-order difference equation in a vector ξ_t . Define the $(p \times 1)$ vector ξ_t by

$$\xi_t = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}. \quad [1.2.2]$$

That is, the first element of the vector ξ at date t is the value y took on at date t . The second element of ξ_t is the value y took on at date $t - 1$, and so on. Define the $(p \times p)$ matrix \mathbf{F} by

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad [1.2.3]$$

For example, for $p = 4$, \mathbf{F} refers to the following 4×4 matrix:

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $p = 1$ (the first-order difference equation [1.1.1]), \mathbf{F} is just the scalar ϕ . Finally, define the $(p \times 1)$ vector \mathbf{v}_t by

$$\mathbf{v}_t = \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad [1.2.4]$$

Consider the following first-order vector difference equation:

$$\xi_t = \mathbf{F}\xi_{t-1} + \mathbf{v}_t, \quad [1.2.5]$$

or

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is a system of p equations. The first equation in this system is identical to equation [1.2.1]. The second equation is simply the identity

$$y_{t-1} = y_{t-1},$$

owing to the fact that the second element of ξ_t is the same as the first element of ξ_{t-1} . The third equation in [1.2.5] states that $y_{t-2} = y_{t-2}$; the p th equation states that $y_{t-p+1} = y_{t-p+1}$.

Thus, the first-order vector system [1.2.5] is simply an alternative representation of the p th-order scalar system [1.2.1]. The advantage of rewriting the p th-order system [1.2.1] in the form of a first-order system [1.2.5] is that first-order systems are often easier to work with than p th-order systems.

A dynamic multiplier for [1.2.5] can be found in exactly the same way as was done for the first-order scalar system of Section 1.1. If we knew the value of the vector ξ for date $t = -1$ and of \mathbf{v} for date $t = 0$, we could find the value of ξ for date 0 from

$$\xi_0 = \mathbf{F}\xi_{-1} + \mathbf{v}_0.$$

The value of ξ for date 1 is

$$\xi_1 = \mathbf{F}\xi_0 + \mathbf{v}_1 = \mathbf{F}(\mathbf{F}\xi_{-1} + \mathbf{v}_0) + \mathbf{v}_1 = \mathbf{F}^2\xi_{-1} + \mathbf{F}\mathbf{v}_0 + \mathbf{v}_1.$$

Proceeding recursively in this fashion produces a generalization of [1.1.7]:

$$\xi_t = \mathbf{F}^{t+1}\xi_{-1} + \mathbf{F}^t\mathbf{v}_0 + \mathbf{F}^{t-1}\mathbf{v}_1 + \mathbf{F}^{t-2}\mathbf{v}_2 + \dots + \mathbf{F}\mathbf{v}_{t-1} + \mathbf{v}_t. \quad [1.2.6]$$

Writing this out in terms of the definitions of ξ and \mathbf{v} ,

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \mathbf{F}^{t+1} \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \\ y_{-p} \end{bmatrix} + \mathbf{F}^t \begin{bmatrix} w_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \mathbf{F}^{t-1} \begin{bmatrix} w_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \mathbf{F}^1 \begin{bmatrix} w_{t-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad [1.2.7]$$

Consider the first equation of this system, which characterizes the value of y_t . Let $f_{11}^{(t)}$ denote the $(1, 1)$ element of \mathbf{F}^t , $f_{12}^{(t)}$ the $(1, 2)$ element of \mathbf{F}^t , and so on. Then the first equation of [1.2.7] states that

$$y_t = f_{11}^{(t+1)}y_{-1} + f_{12}^{(t+1)}y_{-2} + \cdots + f_{1p}^{(t+1)}y_{-p} + f_{11}^{(t)}w_0 + f_{11}^{(t-1)}w_1 + \cdots + f_{11}^{(1)}w_{t-1} + w_t. \quad [1.2.8]$$

This describes the value of y at date t as a linear function of p initial values of y ($y_{-1}, y_{-2}, \dots, y_{-p}$) and the history of the input variable w since time 0 (w_0, w_1, \dots, w_t). Note that whereas only one initial value for y (the value y_{-1}) was needed in the case of a first-order difference equation, p initial values for y (the values $y_{-1}, y_{-2}, \dots, y_{-p}$) are needed in the case of a p th-order difference equation.

The obvious generalization of [1.1.9] is

$$\begin{aligned} \xi_{t+j} &= \mathbf{F}^{j+1}\xi_{t-1} + \mathbf{F}^j\mathbf{v}_t + \mathbf{F}^{j-1}\mathbf{v}_{t+1} + \mathbf{F}^{j-2}\mathbf{v}_{t+2} + \cdots \\ &\quad + \mathbf{F}\mathbf{v}_{t+j-1} + \mathbf{v}_{t+j} \end{aligned} \quad [1.2.9]$$

from which

$$\begin{aligned} y_{t+j} &= f_{11}^{(j+1)}y_{t-1} + f_{12}^{(j+1)}y_{t-2} + \cdots + f_{1p}^{(j+1)}y_{t-p} + f_{11}^{(j)}w_t \\ &\quad + f_{11}^{(j-1)}w_{t+1} + f_{11}^{(j-2)}w_{t+2} + \cdots + f_{11}^{(1)}w_{t+j-1} + w_{t+j}. \end{aligned} \quad [1.2.10]$$

Thus, for a p th-order difference equation, the dynamic multiplier is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)} \quad [1.2.11]$$

where $f_{11}^{(j)}$ denotes the $(1, 1)$ element of \mathbf{F}^j . For $j = 1$, this is simply the $(1, 1)$ element of \mathbf{F} , or the parameter ϕ_1 . Thus, for any p th-order system, the effect on y_{t+1} of a one-unit increase in w_t is given by the coefficient relating y_t to y_{t-1} in equation [1.2.1]:

$$\frac{\partial y_{t+1}}{\partial w_t} = \phi_1.$$

Direct multiplication of [1.2.3] reveals that the (1, 1) element of \mathbf{F}^2 is $(\phi_1^2 + \phi_2)$, so

$$\frac{\partial y_{t+2}}{\partial w_t} = \phi_1^2 + \phi_2$$

in a p th-order system.

For larger values of j , an easy way to obtain a numerical value for the dynamic multiplier $\partial y_{t+j}/\partial w_t$ is to simulate the system. This is done as follows. Set $y_{-1} = y_{-2} = \dots = y_{-p} = 0$, $w_0 = 1$, and set the value of w for all other dates to 0. Then use [1.2.1] to calculate the value of y_t for $t = 0$ (namely, $y_0 = 1$). Next substitute this value along with $y_{t-1}, y_{t-2}, \dots, y_{t-p+1}$ back into [1.2.1] to calculate y_{t+1} , and continue recursively in this fashion. The value of y at step t gives the effect of a one-unit change in w_0 on y_t .

Although numerical simulation may be adequate for many circumstances, it is also useful to have a simple analytical characterization of $\partial y_{t+j}/\partial w_t$, which, we know from [1.2.11], is given by the (1, 1) element of \mathbf{F}^j . This is fairly easy to obtain in terms of the eigenvalues of the matrix \mathbf{F} . Recall that the eigenvalues of a matrix \mathbf{F} are those numbers λ for which

$$|\mathbf{F} - \lambda \mathbf{I}_p| = 0. \quad [1.2.12]$$

For example, for $p = 2$ the eigenvalues are the solutions to

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} (\phi_1 - \lambda) & \phi_2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \phi_1\lambda - \phi_2 = 0. \quad [1.2.13]$$

The two eigenvalues of \mathbf{F} for a second-order difference equation are thus given by

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad [1.2.14]$$

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}. \quad [1.2.15]$$

For a general p th-order system, the determinant in [1.2.12] is a p th-order polynomial in λ whose p solutions characterize the p eigenvalues of \mathbf{F} . This polynomial turns out to take a very similar form to [1.2.13]. The following result is proved in Appendix 1.A at the end of this chapter.

Proposition 1.1: *The eigenvalues of the matrix \mathbf{F} defined in equation [1.2.3] are the values of λ that satisfy*

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \dots - \phi_{p-1}\lambda - \phi_p = 0. \quad [1.2.16]$$

Once we know the eigenvalues, it is straightforward to characterize the dynamic behavior of the system. First we consider the case when the eigenvalues of \mathbf{F} are distinct; for example, we require that λ_1 and λ_2 in [1.2.14] and [1.2.15] be different numbers.

General Solution of a p th-Order Difference Equation with Distinct Eigenvalues

Recall² that if the eigenvalues of a $(p \times p)$ matrix \mathbf{F} are distinct, there exists a nonsingular $(p \times p)$ matrix \mathbf{T} such that

$$\mathbf{F} = \mathbf{T}\Lambda\mathbf{T}^{-1} \quad [1.2.17]$$

where Λ is a $(p \times p)$ matrix with the eigenvalues of \mathbf{F} along the principal diagonal and zeros elsewhere:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p \end{bmatrix}. \quad [1.2.18]$$

This enables us to characterize the dynamic multiplier (the $(1, 1)$ element of \mathbf{F}^j in [1.2.11]) very easily. For example, from [1.2.17] we can write \mathbf{F}^2 as

$$\begin{aligned} \mathbf{F}^2 &= \mathbf{T}\Lambda\mathbf{T}^{-1} \times \mathbf{T}\Lambda\mathbf{T}^{-1} \\ &= \mathbf{T} \times \Lambda \times (\mathbf{T}^{-1}\mathbf{T}) \times \Lambda \times \mathbf{T}^{-1} \\ &= \mathbf{T} \times \Lambda \times \mathbf{I}_p \times \Lambda \times \mathbf{T}^{-1} \\ &= \mathbf{T}\Lambda^2\mathbf{T}^{-1}. \end{aligned}$$

The diagonal structure of Λ implies that Λ^2 is also a diagonal matrix whose elements are the squares of the eigenvalues of \mathbf{F} :

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^2 \end{bmatrix}.$$

More generally, we can characterize \mathbf{F}^j in terms of the eigenvalues of \mathbf{F} as

$$\begin{aligned} \mathbf{F}^j &= \underbrace{\mathbf{T}\Lambda\mathbf{T}^{-1} \times \mathbf{T}\Lambda\mathbf{T}^{-1} \times \cdots \times \mathbf{T}\Lambda\mathbf{T}^{-1}}_{j \text{ terms}} \\ &= \mathbf{T} \times \Lambda \times (\mathbf{T}^{-1}\mathbf{T}) \times \Lambda \times (\mathbf{T}^{-1}\mathbf{T}) \times \cdots \times \Lambda \times \mathbf{T}^{-1}, \end{aligned}$$

which simplifies to

$$\mathbf{F}^j = \mathbf{T}\Lambda^j\mathbf{T}^{-1} \quad [1.2.19]$$

where

$$\Lambda^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix}.$$

²See equation [A.4.24] in the Mathematical Review (Appendix A) at the end of the book.

Let t_{ij} denote the row i , column j element of \mathbf{T} and let t^{ij} denote the row i , column j element of \mathbf{T}^{-1} . Equation [1.2.19] written out explicitly becomes

$$\begin{aligned}\mathbf{F}^j &= \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix} \\ &= \begin{bmatrix} t_{11}\lambda_1^j & t_{12}\lambda_2^j & \cdots & t_{1p}\lambda_p^j \\ t_{21}\lambda_1^j & t_{22}\lambda_2^j & \cdots & t_{2p}\lambda_p^j \\ \vdots & \vdots & \cdots & \vdots \\ t_{p1}\lambda_1^j & t_{p2}\lambda_2^j & \cdots & t_{pp}\lambda_p^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \cdots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{bmatrix}\end{aligned}$$

from which the (1, 1) element of \mathbf{F}^j is given by

$$f_{11}^{(j)} = [t_{11}t^{11}]\lambda_1^j + [t_{12}t^{21}]\lambda_2^j + \cdots + [t_{1p}t^{p1}]\lambda_p^j$$

or

$$f_{11}^{(j)} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j \quad [1.2.20]$$

where

$$c_i = [t_{1i}t^{i1}]. \quad [1.2.21]$$

Note that the sum of the c_i terms has the following interpretation:

$$c_1 + c_2 + \cdots + c_p = [t_{11}t^{11}] + [t_{12}t^{21}] + \cdots + [t_{1p}t^{p1}], \quad [1.2.22]$$

which is the (1, 1) element of $\mathbf{T} \cdot \mathbf{T}^{-1}$. Since $\mathbf{T} \cdot \mathbf{T}^{-1}$ is just the $(p \times p)$ identity matrix, [1.2.22] implies that the c_i terms sum to unity:

$$c_1 + c_2 + \cdots + c_p = 1. \quad [1.2.23]$$

Substituting [1.2.20] into [1.2.11] gives the form of the dynamic multiplier for a p th-order difference equation:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j. \quad [1.2.24]$$

Equation [1.2.24] characterizes the dynamic multiplier as a weighted average of each of the p eigenvalues raised to the j th power.

The following result provides a closed-form expression for the constants (c_1, c_2, \dots, c_p) .

Proposition 1.2: *If the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of the matrix \mathbf{F} in [1.2.3] are distinct, then the magnitude c_i in [1.2.21] can be written*

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{\substack{k=1 \\ k \neq i}}^p (\lambda_i - \lambda_k)}. \quad [1.2.25]$$

To summarize, the p th-order difference equation [1.2.1] implies that

$$\begin{aligned}y_{t+j} &= f_{11}^{(j+1)}y_{t-1} + f_{12}^{(j+1)}y_{t-2} + \cdots + f_{1p}^{(j+1)}y_{t-p} \\ &\quad + w_{t+j} + \psi_1 w_{t+j-1} + \psi_2 w_{t+j-2} + \cdots + \psi_{j-1} w_{t+1} + \psi_j w_t.\end{aligned} \quad [1.2.26]$$

The dynamic multiplier

$$\frac{\partial y_{t+j}}{\partial w_t} = \psi_j \quad [1.2.27]$$

is given by the $(1, 1)$ element of \mathbf{F}^j :

$$\psi_j = f_{11}^{(j)}. \quad [1.2.28]$$

A closed-form expression for ψ_j can be obtained by finding the eigenvalues of \mathbf{F} , or the values of λ satisfying [1.2.16]. Denoting these p values by $(\lambda_1, \lambda_2, \dots, \lambda_p)$ and assuming them to be distinct, the dynamic multiplier is given by

$$\psi_j = c_1 \lambda_1^j + c_2 \lambda_2^j + \dots + c_p \lambda_p^j \quad [1.2.29]$$

where (c_1, c_2, \dots, c_p) is a set of constants summing to unity given by expression [1.2.25].

For a first-order system ($p = 1$), this rule would have us solve [1.2.16],

$$\lambda - \phi_1 = 0,$$

which has the single solution

$$\lambda_1 = \phi_1. \quad [1.2.30]$$

According to [1.2.29], the dynamic multiplier is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1 \lambda_1^j. \quad [1.2.31]$$

From [1.2.23], $c_1 = 1$. Substituting this and [1.2.30] into [1.2.31] gives

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi_1^j,$$

or the same result found in Section 1.1.

For higher-order systems, [1.2.29] allows a variety of more complicated dynamics. Suppose first that all the eigenvalues of \mathbf{F} (or solutions to [1.2.16]) are real. This would be the case, for example, if $p = 2$ and $\phi_1^2 + 4\phi_2 > 0$ in the solutions [1.2.14] and [1.2.15] for the second-order system. If, furthermore, all of the eigenvalues are less than 1 in absolute value, then the system is stable, and its dynamics are represented as a weighted average of decaying exponentials or decaying exponentials oscillating in sign. For example, consider the following second-order difference equation:

$$y_t = 0.6y_{t-1} + 0.2y_{t-2} + w_t.$$

From equations [1.2.14] and [1.2.15], the eigenvalues of this system are given by

$$\lambda_1 = \frac{0.6 + \sqrt{(0.6)^2 + 4(0.2)}}{2} = 0.84$$

$$\lambda_2 = \frac{0.6 - \sqrt{(0.6)^2 + 4(0.2)}}{2} = -0.24.$$

From [1.2.25], we have

$$c_1 = \lambda_1 / (\lambda_1 - \lambda_2) = 0.778$$

$$c_2 = \lambda_2 / (\lambda_2 - \lambda_1) = 0.222.$$

The dynamic multiplier for this system,

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1 \lambda_1^j + c_2 \lambda_2^j,$$

is plotted as a function of j in panel (a) of Figure 1.4.³ Note that as j becomes larger, the pattern is dominated by the larger eigenvalue (λ_1), approximating a simple geometric decay at rate λ_1 .

If the eigenvalues (the solutions to [1.2.16]) are real but at least one is greater than unity in absolute value, the system is explosive. If λ_1 denotes the eigenvalue that is largest in absolute value, the dynamic multiplier is eventually dominated by an exponential function of that eigenvalue:

$$\lim_{j \rightarrow \infty} \frac{\partial y_{t+j}}{\partial w_t} \cdot \frac{1}{\lambda_1^j} = c_1.$$

Other interesting possibilities arise if some of the eigenvalues are complex. Whenever this is the case, they appear as complex conjugates. For example, if $p = 2$ and $\phi_1^2 + 4\phi_2 < 0$, then the solutions λ_1 and λ_2 in [1.2.14] and [1.2.15] are complex conjugates. Suppose that λ_1 and λ_2 are complex conjugates, written as

$$\lambda_1 = a + bi \quad [1.2.32]$$

$$\lambda_2 = a - bi. \quad [1.2.33]$$

For the $p = 2$ case of [1.2.14] and [1.2.15], we would have

$$a = \phi_1/2 \quad [1.2.34]$$

$$b = (1/2)\sqrt{-\phi_1^2 - 4\phi_2}. \quad [1.2.35]$$

Our goal is to characterize the contribution to the dynamic multiplier $c_1\lambda_1^j$ when λ_1 is a complex number as in [1.2.32]. Recall that to raise a complex number to a power, we rewrite [1.2.32] in polar coordinate form:

$$\lambda_1 = R[\cos(\theta) + i\sin(\theta)], \quad [1.2.36]$$

where θ and R are defined in terms of a and b by the following equations:

$$R = \sqrt{a^2 + b^2}$$

$$\cos(\theta) = a/R$$

$$\sin(\theta) = b/R.$$

Note that R is equal to the modulus of the complex number λ_1 .

The eigenvalue λ_1 in [1.2.36] can be written as⁴

$$\lambda_1 = R[e^{i\theta}],$$

and so

$$\lambda_1^j = R^j[e^{i\theta j}] = R^j[\cos(\theta j) + i\sin(\theta j)]. \quad [1.2.37]$$

Analogously, if λ_2 is the complex conjugate of λ_1 , then

$$\lambda_2 = R[\cos(\theta) - i\sin(\theta)],$$

which can be written⁵

$$\lambda_2 = R[e^{-i\theta}].$$

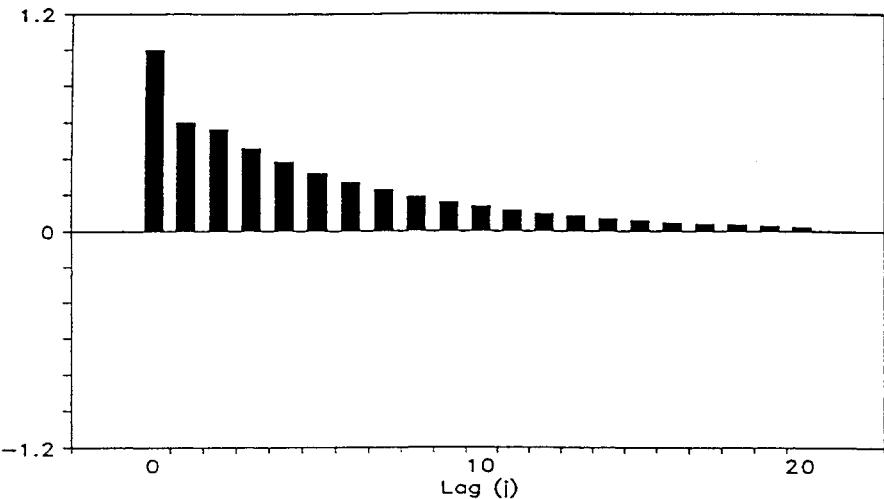
Thus

$$\lambda_2^j = R^j[e^{-i\theta j}] = R^j[\cos(\theta j) - i\sin(\theta j)]. \quad [1.2.38]$$

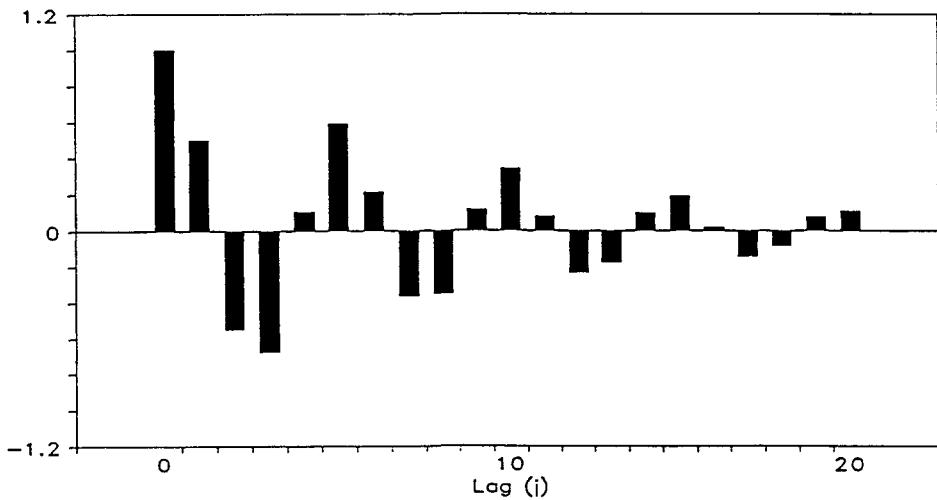
³Again, if one's purpose is solely to generate a numerical plot as in Figure 1.4, the easiest approach is numerical simulation of the system.

⁴See equation [A.3.25] in the Mathematical Review (Appendix A) at the end of the book.

⁵See equation [A.3.26].



(a) $\phi_1 = 0.6, \phi_2 = 0.2$



(b) $\phi_1 = 0.5, \phi_2 = -0.8$

FIGURE 1.4 Dynamic multiplier for second-order difference equation for different values of ϕ_1 and ϕ_2 (plot of $\partial y_{t+j}/\partial w_t$ as a function of the lag j).

Substituting [1.2.37] and [1.2.38] into [1.2.29] gives the contribution of the complex conjugates to the dynamic multiplier $\partial y_{t+j}/\partial w_t$:

$$\begin{aligned} c_1 \lambda_1^j + c_2 \lambda_2^j &= c_1 R^j [\cos(\theta j) + i \cdot \sin(\theta j)] + c_2 R^j [\cos(\theta j) - i \cdot \sin(\theta j)] \\ &= [c_1 + c_2] \cdot R^j \cdot \cos(\theta j) + i \cdot [c_1 - c_2] \cdot R^j \cdot \sin(\theta j). \end{aligned} \quad [1.2.39]$$

The appearance of the imaginary number i in [1.2.39] may seem a little troubling. After all, this calculation was intended to give the effect of a change in the real-valued variable w_t on the real-valued variable y_{t+j} as predicted by the real-valued system [1.2.1], and it would be odd indeed if the correct answer involved the imaginary number i ! Fortunately, it turns out from [1.2.25] that if λ_1 and λ_2 are complex conjugates, then c_1 and c_2 are complex conjugates; that is, they can

be written as

$$\begin{aligned}c_1 &= \alpha + \beta i \\c_2 &= \alpha - \beta i\end{aligned}$$

for some real numbers α and β . Substituting these expressions into [1.2.39] yields

$$\begin{aligned}c_1 \lambda_1^j + c_2 \lambda_2^j &= [(\alpha + \beta i) + (\alpha - \beta i)] \cdot R^j \cos(\theta j) + i \cdot [(\alpha + \beta i) - (\alpha - \beta i)] \cdot R^j \sin(\theta j) \\&= [2\alpha] \cdot R^j \cos(\theta j) + i \cdot [2\beta i] \cdot R^j \sin(\theta j) \\&= 2\alpha R^j \cos(\theta j) - 2\beta R^j \sin(\theta j),\end{aligned}$$

which is strictly real.

Thus, when some of the eigenvalues are complex, they contribute terms proportional to $R^j \cos(\theta j)$ and $R^j \sin(\theta j)$ to the dynamic multiplier $\partial y_{t+j}/\partial w_t$. Note that if $R = 1$ —that is, if the complex eigenvalues have unit modulus—the multipliers are periodic sine and cosine functions of j . A given increase in w_t increases y_{t+j} for some ranges of j and decreases y_{t+j} over other ranges, with the impulse never dying out as $j \rightarrow \infty$. If the complex eigenvalues are less than 1 in modulus ($R < 1$), the impulse again follows a sinusoidal pattern though its amplitude decays at the rate R^j . If the complex eigenvalues are greater than 1 in modulus ($R > 1$), the amplitude of the sinusoids explodes at the rate R^j .

For an example of dynamic behavior characterized by decaying sinusoids, consider the second-order system

$$y_t = 0.5y_{t-1} - 0.8y_{t-2} + w_t.$$

The eigenvalues for this system are given from [1.2.14] and [1.2.15]:

$$\begin{aligned}\lambda_1 &= \frac{0.5 + \sqrt{(0.5)^2 - 4(0.8)}}{2} = 0.25 + 0.86i \\ \lambda_2 &= \frac{0.5 - \sqrt{(0.5)^2 - 4(0.8)}}{2} = 0.25 - 0.86i,\end{aligned}$$

with modulus

$$R = \sqrt{(0.25)^2 + (0.86)^2} = 0.9.$$

Since $R < 1$, the dynamic multiplier follows a pattern of damped oscillation plotted in panel (b) of Figure 1.4. The frequency⁶ of these oscillations is given by the parameter θ in [1.2.39], which was defined implicitly by

$$\cos(\theta) = a/R = (0.25)/(0.9) = 0.28$$

or

$$\theta = 1.29.$$

The cycles associated with the dynamic multiplier function [1.2.39] thus have a period of

$$\frac{2\pi}{\theta} = \frac{(2)(3.14159)}{1.29} = 4.9;$$

that is, the peaks in the pattern in panel (b) of Figure 1.4 appear about five periods apart.

⁶See Section A.1 of the Mathematical Review (Appendix A) at the end of the book for a discussion of the frequency and period of a sinusoidal function.

Solution of a Second-Order Difference Equation with Distinct Eigenvalues

The second-order difference equation ($p = 2$) comes up sufficiently often that it is useful to summarize the properties of the solution as a general function of ϕ_1 and ϕ_2 , which we now do.⁷

The eigenvalues λ_1 and λ_2 in [1.2.14] and [1.2.15] are complex whenever

$$\phi_1^2 + 4\phi_2 < 0,$$

or whenever (ϕ_1, ϕ_2) lies below the parabola indicated in Figure 1.5. For the case of complex eigenvalues, the modulus R satisfies

$$R^2 = a^2 + b^2,$$

or, from [1.2.34] and [1.2.35],

$$R^2 = (\phi_1/2)^2 - (\phi_1^2 + 4\phi_2)/4 = -\phi_2.$$

Thus, a system with complex eigenvalues is explosive whenever $\phi_2 < -1$. Also, when the eigenvalues are complex, the frequency of oscillations is given by

$$\theta = \cos^{-1}(a/R) = \cos^{-1}[\phi_1/(2\sqrt{-\phi_2})],$$

where “ $\cos^{-1}(x)$ ” denotes the inverse of the cosine function, or the radian measure of an angle whose cosine is x .

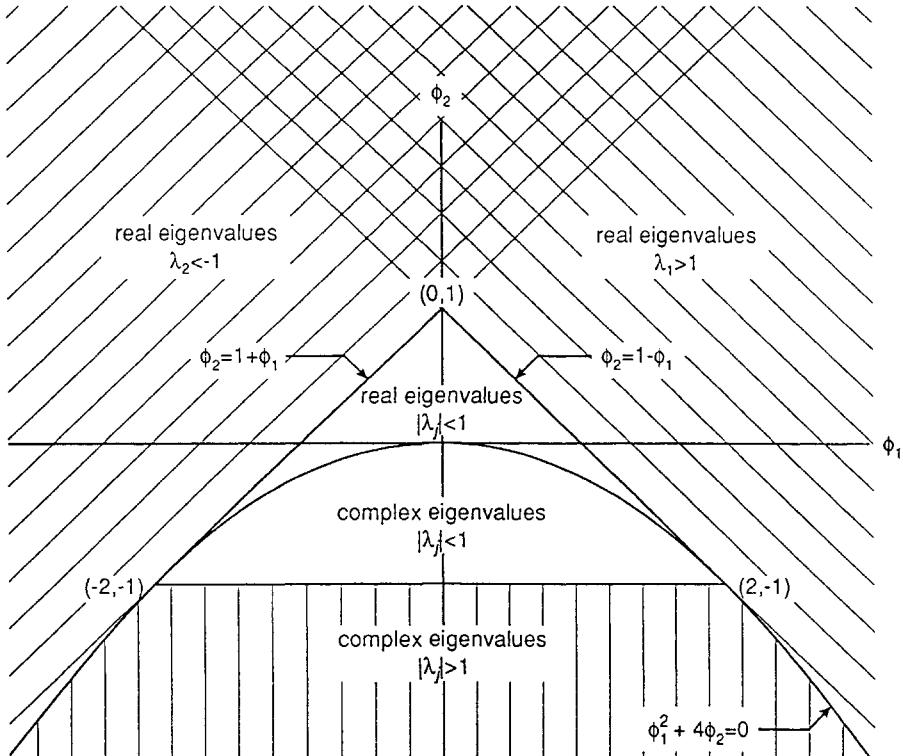


FIGURE 1.5 Summary of dynamics for a second-order difference equation.

⁷This discussion closely follows Sargent (1987, pp. 188–89).

For the case of real eigenvalues, the arithmetically larger eigenvalue (λ_1) will be greater than unity whenever

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} > 1$$

or

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 - \phi_1.$$

Assuming that λ_1 is real, the left side of this expression is a positive number and the inequality would be satisfied for any value of $\phi_1 > 2$. If, on the other hand, $\phi_1 < 2$, we can square both sides to conclude that λ_1 will exceed unity whenever

$$\phi_1^2 + 4\phi_2 > 4 - 4\phi_1 + \phi_1^2$$

or

$$\phi_2 > 1 - \phi_1.$$

Thus, in the real region, λ_1 will be greater than unity either if $\phi_1 > 2$ or if (ϕ_1, ϕ_2) lies northeast of the line $\phi_2 = 1 - \phi_1$ in Figure 1.5. Similarly, with real eigenvalues, the arithmetically smaller eigenvalue (λ_2) will be less than -1 whenever

$$\begin{aligned}\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} &< -1 \\ -\sqrt{\phi_1^2 + 4\phi_2} &< -2 - \phi_1 \\ \sqrt{\phi_1^2 + 4\phi_2} &> 2 + \phi_1.\end{aligned}$$

Again, if $\phi_1 < -2$, this must be satisfied, and in the case when $\phi_1 > -2$, we can square both sides:

$$\begin{aligned}\phi_1^2 + 4\phi_2 &> 4 + 4\phi_1 + \phi_1^2 \\ \phi_2 &> 1 + \phi_1.\end{aligned}$$

Thus, in the real region, λ_2 will be less than -1 if either $\phi_1 < -2$ or (ϕ_1, ϕ_2) lies to the northwest of the line $\phi_2 = 1 + \phi_1$ in Figure 1.5.

The system is thus stable whenever (ϕ_1, ϕ_2) lies within the triangular region of Figure 1.5.

General Solution of a pth-Order Difference Equation with Repeated Eigenvalues

In the more general case of a difference equation for which \mathbf{F} has repeated eigenvalues and $s < p$ linearly independent eigenvectors, result [1.2.17] is generalized by using the Jordan decomposition,

$$\mathbf{F} = \mathbf{M}\mathbf{J}\mathbf{M}^{-1} \quad [1.2.40]$$

where \mathbf{M} is a $(p \times p)$ matrix and \mathbf{J} takes the form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s \end{bmatrix}$$

with

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \quad [1.2.41]$$

for λ_i an eigenvalue of \mathbf{F} . If [1.2.17] is replaced by [1.2.40], then equation [1.2.19] generalizes to

$$\mathbf{F}^j = \mathbf{M} \mathbf{J}^j \mathbf{M}^{-1} \quad [1.2.42]$$

where

$$\mathbf{J}^j = \begin{bmatrix} \mathbf{J}_1^j & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^j & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s^j \end{bmatrix}.$$

Moreover, from [1.2.41], if \mathbf{J}_i is of dimension $(n_i \times n_i)$, then⁸

$$\mathbf{J}_i^j = \begin{bmatrix} \lambda_i^j & (\binom{j}{1})\lambda_i^{j-1} & (\binom{j}{2})\lambda_i^{j-2} & \cdots & (\binom{j}{n_i-1})\lambda_i^{j-n_i+1} \\ 0 & \lambda_i^j & (\binom{j}{1})\lambda_i^{j-1} & \cdots & (\binom{j}{n_i-2})\lambda_i^{j-n_i+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i^j \end{bmatrix} \quad [1.2.43]$$

where

$$\binom{j}{n} \equiv \begin{cases} \frac{j(j-1)(j-2) \cdots (j-n+1)}{n(n-1) \cdots 3 \cdot 2 \cdot 1} & \text{for } j \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Equation [1.2.43] may be verified by induction by multiplying [1.2.41] by [1.2.43] and noticing that $\binom{j}{n} + \binom{j-1}{n-1} = \binom{j+1}{n}$.

For example, consider again the second-order difference equation, this time with repeated roots. Then

$$\mathbf{F}^j = \mathbf{M} \begin{bmatrix} \lambda^j & j\lambda^{j-1} \\ 0 & \lambda^j \end{bmatrix} \mathbf{M}^{-1},$$

so that the dynamic multiplier takes the form

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)} = k_1 \lambda^j + k_2 j \lambda^{j-1}.$$

Long-Run and Present-Value Calculations

If the eigenvalues are all less than 1 in modulus, then \mathbf{F}^j in [1.2.9] goes to zero as j becomes large. If all values of w and y are taken to be bounded, we can

⁸This expression is taken from Chiang (1980, p. 444).

think of a “solution” of y_t in terms of the infinite history of w ,

$$y_t = w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \dots, \quad [1.2.44]$$

where ψ_j is given by the $(1, 1)$ element of \mathbf{F}^j and takes the particular form of [1.2.29] in the case of distinct eigenvalues.

It is also straightforward to calculate the effect on the present value of y of a transitory increase in w . This is simplest to find if we first consider the slightly more general problem of the hypothetical consequences of a change in any element of the vector \mathbf{v}_t on any element of ξ_{t+j} in a general system of the form of [1.2.5]. The answer to this more general problem can be inferred immediately from [1.2.9]:

$$\frac{\partial \xi_{t+j}}{\partial \mathbf{v}'_t} = \mathbf{F}^j. \quad [1.2.45]$$

The true dynamic multiplier of interest, $\partial y_{t+j}/\partial w_t$, is just the $(1, 1)$ element of the $(p \times p)$ matrix in [1.2.45]. The effect on the present value of ξ of a change in \mathbf{v} is given by

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j \xi_{t+j}}{\partial \mathbf{v}'_t} = \sum_{j=0}^{\infty} \beta^j \mathbf{F}^j = (\mathbf{I}_p - \beta \mathbf{F})^{-1}, \quad [1.2.46]$$

provided that the eigenvalues of \mathbf{F} are all less than β^{-1} in modulus. The effect on the present value of y of a change in w ,

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t},$$

is thus the $(1, 1)$ element of the $(p \times p)$ matrix in [1.2.46]. This value is given by the following proposition.

Proposition 1.3: *If the eigenvalues of the $(p \times p)$ matrix \mathbf{F} defined in [1.2.3] are all less than β^{-1} in modulus, then the matrix $(\mathbf{I}_p - \beta \mathbf{F})^{-1}$ exists and the effect of w on the present value of y is given by its $(1, 1)$ element:*

$$1/(1 - \phi_1 \beta - \phi_2 \beta^2 - \dots - \phi_{p-1} \beta^{p-1} - \phi_p \beta^p).$$

Note that Proposition 1.3 includes the earlier result for a first-order system (equation [1.1.14]) as a special case.

The cumulative effect of a one-time change in w_t on y_t, y_{t+1}, \dots can be considered a special case of Proposition 1.3 with no discounting. Setting $\beta = 1$ in Proposition 1.3 shows that, provided the eigenvalues of \mathbf{F} are all less than 1 in modulus, the cumulative effect of a one-time change in w on y is given by

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = 1/(1 - \phi_1 - \phi_2 - \dots - \phi_p). \quad [1.2.47]$$

Notice again that [1.2.47] can alternatively be interpreted as giving the eventual long-run effect on y of a permanent change in w :

$$\lim_{j \rightarrow \infty} \frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \dots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = 1/(1 - \phi_1 - \phi_2 - \dots - \phi_p).$$

APPENDIX 1.A. Proofs of Chapter 1 Propositions

■ **Proof of Proposition 1.1.** The eigenvalues of \mathbf{F} satisfy

$$|\mathbf{F} - \lambda \mathbf{I}_p| = 0. \quad [1.A.1]$$

For the matrix \mathbf{F} defined in equation [1.2.3], this determinant would be

$$\begin{aligned} & \left| \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} (\phi_1 - \lambda) & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix} \right|. \quad [1.A.2] \end{aligned}$$

Recall that if we multiply a column of a matrix by a constant and add the result to another column, the determinant of the matrix is unchanged. If we multiply the p th column of the matrix in [1.A.2] by $(1/\lambda)$ and add the result to the $(p-1)$ th column, the result is a matrix with the same determinant as that in [1.A.2]:

$$|\mathbf{F} - \lambda \mathbf{I}_p| = \left| \begin{bmatrix} \phi_1 - \lambda & \phi_2 & \phi_3 & \cdots & \phi_{p-2} & \phi_{p-1} + (\phi_p/\lambda) & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda \end{bmatrix} \right|.$$

Next, multiply the $(p-1)$ th column by $(1/\lambda)$ and add the result to the $(p-2)$ th column:

$$\begin{aligned} & |\mathbf{F} - \lambda \mathbf{I}_p| \\ &= \left| \begin{bmatrix} \phi_1 - \lambda & \phi_2 & \phi_3 & \cdots & \phi_{p-2} + \phi_{p-1}/\lambda + \phi_p/\lambda^2 & \phi_{p-1} + \phi_p/\lambda & \phi_p \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda \end{bmatrix} \right|. \end{aligned}$$

Continuing in this fashion shows [1.A.1] to be equivalent to the determinant of the following upper triangular matrix:

$$\begin{aligned} & |\mathbf{F} - \lambda \mathbf{I}_p| \\ &= \left| \begin{bmatrix} \phi_1 - \lambda + \phi_2/\lambda + \phi_3/\lambda^2 + \cdots + \phi_p/\lambda^{p-1} & \phi_2 + \phi_3/\lambda + \phi_4/\lambda^2 + \cdots + \phi_p/\lambda^{p-2} & \cdots & \phi_{p-1} + \phi_p/\lambda & \phi_p \\ 0 & -\lambda & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix} \right|. \end{aligned}$$

But the determinant of an upper triangular matrix is simply the product of the terms along the principal diagonal:

$$\begin{aligned} |\mathbf{F} - \lambda \mathbf{I}_p| &= [\phi_1 - \lambda + \phi_2/\lambda + \phi_3/\lambda^2 + \cdots + \phi_p/\lambda^{p-1}] \cdot [-\lambda]^{p-1} \\ &= (-1)^p \cdot [\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_p]. \quad [1.A.3] \end{aligned}$$

The eigenvalues of \mathbf{F} are thus the values of λ for which [1.A.3] is zero, or for which

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0,$$

as asserted in Proposition 1.1. ■

■ **Proof of Proposition 1.2.** Assuming that the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$ are distinct, the matrix \mathbf{T} in equation [1.2.17] can be constructed from the eigenvectors of \mathbf{F} . Let \mathbf{t}_i denote the following $(p \times 1)$ vector,

$$\mathbf{t}_i = \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}, \quad [1.A.4]$$

where λ_i denotes the i th eigenvalue of \mathbf{F} . Notice

$$\begin{aligned} \mathbf{F}\mathbf{t}_i &= \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \phi_1\lambda_i^{p-1} + \phi_2\lambda_i^{p-2} + \phi_3\lambda_i^{p-3} + \cdots + \phi_{p-1}\lambda_i + \phi_p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^2 \\ \lambda_i \end{bmatrix}. \end{aligned} \quad [1.A.5]$$

Since λ_i is an eigenvalue of \mathbf{F} , it satisfies [1.2.16]:

$$\lambda_i^p - \phi_1\lambda_i^{p-1} - \phi_2\lambda_i^{p-2} - \cdots - \phi_{p-1}\lambda_i - \phi_p = 0. \quad [1.A.6]$$

Substituting [1.A.6] into [1.A.5] reveals

$$\mathbf{F}\mathbf{t}_i = \begin{bmatrix} \lambda_i^p \\ \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i^2 \\ \lambda_i \end{bmatrix} = \lambda_i \begin{bmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \lambda_i^{p-3} \\ \vdots \\ \lambda_i^1 \\ 1 \end{bmatrix}$$

or

$$\mathbf{F}\mathbf{t}_i = \lambda_i \mathbf{t}_i. \quad [1.A.7]$$

Thus \mathbf{t}_i is an eigenvector of \mathbf{F} associated with the eigenvalue λ_i .

We can calculate the matrix \mathbf{T} by combining the eigenvectors $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_p)$ into a $(p \times p)$ matrix

$$\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \cdots \ \mathbf{t}_p]. \quad [1.A.8]$$

To calculate the particular values for c_i in equation [1.2.21], recall that \mathbf{T}^{-1} is characterized by

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}_p, \quad [1.A.9]$$

where \mathbf{T} is given by [1.A.4] and [1.A.8]. Writing out the first column of the matrix system of equations [1.A.9] explicitly, we have

$$\begin{bmatrix} \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \\ \lambda_1^{p-2} & \lambda_2^{p-2} & \cdots & \lambda_p^{p-2} \\ \lambda_1^{p-3} & \lambda_2^{p-3} & \cdots & \lambda_p^{p-3} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_p^1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} t^{11} \\ t^{21} \\ t^{31} \\ \vdots \\ t^{p-1,1} \\ t^{p1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

This gives a system of p linear equations in the p unknowns $(t^{11}, t^{21}, \dots, t^{p1})$. Provided that the λ_i are all distinct, the solution can be shown to be⁹

$$\begin{aligned} t^{11} &= \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_p)} \\ t^{21} &= \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \cdots (\lambda_2 - \lambda_p)} \\ &\vdots \\ t^{p1} &= \frac{1}{(\lambda_p - \lambda_1)(\lambda_p - \lambda_2) \cdots (\lambda_p - \lambda_{p-1})}. \end{aligned}$$

Substituting these values into [1.2.21] gives equation [1.2.25]. ■

■ **Proof of Proposition 1.3.** The first claim in this proposition is that if the eigenvalues of \mathbf{F} are less than β^{-1} in modulus, then the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ exists. Suppose the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ did not exist. Then the determinant $|\mathbf{I}_p - \beta\mathbf{F}|$ would have to be zero. But

$$|\mathbf{I}_p - \beta\mathbf{F}| = |-\beta \cdot [\mathbf{F} - \beta^{-1}\mathbf{I}_p]| = (-\beta)^p |\mathbf{F} - \beta^{-1}\mathbf{I}_p|,$$

so that $|\mathbf{F} - \beta^{-1}\mathbf{I}_p|$ would have to be zero whenever the inverse of $(\mathbf{I}_p - \beta\mathbf{F})$ fails to exist. But this would mean that β^{-1} is an eigenvalue of \mathbf{F} , which is ruled out by the assumption that all eigenvalues of \mathbf{F} are strictly less than β^{-1} in modulus. Thus, the matrix $\mathbf{I}_p - \beta\mathbf{F}$ must be nonsingular.

Since $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$ exists, it satisfies the equation

$$[\mathbf{I}_p - \beta\mathbf{F}]^{-1}[\mathbf{I}_p - \beta\mathbf{F}] = \mathbf{I}_p. \quad [1.A.10]$$

Let x_{ij} denote the row i , column j element of $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$, and write [1.A.10] as

$$\begin{aligned} &\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{bmatrix} \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \end{aligned} \quad [1.A.11]$$

The task is then to find the $(1, 1)$ element of $[\mathbf{I}_p - \beta\mathbf{F}]^{-1}$, that is, to find the value of x_{11} . To do this we need only consider the first row of equations in [1.A.11]:

$$\begin{aligned} &[x_{11} \ x_{12} \ \cdots \ x_{1p}] \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & 1 \end{bmatrix} \\ &= [1 \ 0 \ \cdots \ 0 \ 0]. \quad [1.A.12] \end{aligned}$$

⁹See Lemma 2 of Chiang (1980, p. 144).

Consider postmultiplying this system of equations by a matrix with 1s along the principal diagonal, β in the row $p - 1$ position, and 0s elsewhere:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 1 \end{bmatrix}.$$

The effect of this operation is to multiply the p th column of a matrix by β and add the result to the $(p - 1)$ th column:

$$[x_{11} \ x_{12} \ \cdots \ x_{1p}] \begin{bmatrix} 1 - \beta\phi_1 & -\beta\phi_2 & \cdots & -\beta\phi_{p-1} & -\beta^2\phi_p & -\beta\phi_p \\ -\beta & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \end{bmatrix} = [1 \ 0 \ \cdots \ 0 \ 0].$$

Next multiply the $(p - 1)$ th column by β and add the result to the $(p - 2)$ th column. Proceeding in this fashion, we arrive at

$$[x_{11} \ x_{12} \ \cdots \ x_{1p}] \times \begin{bmatrix} 1 - \beta\phi_1 - \beta^2\phi_2 - \cdots - \beta^{p-1}\phi_{p-1} - \beta^p\phi_p & -\beta\phi_2 - \beta^2\phi_3 - \cdots - \beta^{p-1}\phi_p & \cdots & -\beta\phi_{p-1} - \beta^2\phi_p & -\beta\phi_p \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = [1 \ 0 \ \cdots \ 0 \ 0]. \quad [1.A.13]$$

The first equation in [1.A.13] states that

$$x_{11} \cdot (1 - \beta\phi_1 - \beta^2\phi_2 - \cdots - \beta^{p-1}\phi_{p-1} - \beta^p\phi_p) = 1$$

or

$$x_{11} = 1/(1 - \beta\phi_1 - \beta^2\phi_2 - \cdots - \beta^p\phi_p),$$

as claimed in Proposition 1.3. ■

Chapter 1 References

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Lag Operators

2.1. Introduction

The previous chapter analyzed the dynamics of linear difference equations using matrix algebra. This chapter develops some of the same results using time series operators. We begin with some introductory remarks on some useful time series operators.

A time series is a collection of observations indexed by the date of each observation. Usually we have collected data beginning at some particular date (say, $t = 1$) and ending at another (say, $t = T$):

$$(y_1, y_2, \dots, y_T).$$

We often imagine that we could have obtained earlier observations ($y_0, y_{-1}, y_{-2}, \dots$) or later observations (y_{T+1}, y_{T+2}, \dots) had the process been observed for more time. The observed sample (y_1, y_2, \dots, y_T) could then be viewed as a finite segment of a doubly infinite sequence, denoted $\{y_t\}_{t=-\infty}^{\infty}$:

$$\{y_t\}_{t=-\infty}^{\infty} = \{\dots, y_{-1}, y_0, \underbrace{y_1, y_2, \dots, y_T}_{\text{observed sample}}, y_{T+1}, y_{T+2}, \dots\}.$$

Typically, a time series $\{y_t\}_{t=-\infty}^{\infty}$ is identified by describing the t th element. For example, a *time trend* is a series whose value at date t is simply the date of the observation:

$$y_t = t.$$

We could also consider a time series in which each element is equal to a constant c , regardless of the date of the observation t :

$$y_t = c.$$

Another important time series is a *Gaussian white noise process*, denoted

$$y_t = \varepsilon_t,$$

where $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of independent random variables each of which has a $N(0, \sigma^2)$ distribution.

We are used to thinking of a function such as $y = f(x)$ or $y = g(x, w)$ as an operation that accepts as input a number (x) or group of numbers (x, w) and produces the output (y). A time series *operator* transforms one time series or group

of time series into a new time series. It accepts as input a sequence such as $\{x_t\}_{t=-\infty}^{\infty}$ or a group of sequences such as $(\{x_t\}_{t=-\infty}^{\infty}, \{w_t\}_{t=-\infty}^{\infty})$ and has as output a new sequence $\{y_t\}_{t=-\infty}^{\infty}$. Again, the operator is summarized by describing the value of a typical element of $\{y_t\}_{t=-\infty}^{\infty}$ in terms of the corresponding elements of $\{x_t\}_{t=-\infty}^{\infty}$.

An example of a time series operator is the multiplication operator, represented as

$$y_t = \beta x_t. \quad [2.1.1]$$

Although it is written exactly the same way as simple scalar multiplication, equation [2.1.1] is actually shorthand for an infinite sequence of multiplications, one for each date t . The operator multiplies the value x takes on at any date t by some constant β to generate the value of y for that date.

Another example of a time series operator is the addition operator:

$$y_t = x_t + w_t.$$

Here the value of y at any date t is the sum of the values that x and w take on for that date.

Since the multiplication or addition operators amount to element-by-element multiplication or addition, they obey all the standard rules of algebra. For example, if we multiply each observation of $\{x_t\}_{t=-\infty}^{\infty}$ by β and each observation of $\{w_t\}_{t=-\infty}^{\infty}$ by β and add the results,

$$\beta x_t + \beta w_t,$$

the outcome is the same as if we had first added $\{x_t\}_{t=-\infty}^{\infty}$ to $\{w_t\}_{t=-\infty}^{\infty}$ and then multiplied each element of the resulting series by β :

$$\beta(x_t + w_t).$$

A highly useful operator is the lag operator. Suppose that we start with a sequence $\{x_t\}_{t=-\infty}^{\infty}$ and generate a new sequence $\{y_t\}_{t=-\infty}^{\infty}$, where the value of y for date t is equal to the value x took on at date $t - 1$:

$$y_t = x_{t-1}. \quad [2.1.2]$$

This is described as applying the *lag operator* to $\{x_t\}_{t=-\infty}^{\infty}$. The operation is represented by the symbol L :

$$Lx_t \equiv x_{t-1}. \quad [2.1.3]$$

Consider the result of applying the lag operator twice to a series:

$$L(Lx_t) = L(x_{t-1}) = x_{t-2}.$$

Such a double application of the lag operator is indicated by “ L^2 ”:

$$L^2x_t = x_{t-2}.$$

In general, for any integer k ,

$$L^k x_t = x_{t-k}. \quad [2.1.4]$$

Notice that if we first apply the multiplication operator and then the lag operator, as in

$$x_t \rightarrow \beta x_t \rightarrow \beta x_{t-1},$$

the result will be exactly the same as if we had applied the lag operator first and then the multiplication operator:

$$x_t \rightarrow x_{t-1} \rightarrow \beta x_{t-1}.$$

Thus the lag operator and multiplication operator are commutative:

$$L(\beta x_t) = \beta \cdot Lx_t.$$

Similarly, if we first add two series and then apply the lag operator to the result,

$$(x_t, w_t) \rightarrow x_t + w_t \rightarrow x_{t-1} + w_{t-1},$$

the result is the same as if we had applied the lag operator before adding:

$$(x_t, w_t) \rightarrow (x_{t-1}, w_{t-1}) \rightarrow x_{t-1} + w_{t-1}.$$

Thus, the lag operator is distributive over the addition operator:

$$L(x_t + w_t) = Lx_t + Lw_t.$$

We thus see that the lag operator follows exactly the same algebraic rules as the multiplication operator. For this reason, it is tempting to use the expression “multiply y_t by L ” rather than “operate on $\{y_t\}_{t=-\infty}^{\infty}$ by L . ” Although the latter expression is technically more correct, this text will often use the former shorthand expression to facilitate the exposition.

Faced with a time series defined in terms of compound operators, we are free to use the standard commutative, associative, and distributive algebraic laws for multiplication and addition to express the compound operator in an alternative form. For example, the process defined by

$$y_t = (a + bL)Lx_t$$

is exactly the same as

$$y_t = (aL + bL^2)x_t = ax_{t-1} + bx_{t-2}.$$

To take another example,

$$\begin{aligned} (1 - \lambda_1 L)(1 - \lambda_2 L)x_t &= (1 - \lambda_1 L - \lambda_2 L + \lambda_1 \lambda_2 L^2)x_t \\ &= (1 - [\lambda_1 + \lambda_2]L + \lambda_1 \lambda_2 L^2)x_t \\ &= x_t - (\lambda_1 + \lambda_2)x_{t-1} + (\lambda_1 \lambda_2)x_{t-2}. \end{aligned} \quad [2.1.5]$$

An expression such as $(aL + bL^2)$ is referred to as a *polynomial in the lag operator*. It is algebraically similar to a simple polynomial $(az + bz^2)$ where z is a scalar. The difference is that the simple polynomial $(az + bz^2)$ refers to a particular number, whereas a polynomial in the lag operator $(aL + bL^2)$ refers to an operator that would be applied to one time series $\{x_t\}_{t=-\infty}^{\infty}$ to produce a new time series $\{y_t\}_{t=-\infty}^{\infty}$.

Notice that if $\{x_t\}_{t=-\infty}^{\infty}$ is just a series of constants,

$$x_t = c \quad \text{for all } t,$$

then the lag operator applied to x_t produces the same series of constants:

$$Lx_t = x_{t-1} = c.$$

Thus, for example,

$$(\alpha L + \beta L^2 + \gamma L^3)c = (\alpha + \beta + \gamma) \cdot c. \quad [2.1.6]$$

2.2. First-Order Difference Equations

Let us now return to the first-order difference equation analyzed in Section 1.1:

$$y_t = \phi y_{t-1} + w_t. \quad [2.2.1]$$

Equation [2.2.1] can be rewritten using the lag operator [2.1.3] as

$$y_t = \phi L y_t + w_t.$$

This equation, in turn, can be rearranged using standard algebra,

$$y_t - \phi L y_t = w_t,$$

or

$$(1 - \phi L) y_t = w_t. \quad [2.2.2]$$

Next consider “multiplying” both sides of [2.2.2] by the following operator:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t). \quad [2.2.3]$$

The result would be

$$\begin{aligned} (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t \\ = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) w_t. \end{aligned} \quad [2.2.4]$$

Expanding out the compound operator on the left side of [2.2.4] results in

$$\begin{aligned} & (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) \\ &= (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) \\ &\quad - (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) \phi L \\ &= (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) \\ &\quad - (\phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t + \phi^{t+1} L^{t+1}) \\ &= (1 - \phi^{t+1} L^{t+1}). \end{aligned} \quad [2.2.5]$$

Substituting [2.2.5] into [2.2.4] yields

$$(1 - \phi^{t+1} L^{t+1}) y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t) w_t. \quad [2.2.6]$$

Writing [2.2.6] out explicitly using [2.1.4] produces

$$y_t - \phi^{t+1} y_{t-(t+1)} = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots + \phi^t w_{t-t}$$

or

$$y_t = \phi^{t+1} y_{-1} + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots + \phi^t w_0. \quad [2.2.7]$$

Notice that equation [2.2.7] is identical to equation [1.1.7]. Applying the operator [2.2.3] is performing exactly the same set of recursive substitutions that were employed in the previous chapter to arrive at [1.1.7].

It is interesting to reflect on the nature of the operator [2.2.3] as t becomes large. We saw in [2.2.5] that

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t = y_t - \phi^{t+1} y_{-1}.$$

That is, $(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t$ differs from y_t by the term $\phi^{t+1} y_{-1}$. If $|\phi| < 1$ and if y_{-1} is a finite number, this residual $\phi^{t+1} y_{-1}$ will become negligible as t becomes large:

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)(1 - \phi L) y_t \approx y_t \quad \text{for } t \text{ large.}$$

A sequence $\{y_t\}_{t=-\infty}^{\infty}$ is said to be *bounded* if there exists a finite number \bar{y} such that

$$|y_t| < \bar{y} \quad \text{for all } t.$$

Thus, when $|\phi| < 1$ and when we are considering applying an operator to a bounded sequence, we can think of

$$(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^t L^t)$$

as approximating the inverse of the operator $(1 - \phi L)$, with this approximation made arbitrarily accurate by choosing j sufficiently large:

$$(1 - \phi L)^{-1} = \lim_{j \rightarrow \infty} (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots + \phi^j L^j). \quad [2.2.8]$$

This operator $(1 - \phi L)^{-1}$ has the property

$$(1 - \phi L)^{-1}(1 - \phi L) = 1,$$

where “1” denotes the identity operator:

$$1y_t = y_t.$$

The following chapter discusses stochastic sequences rather than the deterministic sequences studied here. There we will speak of mean square convergence and stationary stochastic processes in place of limits of bounded deterministic sequences, though the practical meaning of [2.2.8] will be little changed.

Provided that $|\phi| < 1$ and we restrict ourselves to bounded sequences or stationary stochastic processes, both sides of [2.2.2] can be “divided” by $(1 - \phi L)$ to obtain

$$y_t = (1 - \phi L)^{-1}w_t$$

or

$$y_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots. \quad [2.2.9]$$

It should be emphasized that if we were not restricted to considering bounded sequences or stationary stochastic processes $\{w_t\}_{t=-\infty}^{\infty}$ and $\{y_t\}_{t=-\infty}^{\infty}$, then expression [2.2.9] would not be a necessary implication of [2.2.1]. Equation [2.2.9] is consistent with [2.2.1], but adding a term $a_o\phi^t$,

$$y_t = a_o\phi^t + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \cdots, \quad [2.2.10]$$

produces another series consistent with [2.2.1] for any constant a_o . To verify that [2.2.10] is consistent with [2.2.1], multiply [2.2.10] by $(1 - \phi L)$:

$$\begin{aligned} (1 - \phi L)y_t &= (1 - \phi L)a_o\phi^t + (1 - \phi L)(1 - \phi L)^{-1}w_t \\ &= a_o\phi^t - \phi \cdot a_o\phi^{t-1} + w_t \\ &= w_t, \end{aligned}$$

so that [2.2.10] is consistent with [2.2.1] for any constant a_o .

Although any process of the form of [2.2.10] is consistent with the difference equation [2.2.1], notice that since $|\phi| < 1$,

$$|a_o\phi^t| \rightarrow \infty \quad \text{as} \quad t \rightarrow -\infty.$$

Thus, even if $\{w_t\}_{t=-\infty}^{\infty}$ is a bounded sequence, the solution $\{y_t\}_{t=-\infty}^{\infty}$ given by [2.2.10] is unbounded unless $a_o = 0$ in [2.2.10]. Thus, there was a particular reason for defining the operator [2.2.8] to be the inverse of $(1 - \phi L)$ —namely, (1 - ϕL)⁻¹ defined in [2.2.8] is the unique operator satisfying

$$(1 - \phi L)^{-1}(1 - \phi L) = 1$$

that maps a bounded sequence $\{w_t\}_{t=-\infty}^{\infty}$ into a bounded sequence $\{y_t\}_{t=-\infty}^{\infty}$.

The nature of $(1 - \phi L)^{-1}$ when $|\phi| \geq 1$ will be discussed in Section 2.5.

2.3. Second-Order Difference Equations

Consider next a second-order difference equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t. \quad [2.3.1]$$

Rewriting this in lag operator form produces

$$(1 - \phi_1 L - \phi_2 L^2) y_t = w_t. \quad [2.3.2]$$

The left side of [2.3.2] contains a second-order polynomial in the lag operator L . Suppose we factor this polynomial, that is, find numbers λ_1 and λ_2 such that

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) = (1 - [\lambda_1 + \lambda_2]L + \lambda_1 \lambda_2 L^2). \quad [2.3.3]$$

This is just the operation in [2.1.5] in reverse. Given values for ϕ_1 and ϕ_2 , we seek numbers λ_1 and λ_2 with the properties that

$$\lambda_1 + \lambda_2 = \phi_1$$

and

$$\lambda_1 \lambda_2 = -\phi_2.$$

For example, if $\phi_1 = 0.6$ and $\phi_2 = -0.08$, then we should choose $\lambda_1 = 0.4$ and $\lambda_2 = 0.2$:

$$(1 - 0.6L + 0.08L^2) = (1 - 0.4L)(1 - 0.2L). \quad [2.3.4]$$

It is easy enough to see that these values of λ_1 and λ_2 work for this numerical example, but how are λ_1 and λ_2 found in general? The task is to choose λ_1 and λ_2 so as to make sure that the operator on the right side of [2.3.3] is identical to that on the left side. This will be true whenever the following represent the identical functions of z :

$$(1 - \phi_1 z - \phi_2 z^2) = (1 - \lambda_1 z)(1 - \lambda_2 z). \quad [2.3.5]$$

This equation simply replaces the lag operator L in [2.3.3] with a scalar z . What is the point of doing so? With [2.3.5], we can now ask, For what values of z is the right side of [2.3.5] equal to zero? The answer is, if either $z = \lambda_1^{-1}$ or $z = \lambda_2^{-1}$, then the right side of [2.3.5] would be zero. It would not have made sense to ask an analogous question of [2.3.3]— L denotes a particular operator, not a number, and $L = \lambda_1^{-1}$ is not a sensible statement.

Why should we care that the right side of [2.3.5] is zero if $z = \lambda_1^{-1}$ or if $z = \lambda_2^{-1}$? Recall that the goal was to choose λ_1 and λ_2 so that the two sides of [2.3.5] represented the identical polynomial in z . This means that for any particular value z the two functions must produce the same number. If we find a value of z that sets the right side to zero, that same value of z must set the left side to zero as well. But the values of z that set the left side to zero,

$$(1 - \phi_1 z - \phi_2 z^2) = 0, \quad [2.3.6]$$

are given by the quadratic formula:

$$z_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \quad [2.3.7]$$

$$z_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}. \quad [2.3.8]$$

Setting $z = z_1$ or z_2 makes the left side of [2.3.5] zero, while $z = \lambda_1^{-1}$ or λ_2^{-1} sets the right side of [2.3.5] to zero. Thus

$$\lambda_1^{-1} = z_1 \quad [2.3.9]$$

$$\lambda_2^{-1} = z_2. \quad [2.3.10]$$

Returning to the numerical example [2.3.4] in which $\phi_1 = 0.6$ and $\phi_2 = -0.08$, we would calculate

$$z_1 = \frac{0.6 - \sqrt{(0.6)^2 - 4(0.08)}}{2(0.08)} = 2.5$$

$$z_2 = \frac{0.6 + \sqrt{(0.6)^2 - 4(0.08)}}{2(0.08)} = 5.0,$$

and so

$$\lambda_1 = 1/(2.5) = 0.4$$

$$\lambda_2 = 1/(5.0) = 0.2,$$

as was found in [2.3.4].

When $\phi_1^2 + 4\phi_2 < 0$, the values z_1 and z_2 are complex conjugates, and their reciprocals λ_1 and λ_2 can be found by first writing the complex number in polar coordinate form. Specifically, write

$$z_1 = a + bi$$

as

$$z_1 = R \cdot [\cos(\theta) + i \cdot \sin(\theta)] = R \cdot e^{i\theta}.$$

Then

$$z_1^{-1} = R^{-1} \cdot e^{-i\theta} = R^{-1} \cdot [\cos(\theta) - i \cdot \sin(\theta)].$$

Actually, there is a more direct method for calculating the values of λ_1 and λ_2 from ϕ_1 and ϕ_2 . Divide both sides of [2.3.5] by z^2 :

$$(z^{-2} - \phi_1 z^{-1} - \phi_2) = (z^{-1} - \lambda_1)(z^{-1} - \lambda_2) \quad [2.3.11]$$

and define λ to be the variable z^{-1} :

$$\lambda \equiv z^{-1}. \quad [2.3.12]$$

Substituting [2.3.12] into [2.3.11] produces

$$(\lambda^2 - \phi_1 \lambda - \phi_2) = (\lambda - \lambda_1)(\lambda - \lambda_2). \quad [2.3.13]$$

Again, [2.3.13] must hold for all values of λ in order for the two sides of [2.3.5] to represent the same polynomial. The values of λ that set the right side to zero are $\lambda = \lambda_1$ and $\lambda = \lambda_2$. These same values must set the left side of [2.3.13] to zero as well:

$$(\lambda^2 - \phi_1 \lambda - \phi_2) = 0. \quad [2.3.14]$$

Thus, to calculate the values of λ_1 and λ_2 that factor the polynomial in [2.3.3], we can find the roots of [2.3.14] directly from the quadratic formula:

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad [2.3.15]$$

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}. \quad [2.3.16]$$

For the example of [2.3.4], we would thus calculate

$$\lambda_1 = \frac{0.6 + \sqrt{(0.6)^2 - 4(0.08)}}{2} = 0.4$$

$$\lambda_2 = \frac{0.6 - \sqrt{(0.6)^2 - 4(0.08)}}{2} = 0.2.$$

It is instructive to compare these results with those in Chapter 1. There the dynamics of the second-order difference equation [2.3.1] were summarized by calculating the eigenvalues of the matrix \mathbf{F} given by

$$\mathbf{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}. \quad [2.3.17]$$

The eigenvalues of \mathbf{F} were seen to be the two values of λ that satisfy equation [1.2.13]:

$$(\lambda^2 - \phi_1\lambda - \phi_2) = 0.$$

But this is the same calculation as in [2.3.14]. This finding is summarized in the following proposition.

Proposition 2.1: *Factoring the polynomial $(1 - \phi_1L - \phi_2L^2)$ as*

$$(1 - \phi_1L - \phi_2L^2) = (1 - \lambda_1L)(1 - \lambda_2L) \quad [2.3.18]$$

is the same calculation as finding the eigenvalues of the matrix \mathbf{F} in [2.3.17]. The eigenvalues λ_1 and λ_2 of \mathbf{F} are the same as the parameters λ_1 and λ_2 in [2.3.18], and are given by equations [2.3.15] and [2.3.16].

The correspondence between calculating the eigenvalues of a matrix and factoring a polynomial in the lag operator is very instructive. However, it introduces one minor source of possible semantic confusion about which we have to be careful. Recall from Chapter 1 that the system [2.3.1] is stable if both λ_1 and λ_2 are less than 1 in modulus and explosive if either λ_1 or λ_2 is greater than 1 in modulus. Sometimes this is described as the requirement that the roots of

$$(\lambda^2 - \phi_1\lambda - \phi_2) = 0 \quad [2.3.19]$$

lie inside the unit circle. The possible confusion is that it is often convenient to work directly with the polynomial in the form in which it appears in [2.3.2],

$$(1 - \phi_1z - \phi_2z^2) = 0, \quad [2.3.20]$$

whose roots, we have seen, are the reciprocals of those of [2.3.19]. Thus, we could say with equal accuracy that “the difference equation [2.3.1] is stable whenever the roots of [2.3.19] lie *inside* the unit circle” or that “the difference equation [2.3.1] is stable whenever the roots of [2.3.20] lie *outside* the unit circle.” The two statements mean exactly the same thing. Some scholars refer simply to the “roots of the difference equation [2.3.1],” though this raises the possibility of confusion between [2.3.19] and [2.3.20]. This book will follow the convention of using the term “eigenvalues” to refer to the roots of [2.3.19]. Wherever the term “roots” is used, we will indicate explicitly the equation whose roots are being described.

From here on in this section, it is assumed that the second-order difference equation is stable, with the eigenvalues λ_1 and λ_2 distinct and both inside the unit circle. Where this is the case, the inverses

$$\begin{aligned} (1 - \lambda_1L)^{-1} &= 1 + \lambda_1^1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \dots \\ (1 - \lambda_2L)^{-1} &= 1 + \lambda_2^1 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \dots \end{aligned}$$

are well defined for bounded sequences. Write [2.3.2] in factored form:

$$(1 - \lambda_1L)(1 - \lambda_2L)y_t = w_t$$

and operate on both sides by $(1 - \lambda_1L)^{-1}(1 - \lambda_2L)^{-1}$:

$$y_t = (1 - \lambda_1L)^{-1}(1 - \lambda_2L)^{-1}w_t. \quad [2.3.21]$$

Following Sargent (1987, p. 184), when $\lambda_1 \neq \lambda_2$, we can use the following operator:

$$(\lambda_1 - \lambda_2)^{-1} \left\{ \frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L} \right\}. \quad [2.3.22]$$

Notice that this is simply another way of writing the operator in [2.3.21]:

$$\begin{aligned} & (\lambda_1 - \lambda_2)^{-1} \left\{ \frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L} \right\} \\ &= (\lambda_1 - \lambda_2)^{-1} \left\{ \frac{\lambda_1(1 - \lambda_2 L) - \lambda_2(1 - \lambda_1 L)}{(1 - \lambda_1 L) \cdot (1 - \lambda_2 L)} \right\} \\ &= \frac{1}{(1 - \lambda_1 L) \cdot (1 - \lambda_2 L)}. \end{aligned}$$

Thus, [2.3.21] can be written as

$$\begin{aligned} y_t &= (\lambda_1 - \lambda_2)^{-1} \left\{ \frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L} \right\} w_t \\ &= \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_2} [1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \dots] \right. \\ &\quad \left. - \frac{\lambda_2}{\lambda_1 - \lambda_2} [1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \dots] \right\} w_t \end{aligned}$$

or

$$y_t = [c_1 + c_2]w_t + [c_1\lambda_1 + c_2\lambda_2]w_{t-1} + [c_1\lambda_1^2 + c_2\lambda_2^2]w_{t-2} + [c_1\lambda_1^3 + c_2\lambda_2^3]w_{t-3} + \dots, \quad [2.3.23]$$

where

$$c_1 = \lambda_1 / (\lambda_1 - \lambda_2) \quad [2.3.24]$$

$$c_2 = -\lambda_2 / (\lambda_1 - \lambda_2). \quad [2.3.25]$$

From [2.3.23] the dynamic multiplier can be read off directly as

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j,$$

the same result arrived at in equations [1.2.24] and [1.2.25].

2.4. *p*th-Order Difference Equations

These techniques generalize in a straightforward way to a *p*th-order difference equation of the form

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + w_t. \quad [2.4.1]$$

Write [2.4.1] in terms of lag operators as

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) y_t = w_t. \quad [2.4.2]$$

Factor the operator on the left side of [2.4.2] as

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L). \quad [2.4.3]$$

This is the same as finding the values of $(\lambda_1, \lambda_2, \dots, \lambda_p)$ such that the following polynomials are the same for all z :

$$(1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p) = (1 - \lambda_1 z)(1 - \lambda_2 z) \cdots (1 - \lambda_p z).$$

As in the second-order system, we multiply both sides of this equation by z^{-p} and define $\lambda \equiv z^{-1}$:

$$(\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_p). \quad [2.4.4]$$

Clearly, setting $\lambda = \lambda_i$ for $i = 1, 2, \dots, p$ causes the right side of [2.4.4] to equal zero. Thus the values $(\lambda_1, \lambda_2, \dots, \lambda_p)$ must be the numbers that set the left side of expression [2.4.4] to zero as well:

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0. \quad [2.4.5]$$

This expression again is identical to that given in Proposition 1.1, which characterized the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of the matrix \mathbf{F} defined in equation [1.2.3]. Thus, Proposition 2.1 readily generalizes.

Proposition 2.2: *Factoring a p th-order polynomial in the lag operator,*

$$(1 - \phi_1L - \phi_2L^2 - \cdots - \phi_pL^p) = (1 - \lambda_1L)(1 - \lambda_2L) \cdots (1 - \lambda_pL),$$

is the same calculation as finding the eigenvalues of the matrix \mathbf{F} defined in [1.2.3]. The eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of \mathbf{F} are the same as the parameters $(\lambda_1, \lambda_2, \dots, \lambda_p)$ in [2.4.3] and are given by the solutions to equation [2.4.5].

The difference equation [2.4.1] is stable if the eigenvalues (the roots of [2.4.5]) lie inside the unit circle, or equivalently if the roots of

$$1 - \phi_1z - \phi_2z^2 - \cdots - \phi_pz^p = 0 \quad [2.4.6]$$

lie outside the unit circle.

Assuming that the eigenvalues are inside the unit circle and that we are restricting ourselves to considering bounded sequences, the inverses $(1 - \lambda_1L)^{-1}$, $(1 - \lambda_2L)^{-1}, \dots, (1 - \lambda_pL)^{-1}$ all exist, permitting the difference equation

$$(1 - \lambda_1L)(1 - \lambda_2L) \cdots (1 - \lambda_pL)y_t = w_t$$

to be written as

$$y_t = (1 - \lambda_1L)^{-1}(1 - \lambda_2L)^{-1} \cdots (1 - \lambda_pL)^{-1}w_t. \quad [2.4.7]$$

Provided further that the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$ are all distinct, the polynomial associated with the operator on the right side of [2.4.7] can again be expanded with partial fractions:

$$\begin{aligned} & \frac{1}{(1 - \lambda_1z)(1 - \lambda_2z) \cdots (1 - \lambda_pz)} \\ &= \frac{c_1}{(1 - \lambda_1z)} + \frac{c_2}{(1 - \lambda_2z)} + \cdots + \frac{c_p}{(1 - \lambda_pz)}. \end{aligned} \quad [2.4.8]$$

Following Sargent (1987, pp. 192–93), the values of (c_1, c_2, \dots, c_p) that make [2.4.8] true can be found by multiplying both sides by $(1 - \lambda_1z)(1 - \lambda_2z) \cdots (1 - \lambda_pz)$:

$$\begin{aligned} 1 &= c_1(1 - \lambda_2z)(1 - \lambda_3z) \cdots (1 - \lambda_pz) \\ &\quad + c_2(1 - \lambda_1z)(1 - \lambda_3z) \cdots (1 - \lambda_pz) + \cdots \\ &\quad + c_p(1 - \lambda_1z)(1 - \lambda_2z) \cdots (1 - \lambda_{p-1}z). \end{aligned} \quad [2.4.9]$$

Equation [2.4.9] has to hold for all values of z . Since it is a $(p - 1)$ th-order polynomial, if (c_1, c_2, \dots, c_p) are chosen so that [2.4.9] holds for p particular

distinct values of z , then [2.4.9] must hold for all z . To ensure that [2.4.9] holds at $z = \lambda_1^{-1}$ requires that

$$1 = c_1(1 - \lambda_2\lambda_1^{-1})(1 - \lambda_3\lambda_1^{-1}) \cdots (1 - \lambda_p\lambda_1^{-1})$$

or

$$c_1 = \frac{\lambda_1^{p-1}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_p)}. \quad [2.4.10]$$

For [2.4.9] to hold for $z = \lambda_2^{-1}, \lambda_3^{-1}, \dots, \lambda_p^{-1}$ requires

$$c_2 = \frac{\lambda_2^{p-1}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) \cdots (\lambda_2 - \lambda_p)} \quad [2.4.11]$$

⋮

$$c_p = \frac{\lambda_p^{p-1}}{(\lambda_p - \lambda_1)(\lambda_p - \lambda_2) \cdots (\lambda_p - \lambda_{p-1})}. \quad [2.4.12]$$

Note again that these are identical to expression [1.2.25] in Chapter 1. Recall from the discussion there that $c_1 + c_2 + \cdots + c_p = 1$.

To conclude, [2.4.7] can be written

$$\begin{aligned} y_t &= \frac{c_1}{(1 - \lambda_1 L)} w_t + \frac{c_2}{(1 - \lambda_2 L)} w_t + \cdots + \frac{c_p}{(1 - \lambda_p L)} w_t \\ &= c_1(1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \cdots)w_t + c_2(1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \cdots)w_t \\ &\quad + \cdots + c_p(1 + \lambda_p L + \lambda_p^2 L^2 + \lambda_p^3 L^3 + \cdots)w_t \end{aligned}$$

or

$$\begin{aligned} y_t &= [c_1 + c_2 + \cdots + c_p]w_t + [c_1\lambda_1 + c_2\lambda_2 + \cdots + c_p\lambda_p]w_{t-1} \\ &\quad + [c_1\lambda_1^2 + c_2\lambda_2^2 + \cdots + c_p\lambda_p^2]w_{t-2} \\ &\quad + [c_1\lambda_1^3 + c_2\lambda_2^3 + \cdots + c_p\lambda_p^3]w_{t-3} + \cdots \end{aligned} \quad [2.4.13]$$

where (c_1, c_2, \dots, c_p) are given by equations [2.4.10] through [2.4.12]. Again, the dynamic multiplier can be read directly off [2.4.13]:

$$\frac{\partial y_{t+j}}{\partial w_t} = [c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j], \quad [2.4.14]$$

reproducing the result from Chapter 1.

There is a very convenient way to calculate the effect of w on the present value of y using the lag operator representation. Write [2.4.13] as

$$y_t = \psi_0 w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \cdots \quad [2.4.15]$$

where

$$\psi_j = [c_1\lambda_1^j + c_2\lambda_2^j + \cdots + c_p\lambda_p^j]. \quad [2.4.16]$$

Next rewrite [2.4.15] in lag operator notation as

$$y_t = \psi(L)w_t, \quad [2.4.17]$$

where $\psi(L)$ denotes an infinite-order polynomial in the lag operator:

$$\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \cdots.$$

Notice that ψ_j is the dynamic multiplier [2.4.14]. The effect of w_t on the present value of y is given by

$$\begin{aligned}\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} &= \sum_{j=0}^{\infty} \beta^j \frac{\partial y_{t+j}}{\partial w_t} \\ &= \sum_{j=0}^{\infty} \beta^j \psi_j.\end{aligned}\quad [2.4.18]$$

Thinking of $\psi(z)$ as a polynomial in a real number z ,

$$\psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \dots,$$

it appears that the multiplier [2.4.18] is simply this polynomial evaluated at $z = \beta$:

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} = \psi(\beta) = \psi_0 + \psi_1 \beta + \psi_2 \beta^2 + \psi_3 \beta^3 + \dots \quad [2.4.19]$$

But comparing [2.4.17] with [2.4.7], it is apparent that

$$\psi(L) = [(1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L)]^{-1},$$

and from [2.4.3] this means that

$$\psi(L) = [1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p]^{-1}.$$

We conclude that

$$\psi(z) = [1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p]^{-1}$$

for any value of z , so, in particular,

$$\psi(\beta) = [1 - \phi_1 \beta - \phi_2 \beta^2 - \cdots - \phi_p \beta^p]^{-1}. \quad [2.4.20]$$

Substituting [2.4.20] into [2.4.19] reveals that

$$\frac{\partial \sum_{j=0}^{\infty} \beta^j y_{t+j}}{\partial w_t} = \frac{1}{1 - \phi_1 \beta - \phi_2 \beta^2 - \cdots - \phi_p \beta^p}, \quad [2.4.21]$$

reproducing the claim in Proposition 1.3. Again, the long-run multiplier obtains as the special case of [2.4.21] with $\beta = 1$:

$$\lim_{j \rightarrow \infty} \left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} \right] = \frac{1}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}.$$

2.5. Initial Conditions and Unbounded Sequences

Section 1.2 analyzed the following problem. Given a p th-order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t, \quad [2.5.1]$$

p initial values of y ,

$$y_{-1}, y_{-2}, \dots, y_{-p}, \quad [2.5.2]$$

and a sequence of values for the input variable w ,

$$\{w_0, w_1, \dots, w_t\}, \quad [2.5.3]$$

we sought to calculate the sequence of values for the output variable y :

$$\{y_0, y_1, \dots, y_t\}.$$

Certainly there are systems where the question is posed in precisely this form. We may know the equation of motion for the system [2.5.1] and its current state [2.5.2] and wish to characterize the values that $\{y_0, y_1, \dots, y_t\}$ might take on for different specifications of $\{w_0, w_1, \dots, w_t\}$.

However, there are many examples in economics and finance in which a theory specifies just the equation of motion [2.5.1] and a sequence of driving variables [2.5.3]. Clearly, these two pieces of information alone are insufficient to determine the sequence $\{y_0, y_1, \dots, y_t\}$, and some additional theory beyond that contained in the difference equation [2.5.1] is needed to describe fully the dependence of y on w . These additional restrictions can be of interest in their own right and also help give some insight into some of the technical details of manipulating difference equations. For these reasons, this section discusses in some depth an example of the role of initial conditions and their implications for solving difference equations.

Let P_t denote the price of a stock and D_t its dividend payment. If an investor buys the stock at date t and sells it at $t + 1$, the investor will earn a yield of D_t/P_t from the dividend and a yield of $(P_{t+1} - P_t)/P_t$ in capital gains. The investor's total return (r_{t+1}) is thus

$$r_{t+1} = (P_{t+1} - P_t)/P_t + D_t/P_t.$$

A very simple model of the stock market posits that the return investors earn on stocks is constant across time periods:

$$r = (P_{t+1} - P_t)/P_t + D_t/P_t \quad r > 0. \quad [2.5.4]$$

Equation [2.5.4] may seem too simplistic to be of much practical interest; it assumes among other things that investors have perfect foresight about future stock prices and dividends. However, a slightly more realistic model in which *expected* stock returns are constant involves a very similar set of technical issues. The advantage of the perfect-foresight model [2.5.4] is that it can be discussed using the tools already in hand to gain some further insight into using lag operators to solve difference equations.

Multiply [2.5.4] by P_t to arrive at

$$rP_t = P_{t+1} - P_t + D_t$$

or

$$P_{t+1} = (1 + r)P_t - D_t. \quad [2.5.5]$$

Equation [2.5.5] will be recognized as a first-order difference equation of the form of [1.1.1] with $y_t = P_{t+1}$, $\phi = (1 + r)$, and $w_t = -D_t$. From [1.1.7], we know that [2.5.5] implies that

$$\begin{aligned} P_{t+1} &= (1 + r)^{t+1}P_0 - (1 + r)^tD_0 - (1 + r)^{t-1}D_1 - (1 + r)^{t-2}D_2 \\ &\quad - \dots - (1 + r)D_{t-1} - D_t. \end{aligned} \quad [2.5.6]$$

If the sequence $\{D_0, D_1, \dots, D_t\}$ and the value of P_0 were given, then [2.5.6] could determine the values of $\{P_1, P_2, \dots, P_{t+1}\}$. But if only the values $\{D_0, D_1, \dots, D_t\}$ are given, then equation [2.5.6] would not be enough to pin down $\{P_1, P_2, \dots, P_{t+1}\}$. There are an infinite number of possible sequences $\{P_1, P_2, \dots, P_{t+1}\}$ consistent with [2.5.5] and with a given $\{D_0, D_1, \dots, D_t\}$. This infinite number of possibilities is indexed by the initial value P_0 .

A further simplifying assumption helps clarify the nature of these different paths for $\{P_1, P_2, \dots, P_{t+1}\}$. Suppose that dividends are constant over time:

$$D_t = D \quad \text{for all } t.$$

Then [2.5.6] becomes

$$\begin{aligned} P_{t+1} &= (1 + r)^{t+1}P_0 - [(1 + r)^t + (1 + r)^{t-1} \\ &\quad + \dots + (1 + r) + 1]D \\ &= (1 + r)^{t+1}P_0 - \frac{1 - (1 + r)^{t+1}}{1 - (1 + r)} D \\ &= (1 + r)^{t+1}[P_0 - (D/r)] + (D/r). \end{aligned} \quad [2.5.7]$$

Consider first the solution in which $P_0 = D/r$. If the initial stock price should happen to take this value, then [2.5.7] implies that

$$P_t = D/r \quad [2.5.8]$$

for all t . In this solution, dividends are constant at D and the stock price is constant at D/r . With no change in stock prices, investors never have any capital gains or losses, and their return is solely the dividend yield $D/P = r$. In a world with no changes in dividends this seems to be a sensible expression of the theory represented by [2.5.4]. Equation [2.5.8] is sometimes described as the “market fundamentals” solution to [2.5.4] for the case of constant dividends.

However, even with constant dividends, equation [2.5.8] is not the only result consistent with [2.5.4]. Suppose that the initial price exceeded D/r :

$$P_0 > D/r.$$

Investors seem to be valuing the stock beyond the potential of its constant dividend stream. From [2.5.7] this could be consistent with the asset pricing theory [2.5.4] provided that P_1 exceeds D/r by an even larger amount. As long as investors all believe that prices will continue to rise over time, each will earn the required return r from the realized capital gain and [2.5.4] will be satisfied. This scenario has reminded many economists of a speculative bubble in stock prices.

If such bubbles are to be ruled out, additional knowledge about the process for $\{P_t\}_{t=-\infty}^{\infty}$ is required beyond that contained in the theory of [2.5.4]. For example, we might argue that finite world resources put an upper limit on feasible stock prices, as in

$$|P_t| < \bar{P} \quad \text{for all } t. \quad [2.5.9]$$

Then the only sequence for $\{P_t\}_{t=-\infty}^{\infty}$ consistent with both [2.5.4] and [2.5.9] would be the market fundamentals solution [2.5.8].

Let us now relax the assumption that dividends are constant and replace it with the assumption that $\{D_t\}_{t=-\infty}^{\infty}$ is a bounded sequence. What path for $\{P_t\}_{t=-\infty}^{\infty}$ in [2.5.6] is consistent with [2.5.9] in this case? The answer can be found by returning to the difference equation [2.5.5]. We arrived at the form [2.5.6] by recursively substituting this equation backward. That is, we used the fact that [2.5.5] held for dates $t, t-1, t-2, \dots, 0$ and recursively substituted to arrive at [2.5.6] as a logical implication of [2.5.5]. Equation [2.5.5] could equally well be solved recursively *forward*. To do so, equation [2.5.5] is written as

$$P_t = \frac{1}{1+r} [P_{t+1} + D_t]. \quad [2.5.10]$$

An analogous equation must hold for date $t + 1$:

$$P_{t+1} = \frac{1}{1+r} [P_{t+2} + D_{t+1}]. \quad [2.5.11]$$

Substitute [2.5.11] into [2.5.10] to deduce

$$\begin{aligned} P_t &= \frac{1}{1+r} \left[\frac{1}{1+r} [P_{t+2} + D_{t+1}] + D_t \right] \\ &= \left[\frac{1}{1+r} \right]^2 P_{t+2} + \left[\frac{1}{1+r} \right]^2 D_{t+1} + \left[\frac{1}{1+r} \right] D_t. \end{aligned} \quad [2.5.12]$$

Using [2.5.10] for date $t + 2$,

$$P_{t+2} = \frac{1}{1+r} [P_{t+3} + D_{t+2}],$$

and substituting into [2.5.12] gives

$$P_t = \left[\frac{1}{1+r} \right]^3 P_{t+3} + \left[\frac{1}{1+r} \right]^3 D_{t+2} + \left[\frac{1}{1+r} \right]^2 D_{t+1} + \left[\frac{1}{1+r} \right] D_t.$$

Continuing in this fashion T periods into the future produces

$$\begin{aligned} P_t &= \left[\frac{1}{1+r} \right]^T P_{t+T} + \left[\frac{1}{1+r} \right]^T D_{t+T-1} + \left[\frac{1}{1+r} \right]^{T-1} D_{t+T-2} \\ &\quad + \cdots + \left[\frac{1}{1+r} \right]^2 D_{t+1} + \left[\frac{1}{1+r} \right] D_t. \end{aligned} \quad [2.5.13]$$

If the sequence $\{P_t\}_{t=-\infty}^{\infty}$ is to satisfy [2.5.9], then

$$\lim_{T \rightarrow \infty} \left[\frac{1}{1+r} \right]^T P_{t+T} = 0.$$

If $\{D_t\}_{t=-\infty}^{\infty}$ is likewise a bounded sequence, then the following limit exists:

$$\lim_{T \rightarrow \infty} \sum_{j=0}^T \left[\frac{1}{1+r} \right]^{j+1} D_{t+j}.$$

Thus, if $\{P_t\}_{t=-\infty}^{\infty}$ is to be a bounded sequence, then we can take the limit of [2.5.13] as $T \rightarrow \infty$ to conclude

$$P_t = \sum_{j=0}^{\infty} \left[\frac{1}{1+r} \right]^{j+1} D_{t+j}, \quad [2.5.14]$$

which is referred to as the “market fundamentals” solution of [2.5.5] for the general case of time-varying dividends. Notice that [2.5.14] produces [2.5.8] as a special case when $D_t = D$ for all t .

Describing the value of a variable at time t as a function of future realizations of another variable as in [2.5.14] may seem an artifact of assuming a perfect-foresight model of stock prices. However, an analogous set of operations turns out to be appropriate in a system similar to [2.5.4] in which expected returns are constant.¹ In such systems [2.5.14] generalizes to

$$P_t = \sum_{j=0}^{\infty} \left[\frac{1}{1+r} \right]^{j+1} E_t D_{t+j},$$

¹See Sargent (1987) and Whiteman (1983) for an introduction to the manipulation of difference equations involving expectations.

where E_t denotes an expectation of an unknown future quantity based on information available to investors at date t .

Expression [2.5.14] determines the particular value for the initial price P_0 that is consistent with the boundedness condition [2.5.9]. Setting $t = 0$ in [2.5.14] and substituting into [2.5.6] produces

$$\begin{aligned} P_{t+1} &= (1 + r)^{t+1} \left\{ \left[\frac{1}{1+r} \right] D_0 + \left[\frac{1}{1+r} \right]^2 D_1 + \left[\frac{1}{1+r} \right]^3 D_2 \right. \\ &\quad \left. + \cdots + \left[\frac{1}{1+r} \right]^{t+1} D_t + \left[\frac{1}{1+r} \right]^{t+2} D_{t+1} + \cdots \right\} - (1 + r)^t D_0 \\ &\quad - (1 + r)^{t-1} D_1 - (1 + r)^{t-2} D_2 - \cdots - (1 + r) D_{t-1} - D_t \\ &= \left[\frac{1}{1+r} \right] D_{t+1} + \left[\frac{1}{1+r} \right]^2 D_{t+2} + \left[\frac{1}{1+r} \right]^3 D_{t+3} + \cdots. \end{aligned}$$

Thus, setting the initial condition P_0 to satisfy [2.5.14] is sufficient to ensure that it holds for all t . Choosing P_0 equal to any other value would cause the consequences of each period's dividends to accumulate over time so as to lead to a violation of [2.5.9] eventually.

It is useful to discuss these same calculations from the perspective of lag operators. In Section 2.2 the recursive substitution backward that led from [2.5.5] to [2.5.6] was represented by writing [2.5.5] in terms of lag operators as

$$[1 - (1 + r)L]P_{t+1} = -D_t \quad [2.5.15]$$

and multiplying both sides of [2.5.15] by the following operator:

$$[1 + (1 + r)L + (1 + r)^2L^2 + \cdots + (1 + r)^tL^t]. \quad [2.5.16]$$

If $(1 + r)$ were less than unity, it would be natural to consider the limit of [2.5.16] as $t \rightarrow \infty$:

$$[1 - (1 + r)L]^{-1} = 1 + (1 + r)L + (1 + r)^2L^2 + \cdots.$$

In the case of the theory of stock returns discussed here, however, $r > 0$ and this operator is not defined. In this case, a lag operator representation can be sought for the recursive substitution forward that led from [2.5.5] to [2.5.13]. This is accomplished using the inverse of the lag operator,

$$L^{-1}w_t = w_{t+1},$$

which extends result [2.1.4] to negative values of k . Note that L^{-1} is indeed the inverse of the operator L :

$$L^{-1}(Lw_t) = L^{-1}w_{t-1} = w_t.$$

In general,

$$L^{-k}L^j = L^{j-k},$$

with L^0 defined as the identity operator:

$$L^0w_t \equiv w_t.$$

Now consider multiplying [2.5.15] by

$$[1 + (1+r)^{-1}L^{-1} + (1+r)^{-2}L^{-2} + \cdots + (1+r)^{-(T-1)}L^{-(T-1)}] \times [-(1+r)^{-1}L^{-1}] \quad [2.5.17]$$

to obtain

$$\begin{aligned} & [1 + (1+r)^{-1}L^{-1} + (1+r)^{-2}L^{-2} + \cdots + (1+r)^{-(T-1)}L^{-(T-1)}] \\ & \times [1 - (1+r)^{-1}L^{-1}]P_{t+1} \\ & = [1 + (1+r)^{-1}L^{-1} + (1+r)^{-2}L^{-2} + \cdots \\ & + (1+r)^{-(T-1)}L^{-(T-1)}] \times (1+r)^{-1}D_{t+1} \end{aligned}$$

or

$$\begin{aligned} [1 - (1+r)^{-T}L^{-T}]P_{t+1} &= \left[\frac{1}{1+r} \right] D_{t+1} + \left[\frac{1}{1+r} \right]^2 D_{t+2} \\ &+ \left[\frac{1}{1+r} \right]^3 D_{t+3} + \cdots + \left[\frac{1}{1+r} \right]^T D_{t+T}, \end{aligned}$$

which is identical to [2.5.13] with t in [2.5.13] replaced with $t+1$.

When $r > 0$ and $\{P_t\}_{t=-\infty}^{\infty}$ is a bounded sequence, the left side of the preceding equation will approach P_{t+1} as T becomes large. Thus, when $r > 0$ and $\{P_t\}_{t=-\infty}^{\infty}$ and $\{D_t\}_{t=-\infty}^{\infty}$ are bounded sequences, the limit of the operator in [2.5.17] exists and could be viewed as the inverse of the operator on the left side of [2.5.15]:

$$\begin{aligned} [1 - (1+r)L]^{-1} &= -(1+r)^{-1}L^{-1} \\ &\times [1 + (1+r)^{-1}L^{-1} + (1+r)^{-2}L^{-2} + \cdots]. \end{aligned}$$

Applying this limiting operator to [2.5.15] amounts to solving the difference equation forward as in [2.5.14] and selecting the market fundamentals solution among the set of possible time paths for $\{P_t\}_{t=-\infty}^{\infty}$ given a particular time path for dividends $\{D_t\}_{t=-\infty}^{\infty}$.

Thus, given a first-order difference equation of the form

$$(1 - \phi L)y_t = w_t, \quad [2.5.18]$$

Sargent's (1987) advice was to solve the equation "backward" when $|\phi| < 1$ by multiplying by

$$[1 - \phi L]^{-1} = [1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \cdots] \quad [2.5.19]$$

and to solve the equation "forward" when $|\phi| > 1$ by multiplying by

$$\begin{aligned} [1 - \phi L]^{-1} &= \frac{-\phi^{-1}L^{-1}}{1 - \phi^{-1}L^{-1}} \\ &= -\phi^{-1}L^{-1}[1 + \phi^{-1}L^{-1} + \phi^{-2}L^{-2} + \phi^{-3}L^{-3} + \cdots]. \end{aligned} \quad [2.5.20]$$

Defining the inverse of $[1 - \phi L]$ in this way amounts to selecting an operator $[1 - \phi L]^{-1}$ with the properties that

$$[1 - \phi L]^{-1} \times [1 - \phi L] = 1 \quad (\text{the identity operator})$$

and that, when it is applied to a bounded sequence $\{w_t\}_{t=-\infty}^{\infty}$,

$$[1 - \phi L]^{-1}w_t,$$

the result is another bounded sequence.

The conclusion from this discussion is that in applying an operator such as $[1 - \phi L]^{-1}$, we are implicitly imposing a boundedness assumption that rules out

phenomena such as the speculative bubbles of equation [2.5.7] a priori. Where that is our intention, so much the better, though we should not apply the rules [2.5.19] or [2.5.20] without some reflection on their economic content.

Chapter 2 References

- Sargent, Thomas J. 1987. *Macroeconomic Theory*, 2d ed. Boston: Academic Press.
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Stationary ARMA Processes

This chapter introduces univariate *ARMA* processes, which provide a very useful class of models for describing the dynamics of an individual time series. The chapter begins with definitions of some of the key concepts used in time series analysis. Sections 3.2 through 3.5 then investigate the properties of various *ARMA* processes. Section 3.6 introduces the autocovariance-generating function, which is useful for analyzing the consequences of combining different time series and for an understanding of the population spectrum. The chapter concludes with a discussion of invertibility (Section 3.7), which can be important for selecting the *ARMA* representation of an observed time series that is appropriate given the uses to be made of the model.

3.1. *Expectations, Stationarity, and Ergodicity*

Expectations and Stochastic Processes

Suppose we have observed a sample of size T of some random variable Y_t :

$$\{y_1, y_2, \dots, y_T\}. \quad [3.1.1]$$

For example, consider a collection of T independent and identically distributed (i.i.d.) variables ε_t ,

$$\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_T\}, \quad [3.1.2]$$

with

$$\varepsilon_t \sim N(0, \sigma^2).$$

This is referred to as a sample of size T from a *Gaussian white noise* process.

The observed sample [3.1.1] represents T particular numbers, but this set of T numbers is only one possible outcome of the underlying stochastic process that generated the data. Indeed, even if we were to imagine having observed the process for an infinite period of time, arriving at the sequence

$$\{y_t\}_{t=-\infty}^{\infty} = \{\dots, y_{-1}, y_0, y_1, y_2, \dots, y_T, y_{T+1}, y_{T+2}, \dots\},$$

the infinite sequence $\{y_t\}_{t=-\infty}^{\infty}$ would still be viewed as a single realization from a time series process. For example, we might set one computer to work generating an infinite sequence of i.i.d. $N(0, \sigma^2)$ variates, $\{\varepsilon_t^{(1)}\}_{t=-\infty}^{\infty}$, and a second computer generating a separate sequence, $\{\varepsilon_t^{(2)}\}_{t=-\infty}^{\infty}$. We would then view these as two independent realizations of a Gaussian white noise process.

Imagine a battery of I such computers generating sequences $\{y_t^{(1)}\}_{t=-\infty}^{\infty}$, $\{y_t^{(2)}\}_{t=-\infty}^{\infty}, \dots, \{y_t^{(I)}\}_{t=-\infty}^{\infty}$, and consider selecting the observation associated with date t from each sequence:

$$\{y_t^{(1)}, y_t^{(2)}, \dots, y_t^{(I)}\}.$$

This would be described as a sample of I realizations of the random variable Y_t . This random variable has some density, denoted $f_{Y_t}(y_t)$, which is called the *unconditional density* of Y_t . For example, for the Gaussian white noise process, this density is given by

$$f_{Y_t}(y_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-y_t^2}{2\sigma^2}\right].$$

The *expectation* of the t th observation of a time series refers to the mean of this probability distribution, provided it exists:

$$E(Y_t) \equiv \int_{-\infty}^{\infty} y_t f_{Y_t}(y_t) dy_t. \quad [3.1.3]$$

We might view this as the probability limit of the ensemble average:

$$E(Y_t) = \operatorname{plim}_{I \rightarrow \infty} (1/I) \sum_{i=1}^I Y_t^{(i)}. \quad [3.1.4]$$

For example, if $\{Y_t\}_{t=-\infty}^{\infty}$ represents the sum of a constant μ plus a Gaussian white noise process $\{\varepsilon_t\}_{t=-\infty}^{\infty}$,

$$Y_t = \mu + \varepsilon_t, \quad [3.1.5]$$

then its mean is

$$E(Y_t) = \mu + E(\varepsilon_t) = \mu. \quad [3.1.6]$$

If Y_t is a time trend plus Gaussian white noise,

$$Y_t = \beta t + \varepsilon_t, \quad [3.1.7]$$

then its mean is

$$E(Y_t) = \beta t. \quad [3.1.8]$$

Sometimes for emphasis the expectation $E(Y_t)$ is called the *unconditional mean* of Y_t . The unconditional mean is denoted μ_t :

$$E(Y_t) = \mu_t.$$

Note that this notation allows the general possibility that the mean can be a function of the date of the observation t . For the process [3.1.7] involving the time trend, the mean [3.1.8] is a function of time, whereas for the constant plus Gaussian white noise, the mean [3.1.6] is not a function of time.

The *variance* of the random variable Y_t (denoted γ_{0t}) is similarly defined as

$$\gamma_{0t} \equiv E(Y_t - \mu_t)^2 = \int_{-\infty}^{\infty} (y_t - \mu_t)^2 f_{Y_t}(y_t) dy_t. \quad [3.1.9]$$

For example, for the process [3.1.7], the variance is

$$\gamma_{0t} = E(Y_t - \beta t)^2 = E(\varepsilon_t^2) = \sigma^2.$$

Autocovariance

Given a particular realization such as $\{y_t^{(1)}\}_{t=-\infty}^{\infty}$ on a time series process, consider constructing a vector $\mathbf{x}_t^{(1)}$ associated with date t . This vector consists of the $[j + 1]$ most recent observations on y as of date t for that realization:

$$\mathbf{x}_t^{(1)} = \begin{bmatrix} y_t^{(1)} \\ y_{t-1}^{(1)} \\ \vdots \\ y_{t-j}^{(1)} \end{bmatrix}.$$

We think of each realization $\{y_t\}_{t=-\infty}^{\infty}$ as generating one particular value of the vector \mathbf{x}_t and want to calculate the probability distribution of this vector $\mathbf{x}_t^{(i)}$ across realizations i . This distribution is called the *joint distribution* of $(Y_t, Y_{t-1}, \dots, Y_{t-j})$. From this distribution we can calculate the j th *autocovariance* of Y_t (denoted γ_{jt}):

$$\begin{aligned} \gamma_{jt} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (y_t - \mu_t)(y_{t-j} - \mu_{t-j}) \\ &\quad \times f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}) dy_t dy_{t-1} \cdots dy_{t-j} \quad [3.1.10] \\ &= E(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j}). \end{aligned}$$

Note that [3.1.10] has the form of a covariance between two variables X and Y :

$$\text{Cov}(X, Y) = E(X - \mu_X)(Y - \mu_Y).$$

Thus [3.1.10] could be described as the covariance of Y_t with its own lagged value; hence, the term “autocovariance.” Notice further from [3.1.10] that the 0th autocovariance is just the variance of Y_t , as anticipated by the notation γ_{0t} in [3.1.9].

The autocovariance γ_{jt} can be viewed as the $(1, j + 1)$ element of the variance-covariance matrix of the vector \mathbf{x}_t . For this reason, the autocovariances are described as the second moments of the process for Y_t .

Again it may be helpful to think of the j th autocovariance as the probability limit of an ensemble average:

$$\gamma_{jt} = \lim_{I \rightarrow \infty} (1/I) \sum_{i=1}^I [Y_i^{(i)} - \mu_t] \cdot [Y_{i-j}^{(i)} - \mu_{t-j}]. \quad [3.1.11]$$

As an example of calculating autocovariances, note that for the process in [3.1.5] the autocovariances are all zero for $j \neq 0$:

$$\gamma_{jt} = E(Y_t - \mu)(Y_{t-j} - \mu) = E(\varepsilon_t \varepsilon_{t-j}) = 0 \quad \text{for } j \neq 0.$$

Stationarity

If neither the mean μ_t nor the autocovariances γ_{jt} depend on the date t , then the process for Y_t is said to be *covariance-stationary* or *weakly stationary*:

$$E(Y_t) = \mu \quad \text{for all } t$$

$$E(Y_t - \mu)(Y_{t-j} - \mu) = \gamma_j \quad \text{for all } t \text{ and any } j.$$

For example, the process in [3.1.5] is covariance-stationary:

$$E(Y_t) = \mu$$

$$E(Y_t - \mu)(Y_{t-j} - \mu) = \begin{cases} \sigma^2 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0. \end{cases}$$

By contrast, the process of [3.1.7] is not covariance-stationary, because its mean, βt , is a function of time.

Notice that if a process is covariance-stationary, the covariance between Y_t and Y_{t-j} depends only on j , the length of time separating the observations, and not on t , the date of the observation. It follows that for a covariance-stationary process, γ_j and γ_{-j} would represent the same magnitude. To see this, recall the definition

$$\gamma_j = E(Y_t - \mu)(Y_{t-j} - \mu). \quad [3.1.12]$$

If the process is covariance-stationary, then this magnitude is the same for any value of t we might have chosen; for example, we can replace t with $t + j$:

$$\gamma_j = E(Y_{t+j} - \mu)(Y_{[t+j]-j} - \mu) = E(Y_{t+j} - \mu)(Y_t - \mu) = E(Y_t - \mu)(Y_{t+j} - \mu).$$

But referring again to the definition [3.1.12], this last expression is just the definition of γ_{-j} . Thus, for any covariance-stationary process,

$$\gamma_j = \gamma_{-j} \quad \text{for all integers } j. \quad [3.1.13]$$

A different concept is that of *strict stationarity*. A process is said to be strictly stationary if, for any values of j_1, j_2, \dots, j_n , the joint distribution of $(Y_t, Y_{t+j_1}, Y_{t+j_2}, \dots, Y_{t+j_n})$ depends only on the intervals separating the dates (j_1, j_2, \dots, j_n) and not on the date itself (t). Notice that if a process is strictly stationary with finite second moments, then it must be covariance-stationary—if the densities over which we are integrating in [3.1.3] and [3.1.10] do not depend on time, then the moments μ_t and γ_{jt} will not depend on time. However, it is possible to imagine a process that is covariance-stationary but not strictly stationary; the mean and autocovariances could not be functions of time, but perhaps higher moments such as $E(Y_t^3)$ are.

In this text the term “stationary” by itself is taken to mean “covariance-stationary.”

A process $\{Y_t\}$ is said to be *Gaussian* if the joint density

$$f_{Y_t, Y_{t+j_1}, \dots, Y_{t+j_n}}(y_t, y_{t+j_1}, \dots, y_{t+j_n})$$

is Gaussian for any j_1, j_2, \dots, j_n . Since the mean and variance are all that are needed to parameterize a multivariate Gaussian distribution completely, a covariance-stationary Gaussian process is strictly stationary.

Ergodicity

We have viewed expectations of a time series in terms of ensemble averages such as [3.1.4] and [3.1.11]. These definitions may seem a bit contrived, since usually all one has available is a single realization of size T from the process, which we earlier denoted $\{y_1^{(1)}, y_2^{(1)}, \dots, y_T^{(1)}\}$. From these observations we would calculate the sample mean \bar{y} . This, of course, is not an ensemble average but rather a time average:

$$\bar{y} \equiv (1/T) \sum_{t=1}^T y_t^{(1)}. \quad [3.1.14]$$

Whether time averages such as [3.1.14] eventually converge to the ensemble concept $E(Y_t)$ for a stationary process has to do with *ergodicity*. A covariance-stationary process is said to be *ergodic for the mean* if [3.1.14] converges in probability to $E(Y_t)$ as $T \rightarrow \infty$.¹ A process will be ergodic for the mean provided that the autocovariance γ_j goes to zero sufficiently quickly as j becomes large. In Chapter 7 we will see that if the autocovariances for a covariance-stationary process satisfy

$$\sum_{j=0}^{\infty} |\gamma_j| < \infty, \quad [3.1.15]$$

then $\{Y_t\}$ is ergodic for the mean.

Similarly, a covariance-stationary process is said to be ergodic for second moments if

$$[1/(T-j)] \sum_{t=j+1}^T (Y_t - \mu)(Y_{t-j} - \mu) \xrightarrow{P} \gamma_j$$

for all j . Sufficient conditions for second-moment ergodicity will be presented in Chapter 7. In the special case where $\{Y_t\}$ is a stationary Gaussian process, condition [3.1.15] is sufficient to ensure ergodicity for all moments.

For many applications, stationarity and ergodicity turn out to amount to the same requirements. For purposes of clarifying the concepts of stationarity and ergodicity, however, it may be helpful to consider an example of a process that is stationary but not ergodic. Suppose the mean $\mu^{(i)}$ for the i th realization $\{y_t^{(i)}\}_{t=-\infty}^{\infty}$ is generated from a $N(0, \lambda^2)$ distribution, say

$$Y_t^{(i)} = \mu^{(i)} + \varepsilon_t. \quad [3.1.16]$$

Here $\{\varepsilon_t\}$ is a Gaussian white noise process with mean zero and variance σ^2 that is independent of $\mu^{(i)}$. Notice that

$$\mu_t = E(\mu^{(i)}) + E(\varepsilon_t) = 0.$$

Also,

$$\gamma_{0t} = E(\mu^{(i)} + \varepsilon_t)^2 = \lambda^2 + \sigma^2$$

and

$$\gamma_{jt} = E(\mu^{(i)} + \varepsilon_t)(\mu^{(i)} + \varepsilon_{t-j}) = \lambda^2 \quad \text{for } j \neq 0.$$

Thus the process of [3.1.16] is covariance-stationary. It does not satisfy the sufficient condition [3.1.15] for ergodicity for the mean, however, and indeed, the time average

$$(1/T) \sum_{t=1}^T Y_t^{(i)} = (1/T) \sum_{t=1}^T (\mu^{(i)} + \varepsilon_t) = \mu^{(i)} + (1/T) \sum_{t=1}^T \varepsilon_t$$

converges to $\mu^{(i)}$ rather than to zero, the mean of Y_t .

3.2. White Noise

The basic building block for all the processes considered in this chapter is a sequence $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ whose elements have mean zero and variance σ^2 ,

$$E(\varepsilon_t) = 0 \quad [3.2.1]$$

$$E(\varepsilon_t^2) = \sigma^2, \quad [3.2.2]$$

and for which the ε 's are uncorrelated across time:

¹Often “ergodicity” is used in a more general sense; see Anderson and Moore (1979, p. 319) or Hannan (1970, pp. 201–20).

$$E(\varepsilon_t \varepsilon_\tau) = 0 \quad \text{for } t \neq \tau. \quad [3.2.3]$$

A process satisfying [3.2.1] through [3.2.3] is described as a *white noise process*.

We shall on occasion wish to replace [3.2.3] with the slightly stronger condition that the ε 's are independent across time:

$$\varepsilon_t, \varepsilon_\tau \text{ independent for } t \neq \tau. \quad [3.2.4]$$

Notice that [3.2.4] implies [3.2.3] but [3.2.3] does not imply [3.2.4]. A process satisfying [3.2.1] through [3.2.4] is called an *independent white noise process*.

Finally, if [3.2.1] through [3.2.4] hold along with

$$\varepsilon_t \sim N(0, \sigma^2), \quad [3.2.5]$$

then we have the *Gaussian white noise process*.

3.3. Moving Average Processes

The First-Order Moving Average Process

Let $\{\varepsilon_t\}$ be white noise as in [3.2.1] through [3.2.3], and consider the process

$$Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}, \quad [3.3.1]$$

where μ and θ could be any constants. This time series is called a *first-order moving average process*, denoted *MA(1)*. The term “moving average” comes from the fact that Y_t is constructed from a weighted sum, akin to an average, of the two most recent values of ε .

The expectation of Y_t is given by

$$E(Y_t) = E(\mu + \varepsilon_t + \theta \varepsilon_{t-1}) = \mu + E(\varepsilon_t) + \theta E(\varepsilon_{t-1}) = \mu. \quad [3.3.2]$$

We used the symbol μ for the constant term in [3.3.1] in anticipation of the result that this constant term turns out to be the mean of the process.

The variance of Y_t is

$$\begin{aligned} E(Y_t - \mu)^2 &= E(\varepsilon_t + \theta \varepsilon_{t-1})^2 \\ &= E(\varepsilon_t^2 + 2\theta \varepsilon_t \varepsilon_{t-1} + \theta^2 \varepsilon_{t-1}^2) \\ &= \sigma^2 + 0 + \theta^2 \sigma^2 \\ &= (1 + \theta^2) \sigma^2. \end{aligned} \quad [3.3.3]$$

The first autocovariance is

$$\begin{aligned} E(Y_t - \mu)(Y_{t-1} - \mu) &= E(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-1} + \theta \varepsilon_{t-2}) \\ &= E(\varepsilon_t \varepsilon_{t-1} + \theta \varepsilon_{t-1}^2 + \theta \varepsilon_t \varepsilon_{t-2} + \theta^2 \varepsilon_{t-1} \varepsilon_{t-2}) \\ &= 0 + \theta \sigma^2 + 0 + 0. \end{aligned} \quad [3.3.4]$$

Higher autocovariances are all zero:

$$E(Y_t - \mu)(Y_{t-j} - \mu) = E(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-j} + \theta \varepsilon_{t-j-1}) = 0 \quad \text{for } j > 1. \quad [3.3.5]$$

Since the mean and autocovariances are not functions of time, an *MA(1)* process is covariance-stationary regardless of the value of θ . Furthermore, [3.1.15] is clearly satisfied:

$$\sum_{j=0}^{\infty} |\gamma_j| = (1 + \theta^2) \sigma^2 + |\theta \sigma^2|.$$

Thus, if $\{\varepsilon_t\}$ is Gaussian white noise, then the *MA(1)* process [3.3.1] is ergodic for all moments.

The j th *autocorrelation* of a covariance-stationary process (denoted ρ_j) is defined as its j th autocovariance divided by the variance:

$$\rho_j = \gamma_j / \gamma_0. \quad [3.3.6]$$

Again the terminology arises from the fact that ρ_j is the correlation between Y_t and Y_{t-j} :

$$\text{Corr}(Y_t, Y_{t-j}) = \frac{\text{Cov}(Y_t, Y_{t-j})}{\sqrt{\text{Var}(Y_t)} \sqrt{\text{Var}(Y_{t-j})}} = \frac{\gamma_j}{\sqrt{\gamma_0} \sqrt{\gamma_0}} = \rho_j.$$

Since ρ_j is a correlation, $|\rho_j| \leq 1$ for all j , by the Cauchy-Schwarz inequality. Notice also that the 0th autocorrelation ρ_0 is equal to unity for any covariance-stationary process by definition.

From [3.3.3] and [3.3.4], the first autocorrelation for an $MA(1)$ process is given by

$$\rho_1 = \frac{\theta \sigma^2}{(1 + \theta^2) \sigma^2} = \frac{\theta}{(1 + \theta^2)}. \quad [3.3.7]$$

Higher autocorrelations are all zero.

The autocorrelation ρ_j can be plotted as a function of j as in Figure 3.1. Panel (a) shows the autocorrelation function for white noise, while panel (b) gives the autocorrelation function for the $MA(1)$ process:

$$Y_t = \varepsilon_t + 0.8\varepsilon_{t-1}.$$

For different specifications of θ we would obtain different values for the first autocorrelation ρ_1 in [3.3.7]. Positive values of θ induce positive autocorrelation in the series. In this case, an unusually large value of Y_t is likely to be followed by a larger-than-average value for Y_{t+1} , just as a smaller-than-average Y_t may well be followed by a smaller-than-average Y_{t+1} . By contrast, negative values of θ imply negative autocorrelation—a large Y_t might be expected to be followed by a small value for Y_{t+1} .

The values for ρ_1 implied by different specifications of θ are plotted in Figure 3.2. Notice that the largest possible value for ρ_1 is 0.5; this occurs if $\theta = 1$. The smallest value for ρ_1 is -0.5 , which occurs if $\theta = -1$. For any value of ρ_1 between -0.5 and 0.5 , there are two different values of θ that could produce that autocorrelation. This is because the value of $\theta/(1 + \theta^2)$ is unchanged if θ is replaced by $1/\theta$:

$$\rho_1 = \frac{(1/\theta)}{1 + (1/\theta)^2} = \frac{\theta^2 \cdot (1/\theta)}{\theta^2[1 + (1/\theta)^2]} = \frac{\theta}{\theta^2 + 1}.$$

For example, the processes

$$Y_t = \varepsilon_t + 0.5\varepsilon_{t-1}$$

and

$$Y_t = \varepsilon_t + 2\varepsilon_{t-1}$$

would have the same autocorrelation function:

$$\rho_1 = \frac{2}{(1 + 2^2)} = \frac{0.5}{(1 + 0.5^2)} = 0.4.$$

We will have more to say about the relation between two $MA(1)$ processes that share the same autocorrelation function in Section 3.7.

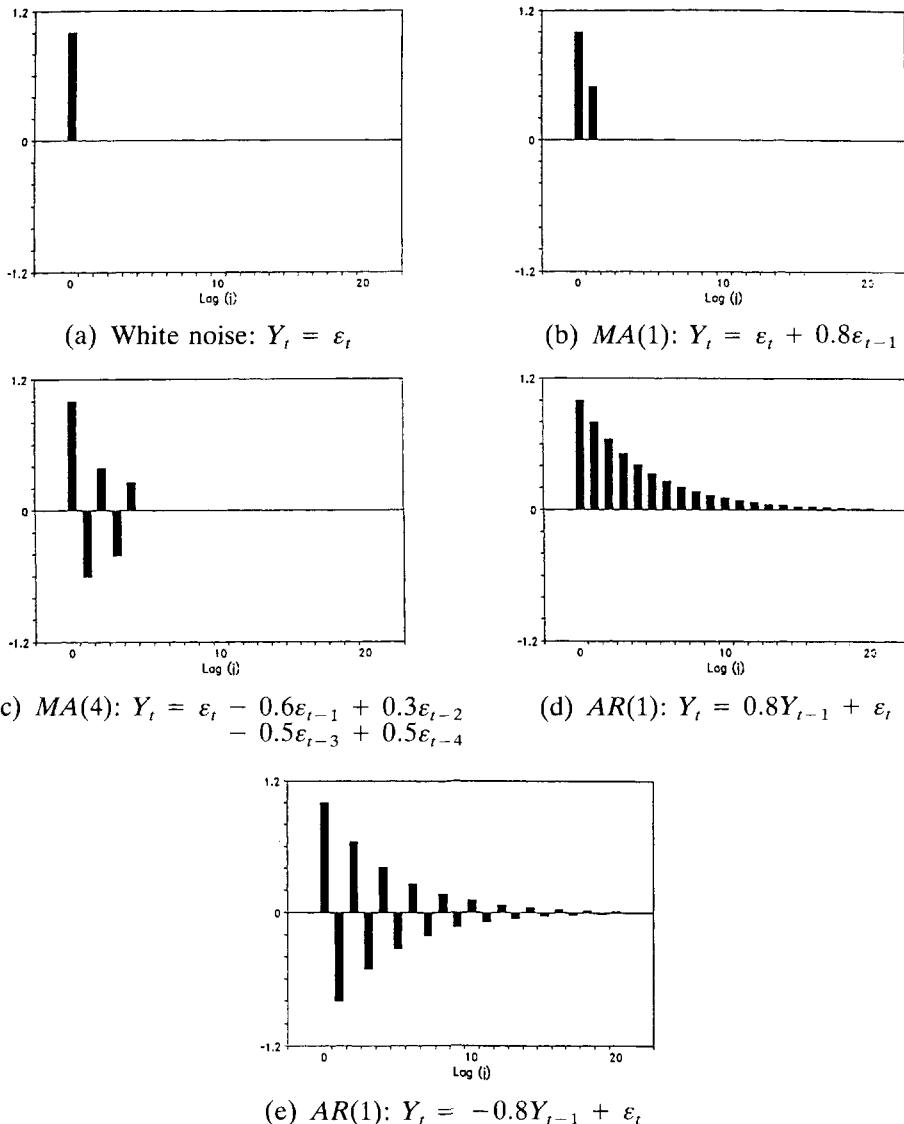


FIGURE 3.1 Autocorrelation functions for assorted ARMA processes.

The qth-Order Moving Average Process

A *qth-order moving average process*, denoted $MA(q)$, is characterized by

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}, \quad [3.3.8]$$

where $\{\varepsilon_t\}$ satisfies [3.2.1] through [3.2.3] and $(\theta_1, \theta_2, \dots, \theta_q)$ could be any real numbers. The mean of [3.3.8] is again given by μ :

$$E(Y_t) = \mu + E(\varepsilon_t) + \theta_1 \cdot E(\varepsilon_{t-1}) + \theta_2 \cdot E(\varepsilon_{t-2}) + \cdots + \theta_q \cdot E(\varepsilon_{t-q}) = \mu.$$

The variance of an $MA(q)$ process is

$$\gamma_0 = E(Y_t - \mu)^2 = E(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q})^2. \quad [3.3.9]$$

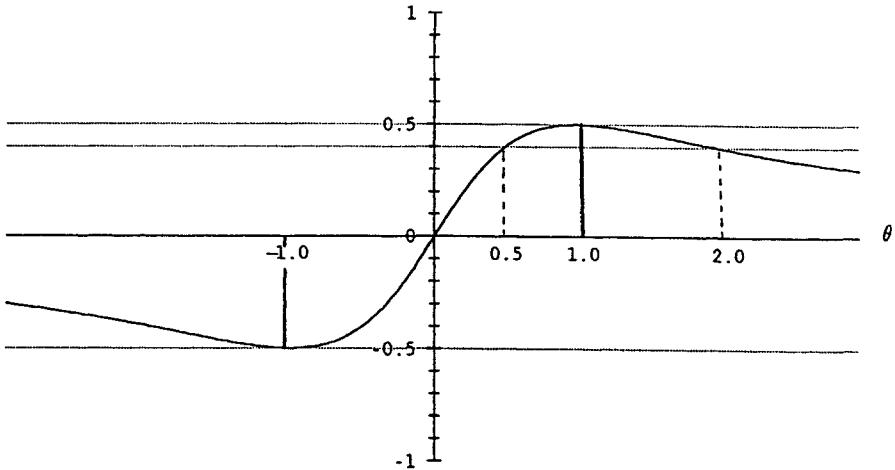


FIGURE 3.2 The first autocorrelation (ρ_1) for an $MA(1)$ process possible for different values of θ .

Since the ε 's are uncorrelated, the variance [3.3.9] is²

$$\gamma_0 = \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 + \cdots + \theta_q^2\sigma^2 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2. \quad [3.3.10]$$

For $j = 1, 2, \dots, q$,

$$\begin{aligned} \gamma_j &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}) \\ &\quad \times (\varepsilon_{t-j} + \theta_1\varepsilon_{t-j-1} + \theta_2\varepsilon_{t-j-2} + \cdots + \theta_q\varepsilon_{t-j-q})] \\ &= E[\theta_j\varepsilon_{t-j}^2 + \theta_{j+1}\theta_1\varepsilon_{t-j-1}^2 + \theta_{j+2}\theta_2\varepsilon_{t-j-2}^2 + \cdots + \theta_q\theta_{q-j}\varepsilon_{t-q}^2]. \end{aligned} \quad [3.3.11]$$

Terms involving ε 's at different dates have been dropped because their product has expectation zero, and θ_0 is defined to be unity. For $j > q$, there are no ε 's with common dates in the definition of γ_j , and so the expectation is zero. Thus,

$$\gamma_j = \begin{cases} [\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots + \theta_q\theta_{q-j}]\cdot\sigma^2 & \text{for } j = 1, 2, \dots, q \\ 0 & \text{for } j > q. \end{cases} \quad [3.3.12]$$

For example, for an $MA(2)$ process,

$$\begin{aligned} \gamma_0 &= [1 + \theta_1^2 + \theta_2^2]\cdot\sigma^2 \\ \gamma_1 &= [\theta_1 + \theta_2\theta_1]\cdot\sigma^2 \\ \gamma_2 &= [\theta_2]\cdot\sigma^2 \\ \gamma_3 &= \gamma_4 = \cdots = 0. \end{aligned}$$

For any values of $(\theta_1, \theta_2, \dots, \theta_q)$, the $MA(q)$ process is thus covariance-stationary. Condition [3.1.15] is satisfied, so for Gaussian ε_t the $MA(q)$ process is also ergodic for all moments. The autocorrelation function is zero after q lags, as in panel (c) of Figure 3.1.

The Infinite-Order Moving Average Process

The $MA(q)$ process can be written

$$Y_t = \mu + \sum_{j=0}^q \theta_j \varepsilon_{t-j}$$

²See equation [A.5.18] in Appendix A at the end of the book.

with $\theta_0 \equiv 1$. Consider the process that results as $q \rightarrow \infty$:

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} = \mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots \quad [3.3.13]$$

This could be described as an $MA(\infty)$ process. To preserve notational flexibility later, we will use ψ 's for the coefficients of an infinite-order moving average process and θ 's for the coefficients of a finite-order moving average process.

Appendix 3.A to this chapter shows that the infinite sequence in [3.3.13] generates a well defined covariance-stationary process provided that

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty. \quad [3.3.14]$$

It is often convenient to work with a slightly stronger condition than [3.3.14]:

$$\sum_{j=0}^{\infty} |\psi_j| < \infty. \quad [3.3.15]$$

A sequence of numbers $\{\psi_j\}_{j=0}^{\infty}$ satisfying [3.3.14] is said to be *square summable*, whereas a sequence satisfying [3.3.15] is said to be *absolutely summable*. Absolute summability implies square-summability, but the converse does not hold—there are examples of square-summable sequences that are not absolutely summable (again, see Appendix 3.A).

The mean and autocovariances of an $MA(\infty)$ process with absolutely summable coefficients can be calculated from a simple extrapolation of the results for an $MA(q)$ process:³

$$E(Y_t) = \lim_{T \rightarrow \infty} E(\mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots + \psi_T \varepsilon_{t-T}) \quad [3.3.16]$$

$$= \mu$$

$$\gamma_0 = E(Y_t - \mu)^2$$

$$= \lim_{T \rightarrow \infty} E(\psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots + \psi_T \varepsilon_{t-T})^2 \quad [3.3.17]$$

$$= \lim_{T \rightarrow \infty} (\psi_0^2 + \psi_1^2 + \psi_2^2 + \dots + \psi_T^2) \cdot \sigma^2$$

$$\gamma_j = E(Y_t - \mu)(Y_{t-j} - \mu) \quad [3.3.18]$$

$$= \sigma^2 (\psi_j \psi_0 + \psi_{j+1} \psi_1 + \psi_{j+2} \psi_2 + \psi_{j+3} \psi_3 + \dots).$$

Moreover, an $MA(\infty)$ process with absolutely summable coefficients has absolutely summable autocovariances:

$$\sum_{j=0}^{\infty} |\gamma_j| < \infty. \quad [3.3.19]$$

Hence, an $MA(\infty)$ process satisfying [3.3.15] is ergodic for the mean (see Appendix 3.A). If the ε 's are Gaussian, then the process is ergodic for all moments.

³Absolute summability of $\{\psi_j\}_{j=0}^{\infty}$ and existence of the second moment $E(\varepsilon_i^2)$ are sufficient conditions to permit interchanging the order of integration and summation. Specifically, if $\{X_T\}_{T=1}^{\infty}$ is a sequence of random variables such that

$$\sum_{T=1}^{\infty} E|X_T| < \infty,$$

then

$$E\left\{ \sum_{T=1}^{\infty} X_T \right\} = \sum_{T=1}^{\infty} E(X_T).$$

See Rao (1973, p. 111).

3.4. Autoregressive Processes

The First-Order Autoregressive Process

A *first-order autoregression*, denoted $AR(1)$, satisfies the following difference equation:

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t. \quad [3.4.1]$$

Again, $\{\varepsilon_t\}$ is a white noise sequence satisfying [3.2.1] through [3.2.3]. Notice that [3.4.1] takes the form of the first-order difference equation [1.1.1] or [2.2.1] in which the input variable w_t is given by $w_t = c + \varepsilon_t$. We know from the analysis of first-order difference equations that if $|\phi| \geq 1$, the consequences of the ε 's for Y accumulate rather than die out over time. It is thus perhaps not surprising that when $|\phi| \geq 1$, there does not exist a covariance-stationary process for Y_t with finite variance that satisfies [3.4.1]. In the case when $|\phi| < 1$, there is a covariance-stationary process for Y_t satisfying [3.4.1]. It is given by the stable solution to [3.4.1] characterized in [2.2.9]:

$$\begin{aligned} Y_t &= (c + \varepsilon_t) + \phi \cdot (c + \varepsilon_{t-1}) + \phi^2 \cdot (c + \varepsilon_{t-2}) + \phi^3 \cdot (c + \varepsilon_{t-3}) + \dots \\ &= [c/(1 - \phi)] + \varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \dots \end{aligned} \quad [3.4.2]$$

This can be viewed as an $MA(\infty)$ process as in [3.3.13] with ψ_j given by ϕ^j . When $|\phi| < 1$, condition [3.3.15] is satisfied:

$$\sum_{j=0}^{\infty} |\psi_j| = \sum_{j=0}^{\infty} |\phi|^j,$$

which equals $1/(1 - |\phi|)$ provided that $|\phi| < 1$. The remainder of this discussion of first-order autoregressive processes assumes that $|\phi| < 1$. This ensures that the $MA(\infty)$ representation exists and can be manipulated in the obvious way, and that the $AR(1)$ process is ergodic for the mean.

Taking expectations of [3.4.2], we see that

$$E(Y_t) = [c/(1 - \phi)] + 0 + 0 + \dots,$$

so that the mean of a stationary $AR(1)$ process is

$$\mu = c/(1 - \phi). \quad [3.4.3]$$

The variance is

$$\begin{aligned} \gamma_0 &= E(Y_t - \mu)^2 \\ &= E(\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \dots)^2 \\ &= (1 + \phi^2 + \phi^4 + \phi^6 + \dots) \cdot \sigma^2 \\ &= \sigma^2/(1 - \phi^2), \end{aligned} \quad [3.4.4]$$

while the j th autocovariance is

$$\begin{aligned} \gamma_j &= E(Y_t - \mu)(Y_{t-j} - \mu) \\ &= E[\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \dots + \phi^j \varepsilon_{t-j} + \phi^{j+1} \varepsilon_{t-j-1} \\ &\quad + \phi^{j+2} \varepsilon_{t-j-2} + \dots] \times [\varepsilon_{t-j} + \phi \varepsilon_{t-j-1} + \phi^2 \varepsilon_{t-j-2} + \dots] \\ &= [\phi^j + \phi^{j+2} + \phi^{j+4} + \dots] \cdot \sigma^2 \\ &= \phi^j [1 + \phi^2 + \phi^4 + \dots] \cdot \sigma^2 \\ &= [\phi^j / (1 - \phi^2)] \cdot \sigma^2. \end{aligned} \quad [3.4.5]$$

It follows from [3.4.4] and [3.4.5] that the autocorrelation function,

$$\rho_j = \gamma_j / \gamma_0 = \phi^j, \quad [3.4.6]$$

follows a pattern of geometric decay as in panel (d) of Figure 3.1. Indeed, the autocorrelation function [3.4.6] for a stationary $AR(1)$ process is identical to the dynamic multiplier or impulse-response function [1.1.10]; the effect of a one-unit increase in ε_t on Y_{t+j} is equal to the correlation between Y_t and Y_{t+j} . A positive value of ϕ , like a positive value of θ for an $MA(1)$ process, implies positive correlation between Y_t and Y_{t+1} . A negative value of ϕ implies negative first-order but positive second-order autocorrelation, as in panel (e) of Figure 3.1.

Figure 3.3 shows the effect on the appearance of the time series $\{y_t\}$ of varying the parameter ϕ . The panels show realizations of the process in [3.4.1] with $c = 0$ and $\varepsilon_t \sim N(0, 1)$ for different values of the autoregressive parameter ϕ . Panel (a) displays white noise ($\phi = 0$). A series with no autocorrelation looks choppy and patternless to the eye; the value of one observation gives no information about the value of the next observation. For $\phi = 0.5$ (panel (b)), the series seems smoother, with observations above or below the mean often appearing in clusters of modest duration. For $\phi = 0.9$ (panel (c)), departures from the mean can be quite prolonged; strong shocks take considerable time to die out.

The moments for a stationary $AR(1)$ were derived above by viewing it as an $MA(\infty)$ process. A second way to arrive at the same results is to assume that the process is covariance-stationary and calculate the moments directly from the difference equation [3.4.1]. Taking expectations of both sides of [3.4.1],

$$E(Y_t) = c + \phi \cdot E(Y_{t-1}) + E(\varepsilon_t). \quad [3.4.7]$$

Assuming that the process is covariance-stationary,

$$E(Y_t) = E(Y_{t-1}) = \mu. \quad [3.4.8]$$

Substituting [3.4.8] into [3.4.7],

$$\mu = c + \phi\mu + 0$$

or

$$\mu = c/(1 - \phi), \quad [3.4.9]$$

reproducing the earlier result [3.4.3].

Notice that formula [3.4.9] is clearly not generating a sensible statement if $|\phi| \geq 1$. For example, if $c > 0$ and $\phi > 1$, then Y_t in [3.4.1] is equal to a positive constant plus a positive number times its lagged value plus a mean-zero random variable. Yet [3.4.9] seems to assert that Y_t would be negative on average for such a process! The reason that formula [3.4.9] is not valid when $|\phi| \geq 1$ is that we assumed in [3.4.8] that Y_t is covariance-stationary, an assumption which is not correct when $|\phi| \geq 1$.

To find the second moments of Y_t in an analogous manner, use [3.4.3] to rewrite [3.4.1] as

$$Y_t = \mu(1 - \phi) + \phi Y_{t-1} + \varepsilon_t$$

or

$$(Y_t - \mu) = \phi(Y_{t-1} - \mu) + \varepsilon_t. \quad [3.4.10]$$

Now square both sides of [3.4.10] and take expectations:

$$E(Y_t - \mu)^2 = \phi^2 E(Y_{t-1} - \mu)^2 + 2\phi E[(Y_{t-1} - \mu)\varepsilon_t] + E(\varepsilon_t^2). \quad [3.4.11]$$

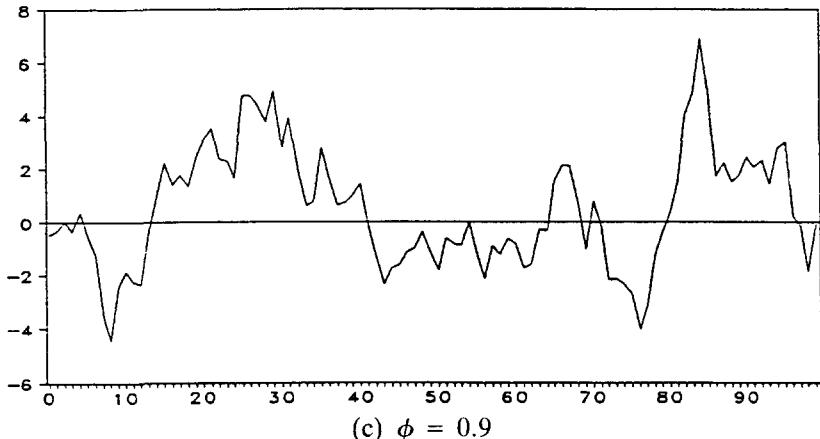
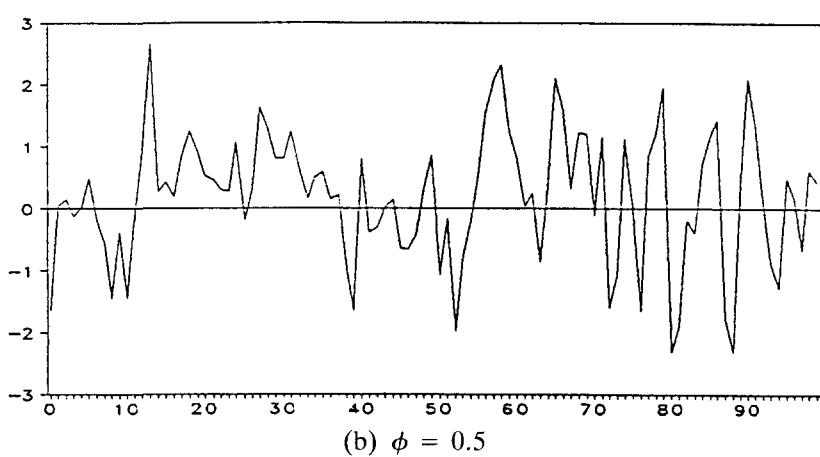
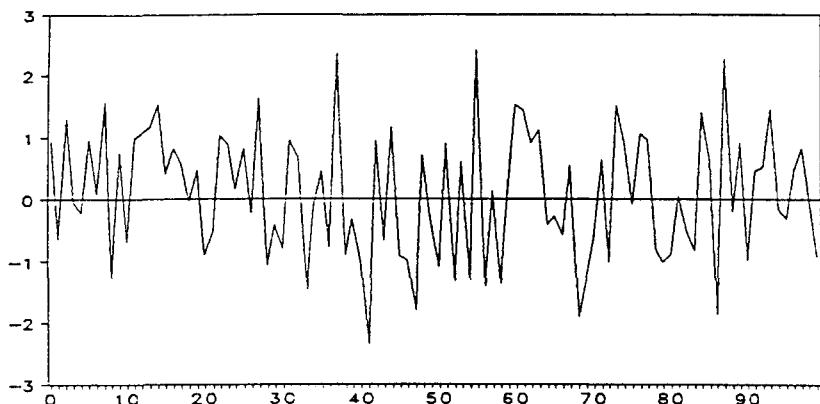


FIGURE 3.3 Realizations of an $AR(1)$ process, $Y_t = \phi Y_{t-1} + \varepsilon_t$, for alternative values of ϕ .

Recall from [3.4.2] that $(Y_{t-1} - \mu)$ is a linear function of $\varepsilon_{t-1}, \varepsilon_{t-2}, \dots$:

$$(Y_{t-1} - \mu) = \varepsilon_{t-1} + \phi\varepsilon_{t-2} + \phi^2\varepsilon_{t-3} + \dots.$$

But ε_t is uncorrelated with $\varepsilon_{t-1}, \varepsilon_{t-2}, \dots$, so ε_t must be uncorrelated with $(Y_{t-1} - \mu)$. Thus the middle term on the right side of [3.4.11] is zero:

$$E[(Y_{t-1} - \mu)\varepsilon_t] = 0. \quad [3.4.12]$$

Again, assuming covariance-stationarity, we have

$$E(Y_t - \mu)^2 = E(Y_{t-1} - \mu)^2 = \gamma_0. \quad [3.4.13]$$

Substituting [3.4.13] and [3.4.12] into [3.4.11],

$$\gamma_0 = \phi^2\gamma_0 + 0 + \sigma^2$$

or

$$\gamma_0 = \sigma^2/(1 - \phi^2),$$

reproducing [3.4.4].

Similarly, we could multiply [3.4.10] by $(Y_{t-j} - \mu)$ and take expectations:

$$\begin{aligned} E[(Y_t - \mu)(Y_{t-j} - \mu)] \\ = \phi \cdot E[(Y_{t-1} - \mu)(Y_{t-j} - \mu)] + E[\varepsilon_t(Y_{t-j} - \mu)]. \end{aligned} \quad [3.4.14]$$

But the term $(Y_{t-j} - \mu)$ will be a linear function of $\varepsilon_{t-j}, \varepsilon_{t-j-1}, \varepsilon_{t-j-2}, \dots$, which, for $j > 0$, will be uncorrelated with ε_t . Thus, for $j > 0$, the last term on the right side in [3.4.14] is zero. Notice, moreover, that the expression appearing in the first term on the right side of [3.4.14],

$$E[(Y_{t-1} - \mu)(Y_{t-j} - \mu)],$$

is the autocovariance of observations on Y separated by $j - 1$ periods:

$$E[(Y_{t-1} - \mu)(Y_{[t-1]-[j-1]} - \mu)] = \gamma_{j-1}.$$

Thus, for $j > 0$, [3.4.14] becomes

$$\gamma_j = \phi\gamma_{j-1}. \quad [3.4.15]$$

Equation [3.4.15] takes the form of a first-order difference equation,

$$y_t = \phi y_{t-1} + w_t,$$

in which the autocovariance γ takes the place of the variable y and in which the subscript j (which indexes the order of the autocovariance) replaces t (which indexes time). The input w_t in [3.4.15] is identically equal to zero. It is easy to see that the difference equation [3.4.15] has the solution

$$\gamma_j = \phi^j\gamma_0,$$

which reproduces [3.4.6]. We now see why the impulse-response function and autocorrelation function for an AR(1) process coincide—they both represent the solution to a first-order difference equation with autoregressive parameter ϕ , an initial value of unity, and no subsequent shocks.

The Second-Order Autoregressive Process

A second-order autoregression, denoted AR(2), satisfies

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t, \quad [3.4.16]$$

or, in lag operator notation,

$$(1 - \phi_1 L - \phi_2 L^2) Y_t = c + \varepsilon_t. \quad [3.4.17]$$

The difference equation [3.4.16] is stable provided that the roots of

$$(1 - \phi_1 z - \phi_2 z^2) = 0 \quad [3.4.18]$$

lie outside the unit circle. When this condition is satisfied, the *AR(2)* process turns out to be covariance-stationary, and the inverse of the autoregressive operator in [3.4.17] is given by

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2)^{-1} = \psi_0 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \dots. \quad [3.4.19]$$

Recalling [1.2.44], the value of ψ_j can be found from the $(1, 1)$ element of the matrix \mathbf{F} raised to the j th power, as in expression [1.2.28]. Where the roots of [3.4.18] are distinct, a closed-form expression for ψ_j is given by [1.2.29] and [1.2.25]. Exercise 3.3 at the end of this chapter discusses alternative algorithms for calculating ψ_j .

Multiplying both sides of [3.4.17] by $\psi(L)$ gives

$$Y_t = \psi(L)c + \psi(L)\varepsilon_t. \quad [3.4.20]$$

It is straightforward to show that

$$\psi(L)c = c/(1 - \phi_1 - \phi_2) \quad [3.4.21]$$

and

$$\sum_{j=0}^{\infty} |\psi_j| < \infty; \quad [3.4.22]$$

the reader is invited to prove these claims in Exercises 3.4 and 3.5. Since [3.4.20] is an absolutely summable *MA*(∞) process, its mean is given by the constant term:

$$\mu = c/(1 - \phi_1 - \phi_2). \quad [3.4.23]$$

An alternative method for calculating the mean is to assume that the process is covariance-stationary and take expectations of [3.4.16] directly:

$$E(Y_t) = c + \phi_1 E(Y_{t-1}) + \phi_2 E(Y_{t-2}) + E(\varepsilon_t),$$

implying

$$\mu = c + \phi_1 \mu + \phi_2 \mu + 0,$$

reproducing [3.4.23].

To find second moments, write [3.4.16] as

$$Y_t = \mu \cdot (1 - \phi_1 - \phi_2) + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$$

or

$$(Y_t - \mu) = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \varepsilon_t. \quad [3.4.24]$$

Multiplying both sides of [3.4.24] by $(Y_{t-j} - \mu)$ and taking expectations produces

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \quad \text{for } j = 1, 2, \dots. \quad [3.4.25]$$

Thus, the autocovariances follow the same second-order difference equation as does the process for Y_t , with the difference equation for γ_j indexed by the lag j . The autocovariances therefore behave just as the solutions to the second-order difference equation analyzed in Section 1.2. An *AR(2)* process is covariance-stationary provided that ϕ_1 and ϕ_2 lie within the triangular region of Figure 1.5.

When ϕ_1 and ϕ_2 lie within the triangular region but above the parabola in that figure, the autocovariance function γ_j is the sum of two decaying exponential functions of j . When ϕ_1 and ϕ_2 fall within the triangular region but below the parabola, γ_j is a damped sinusoidal function.

The autocorrelations are found by dividing both sides of [3.4.25] by γ_0 :

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} \quad \text{for } j = 1, 2, \dots \quad [3.4.26]$$

In particular, setting $j = 1$ produces

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

or

$$\rho_1 = \phi_1 / (1 - \phi_2). \quad [3.4.27]$$

For $j = 2$,

$$\rho_2 = \phi_1 \rho_1 + \phi_2. \quad [3.4.28]$$

The variance of a covariance-stationary second-order autoregression can be found by multiplying both sides of [3.4.24] by $(Y_t - \mu)$ and taking expectations:

$$\begin{aligned} E(Y_t - \mu)^2 &= \phi_1 \cdot E(Y_{t-1} - \mu)(Y_t - \mu) + \phi_2 \cdot E(Y_{t-2} - \mu)(Y_t - \mu) \\ &\quad + E(\varepsilon_t)(Y_t - \mu), \end{aligned}$$

or

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2. \quad [3.4.29]$$

The last term (σ^2) in [3.4.29] comes from noticing that

$$\begin{aligned} E(\varepsilon_t)(Y_t - \mu) &= E(\varepsilon_t)[\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \varepsilon_t] \\ &= \phi_1 \cdot 0 + \phi_2 \cdot 0 + \sigma^2. \end{aligned}$$

Equation [3.4.29] can be written

$$\gamma_0 = \phi_1 \rho_1 \gamma_0 + \phi_2 \rho_2 \gamma_0 + \sigma^2. \quad [3.4.30]$$

Substituting [3.4.27] and [3.4.28] into [3.4.30] gives

$$\gamma_0 = \left[\frac{\phi_1^2}{(1 - \phi_2)} + \frac{\phi_2 \phi_1^2}{(1 - \phi_2)} + \phi_2^2 \right] \gamma_0 + \sigma^2$$

or

$$\gamma_0 = \frac{(1 - \phi_2)\sigma^2}{(1 + \phi_2)[(1 - \phi_2)^2 - \phi_1^2]}.$$

The pth-Order Autoregressive Process

A *p*th-order autoregression, denoted $AR(p)$, satisfies

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \varepsilon_t. \quad [3.4.31]$$

Provided that the roots of

$$1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0 \quad [3.4.32]$$

all lie outside the unit circle, it is straightforward to verify that a covariance-stationary representation of the form

$$Y_t = \mu + \psi(L) \varepsilon_t \quad [3.4.33]$$

exists where

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1}$$

and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Assuming that the stationarity condition is satisfied, one way to find the mean is to take expectations of [3.4.31]:

$$\mu = c + \phi_1\mu + \phi_2\mu + \cdots + \phi_p\mu,$$

or

$$\mu = c/(1 - \phi_1 - \phi_2 - \cdots - \phi_p). \quad [3.4.34]$$

Using [3.4.34], equation [3.4.31] can be written

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t. \quad [3.4.35]$$

Autocovariances are found by multiplying both sides of [3.4.35] by $(Y_{t-j} - \mu)$ and taking expectations:

$$\gamma_j = \begin{cases} \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \cdots + \phi_p\gamma_{j-p} & \text{for } j = 1, 2, \dots \\ \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma^2 & \text{for } j = 0. \end{cases} \quad [3.4.36]$$

Using the fact that $\gamma_{-j} = \gamma_j$, the system of equations in [3.4.36] for $j = 0, 1, \dots, p$ can be solved for $\gamma_0, \gamma_1, \dots, \gamma_p$ as functions of $\sigma^2, \phi_1, \phi_2, \dots, \phi_p$. It can be shown⁴ that the $(p \times 1)$ vector $(\gamma_0, \gamma_1, \dots, \gamma_{p-1})'$ is given by the first p elements of the first column of the $(p^2 \times p^2)$ matrix $\sigma^2[\mathbf{I}_{p^2} - (\mathbf{F} \otimes \mathbf{F})]^{-1}$ where \mathbf{F} is the $(p \times p)$ matrix defined in equation [1.2.3] and \otimes indicates the Kronecker product.

Dividing [3.4.36] by γ_0 produces the *Yule-Walker equations*:

$$\rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + \cdots + \phi_p\rho_{j-p} \quad \text{for } j = 1, 2, \dots \quad [3.4.37]$$

Thus, the autocovariances and autocorrelations follow the same p th-order difference equation as does the process itself [3.4.31]. For distinct roots, their solutions take the form

$$\gamma_j = g_1\lambda_1^j + g_2\lambda_2^j + \cdots + g_p\lambda_p^j, \quad [3.4.38]$$

where the eigenvalues $(\lambda_1, \dots, \lambda_p)$ are the solutions to

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0.$$

3.5. Mixed Autoregressive Moving Average Processes

An $ARMA(p, q)$ process includes both autoregressive and moving average terms:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}, \quad [3.5.1]$$

or, in lag operator form,

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)Y_t = c + (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\varepsilon_t. \quad [3.5.2]$$

Provided that the roots of

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0 \quad [3.5.3]$$

⁴The reader will be invited to prove this in Exercise 10.1 in Chapter 10.

lie outside the unit circle, both sides of [3.5.2] can be divided by $(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$ to obtain

$$Y_t = \mu + \psi(L)\varepsilon_t$$

where

$$\begin{aligned}\psi(L) &= \frac{(1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)}{(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)} \\ \sum_{j=0}^{\infty} |\psi_j| &< \infty \\ \mu &= c/(1 - \phi_1 - \phi_2 - \cdots - \phi_p).\end{aligned}$$

Thus, stationarity of an *ARMA* process depends entirely on the autoregressive parameters $(\phi_1, \phi_2, \dots, \phi_p)$ and not on the moving average parameters $(\theta_1, \theta_2, \dots, \theta_q)$.

It is often convenient to write the *ARMA* process [3.5.1] in terms of deviations from the mean:

$$\begin{aligned}Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \cdots \\ &\quad + \phi_p(Y_{t-p} - \mu) + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q}.\end{aligned}\quad [3.5.4]$$

Autocovariances are found by multiplying both sides of [3.5.4] by $(Y_{t-j} - \mu)$ and taking expectations. For $j > q$, the resulting equations take the form

$$\gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \cdots + \phi_p\gamma_{j-p} \quad \text{for } j = q+1, q+2, \dots \quad [3.5.5]$$

Thus, after q lags the autocovariance function γ_j (and the autocorrelation function ρ_j) follow the p th-order difference equation governed by the autoregressive parameters.

Note that [3.5.5] does not hold for $j \leq q$, owing to correlation between $\theta_j\varepsilon_{t-j}$ and Y_{t-j} . Hence, an *ARMA*(p, q) process will have more complicated autocovariances for lags 1 through q than would the corresponding *AR*(p) process. For $j > q$ with distinct autoregressive roots, the autocovariances will be given by

$$\gamma_j = h_1\lambda_1^j + h_2\lambda_2^j + \cdots + h_p\lambda_p^j. \quad [3.5.6]$$

This takes the same form as the autocovariances for an *AR*(p) process [3.4.38], though because the initial conditions $(\gamma_0, \gamma_1, \dots, \gamma_q)$ differ for the *ARMA* and *AR* processes, the parameters h_k in [3.5.6] will not be the same as the parameters g_k in [3.4.38].

There is a potential for redundant parameterization with *ARMA* processes. Consider, for example, a simple white noise process,

$$Y_t = \varepsilon_t. \quad [3.5.7]$$

Suppose both sides of [3.5.7] are multiplied by $(1 - \rho L)$:

$$(1 - \rho L)Y_t = (1 - \rho L)\varepsilon_t. \quad [3.5.8]$$

Clearly, if [3.5.7] is a valid representation, then so is [3.5.8] for any value of ρ . Thus, [3.5.8] might be described as an *ARMA*(1, 1) process, with $\phi_1 = \rho$ and $\theta_1 = -\rho$. It is important to avoid such a parameterization. Since any value of ρ in [3.5.8] describes the data equally well, we will obviously get into trouble trying to estimate the parameter ρ in [3.5.8] by maximum likelihood. Moreover, theoretical manipulations based on a representation such as [3.5.8] may overlook key cancellations. If we are using an *ARMA*(1, 1) model in which θ_1 is close to $-\phi_1$, then the data might better be modeled as simple white noise.

A related overparameterization can arise with an $ARMA(p, q)$ model. Consider factoring the lag polynomial operators in [3.5.2] as in [2.4.3]:

$$(1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L)(Y_t - \mu) = (1 - \eta_1 L)(1 - \eta_2 L) \cdots (1 - \eta_q L)\varepsilon_t. \quad [3.5.9]$$

We assume that $|\lambda_i| < 1$ for all i , so that the process is covariance-stationary. If the autoregressive operator $(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$ and the moving average operator $(1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)$ have any roots in common, say, $\lambda_i = \eta_j$ for some i and j , then both sides of [3.5.9] can be divided by $(1 - \lambda_i L)$:

$$\prod_{\substack{k=1 \\ k \neq i}}^p (1 - \lambda_k L)(Y_t - \mu) = \prod_{\substack{k=1 \\ k \neq j}}^q (1 - \eta_k L)\varepsilon_t,$$

or

$$(1 - \phi_1^* L - \phi_2^* L^2 - \cdots - \phi_{p-1}^* L^{p-1})(Y_t - \mu) = (1 + \theta_1^* L + \theta_2^* L^2 + \cdots + \theta_{q-1}^* L^{q-1})\varepsilon_t, \quad [3.5.10]$$

where

$$\begin{aligned} (1 - \phi_1^* L - \phi_2^* L^2 - \cdots - \phi_{p-1}^* L^{p-1}) \\ \equiv (1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_{i-1} L)(1 - \lambda_{i+1} L) \cdots (1 - \lambda_p L) \\ (1 + \theta_1^* L + \theta_2^* L^2 + \cdots + \theta_{q-1}^* L^{q-1}) \\ \equiv (1 - \eta_1 L)(1 - \eta_2 L) \cdots (1 - \eta_{j-1} L)(1 - \eta_{j+1} L) \cdots (1 - \eta_q L). \end{aligned}$$

The stationary $ARMA(p, q)$ process satisfying [3.5.2] is clearly identical to the stationary $ARMA(p-1, q-1)$ process satisfying [3.5.10].

3.6. The Autocovariance-Generating Function

For each of the covariance-stationary processes for Y_t considered so far, we calculated the sequence of autocovariances $\{\gamma_j\}_{j=-\infty}^{\infty}$. If this sequence is absolutely summable, then one way of summarizing the autocovariances is through a scalar-valued function called the *autocovariance-generating function*:

$$g_Y(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j. \quad [3.6.1]$$

This function is constructed by taking the j th autocovariance and multiplying it by some number z raised to the j th power, and then summing over all the possible values of j . The argument of this function (z) is taken to be a complex scalar.

Of particular interest as an argument for the autocovariance-generating function is any value of z that lies on the complex unit circle,

$$z = \cos(\omega) - i \sin(\omega) = e^{-i\omega},$$

where $i = \sqrt{-1}$ and ω is the radian angle that z makes with the real axis. If the autocovariance-generating function is evaluated at $z = e^{-i\omega}$ and divided by 2π , the resulting function of ω ,

$$s_Y(\omega) = \frac{1}{2\pi} g_Y(e^{-i\omega}) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j},$$

is called the *population spectrum* of Y . The population spectrum will be discussed

in detail in Chapter 6. There it will be shown that for a process with absolutely summable autocovariances, the function $s_Y(\omega)$ exists and can be used to calculate all of the autocovariances. This means that if two different processes share the same autocovariance-generating function, then the two processes exhibit the identical sequence of autocovariances.

As an example of calculating an autocovariance-generating function, consider the $MA(1)$ process. From equations [3.3.3] to [3.3.5], its autocovariance-generating function is

$$g_Y(z) = [\theta\sigma^2]z^{-1} + [(1 + \theta^2)\sigma^2]z^0 + [\theta\sigma^2]z^1 = \sigma^2[\theta z^{-1} + (1 + \theta^2) + \theta z].$$

Notice that this expression could alternatively be written

$$g_Y(z) = \sigma^2(1 + \theta z)(1 + \theta z^{-1}). \quad [3.6.2]$$

The form of expression [3.6.2] suggests that for the $MA(q)$ process,

$$Y_t = \mu + (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q) \varepsilon_t,$$

the autocovariance-generating function might be calculated as

$$\begin{aligned} g_Y(z) &= \sigma^2(1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q) \\ &\quad \times (1 + \theta_1 z^{-1} + \theta_2 z^{-2} + \cdots + \theta_q z^{-q}). \end{aligned} \quad [3.6.3]$$

This conjecture can be verified by carrying out the multiplication in [3.6.3] and collecting terms by powers of z :

$$\begin{aligned} &(1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q) \times (1 + \theta_1 z^{-1} + \theta_2 z^{-2} + \cdots + \theta_q z^{-q}) \\ &= (\theta_q)z^q + (\theta_{q-1} + \theta_q \theta_1)z^{(q-1)} + (\theta_{q-2} + \theta_{q-1}\theta_1 + \theta_q \theta_2)z^{(q-2)} \\ &\quad + \cdots + (\theta_1 + \theta_2 \theta_1 + \theta_3 \theta_2 + \cdots + \theta_q \theta_{q-1})z^1 \\ &\quad + (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)z^0 \\ &\quad + (\theta_1 + \theta_2 \theta_1 + \theta_3 \theta_2 + \cdots + \theta_q \theta_{q-1})z^{-1} + \cdots + (\theta_q)z^{-q}. \end{aligned} \quad [3.6.4]$$

Comparison of [3.6.4] with [3.3.10] or [3.3.12] confirms that the coefficient on z^j in [3.6.3] is indeed the j th autocovariance.

This method for finding $g_Y(z)$ extends to the $MA(\infty)$ case. If

$$Y_t = \mu + \psi(L) \varepsilon_t \quad [3.6.5]$$

with

$$\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots \quad [3.6.6]$$

and

$$\sum_{j=0}^{\infty} |\psi_j| < \infty, \quad [3.6.7]$$

then

$$g_Y(z) = \sigma^2 \psi(z) \psi(z^{-1}). \quad [3.6.8]$$

For example, the stationary $AR(1)$ process can be written as

$$Y_t - \mu = (1 - \phi L)^{-1} \varepsilon_t,$$

which is in the form of [3.6.5] with $\psi(L) = 1/(1 - \phi L)$. The autocovariance-generating function for an $AR(1)$ process could therefore be calculated from

$$g_Y(z) = \frac{\sigma^2}{(1 - \phi z)(1 - \phi z^{-1})}. \quad [3.6.9]$$

To verify this claim directly, expand out the terms in [3.6.9]:

$$\begin{aligned}\frac{\sigma^2}{(1 - \phi z)(1 - \phi z^{-1})} &= \sigma^2(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \dots) \\ &\quad \times (1 + \phi z^{-1} + \phi^2 z^{-2} + \phi^3 z^{-3} + \dots),\end{aligned}$$

from which the coefficient on z^j is

$$\sigma^2(\phi^j + \phi^{j+1}\phi + \phi^{j+2}\phi^2 + \dots) = \sigma^2\phi^j/(1 - \phi^2).$$

This indeed yields the j th autocovariance as earlier calculated in equation [3.4.5].

The autocovariance-generating function for a stationary $ARMA(p, q)$ process can be written

$$g_Y(z) = \frac{\sigma^2(1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q)(1 + \theta_1 z^{-1} + \theta_2 z^{-2} + \dots + \theta_q z^{-q})}{(1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p)(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \dots - \phi_p z^{-p})}. \quad [3.6.10]$$

Filters

Sometimes the data are *filtered*, or treated in a particular way before they are analyzed, and we would like to summarize the effects of this treatment on the autocovariances. This calculation is particularly simple using the autocovariance-generating function. For example, suppose that the original data Y_t were generated from an $MA(1)$ process,

$$Y_t = (1 + \theta L)\varepsilon_t, \quad [3.6.11]$$

with autocovariance-generating function given by [3.6.2]. Let's say that the data as actually analyzed, X_t , represent the change in Y_t over its value the previous period:

$$X_t = Y_t - Y_{t-1} = (1 - L)Y_t. \quad [3.6.12]$$

Substituting [3.6.11] into [3.6.12], the observed data can be characterized as the following $MA(2)$ process,

$$X_t = (1 - L)(1 + \theta L)\varepsilon_t = [1 + (\theta - 1)L - \theta L^2]\varepsilon_t = [1 + \theta_1 L + \theta_2 L^2]\varepsilon_t, \quad [3.6.13]$$

with $\theta_1 \equiv (\theta - 1)$ and $\theta_2 \equiv -\theta$. The autocovariance-generating function of the observed data X_t can be calculated by direct application of [3.6.3]:

$$g_X(z) = \sigma^2(1 + \theta_1 z + \theta_2 z^2)(1 + \theta_1 z^{-1} + \theta_2 z^{-2}). \quad [3.6.14]$$

It is often instructive, however, to keep the polynomial $(1 + \theta_1 z + \theta_2 z^2)$ in its factored form of the first line of [3.6.13],

$$(1 + \theta_1 z + \theta_2 z^2) = (1 - z)(1 + \theta z),$$

in which case [3.6.14] could be written

$$\begin{aligned}g_X(z) &= \sigma^2(1 - z)(1 + \theta z)(1 - z^{-1})(1 + \theta z^{-1}) \\ &= (1 - z)(1 - z^{-1}) \cdot g_Y(z).\end{aligned} \quad [3.6.15]$$

Of course, [3.6.14] and [3.6.15] represent the identical function of z , and which way we choose to write it is simply a matter of convenience. Applying the filter

$(1 - L)$ to Y_t thus results in multiplying its autocovariance-generating function by $(1 - z)(1 - z^{-1})$.

This principle readily generalizes. Suppose that the original data series $\{Y_t\}$ satisfies [3.6.5] through [3.6.7]. Let's say the data are filtered according to

$$X_t = h(L)Y_t \quad [3.6.16]$$

with

$$\begin{aligned} h(L) &= \sum_{j=-\infty}^{\infty} h_j L^j \\ \sum_{j=-\infty}^{\infty} |h_j| &< \infty. \end{aligned}$$

Substituting [3.6.5] into [3.6.16], the observed data X_t are then generated by

$$X_t = h(1)\mu + h(L)\psi(L)\varepsilon_t \equiv \mu^* + \psi^*(L)\varepsilon_t,$$

where $\mu^* \equiv h(1)\mu$ and $\psi^*(L) \equiv h(L)\psi(L)$. The sequence of coefficients associated with the compound operator $\{\psi_j^*\}_{j=-\infty}^{\infty}$ turns out to be absolutely summable,⁵ and the autocovariance-generating function of X_t can accordingly be calculated as

$$g_X(z) = \sigma^2 \psi^*(z) \psi^*(z^{-1}) = \sigma^2 h(z) \psi(z) \psi(z^{-1}) h(z^{-1}) = h(z) h(z^{-1}) g_Y(z). \quad [3.6.17]$$

Applying the filter $h(L)$ to a series thus results in multiplying its autocovariance-generating function by $h(z)h(z^{-1})$.

3.7. Invertibility

Invertibility for the MA(1) Process

Consider an MA(1) process,

$$Y_t - \mu = (1 + \theta L)\varepsilon_t, \quad [3.7.1]$$

with

$$E(\varepsilon_t \varepsilon_\tau) = \begin{cases} \sigma^2 & \text{for } t = \tau \\ 0 & \text{otherwise.} \end{cases}$$

⁵Specifically,

$$\begin{aligned} \psi^*(z) &= \left(\sum_{j=-\infty}^{\infty} h_j z^j \right) \left(\sum_{k=0}^{\infty} \psi_k z^k \right) \\ &= (\dots + h_{-j} z^{-j} + h_{-j+1} z^{-j+1} + \dots + h_{-1} z^{-1} + h_0 z^0 + h_1 z^1 + \dots \\ &\quad + h_j z^j + h_{j+1} z^{j+1} + \dots)(\psi_0 z^0 + \psi_1 z^1 + \psi_2 z^2 + \dots), \end{aligned}$$

from which the coefficient on z^j is

$$\psi_j^* = h_j \psi_0 + h_{j-1} \psi_1 + h_{j-2} \psi_2 + \dots = \sum_{v=0}^{\infty} h_{j-v} \psi_v.$$

Then

$$\sum_{j=-\infty}^{\infty} |\psi_j^*| = \sum_{j=-\infty}^{\infty} \left| \sum_{v=0}^{\infty} h_{j-v} \psi_v \right| \leq \sum_{j=-\infty}^{\infty} \sum_{v=0}^{\infty} |h_{j-v}| |\psi_v| = \sum_{v=0}^{\infty} |\psi_v| \sum_{j=-\infty}^{\infty} |h_{j-v}| = \sum_{v=0}^{\infty} |\psi_v| \sum_{j=-\infty}^{\infty} |h_j| < \infty.$$