Summary of Physics equations

Me

1 Math

Factorial

$$n! = n(n-1)(n-2)...1 = \prod_{i=1}^{n} i \tag{1}$$

Combinatory

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \tag{2}$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \tag{3}$$

Pascal Triangle

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Property of Combinatory (deduced by using Pascal Triangle)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{4}$$

2 Linear Algebra

Vector (v_i is the i^{th} element of $\vec{v}$)

$$\vec{v} = [v_i] = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{5}$$

Matrix

$$A = [A_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{6}$$

Norm of a vector ("length" of the vector)

$$\|\vec{u}\| = \sqrt{\sum_i u_i^2} \tag{7}$$

Dot product (θ is the angle between u and v)

$$\vec{u} \cdot \vec{v} = \sum_i u_i v_i = \|\vec{u}\| \|\vec{v}\| \cos(\theta) \tag{8}$$

Unit Vector

$$\hat{\mathbf{u}} = \frac{\vec{u}}{\|\vec{u}\|} \tag{9}$$

Cross product ($\hat{\mathbf{i}}$ is the unit vector in the x-axis,$\hat{\mathbf{j}}$ is the unit vector in the y-axis,$\hat{\mathbf{k}}$ is the unit vector in the z-axis)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{10}$$

Norm of the cross product (θ is the angle between u and v)

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin(\theta) \tag{11}$$

Product Matrix - Vector

$$\vec{v} = A\vec{u} \implies [v_i] = [\sum_k A_{ik} u_k] \tag{12}$$

Product Matrix - Matrix

$$A = BC \implies [A_{ij}] = [\sum_k B_{ik} C_{kj}] \tag{13}$$

Triple scalar product (Proof by using a paralelepiped construct using the vectors $\vec{a}$,$\vec{b}$ and $\vec{c}$)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a}) \tag{14}$$

Triple cross product (This is by analyzing that $\vec{a} \times (\vec{b} \times \vec{c})$ is in the plane made by these two vectors $\vec{a}$ and $\vec{b}$)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{15}$$

Line Equation ($\vec{\alpha}_0$ is a reference point to the line, $\mathbf{\hat{u}}$ is a unit vector in the direction of the line and t is a just parameter)

$$\vec{r} = \vec{\alpha}_0 + t\mathbf{\hat{u}} \tag{16}$$

Plane Equation ($\vec{r}_0$ is a point of reference in the plane, $\mathbf{\hat{n}}$ is a unit vector normal to the plane)

$$\mathbf{\hat{n}} \cdot (\vec{r} - \vec{r}_0) = \vec{0} \tag{17}$$

3 Calculus

Derivative

$$\frac{df}{dx} = f'(x) = lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{18}$$

Integrative

$$F(x) = \int f(x)dx \tag{19}$$

$$f(x) = \frac{dF}{dx} \tag{20}$$

Area (A) under the curve f from a to b

$$A_{a \to b} = \int_a^b f(x)dx = F(b) - F(a) \tag{21}$$

Rapid Proof

$$f(a)h = A_{a \to a+h} = \int_a^{a+h} f(x)dx = F(a+h) - F(a) \tag{22}$$

when $h \to 0$

$$f(a) = \frac{F(a+h) - F(a)}{h} = \frac{dF}{dx}\bigg|_{x=a} = F'(x=a) = f(a) \tag{23}$$

Neperian number (e)

$$e = lim_{n \to \infty} (1 + \frac{1}{n})^n \tag{24}$$

4 Multivariable Calculus

Gradient of a function

$$\nabla f(\vec{r}) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \tag{25}$$

$$df = \nabla f \cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{26}$$

when $\vec{h} \to \vec{0}$

$$df = f(\vec{r} + \vec{h}) - f(\vec{r}) = \nabla f \cdot \vec{h} \tag{27}$$

Divergence

$$\nabla \cdot \vec{F}(\vec{r}) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \tag{28}$$

Rotational

$$\nabla \times \vec{F}(\vec{r}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{29}$$

Gauss Theorem

$$\oint_{\partial V} \vec{F} \cdot d\vec{S} = \int \nabla \cdot \vec{F} dV \tag{30}$$

Stokes Theorem

$$\oint_{\partial A} \vec{F} \cdot d\vec{l} = \int \nabla \times \vec{F} \cdot d\vec{S} \tag{31}$$

5 Complex Numbers

Definition of complex unit (i and sometimes j)

$$i^2 = -1 \tag{32}$$

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{33}$$

Euler's identity

$$e^{i\pi} + 1 = 0 \tag{34}$$

Arithmetic with complex numbers ($z_1 = x_1 + iy_i$ and $z_2 = x_2 + iy_2$ with $x_1, y_1, x_2, y_2 \in \mathbb{R}$)

$$\begin{aligned} z_1 \pm z_2 &= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ z_1 \times z_2 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned} \tag{35}$$

Poles of a function f(z) (z_i, in general it is a singularity)

$$lim_{z \to z_i} f(z) \to \infty \tag{36}$$

Residue Theorem

$$\oint_{\mathscr{C}} f(z)dz = \frac{1}{2\pi i}(\sum_i lim_{z \to z_i} f(z)(z - z_i)) \tag{37}$$

6 Statistics

Mean

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n} \tag{38}$$

Standard Deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}} \tag{39}$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{40}$$

Binomial Distribution (p is the probability that the positive event happened, this distribution is used when there is just 2 possible outcomes)

$$B(p, x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{41}$$

Poisson Distribution

$$P_o(x) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{42}$$

7 Probability

Sample of Events

$$S = \{E_1, E_2, E_3, \ldots, E_n\} \tag{43}$$

Probability of an event E_i is $P(E_i)$

$$\sum_i P(E_i) = 1 \tag{44}$$

In continuous

$$\int_\Omega P(x)dx = 1 \tag{45}$$

Expected Value

$$x = \sum_i x_i P(x_i) \tag{46}$$

$$x = \int_\Omega xP(x)dx \tag{47}$$

Example: Dices - Expected value ($P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$)

$$1P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6) = \frac{21}{6} = 3.5 \tag{48}$$

If A and B are indepents:

$$P(A \cup B) = P(A) + P(B) \tag{49}$$

If A and B are not indepents:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{50}$$

Conditional Probability (probability that A happens if B happened)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{51}$$

Bayes Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{52}$$

8 Transformations and Series

All functions can be expressed as a polynomial of infinity degree (This is used to solve ordinary differential equations, just replace this in the equation and find the coefficients a_i)

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \tag{53}$$

Taylor Series (expansion around x_0, f^i is the i^{th}derivative of f)

$$f(x) = \sum_{i=0}^{\infty} \frac{f^i(x_0)(x - x_0)^i}{i!} \tag{54}$$

Laplace (To solve linear differential equations)

$$F(p) = \mathscr{L}\{f(x)\} = \int_0^{\infty} f(x)e^{-px}dx \tag{55}$$

Fourier Series for periodic functions (To solve partial differential equations)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n sin(\frac{2\pi n}{T}x) + b_n cos(\frac{2\pi n}{T}x) \tag{56}$$

Fourier Transform

$$F(w) = \mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-jwx}dx \tag{57}$$

$$f(x) = \mathscr{F}^{-1}\{F(w)\} = \int_{-\infty}^{\infty} F(w)e^{jwx}dx \tag{58}$$

9 Mechanics

Linear Momentum definition

$$\vec{P} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{59}$$

Newton's First Law

$$\sum \vec{F}_i^{ext} = \vec{0} \implies \sum \vec{P}_i^{sys} = constant \tag{60}$$

Newton's Second Law

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{61}$$

$$m = constant \implies \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} \tag{62}$$

Newton's Third Law

$$\vec{F}_{ij} = -\vec{F}_{ij} \tag{63}$$

Center of Mass

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \tag{64}$$

Rigid Body ($\vec{r'}_i$ is the position of any particle of the rigid body with respect of the center of mass)

$$\vec{r}_i = \vec{r}_{CM} + \vec{r'}_i \tag{65}$$

Since it is a rigid body then:

$$\|\vec{r'}_i\| = constant \tag{66}$$

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times m\vec{v} = \vec{r} \times m(\vec{w} \times \vec{r}) = m(\vec{r} \cdot \vec{r})\vec{w} = mr^2\vec{w} \tag{67}$$

Inertia Moment (where r_i is the perpendicular distance from the position of mass m_i to the rotation axis)

$$I = \sum_i m_i r_i^2 \tag{68}$$

$$I = \int_V r^2 dm \tag{69}$$

Angular Momentum (General Definition)

$$\vec{L} = I\vec{w} = I\frac{d\vec{\theta}}{dt} \tag{70}$$

Angular Momentum Conservation

$$\sum \vec{\tau}_i^{ext} = 0 \implies \sum \vec{L}_i^{sys} \tag{71}$$

Torque

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \tag{72}$$

Gravitation Law (Force exerted on m_i by m_j)

$$\vec{F}_g = -Gm_i m_j \frac{\vec{r}_i - \vec{r}_j}{\|\vec{r}_i - \vec{r}_j\|^3} \tag{73}$$

Work definition

$$W_{\vec{r}_0 \Longrightarrow \vec{r}_f} = \int_{\vec{r}_0}^{\vec{r}_f} \vec{F} \cdot d\vec{r} \tag{74}$$

This lead to energy concept and energy conservation:

$$m\vec{a} = \vec{F} \implies \int (m\vec{a} - \vec{F}) \cdot d\vec{r} = \int \vec{0} \cdot d\vec{r} = 0 \tag{75}$$

Kinetic Energy

$$\Delta E_k = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_{v_i}^{v_f} m\vec{v} \cdot d\vec{v} = \frac{mv_f^2}{2} - \frac{mv_0^2}{2} \tag{76}$$

$$E_k = \frac{mv^2}{2} \tag{77}$$

Potential Energy (E_p or U)

$$\Delta E_p = E_p(r_f) - E_p(r_0) = \int_{\vec{r}_0}^{\vec{r}_f} \nabla U(\vec{r}) \cdot d\vec{r} \tag{78}$$

For the case of Gravitational forces ($\vec{r}_j = \vec{0}$ and $\vec{r}_i = \vec{r}$)

$$\Delta E_p = \int_{\vec{r}_0}^{\vec{r}_f} -\vec{F} \cdot d\vec{r} = \int_{\vec{r}_0}^{\vec{r}_f} Gm_i m_j \frac{\vec{r}}{\|\vec{r}\|^3} \cdot d\vec{r} = -\frac{Gm_i m_j}{\|\vec{r}_f\|} + \frac{Gm_i m_j}{\|\vec{r}_0\|} \tag{79}$$

In case of the gravitation force, the potential energy would be:

$$U(\vec{r}) = -\frac{Gm_i m_j}{\|\vec{r}\|} \tag{80}$$

Conservative Forces

$$\vec{F}_c = -\nabla\phi \tag{81}$$

Rotational Energy

$$\begin{gathered} \Delta E_r = \int_{\vec{r}_0}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_f} \vec{F} \cdot \vec{v} dt = \int_{t_0}^{t_f} \vec{F} \cdot (\vec{r} \times \vec{w}) dt = \\ \int_{t_0}^{t_f} (\vec{r} \times \vec{F}) \cdot \vec{w} dt = \int_{t_0}^{t_f} (\vec{r} \times \vec{F}) \cdot \vec{w} dt = \int_{t_0}^{t_f} \vec{\tau} \cdot d\vec{\theta} = \int_{\theta_0}^{\theta_f} \tau d\theta \end{gathered} \tag{82}$$

$$\begin{gathered} \Delta E_r = \int_{t_0}^{t_f} \vec{\tau} \cdot d\vec{\theta} = \int_{t_0}^{t_f} I\frac{d\vec{w}}{dt} \cdot d\vec{\theta} = \\ \int_{t_0}^{t_f} I\vec{w} \cdot d\vec{w} = \int_{w_0}^{w_f} Iw dw = \frac{Iw_f^2}{2} - \frac{Iw_0^2}{2} \end{gathered} \tag{83}$$

$$E_r = \frac{Iw^2}{2} \tag{84}$$

Energy Conservation (When there is just conservative forces)

$$\Delta E_k + \Delta E_p + \Delta E_r = 0 \tag{85}$$

$$E_k + E_p + E_r = constant \tag{86}$$

For Rigid Bodies - Kinetic Energy

$$E_k = \sum_i \frac{m_i v_i^2}{2} = \sum_i \frac{m_i \|\dot{\vec{r}}_i^2\|}{2} = \sum_i \frac{m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}{2} \tag{87}$$

Since $\vec{r}_i = \vec{r}_{CM} + \vec{r'}_i$ and $\vec{r'}_i \cdot \vec{r'}_i = constant \implies \vec{r'}_i \cdot \dot{\vec{r}'_i} = 0$ and also $M = \sum_i m_i$:

$$\vec{r'}_i \cdot \dot{\vec{r}'_i} = 0 \implies \dot{\vec{r}'_i} = \vec{w} \times \vec{r'}_i \tag{88}$$

$$E_k = \sum_i \frac{m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}{2} = \sum_i \frac{m_i(\dot{\vec{r}}_{CM} + \dot{\vec{r}'_i}) \cdot (\dot{\vec{r}}_{CM} + \dot{\vec{r}'_i})}{2} = \sum_i \frac{m_i \dot{\vec{r}}_{CM} \cdot \dot{\vec{r}}_{CM}}{2} + \sum_i \frac{m_i \dot{\vec{r}'_i} \cdot \dot{\vec{r}'_i}}{2} + \dot{\vec{r}}_{CM} \cdot \sum_i m_i \dot{\vec{r}'_i} = \frac{M \dot{\vec{r}}_{CM} \cdot \dot{\vec{r}}_{CM}}{2} + \sum_i \frac{m_i \dot{\vec{r}'_i} \cdot \dot{\vec{r}'_i}}{2} \tag{89}$$

The first term of the previous equation is the kinetic energy of traslation

$$K = \frac{M \dot{\vec{r}}_{CM} \cdot \dot{\vec{r}}_{CM}}{2} \tag{90}$$

The second term can be viewd as (d_i is the distance from that point to the axis of rotation):

$$R = \sum_i \frac{m_i \dot{\vec{r}'_i} \cdot \dot{\vec{r}'_i}}{2} = \sum_i \frac{m_i(\vec{w} \times \vec{r'}_i) \cdot \vec{w} \times \vec{r'}_i)}{2} = \sum_i \frac{m_i \vec{r'}_i \cdot (-\vec{w} \times (\vec{w} \times \vec{r'}_i))}{2} = \sum_i \frac{m_i \vec{r'}_i \cdot ((\vec{w} \cdot \vec{w})\vec{r'}_i - (\vec{w} \cdot \vec{r'}_i)\vec{w})}{2} = w^2 \sum_i \frac{m_i d_i^2}{2} = \frac{Iw^2}{2} \tag{91}$$

For rigid bodies (K is the kinetic energy of traslation and R is the rotational energy respect to an axis that passes the center of mass):

$$E_k = K + R \tag{92}$$

Wave Equation

$$\frac{d^2y}{dx^2} + w^2 y = 0 \tag{93}$$

Solution of the wave equation

$$y = A_1cos(wx) + A_2sin(wx) = Bcos(wx + \phi) \tag{94}$$

Spring (just in x axis, k: Spring constant)

$$F(x) = -kx \tag{95}$$

Spring - Movement equation

$$F(x) = m\frac{d^2x}{dt^2} = -kx \implies \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{96}$$

Spring - Solution ($w = \sqrt{\frac{k}{m}}$)

$$x = Acos(wt + \phi) \tag{97}$$

Pendulum simple (length l, mass m, gravity acceleration)

$$\tau = \vec{r} \times \vec{F} = -mgl\sin(\theta) \tag{98}$$

If $\theta \to 0 \implies \sin(\theta) \sim \theta$

$$\tau = -mgl\theta = \frac{dL}{dt} = \frac{d(ml^2w)}{dt} = ml^2\frac{d^2\theta}{dt^2} \tag{99}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{100}$$

Solution ($w = \sqrt{\frac{g}{l}}$)

$$\theta = A\cos(wt + \phi) \tag{101}$$

Mechanical Wave in a string ($y = y(x,t)$ is the amplitude of the wave in the x position at the time t) - Equation

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{102}$$

Solution of the wave equation (w : angular frequency, k : wave number)

$$y = Acos(wt + kx + \phi) \tag{103}$$

In general (3D case): Wave Equation

$$\frac{\partial^2 F}{\partial t^2} = v^2\nabla^2 F \tag{104}$$

Solution of the Wave Equation (3D) ($\vec{k}$: Wave vector (points in the direction of the propagation of the wave), w : angular frequency)

$$F(\vec{r}, t) = Acos(wt + \vec{k} \cdot \vec{r} + \phi) \tag{105}$$

Lagrangian (T : Kinetic Energy, U : Potential Energy)

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - U(q, \dot{q}, t) \tag{106}$$

Action

$$S = \int_{t_0}^{t_f} L dt \tag{107}$$

Euler-Lagrange Equation (obatined when minimizing the action S)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = 0 \tag{108}$$

Example of Euler-Lagrange Equation ($T = \frac{m\dot{x}^2}{2}$, $U = mgx$)

$$\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = -mg - m\ddot{x} = 0 \implies a = -g \tag{109}$$

10 Fluid Mechanics

Bernoulli Equations (which is just conservation of energy)

$$\rho g h + \frac{\rho v^2}{2} + p = constant \tag{110}$$

Navier-Stokes Equation ($\vec{v} = \vec{v}(\vec{r}, t)$ is the velocity of the fluid at the position $\vec{r}$ and the time t)

$$\rho \frac{D\vec{v}}{Dt} = \rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}) = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{v} \tag{111}$$

Continuity Equation

$$\nabla \cdot \vec{v} = \vec{0} \tag{112}$$

11 Thermodynamics

Pressure (F is the magnitude of the force perpendicular to A)

$$P = \frac{F}{A} \tag{113}$$

Ideal Gas Law (P : pressure, V : Volume, n : Number of moles, R : Ideal gas constant, T : Temperature)

$$PV = nRT \tag{114}$$

Work done by the system

$$W = \int_{V_0}^{V_f} P dV \tag{115}$$

First Law of Thermodynamics (Q is heat, W is work and ΔU is the change of internal energy), this law meas conservation of energy

$$Q = W + \Delta U \tag{116}$$

Second Law of thermodynamics (ΔS is change of entropy)

$$\Delta S \geq 0 \tag{117}$$

Isothermic Process ($T = constant$)

$$W = \int_{V_0}^{V_f} PdV = \int_{V_0}^{V_f} \frac{nRT}{V} dV = nRT \ln(\frac{V_f}{V_0}) \tag{118}$$

Since $U = U(T)$, then for $T = constant \implies \Delta U = 0$

$$Q = nRT \ln(\frac{V_f}{V_0}) \tag{119}$$

Isobaric Process ($P = P_0 = constant$)

$$W = \int_{V_0}^{V_f} PdV = P_0(V_f - V_0) \tag{120}$$

In ideal case: $\Delta U(T) = C_P \Delta T$, where C_P is the Calorific Capacity at constant pressure

$$Q = P_0(V_f - V_0) + C_P(T_f - T_0) \tag{121}$$

Isocoric Process ($V = V_0 = constant$)

$$W = \int_{V_0}^{V_f} PdV = 0 \tag{122}$$

In ideal case: $\Delta U(T) = C_V \Delta T$, where C_V is the Calorific Capacity at constant volume

$$Q = C_V \Delta T = C_V(T_f - T_0) \tag{123}$$

Heat Capacity (Q is heat)

$$Q = \int_V CdT \tag{124}$$

Heat Transfer by convection (A is the area of the surface, T_f is the fluid temperature (like air for example), T is the temperature of the surface and h is the convection constant)

$$\dot{Q}(t) = hA(T_f - T) \tag{125}$$

Heat Transfer by conduction (Fourier's Law, q is the heat flux)

$$q_x = -K \frac{dT}{dx} \tag{126}$$

$$\vec{q} = -K \nabla T \tag{127}$$

Diffusion Equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{128}$$

12 Electromagnetism

Current definition

$$I = \frac{dq}{dt} \tag{129}$$

Current density definition (A is the transversal area through which I passes)

$$J = \frac{I}{A} \tag{130}$$

$$I = \int_A \vec{J} \cdot d\vec{S} \tag{131}$$

Coulomb's Law

$$\vec{F}_E = kq_iq_j \frac{\vec{r}_i - \vec{r}_j}{\|\vec{r}_i - \vec{r}_j\|^3} \tag{132}$$

Electric Field

$$\vec{E} = lim_{q_i \to 0} \frac{\vec{F}_e}{q_i} = kq_j \frac{\vec{r}_i - \vec{r}_j}{\|\vec{r}_i - \vec{r}_j\|^3} \tag{133}$$

Electric Force

$$\vec{F}_E = q\vec{E} \tag{134}$$

Magnetic Force

$$\vec{F_B} = q\vec{v} \times \vec{B} \tag{135}$$

$$\vec{F_B} = \int Id\vec{l} \times \vec{B} \tag{136}$$

$$\vec{F_B} = \int_V \vec{J} \times \vec{B} dV \tag{137}$$

Magnetic Field

$$\vec{B} = \int \frac{\mu_0 I d\vec{l} \times \vec{r}}{4\pi \|\vec{r}\|^3} \tag{138}$$

$$\vec{B} = \int \frac{\mu_0 \vec{J} \times \vec{r} dV}{4\pi \|\vec{r}\|^3} \tag{139}$$

Gauss Law for Electric Field and Magnetic Field ($\rho = \frac{dq}{dV}$ is charge density)

$$\oint_{\partial V} \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0} \tag{140}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{141}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{142}$$

$$\nabla \cdot \vec{B} = 0 \tag{143}$$

Ampere's Law

$$\oint_{\partial A} \vec{B} \cdot d\vec{l} = \mu_0 I \tag{144}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{145}$$

Ampere's Law (with the displacement current)

$$\oint_{\partial A} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d \int_{\partial V} \vec{E} \cdot d\vec{S}}{dt} \tag{146}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{147}$$

Faraday's Law

$$\oint_{\partial A} \vec{E} \cdot d\vec{l} = -\frac{d \int_{\partial V} \vec{B} \cdot d\vec{S}}{dt} \tag{148}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{149}$$

Electromagnetic Waves in vaccum ($\rho = 0$ (no charge), $\vec{J} = \vec{0}$ (no current))

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial \nabla \times \vec{B}}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{150}$$

Since $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{151}$$

Since $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \tag{152}$$

In the same way:

$$\nabla \times \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \nabla \times \vec{E}}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{153}$$

Since $\nabla \times \nabla \times \vec{B} = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$:

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{154}$$

Solution of the wave equation mentioned before (where $c = wk$, w : angular frequency and k : wave number)

$$\vec{E}(\vec{r}, t) = E_0 cos(wt + \vec{k} \cdot \vec{r} + \phi) \tag{155}$$

$$\vec{B}(\vec{r}, t) = B_0 cos(wt + \vec{k} \cdot \vec{r} + \phi) \tag{156}$$

13 Quantum Mechanics

Wave Function $\psi(\vec{r})$:

$$\int_V \psi^*(\vec{r})\psi(\vec{r})d\vec{r} = 1 \tag{157}$$

Probability to find the particle in the region Ω:

$$P(\Omega) = \int_\Omega \psi^*(\vec{r})\psi(\vec{r})d\vec{r} \tag{158}$$

Position Operator

$$\widehat{\vec{r}}\psi(\vec{r}) = \vec{r}\psi(\vec{r}) \tag{159}$$

Expected position

$$\vec{R} = \int_V \psi^*(\vec{r})\vec{r}\psi(\vec{r})d\vec{r} \tag{160}$$

Momentum Operator

$$\widehat{P} = \frac{\hbar}{i}\nabla \tag{161}$$

Schrodinguer Equation

$$\widehat{H}\psi = \frac{\widehat{P}^2}{2m}\psi + \widehat{V}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi = E\psi \tag{162}$$

14 Einstein Relativity

Time dilatation (t is the time in a reference frame which is in rest and t' is the time in a reference frame which is moving with constant velocity v, all of this in the x axis). (The idea is to decipher this is just using a mirror in the roof and the soil of a train moving and a ray of light bouncing between those mirrors, also using the idea that c is the same value in all reference systems)

$$\Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}} \tag{163}$$

Length Contraction (by measuring using a laser, in other words using a ray of light)

$$L' = c\Delta t' = c\Delta t\sqrt{1 - \frac{v^2}{c^2}} = L\sqrt{1 - \frac{v^2}{c^2}} \tag{164}$$

Lorentz Transformations

$$
\begin{aligned}
x &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \\
x' &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \\
t &= \frac{t' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \\
t' &= \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned} \tag{165}
$$

Lorentz Constant

$$
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{166}
$$

In general:

$$
\gamma(\vec{v}) = \frac{1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}} \tag{167}
$$

In order to generalize Lorentz Transformations to any direction, we can use projections of the vector position along the axis of movement $(\vec{r'} \cdot \vec{v})$. So $\vec{r'} = \vec{r'}_1 + \vec{r'}_2$ where $\vec{r'}_1$ is parallel to $\vec{v}$ and $\vec{r'}_2$ is orthogonal to $\vec{v}$

$$
\vec{r'}_1 = (\vec{r'} \cdot \frac{\vec{v}}{\|\vec{v}\|})\frac{\vec{v}}{\|\vec{v}\|} \tag{168}
$$

$$
\vec{r'}_2 = \vec{r'} - \vec{r'}_1 \tag{169}
$$

From both equations about we get that (since we just get contraction length in the direction of $\vec{v}$)

$$
\begin{aligned}
\vec{r} = \vec{v}t + \vec{r'}_2 + \frac{\vec{r'}_1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}} = \vec{v}t + \vec{r'} - \vec{r'}_1 + \frac{\vec{r'}_1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}} = \vec{v}t + \vec{r'} - \vec{r'}_1(1-\gamma) = \\
\vec{r'} + \vec{v}\gamma t' - (\vec{r'} \cdot \frac{\vec{v}}{\|\vec{v}\|})\frac{\vec{v}}{\|\vec{v}\|}(1-\gamma) = \vec{r'} + (\gamma t' + \frac{\vec{r'}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}(\gamma - 1))\vec{v}
\end{aligned} \tag{170}
$$

In the same way for time would be:

$$
t = \frac{t' + \frac{\vec{r'}\cdot v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(t' + \frac{\vec{r'}\cdot v}{c^2}) \tag{171}
$$

And also, due to the invariance:

$$\vec{r'} = \vec{r} - (\gamma t - \frac{\vec{r} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}(\gamma - 1))\vec{v} \tag{172}$$

$$t' = \gamma(t - \frac{\vec{r} \cdot v}{c^2}) \tag{173}$$

From the previous equations we obtain that:

$$d\vec{r} \cdot d\vec{r} = d\vec{r'} \cdot d\vec{r'} - c^2 dt'^2 + c^2 dt^2 \tag{174}$$

The previous equations can be writen as follows (which is called the invariance under Lorentz Transformation):

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 \tag{175}$$

If the frame reference we choose is situated in the particle that is moving (which means that $dx' = dy' = dz' = 0$), then:

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2 = c^2 d\tau^2 \tag{176}$$

We change t' by τ because this is a special time called "proper time" (because is the time measured by the moving particle). Now let's find the four-position and four-velocity of a moving particle.

$$\vec{R} = (ct, \vec{r}) = (ct, x, y, z) \tag{177}$$

Introducing the inner product that we will use in this space (Minkowsi):

$$d\vec{R} \cdot d\vec{R} = cdt^2 - d\vec{r} \cdot d\vec{r} = cdt^2 - dx^2 - dy^2 - dz^2 \tag{178}$$

Now let's find the speed (Remember that $\frac{dt}{d\tau} = \gamma$):

$$\vec{U} = \frac{d}{d\tau}\vec{R} = (c\gamma, \frac{d\vec{r}}{d\tau}) = (c\gamma, \frac{d\vec{r}}{dt}\frac{dt}{d\tau}) = (c\gamma, \frac{d\vec{r}}{dt}\gamma) = \gamma(c, \vec{v}) \tag{179}$$

where $\vec{v}$ is the velocity of the particle with respect to the frame reference in rest. We can check that $\vec{U} \cdot \vec{U}$ is an invariant under any frame reference.

$$\vec{U} \cdot \vec{U} = \gamma^2(c^2 - \vec{v} \cdot \vec{v}) = c^2 \tag{180}$$

Now let's find the four-momentum:

$$\vec{P} = m\vec{U} = (\gamma mc, \gamma m\vec{v}) = (\frac{E}{c}, \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}) = (\gamma mc, \vec{p}) \tag{181}$$