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# Approximation and Parameterized Algorithms for Segment Set Cover

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6

Master's thesis

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in **COMPUTER SCIENCE**

8

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
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## Abstract

23 In this thesis we analyze approximation and parametrization of GEOMETRIC SET COVER  
 24 with segments. To achieve interesting results we define a concept of GEOMETRIC SET COVER  
 25 with  $\delta$ -extension, where we need to cover the points by geometric objects, which are extended  
 26 by a tiny fraction, but we compare the solution size to the optimum solution without ex-  
 27 tension. We prove that SEGMENT SET COVER is APX-hard even if we restrict segments to  
 28 be axis-parallel and allow  $\frac{1}{2}$ -extension (Chapter 3). We provide an FPT algorithm for un-  
 29 weighted SEGMENT SET COVER parametrized by the size of the solution  $k$  (Section 4.1) and  
 30 for WEIGHTED SEGMENT SET COVER with  $\delta$ -extension (Section 4.2). Finally, we prove that  
 31 WEIGHTED SEGMENT SET COVER is W[1]-hard and there does not exist an algorithm run-  
 32 ning in time  $f(k) \cdot n^{o(\sqrt{k})}$  solving this problem even if we restrict the segments to 3 directions  
 33 (Chapter 5).

34

## Keywords

35 set cover, geometric set cover, weighted set cover, FPT, W[1]-hard, APX-hard, ETH, grid  
 36 tiling, MAX-(3,3)-SAT,  $\delta$ -extension

37

## Thesis domain (Socrates-Erasmus subject area codes)

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39

40

## Subject classification

41 Theory of computation  $\rightarrow$  Design and analysis of algorithms  $\rightarrow$  Parameterized complexity  
 42 and exact algorithms

43 Theory of computation  $\rightarrow$  Design and analysis of algorithms  $\rightarrow$  Approximation algorithms  
 44 analysis  $\rightarrow$  Packing and covering problems

45

46

## Tytuł pracy w języku polskim

47 Algoritmy aproksymacyjne i parametryzowane dla problemu pokrywania punktów  
 48 odcinkami na płaszczyźnie



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# Chapter 1

## Introduction

Some problems in Computer Science are known to be NP-complete, meaning that assuming  $P \neq NP$  there is no polynomial time algorithm that can solve these problems. Even so, they still can be amenable to different approaches, such as approximation or parameterization.

**Definition 1.1.** In the **SET COVER** problem we are given a set of elements (universe)  $\mathcal{C}$  and a family of sets  $\mathcal{P}$  that are subsets of the universe  $\mathcal{C}$  and sum up to the whole  $\mathcal{C}$ . Our task is to find a set  $\mathcal{R} \subseteq \mathcal{P}$  such that  $\bigcup \mathcal{R} = \mathcal{C}$  and the size of  $\mathcal{R}$  is minimum possible.

SET COVER is a classical example of an NP-complete problem, which has been proven in [Dinur and Steurer, 2014] to be inapproximable with factor  $(1 - o(1)) \ln n$  assuming  $P \neq NP$  (which is a stronger result than APX-hardness), and W[2]-complete with the natural parameterization, see Theorem 13.21 in [Cygan et al., 2015]. However restricting the problem to various specialized settings can lead to more tractable special cases. In this thesis we take a closer look at the GEOMETRIC SET COVER problem in the plane, where elements to cover are points in the plane and sets to cover them with are geometric objects.

**Definition 1.2.** **SEGMENT SET COVER** is GEOMETRIC SET COVER where objects that we cover the points with are segments in the plane.

**Approximation** Over the years there has been a lot of work related to approximation algorithms for GEOMETRIC SET COVER. Notably, GEOMETRIC SET COVER with unweighted unit disks admits a PTAS (see Corollary 1.1 in [Mustafa and Ray, 2010]). When we consider the same problem with weighted unit disks (or unit squares), the problem admits a QPTAS [Mustafa et al., 2014], see also [Pilipczuk et al., 2020]. On the other hand, [Chan and Grant, 2014] proves that GEOMETRIC SET COVER with unweighted axis-parallel fat rectangles is APX-hard; they also show similar hardness for GEOMETRIC SET COVER with many other standard geometric objects.

**Parameterization** We consider GEOMETRIC SET COVER parameterized by the size of solution. GEOMETRIC SET COVER with unit squares was first proven to be W[1]-hard in [Marx, 2005] (Theorem 5). A later follow-up work [Marx and Pilipczuk, 2015] shows that there is an algorithm running in time  $n^{\mathcal{O}(\sqrt{k})}$  that solves GEOMETRIC SET COVER with unit squares or disks and that there is no algorithm running in time  $f(k) \cdot n^{o(\sqrt{k})}$  for any computable  $f$  under the Exponential-Time Hypothesis, so this is a tight bound for this problem.

We also consider parameterization of weighted problems. There does not seem to be a consensus of what parameterization in the weighted setting is exactly; there was an at-

tempt to introduce a quite complicated general framework of weighted parameterized setting in [Shachnai and Zehavi, 2017]. Kernels for several well-known weighted problems such as WEIGHTED SUBSET SUM or WEIGHTED KNAPSACK are presented in [Etscheid et al., 2017]. Another work [Kim et al., 2021] considers weighted parameterization of WEIGHTED DIRECTED FEEDBACK SET and WEIGHTED *st*-CUT.

**$\delta$ -extension** In this paper, we focus on SEGMENT SET COVER with  $\delta$ -extension.  $\delta$ -extension is a problem relaxation method based on the  $\delta$ -shrinking model which was introduced in [Adamaszek et al., 2015] to provide an interesting result for the MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES problem. In this problem one needs to find a set of non-overlapping weighted rectangles with maximum sum of weight possible. In the  $\delta$ -shrinking relaxed problem the returned set of rectangles must be non-overlapping after all the rectangles are shrunk by a tiny fraction  $\delta$  towards the centre of symmetry. This problem is easier, because we compare this result to the optimum result before the shrinking. It might even lead to finding a set with result better than the optimum for the original problem. The author in [Adamaszek et al., 2015] presents a PTAS for MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES with  $\delta$ -shrinking, which is later improved to EPTAS in [Pilipczuk et al., 2016] alongside presenting a new FPT result for this problem with the natural parameterization. Later the similar  $\delta$ -shrinking model was used in [Wiese, 2018] to present a PTAS for MAXIMUM WEIGHT INDEPENDENT SET OF POLYGONS with  $\delta$ -shrinking.

**Definition 1.3.** For any  $\delta > 0$  and a centre-symmetric convex object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  **$\delta$ -extension** of  $L$  is the open set of points:

$$L^{+\delta} = \text{Int}\{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\},$$

where  $\text{Int}$  denotes the interior of a set of points. That is,  $L^{+\delta}$  is interior of the image of  $L$  under homothety centred at  $S$  with scale  $(1 + \delta)$ .

Analogous to  $\delta$ -shrinking,  $\delta$ -extension provides a framework for relaxing GEOMETRIC SET COVER problems, where we allow the returned set of objects  $\mathcal{R}$  to *almost* cover the points in the universe by requiring that they are covered by  $\mathcal{R}$  after  $\delta$ -extension, i.e. by set  $\mathcal{R}^{+\delta}$ . The same concept could be used for GEOMETRIC SET HITTING problems.

For a longer discussion of this concept see Section 2.4.

Similar model is used to prove that GEOMETRIC SET COVER with fat polygons relaxed with  $\delta$ -extension admits EPTAS [Har-Peled and Lee, 2009]. The  $\delta$ -extension model presented there is well-defined only for fat polygons. It extends the object by all the points that have distance to the closest point in the object  $P$  no larger than  $\delta \cdot \text{rad}(P)$ , where  $\text{rad}(P)$  is a radius of a circle inscribed into  $P$ . Since segments do not have any circle inscribed into them, the definition presented there cannot be utilized for this problem. Polygons extended by  $\delta$ -extension defined in Definition 1.3 covers a superset of set of points that object extended by  $\delta$ -extension defined in [Har-Peled and Lee, 2009]. Since our definition is more permissive for any polygon, the EPTAS from [Har-Peled and Lee, 2009] also works for polygons extended by our  $\delta$ -extension.

## Our contribution

In this paper we make the following contributions.

We show that approximation of SEGMENT SET COVER, even if segments are axis-parallel and we relax the problem with  $\frac{1}{2}$ -extension, is APX-hard (Theorem 1.1).



151 **Theorem 1.1.** (**SEGMENT SET COVER with axis-parallel segments and  $\frac{1}{2}$ -extension**  
 152 **is APX-hard**). SEGMENT SET COVER with axis-parallel segments in the 2D plane is APX-  
 153 hard (even with  $\frac{1}{2}$ -extension). That is, assuming  $P \neq NP$ , there does not exist a PTAS for  
 154 this problem.

155 Theorem 1.1 implies the following. Note that segments are just degenerated rectangles.

156 **Corollary 1.1.** (**GEOMETRIC SET COVER with rectangles is APX-hard**). GEOMET-  
 157 RIC SET COVER with axis-parallel rectangles is APX-hard (even with  $\frac{1}{2}$ -extension).

158 This expands the previous result of [Chan and Grant, 2014] that GEOMETRIC SET COVER  
 159 with axis-parallel fat rectangles is APX-hard, we improved the result that rectangles no longer  
 160 have to be fat (Corollary 1.1). This also proves that the assumption in [Har-Peled and Lee,  
 161 2009] for EPTAS about polygons being fat is necessary, because cover with arbitrary polygons  
 162 with  $\delta$ -extension is APX-hard.

163 We also provide two FPT algorithms for parameterized SEGMENT SET COVER (Theo-  
 164 rem 1.2) and WEIGHTED SEGMENT SET COVER relaxed with  $\delta$ -extension (Theorem 1.3).

165 **Theorem 1.2.** (**FPT for SEGMENT SET COVER**). There exists an algorithm that given  
 166 a family  $\mathcal{P}$  of segments (in any direction), a set of points  $\mathcal{C}$  and a parameter  $k$ , runs in time  
 167  $k^{O(k)}(|\mathcal{C}| \cdot |\mathcal{P}|)^2$ , and outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points  
 168 in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.

169 **Theorem 1.3.** (**FPT for WEIGHTED SEGMENT SET COVER with  $\delta$ -extension**). There  
 170 exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  
 171  $m$  points  $\mathcal{C}$ , and parameters  $k$  and  $\delta > 0$ , runs in time  $f(k, \delta) \cdot (nm)^c$  for some computable  
 172 function  $f$  and a constant  $c$  and outputs a set  $\mathcal{R}$  such that:

- 173 •  $\mathcal{R} \subseteq \mathcal{P}$ ,
- 174 •  $|\mathcal{R}| \leq k$ ,
- 175 •  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ ,
- 176 • the weight of  $\mathcal{R}$  is not greater than the weight of an optimum solution of size at most  $k$   
 177 for this problem without  $\delta$ -extension,

178 or determines that there is no set  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  such that  $\mathcal{R}$  covers all points in  $\mathcal{C}$ .

179 On the other hand, we prove that WEIGHTED SEGMENT SET COVER is W[1]-hard even  
 180 if segments are limited to 3 directions (Theorem 1.4) and assuming ETH there does not  
 181 exist algorithm for this problem that runs in time  $f(k)(|\mathcal{C}| + |\mathcal{P}|)^{O(\sqrt{k})}$ . See Figure 1.1 for a  
 182 summary of parameterized results for WEIGHTED SEGMENT SET COVER. Similar table for  
 183 unweighted problem is present in Figure 1.2.

184 **Theorem 1.4.** Consider the problem of covering a set  $\mathcal{C}$  of points by selecting at most  $k$   
 185 segments from a set of segments  $\mathcal{P}$  with non-negative weights  $w : \mathcal{P} \rightarrow \mathbb{R}^+$  so that the weight  
 186 of the cover is minimal. Then this problem is W[1]-hard when parameterized by  $k$  and assuming  
 187 ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{O(\sqrt{k})}$  for any  
 188 computable function  $f$ . Moreover, this holds even if all segments in  $\mathcal{P}$  are axis-parallel or  
 189 right-diagonal.

See Section 2.1 for exact definitions of axis-parallel and right-diagonal segments.

This result is particularly interesting, because problem without weights is FPT and weighted problem is W[1]-hard. Moreover  $\delta$ -extension allowed us to provide an FPT algorithm for a problem, which is W[1]-hard otherwise.

Note that the result of Theorem 1.4 is not tight: there exists a simple algorithm running in time  $\mathcal{O}(f(k)(|\mathcal{C}| + |\mathcal{P}|)^k)$ . So the question whether there exists an algorithm for this problem running in time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(k)}$  is still open.

Permissive FPT is a relaxed FPT problem, where we need to find solution of *any* size in FPT-time, but we compare it to the optimum solution of size at most  $k$ . Idea for permissive FPT in local search was presented in [Marx and Schlotter, 2011], [Gaspers et al., 2012].

Theorem 1.4 can be improved to show that a permissive FPT algorithm does not exist. This is formulated precisely in Theorem 5.2.

	exact	$\delta$ -extension
axis-parallel	?	FPT*
3 directions	W[1]-hard	FPT*
any direction	W[1]-hard*	FPT

Figure 1.1: Our results for WEIGHTED SEGMENT SET COVER parameterized by the size of solution. Results marked with \* are not explicitly given in this thesis, but they trivially follow from stronger results shown in the other cells of the table.

	exact	$\delta$ -extension
axis-parallel	FPT*	FPT*
3 directions	FPT*	FPT*
any direction	FPT	FPT*

Figure 1.2: Our results for unweighted SEGMENT SET COVER parameterized by the size of solution. Results marked with \* are not explicitly given in this thesis, but they trivially follow from stronger results shown in the other cells of the table.

**Future work.** There are two aforementioned problems that relate to Theorem 1.4 and were not solved in this thesis. We have not presented W[1]-hardness proof of WEIGHTED SEGMENT SET COVER where segments are limited to 3 directions, but it may be possible to improve this construction to use segments in 2 directions instead of 3 directions. The other question is what is the tight bound for this problem. The simple algorithm solving this problem is running in time  $\mathcal{O}(f(k)(|\mathcal{C}| + |\mathcal{P}|)^k)$ .

Another problem to consider is whether GEOMETRIC SET HITTING relaxed with  $\delta$ -extension can yield some better results.

## 210 Chapter 2

## 211 Preliminaries

212 In this chapter we present some basic definitions that will be used later.

### 213 2.1. Geometric Set Cover

214 Whenever speaking about GEOMETRIC SET COVER, we consider it in the 2-dimensional  
215 plane.

216 In the GEOMETRIC SET COVER problem we are given  $\mathcal{P}$  — a set of objects, which  
217 are connected subsets of the plane and  $\mathcal{C}$  — a set of points in the plane. The task is to choose  
218  $\mathcal{R} \subseteq \mathcal{P}$  such that every point in  $\mathcal{C}$  is inside some object from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized. We  
219 will mostly consider the case where  $\mathcal{P}$  consists of segments in the plane.

220 In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$  and we would  
221 like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

222 **Definition 2.1.** A segment is **axis-parallel** if it lies on a line that is either horizontal  $y = c$   
223 or vertical  $x = c$ .

224 **Definition 2.2.** A line is **right-diagonal** if it is described by the linear function  $x + y = d$   
225 for some  $d \in \mathbb{R}$ . Segment is **right-diagonal** if its direction is a right-diagonal line.

### 226 2.2. Parameterization

227 In the parameterized setting of the GEOMETRIC SET COVER for a given  $k$ , our task is to  
228 either find a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$  or decide that there is no such solution.

229 **Definition 2.3.** A **Fixed-parameter Tractable (FPT)** algorithm for a problem with pa-  
230 rameter  $k$  and instance size  $n$  is an algorithm running in time  $f(k) \cdot n^c$  for some constant  $c$   
231 and some computable function  $f$ .

232 **Definition 2.4.** Boolean formula is in **conjunctive normal form (CNF)** if it is a con-  
233 junction of one or more formulas, which are disjunction of literals.  **$k$ -CNF** formula is a CNF  
234 formula, where every disjunction consists of at most  $k$  literals.

235 **Definition 2.5.**  **$k$ -SAT** problem is a boolean satisfiability problem of  $k$ -CNF formulas.  
236 Given  $k$ -CNF formula, one must answer if there exists any variable assignment that satisfies  
237 the formula.

**Definition 2.6.** For  $k \geq 3$  set us define  $S_k$  as a set of constants  $\sigma$  such that there exists an algorithm solving  $k$ -SAT running in time  $\mathcal{O}^*(2^{\sigma n})$ . Let  $s_k$  be the infimum of the set  $S_k$ .

**Exponential Time Hypothesis (ETH)** asserts that  $s_3 > 0$ . This conjecture implies that there does not exist an algorithm solving 3-SAT running in time  $2^{o(n)}$ .

To see the definition of a  $W[1]$ -hard problem and  $W$  hierarchy, see Chapter 13.3 of [Cygan et al., 2015]. When proving that a problem is  $W[1]$ -hard, we are going to use Theorem 5.1, which was proved in [Marx, 2007].

## 2.3. Approximation

Let us recall some definitions related to optimization problems.

**Definition 2.7.** A **polynomial-time approximation scheme (PTAS)** for a minimization problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$  and in polynomial time finds a solution that is within a factor of  $(1 + \epsilon)$  of being optimal. This means that the reported solution has weight at most  $(1 + \epsilon)\text{opt}(I)$ , where  $\text{opt}(I)$  is the weight of an optimal solution to  $I$ .

**Definition 2.8.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

## 2.4. $\delta$ -extension

Another idea presented here, which can be utilized only when considering the problems with geometric objects, is  $\delta$ -extension. We define it specifically for the GEOMETRIC SET COVER problem with convex centre-symmetric objects.

Intuitively, we consider a problem with slightly larger objects, which makes the instance more permissive. However, we aim to find a solution that is not larger than the optimum solution to the original problem, so this is substantially easier than just solving the problem for the larger objects. It may even be the case that we are able to find a solution of size smaller than the optimum solution to the original problem.

Formal definition of  $\delta$ -extended objects is present in Definition 1.3.

The GEOMETRIC SET COVER with  $\delta$ -extension is a version of GEOMETRIC SET COVER with the following modifications.

- We need to cover all the points in  $\mathcal{C}$  by selecting objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extension).
- We look for a solution that is not larger than the optimum solution to the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 2.9.** The **GEOMETRIC SET COVER problem with  $\delta$ -extension** is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$  of GEOMETRIC SET COVER, the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is not larger than the optimal solution to the problem without extension, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

At last, we formulate a definition of the polynomial-time approximation scheme (PTAS) for a problem with  $\delta$ -extension.

277 **Definition 2.10.** A PTAS for GEOMETRIC SET COVER with  $\delta$ -extension is a family  
 278 of algorithms  $\{\mathcal{A}_{\delta,\epsilon}\}_{\delta,\epsilon>0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$  of GEOMETRIC SET  
 279 COVER where objects are centre-symmetric and convex, and in polynomial-time outputs a  
 280 solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$   
 281 factor of the optimal solution to this problem without extension, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## 282 2.5. Weighted Geometric Set Cover

283 In this thesis we also consider a WEIGHTED GEOMETRIC SET COVER problem, which is a  
 284 combination of the weighted and parameterized setting described in Section 2.1. We already  
 285 argued in the introduction that there is no consensus of how it is defined, but when we discuss  
 286 the weighted parameterized setting we will consider the following definition. There is a given  
 287 weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$  and we would like to find a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$  and  
 288  $\sum_{R \in \mathcal{R}} f(R)$  is minimum possible among such sets  $\mathcal{R}$ .

289 **Definition 2.11.** The WEIGHTED GEOMETRIC SET COVER problem with  $\delta$ -extension  
 290 is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C}, f)$  of WEIGHTED GEOMETRIC SET  
 291 COVER, the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$   
 292 covers  $\mathcal{C}$  and it has weight not larger than the optimal solution to the problem without ex-  
 293 tension, i.e.  $\sum_{R \in \mathcal{R}} f(R) \leq |\text{opt}(I)|$ .

294 We also consider weighted parameterized setting with  $\delta$ -extension, which we formally  
 295 define below.

296 **Definition 2.12.** The WEIGHTED GEOMETRIC SET COVER problem with  $\delta$ -extension  
 297 parameterized by the size of a solution is a problem where for an input instance  
 298  $I = (\mathcal{P}, \mathcal{C}, f, k)$  of WEIGHTED GEOMETRIC SET COVER parameterized by the size of a so-  
 299 lution  $k$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$   
 300 covers  $\mathcal{C}$ , uses no more than  $k$  sets, i.e.  $|\mathcal{R}| \leq k$  and it has weight not larger than the optimal  
 301 solution to the problem without extension, i.e.  $\sum_{R \in \mathcal{R}} f(R) \leq |\text{opt}(I)|$ .



## Chapter 3

# APX-hardness of geometric set cover

In this section we analyze whether there exists a PTAS for geometric set cover for rectangles. We show that Segment Set Cover is APX-hard even if we can restrict this problem to a very simple setting: segments parallel to axes and allow  $\frac{1}{2}$ -extension.

Our result can be summarized in the following theorem and this section aims to prove it.

**Theorem 1.1.** (**SEGMENT SET COVER with axis-parallel segments and  $\frac{1}{2}$ -extension is APX-hard**). SEGMENT SET COVER with axis-parallel segments in the 2D plane is APX-hard (even with  $\frac{1}{2}$ -extension). That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.

We prove Theorem 1.1 by taking a problem that is APX-hard and showing a reduction. For this problem we choose MAX-(3,3)-SAT which we define below.

### 3.1. MAX-(3,3)-SAT

See Definition 2.4 for the definition of the  $k$ -CNF formula.

**Definition 3.1.** MAX-3SAT is the following maximization problem. We are given a 3-CNF formula, and we need to find a boolean assignment of variables that satisfies the most clauses.

**Definition 3.2.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction that every variable appears in exactly 3 clauses and every clause contains exactly 3 literals of 3 different variables. Note that thus, the number of clauses is equal to the number of variables.

In our proof of Theorem 1.1 we use hardness of approximation of MAX-(3,3)-SAT proved in [Håstad, 2001] and described in Theorem 3.1 below.

**Definition 3.3.** MAX-3SAT formula with  $m$  clauses is **at most  $\alpha$ -satisfiable**, if every assignment of variables satisfies no more than  $\alpha m$  clauses.

**Theorem 3.1** ([Håstad, 2001]). For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most  $(\frac{7}{8} + \epsilon)$ -satisfiable (3,3)-SAT formulas.

### 3.2. Reduction

The following lemma encapsulates the properties of the reduction described in this section, and it allows us to prove Theorem 1.1.

**Lemma 3.1.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value  $\text{opt}(S)$ , we can construct an instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover with axis-parallel segments in 2D such that:*

(1) *For every solution to instance  $S$  that satisfies  $k$  clauses, there exists a solution to  $(\mathcal{C}, \mathcal{P})$  of size  $15n - k$ .*

(2) *For every solution  $\mathcal{R}$  to instance  $(\mathcal{C}, \mathcal{P})$ , there exists a solution to  $S$  that satisfies at least  $15n - |\mathcal{R}|$  clauses.*

(3) *For every  $\mathcal{R} \subseteq \mathcal{P}$ , if  $\mathcal{R}^{+\frac{1}{2}}$  is a solution to  $(\mathcal{C}, \mathcal{P})$ , then  $\mathcal{R}$  is also a solution to  $(\mathcal{C}, \mathcal{P})$ .*

*Therefore, the optimum size of a solution to  $(\mathcal{C}, \mathcal{P})$  is  $\text{opt}((\mathcal{C}, \mathcal{P})) = 15n - \text{opt}(S)$ .*

We prove Lemma 3.1 in subsequent sections. Section 3.3 describes the proposed instance  $(\mathcal{C}, \mathcal{P})$ . Property (1) is proved by Lemma 3.11, (2) by Lemma 3.13, and finally (3) trivially follows from Lemma 3.10. Firstly let us prove Theorem 1.1 using Lemma 3.1 and Theorem 3.1.

*Proof of Theorem 1.1.* Consider any  $0 < \epsilon < \frac{1}{15.8}$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in 2D with  $\frac{1}{2}$ -extension. We construct an algorithm that solves the problem stated in Theorem 3.1, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $(\mathcal{C}, \mathcal{P})$  using Lemma 3.1. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover relaxed with  $\frac{1}{2}$ -extensions on  $(\mathcal{C}, \mathcal{P})$ . Denote the size of the solution returned by this algorithm as  $\text{approx}^*((\mathcal{C}, \mathcal{P}))$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $\text{approx}^*((\mathcal{C}, \mathcal{P})) \geq 15n - (\frac{7}{8} + \epsilon)n$ , and if  $S$  is satisfiable, then  $\text{approx}^*((\mathcal{C}, \mathcal{P})) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$\text{opt}(S) = n.$$

From Lemma 3.1 we have:

$$\text{opt}((\mathcal{C}, \mathcal{P})) = 14n.$$

Therefore,

$$\begin{aligned} \text{approx}^*((\mathcal{C}, \mathcal{P})) &\leq (1 + \epsilon)\text{opt}((\mathcal{C}, \mathcal{P})) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n. \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$\text{opt}(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3.1 we have:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$



355 Since a solution to  $(\mathcal{C}, \mathcal{P})$  with  $\frac{1}{2}$ -extension is also a solution without any extension, by  
 356 Lemma 3.1 (3), we have:

$$\text{approx}^*((\mathcal{C}, \mathcal{P})) \geq \text{opt}((\mathcal{C}, \mathcal{P})) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

357 Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to distin-  
 358 guish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable: it  
 359 suffices to compare  $\text{approx}^*((\mathcal{C}, \mathcal{P}))$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation  
 360 algorithm cannot exist, unless  $P = NP$ .  $\square$

### 361 3.3. Construction of the Geometric Set Cover instance

362 We proceed to the proof of Lemma 3.1. That is, we show a reduction from the MAX-(3,3)-  
 363 SAT problem to geometric set cover with segments parallel to axes. Moreover, the obtained  
 364 instance of geometric set cover will be robust to  $\frac{1}{2}$ -extension (have the same optimal solution  
 365 after  $\frac{1}{2}$ -extension).

366 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and  
 367 **CLAUSE-gadgets**. **CLAUSE-gadgets** will be constructed using two **OR-gadgets** connected  
 368 together.

#### 369 3.3.1. VARIABLE-gadget

370 **VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It  
 371 allows two minimum solutions of size 3 each. These two choices correspond to the two Boolean  
 372 values of the variable corresponding to this gadget.

373 **Points.** Define points  $a, b, c, d, e, f, g, h$  as follows, where  $L = 22n$ :



Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $\text{pointsVariable}_i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $\text{chooseVariable}_i^{\text{false}}$  and the set of blue segments as  $\text{chooseVariable}_i^{\text{true}}$ .

374

$$\begin{array}{llll} a := (-3L, 0) & b := (-2L, 0) & c := (-L, 0) & d := (-3L, 1) \\ e := (-2L, 1) & f := (-2L, 2) & g := (L, 0) & h := (L, 2) \end{array}$$

Let us define:

$$\text{pointsVariable} := \{a, b, c, d, e, f\}$$

and, for any  $1 \leq i \leq n$ ,

$$\text{pointsVariable}_i := \text{pointsVariable} + (0, 4i).$$

375 We denote  $a_i := a + (0, 4i)$  etc.

376 **Segments.** Let us define:

$$\text{chooseVariable}_i^{\text{true}} := \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\},$$

$$\text{chooseVariable}_i^{\text{false}} := \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\},$$

$$\text{segmentsVariable}_i := \text{chooseVariable}_i^{\text{true}} \cup \text{chooseVariable}_i^{\text{false}}.$$

377 We also name two of these segments for future reference:  $\text{xTrueSegment}_i := (c_i, g_i)$ ,  
378  $\text{xFalseSegment}_i := (f_i, h_i)$ .

379 **Lemma 3.2.** *For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can be covered using 3 segments*  
380 *from  $\text{segmentsVariable}_i$ .*

381 *Proof.* We can use either set  $\text{chooseVariable}_i^{\text{true}}$  or  $\text{chooseVariable}_i^{\text{false}}$ . □

382 **Lemma 3.3.** *For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can not be covered with fewer than*  
383 *3 segments from  $\text{segmentsVariable}_i$ .*

384 *Proof.* No segment of  $\text{segmentsVariable}_i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  
385  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments. □

386 **Lemma 3.4.** *For every set  $A \subseteq \text{segmentsVariable}_i$  such that  $A$  covers  $\text{pointsVariable}_i$  and*  
387  *$\text{xTrueSegment}_i, \text{xFalseSegment}_i \in A$ , it holds that  $|A| \geq 4$ .*

388 *Proof.* No segment from  $\text{segmentsVariable}_i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  
389  $\text{pointsVariable}_i - \{c_i, f_i\}$  can not be covered with fewer than 2 segments. □

### 390 3.3.2. OR-gadget

391 An OR-gadget connects input and output segments (see Figure 3.2) in a way that is supposed  
392 to simulate the binary disjunction.

393 Input segments are the only segments that cover points outside of the gadget, as their left  
394 ends lie outside of it. Point  $v_{i,j}$  is the only one that can be covered by segments that do not  
395 belong to the gadget.

396 The OR-gadget has the property that every set of segments that covers all the points in  
397 the gadget uses at least 3 segments from it. Moreover, the output segment belongs to the  
398 solution of size 3 only if at least one of the input segments belongs to the solution. Therefore,  
399 optimum solutions restricted to the OR-gadget behave like a binary disjunction for the input  
400 segments.



Figure 3.2: **OR-gadget**. Segments from  $\text{chooseOr}_{i,j}^{\text{false}}$  are **red**, segments from  $\text{chooseOr}_{i,j}^{\text{true}}$  are blue (both **light blue** and **dark blue**), segments from  $\text{orMoveVariable}_{i,j}$  are **green** and **yellow**. **Dark blue** segment is the *output* segment. Grey segments  $\text{input}_x$  and  $\text{input}_y$  are input segments that are not part of  $\text{segmentsOr}_{i,j}$ .

401 **Points.** We define

$$\begin{aligned}
 l_0 &:= (0, 0) & m_0 &:= (0, 1) & n_0 &:= (0, 2) & o_0 &:= (0, 3) \\
 p_0 &:= (0, 4) & q_0 &:= (1, 1) & r_0 &:= (1, 3) & s_0 &:= (2, 1) \\
 t_0 &:= (2, 2) & u_0 &:= (2, 3) & v_0 &:= (3, 2)
 \end{aligned}$$

$$\text{vec}_{i,j} := (20i + 3 + 3j, 4(n + 1) + 2j)$$

403 For integers  $i, j$ , define  $\{l_{i,j}, m_{i,j}, \dots, v_{i,j}\}$  as  $\{l_0, m_0, \dots, v_0\}$  shifted by  $\text{vec}_{i,j}$ , i.e.  $l_{i,j} = l_0 + \text{vec}_{i,j}$   
 404 etc.

405 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3). Next, let

$$\text{pointsOr}_{i,j} := \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

406 Note that  $\text{pointsOr}_{i,j}$  does not include the point  $v_{i,j}$ .

407 **Segments.** We define the set of segments in several parts:

$$\begin{aligned}
 \text{chooseOr}_{i,j}^{\text{false}} &:= \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}, \\
 \text{chooseOr}_{i,j}^{\text{true}} &:= \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}, \\
 \text{orMoveVariable}_{i,j} &:= \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}.
 \end{aligned}$$

408 Finally all segments on OR-gadget are defined as:

$$\text{segmentsOr}_{i,j} := \text{chooseOr}_{i,j}^{\text{false}} \cup \text{chooseOr}_{i,j}^{\text{true}} \cup \text{orMoveVariable}_{i,j}$$

409 **Lemma 3.5.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$ , points in  $\text{pointsOr}_{i,j} - \{x\} \cup \{v_{i,j}\}$   
 410 can be covered with 4 segments from  $\text{segmentsOr}_{i,j}$ .

411 *Proof.* We can do this using one segment from  $\text{orMoveVariable}_{i,j}$ , the one that does not cover  
 412  $x$ , and all segments from  $\text{chooseOr}_{i,j}^{\text{true}}$ .  $\square$

413 **Lemma 3.6.** For any  $1 \leq i \leq n, j \in \{0,1\}$ , points in  $\text{pointsOr}_{i,j}$  can be covered with 4  
 414 segments from  $\text{segmentsOr}_{i,j}$ .

415 *Proof.* We can do this using segments from  $\text{orMoveVariable}_{i,j} \cup \text{chooseOr}_{i,j}^{\text{false}}$ .  $\square$

### 416 3.3.3. CLAUSE-gadget

417 A CLAUSE-gadget is responsible for determining whether variable values assigned in variable  
 418 gadgets satisfy the corresponding clause in the input formula  $\phi$ . It has a minimum solution  
 419 of size  $w$  if and only if the clause is satisfied, i.e. at least one of the respective variables is  
 420 assigned the correct value. Otherwise, its minimum solution has size  $w + 1$ . In this way, by  
 421 analyzing the size of the minimum solution to the entire constructed instance, we will be able  
 422 to tell how many clauses it is possible to satisfy in an optimum solution to  $\phi$ .



Figure 3.3: **CLAUSE-gadget for a clause  $a \vee b \vee \neg c$ .** Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the variables in the  $i$ -th clause. Grey segments corresponds to the values of variables satisfying the  $i$ -th clause.

423 **Points.** First, we define auxiliary functions for literals. For a literal  $w$ , let  $\text{idx}(w)$  be the  
 424 index of the variable in  $w$ , and  $\text{neg}(w)$  be the Boolean value (0 or 1) whether the variable is  
 425 negated in  $w$  or not.

$$\begin{aligned} \text{idx}(w) &:= i \text{ when } w = x_i \\ \text{neg}(w) &:= \begin{cases} 0 & \text{if } w = x_i \\ 1 & \text{if } w = \neg x_i \end{cases} \end{aligned}$$

Let us assume that clause  $C_i = a \vee b \vee c$  for any literals  $a, b, c$ . Then, we define points in the gadget as:

$$\begin{aligned} x_{i,0} &:= (20i, 4 \cdot \text{id}\mathbf{x}(a) + 2 \cdot \text{neg}(c)), & x_{i,1} &:= (20i, 4(n+1)), \\ y_{i,0} &:= (20i+1, 4 \cdot \text{id}\mathbf{x}(b) + 2 \cdot \text{neg}(b)), & y_{i,1} &:= (20i+1, 4(n+1)+4), \\ z_{i,0} &:= (20i+2, 4 \cdot \text{id}\mathbf{x}(c) + 2 \cdot \text{neg}(c)), & z_{i,1} &:= (20i+2, 4(n+1)+6). \end{aligned}$$

We are now ready to define the set of points in a clause gadget:

$$\begin{aligned} \text{moveVariable}_i &:= \{x_{i,j} : j \in \{0,1\}\} \cup \{y_{i,j} : j \in \{0,1\}\} \cup \{z_{i,j} : j \in \{0,1\}\}, \\ \text{pointsClause}_i &:= \text{moveVariable}_i \cup \text{pointsOr}_{i,0} \cup \text{pointsOr}_{i,1} \cup \{v_{i,1}\}. \end{aligned}$$

Note that these two points are equal:  $v_{i,0} = l_{i,1}$ . This translates to the fact that the output of the first OR-gadget is an input to the second OR-gadget. This creates an *or* of 3 boolean values.

**Segments.** We also define segments for the clause gadget as below:

$$\begin{aligned} \text{segmentsClause}_i &:= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\ &\cup \text{segmentsOr}_{i,0} \cup \text{segmentsOr}_{i,1}. \end{aligned}$$

The CLAUSE-gadgets consist of two OR-gadgets. Ideally, we would place the  $i$ -th CLAUSE-gadget close to the  $\mathbf{xTrueSegment}_{j_1}$  or  $\mathbf{xFalseSegment}_{j_1}$  segments corresponding to the literals that occur in the  $i$ -th clause. It would be inconvenient to position them there, because between these segments there may be additional  $\mathbf{xTrueSegment}_{j_2}$  or  $\mathbf{xFalseSegment}_{j_2}$  segments corresponding to the other literals.

Instead, we use simple auxiliary gadgets to *transfer* whether the segment is in a solution, i.e. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ . Each transfer gadget consists of two segments  $(x_{i,0}, x_{i,1}), (x_{i,1}, a)$ . These are the only segments that can cover  $x_{i,1}$ . We place  $x_{i,0}$  on a segment that we want to transfer (i.e. segment responsible for choosing the variable value satisfying the corresponding literal). If in some solution  $x_{i,0}$  is already covered by this segment, then we can cover  $x_{i,1}$  by  $(x_{i,1}, a)$ , thus also covering  $a$ . If  $x_{i,0}$  is not covered by this segment, then the only way to cover  $x_{i,0}$  is to use segment  $(x_{i,0}, x_{i,1})$ . Intuitively, in any optimal solution the two segments *transfer* the state of whether  $x_{i,0}$  is covered onto whether  $a$  is covered. Therefore, the number of segments in the optimal solution is increased by one, and we get a point  $a$  that was effectively placed on some segment  $s$ , but it can be placed anywhere in the plane instead, consequently simplifying the construction.

**Lemma 3.7.** *For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , there is a set  $\text{solClause}_i^{\text{true},a} \subseteq \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{true},a}| = 11$  that covers all points in  $\text{pointsClause}_i - \{a\}$ .*

*Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 3.5 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments in  $\text{chooseOr}_{i,0}^{\text{true}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$ .

For  $a = z_{0,i}$ : Using Lemma 3.6 and Lemma 3.5 with  $x = p_{i,1}$ , we obtain 8 segments in  $\text{chooseOr}_{i,0}^{\text{false}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 3.8.** *For any  $1 \leq i \leq n$  there is a set  $\text{solClause}_i^{\text{false}} \subseteq \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{false}}| = 12$  that covers all points in  $\text{pointsClause}_i$ .*

*Proof.* Using Lemma 3.6 twice we can cover  $\text{pointsOr}_{i,0}$  and  $\text{pointsOr}_{i,1}$  with 8 segments. To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$ .

□

**Lemma 3.9.** *For any  $1 \leq i \leq n$ :*

(1) *points in  $\text{pointsClause}_i$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 12;*

(2) *points in  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 11.*

*Proof of (1).* No segment in  $\text{segmentsClause}_i$  covers more than 1 point from

$$\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}.$$

Therefore we need to use at least 12 segments.

□

*Proof of (2).* We can define disjoint sets  $X, Y, Z$  such that

$$X \cup Y \cup Z \subseteq \text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$$

and there are no segments in  $\text{segmentsClause}_i$  covering points from different sets. And we prove a lower bound for each of these sets. First, let:

$$X := \{x_{i,1}, y_{i,1}, z_{i,1}\}.$$

No two points in  $X$  can be covered with one segment of  $\text{segmentsClause}_i$ , so it must be covered with 3 different segments. Next we define other sets:

$$Y := \text{pointsOr}_{i,0} - \{l_{i,0}, p_{i,0}\},$$

$$Z := \text{pointsOr}_{i,1} - \{l_{i,1}, p_{i,1}\}.$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments of  $\text{segmentsClause}_i$  to conclude that none of them cover the considered, so both  $Y$  and  $Z$  have to be covered with disjoint sets of 4 segments each.

Therefore,  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  must be covered with at least  $3 + 4 + 4 = 11$  segments from  $\text{segmentsClause}_i$ .

□

### 3.3.4. Summary

Finally we define the set of points and segments for the constructed instance:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} \text{pointsVariable}_i \cup \text{pointsClause}_i,$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} \text{segmentsVariable}_i \cup \text{segmentsClause}_i.$$

**Lemma 3.10** (Robustness to  $\frac{1}{2}$ -extension). *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+\frac{1}{2}}$  cover the same points from  $\mathcal{C}$ .*



Figure 3.4: **Scheme of the whole construction.**  
General layout of VARIABLE-gadgets and CLAUSE-gadgets and how they interact with each other.

In order to prove this lemma we will define a claim about bounding rectangle for every gadget, which fits both segments and points from this gadget and after  $\frac{1}{2}$ -extension does not cover any points from outside of this rectangle. Checking that the property from the above lemma holds for points and segments within the same gadget can be easily done using the figures above as references.

Note that the lemmas stated below also encapsulate the interactions between the gadgets, which are also mentioned in the helper lemmas above, and prove that gadgets are independent otherwise.

First let us define points to cover inside of rectangle  $R$  as:

$$\text{points}(R) := \text{points from } \mathcal{C} \text{ that lie in rectangle } R.$$

**Claim 3.1.** For any  $1 \leq i \leq n$ ,  $\text{segmentsVariable}_i$  fit in rectangle defined by points  $a_i$  and  $h_i$  from VARIABLE-gadget:  $R := [-3L, L] \times [4i, 4i + 2]$ .

- (1) The only points in  $R$  are  $\text{pointsVariable}_i$  and  $x_{j,0}, y_{j,0}$  or  $z_{j,0}$  points from CLAUSE-gadget for some clause  $c_j$ :

$$\text{pointsVariable}_i \subseteq \text{points}(R) \subseteq \text{pointsVariable}_i \cup \{x_{j,0}, y_{j,0}, z_{j,0} : 1 \leq j \leq n\}.$$

- (2)  $R$  covers the same points from  $\mathcal{C}$  before and after  $\frac{1}{2}$ -extension, i.e.  $\text{points}(R) = \text{points}(R^{+\delta})$ .

- (3) All segments  $\text{segmentsVariable}_i$  fit fully inside of  $R$ .

**Claim 3.2.** For any  $1 \leq i \leq n$ ,  $\text{pointsVariable}_i$  fit in rectangle defined by points  $a_i$  and  $h_i$  from VARIABLE-gadget:  $R := [-3L, -L] \times [4i, 4i + 2]$ .

- (1) The only points in  $R$  are  $\text{pointsVariable}_i$ :  $\text{points}(R) = \text{pointsVariable}_i$ .

- (2)  $R$  covers the same points from  $\mathcal{C}$  before and after  $\frac{1}{2}$ -extension, i.e.  $\text{points}(R) = \text{points}(R^{+\delta})$ .

- (3) All segments  $\text{segmentsVariable}_i - \{\text{xTrueSegment}_i, \text{xFalseSegment}_i\}$  fit fully inside of  $R$ .

**Claim 3.3.** For any  $1 \leq i \leq n$  and  $j \in \{0, 1\}$ , points from OR-gadget point  $\text{pointsOr}_{i,j}$  and segments  $\text{segmentsOr}_{i,j} - \{(t_{i,j}, v_{i,j})\}$  fit in rectangle defined as:

$$R := [x, x + 2] \times [y, y + 4], \text{ where } x = 20i + 3j + 3, y = 4(n + 1) + 2j.$$

- (1)  $R$  covers only  $\text{pointsOr}_{i,j}$ , i.e.  $\text{points}(R) = \text{pointsOr}_{i,j}$ .

- (2)  $R$  covers the same points from  $\mathcal{C}$  before and after  $\frac{1}{2}$ -extension, i.e.  $\text{points}(R) = \text{points}(R^{+\delta})$ .

- (3) All segments  $\text{segmentsOr}_{i,j} - \{(t_{i,j}, v_{i,j})\}$  fit fully inside of  $R$ .

**Claim 3.4.** For any  $1 \leq i \leq n$ ,  $\text{segmentsClause}_i$  and  $\text{pointsClause}_i$  fit in rectangle:

$$R := [20i, 20i + 8] \times [0, 6(n + 1)].$$

- (1)  $R$  covers only  $\text{pointsClause}_i$ , i.e.  $\text{points}(R) = \text{pointsClause}_i$ .

- (2)  $R$  covers the same points from  $\mathcal{C}$  before and after  $\frac{1}{2}$ -extension, i.e.  $\text{points}(R) = \text{points}(R^{+\delta})$ .

- (3) All segments  $\text{segmentsClause}_i$  fit fully inside of  $R$ .

*Proof of Lemma 3.10.* First, we check one-by-one for every segment within every gadget that if it covers some point after  $\frac{1}{2}$ -extension, then it covered that point before extension. In other words, every segments does not cover any new point from the same gadget after  $\frac{1}{2}$ -extension.



**VARIABLE-gadget** We use Claim 3.3 and name this rectangle  $R_1$ . For any  $1 \leq i \leq n$  segmentsVariable $_i$  do not cover any point outside of  $R_i$  after  $\frac{1}{2}$ -extension. Although some points from pointsClause $_j$  for any  $j$  can lie within  $R_1$ , hence we use Claim 3.2 and name this rectangle  $R_2$ .  $R_2$  covers only points from pointsVariable $_i$  (even after  $\frac{1}{2}$ -extension), then any points CLAUSE-gadgets are covered by xTrueSegment $_i$  or xFalseSegment $_i$  and it is enough to check that they cover exactly the same points from CLAUSE-gadget before and after  $\frac{1}{2}$ -extension. They both cover all points from CLAUSE-gadget that are collinear with these segments, so they cover exactly the same set of points after extension.

**CLAUSE-gadget** We use Claims 3.3 and 3.4. For any  $1 \leq i \leq n$  and  $j \in \{0, 1\}$ , we need to check whether segment  $(t_{i,j}, v_{i,j})$  does not cover any new points from segmentsClause $_i$  after  $\frac{1}{2}$ -extension, because it does not fit in the OR-gadget's boundary rectangle.  $\square$

### 3.4. Proof that the construction is correct and sound

In order to prove Lemma 3.1 we introduce several auxiliary lemmas proving properties of the construction described in the previous section.

Consider an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution satisfying  $k$  clauses. Let us construct an instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover as described in Section 3.3 for the instance  $S$  of MAX-(3,3)-SAT.

**Lemma 3.11.** *The instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover admits a solution of size  $15n - k$ .*

*Proof.* Let the clauses in  $S$  be  $c_1, c_2, \dots, c_n$  and the variables be  $x_1, x_2, \dots, x_n$ . Let the variable assignment in the optimum solution to  $S$  be  $\phi : \{x_1, x_2, \dots, x_n\} \rightarrow \{\text{true}, \text{false}\}$ .

We cover every VARIABLE-gadget with solution described in Lemma 3.2, where in the  $i$ -th gadget we choose the set of segments corresponding to the value of  $\phi(x_i)$ .

For every clause that is satisfied, say  $c_i$ , let us name the variable that is **true** in it as  $x_i$  and the point corresponding to  $x_i$  in pointsClause $_i$  as  $a$ . Points in pointsClause $_i$  are covered with set solClause $_i^{\text{true}, a}$  described in Lemma 3.7. For every clause that is not satisfied, say  $c_j$ , points in pointsClause $_j$  are covered with set solClause $_j^{\text{false}}$  described in Lemma 3.8.

Formally, we define sets responsible for choosing variable assignment and satisfying clauses,  $R_i$  and  $C_i$  respectively, as following:

$$R_i := \begin{cases} \text{chooseVariable}_i^{\text{true}} & \text{if } \phi(x_i) = \text{true} \\ \text{chooseVariable}_i^{\text{false}} & \text{if } \phi(x_i) = \text{false} \end{cases}$$

$$C_i := \begin{cases} \text{solClause}_i^{\text{true}, a} & \text{if } c_i \text{ satisfied by the literal corresponding to point } a \\ \text{solClause}_i^{\text{false}} & \text{if } c_i \text{ not satisfied} \end{cases}$$

$$\mathcal{R} := \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}.$$

This set covers all the points from  $\mathcal{C}$ , because the sets  $R_i$ ,  $C_i$  individually cover their corresponding gadgets, as proved in the respective lemmas.

All of these sets are disjoint, so the size of the obtained solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k. \quad \square$$

**Lemma 3.12.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover. Then there exists a solution  $\mathcal{R}'$  such that  $|\mathcal{R}'| \leq |\mathcal{R}|$  and  $\mathcal{R}'$  contains at most one of the segments  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$  from each VARIABLE-gadget.*

*Proof.* Assume that we have  $\{\text{xTrueSegment}_i, \text{xFalseSegment}_i\} \subseteq \mathcal{R}$  for some  $i$ . We will show how to modify  $\mathcal{R}$  into  $\mathcal{R}'$ , such that the number of such  $i$  decreases, while  $\mathcal{R}'$  is still a valid solution to  $(\mathcal{C}, \mathcal{P})$ , and  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Then, by repeating this procedure, we can eventually construct a solution satisfying the property from the Lemma.

To construct  $\mathcal{R}'$ , we first remove from  $\mathcal{R}$  all segments belonging to  $\text{segmentsVariable}_i$ . Recall that the  $i$ -th VARIABLE-gadget corresponds to variable  $x_i$  in  $S$ . As every variable in  $S$  is used in exactly 3 clauses, then one literal  $x_i$  or  $\neg x_i$  must appear in at least 2 clauses. If that literal is  $x_i$ , then we add to the constructed solution all segments from  $\text{chooseVariable}_i^{\text{true}}$ , otherwise we add all segments from  $\text{chooseVariable}_i^{\text{false}}$ .

Now, there exists at most one CLAUSE-gadget which needs adjustment to make  $\mathcal{R}'$  valid; assuming it is the  $j$ -th clause, then one of the points  $x_{j,0}, y_{j,0}$  or  $z_{j,0}$  for this CLAUSE-gadget might be not covered, say  $y_{j,0}$ . We amend the solution by adding  $(y_{j,0}, y_{j,1})$  to  $\mathcal{R}'$ .

By Lemma 3.4 we know that  $\mathcal{R}$  used at least 4 segments from  $\text{segmentsVariable}_i$ . Therefore, we removed at least 4 segments and added at most 4 segments, so  $|\mathcal{R}'| \leq |\mathcal{R}|$ .  $\square$

**Lemma 3.13.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover. Then there exists a solution to  $S$  that satisfies at least  $15n - |\mathcal{R}|$  clauses.*

*Proof.* Let the clauses in  $S$  be  $c_1, c_2, \dots, c_n$  and the variables be  $x_1, x_2, \dots, x_n$ . Given a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover, we use Lemma 3.12 to modify  $\mathcal{R}$  so that for any  $i$ ,  $\mathcal{R}$  contains at most one of  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$ ; this may decrease the size of  $\mathcal{R}$ , but that does not matter in the subsequent construction. To simplify notation, in the remainder of this proof we use  $\mathcal{R}$  to refer to the modified solution.

Given  $\mathcal{R}$ , we construct a solution to  $S$  by defining an assignment of variables:

$$\phi : \{x_1, x_2, \dots, x_n\} \rightarrow \{\text{true}, \text{false}\}$$

that satisfies at least  $15n - |\mathcal{R}|$  clauses in  $S$ .

**Definition of  $\phi$ .** Recall that due to Lemma 3.12,  $\mathcal{R}$  contains at most one of  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$ .

We define the value  $\phi(x_i)$  for the variable  $x_i$  as follows:

$$\phi(x_i) := \begin{cases} \text{true} & \text{if } \text{xTrueSegment}_i \in \mathcal{R}, \\ \text{false} & \text{otherwise} \end{cases}$$

Moreover, from Lemma 3.3 we get  $|\text{segmentsVariable}_i \cap \mathcal{R}| \geq 3$  for every  $i$ .

**Clauses satisfied with the chosen variable assignment.** For a clause  $c_i$ ,  $\mathcal{R}$  needs to use at least 11 segments to cover  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  in the  $i$ -th CLAUSE-gadget (Lemma 3.9).

Moreover, if none of the points  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  are covered by the segments from  $\mathcal{R} \cap \text{segmentsVariable}_i$ , then  $\mathcal{R}$  needs to cover  $\text{pointsClause}_i$  with at least 12 segments by Lemma 3.9.

Let  $a$  be the number of clauses  $c_i$  for which none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  are covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$  for any  $1 \leq j \leq n$ .

Consider a clause  $c_i$  for which at least one of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  is covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$  for some  $1 \leq j \leq n$ . Denote this point

576 as  $t$  and say it corresponds to literal  $q$  and variable  $x_j$ . Point  $t$  can be only covered in  
 577 `segmentsVariablej` by a corresponding segment `xTrueSegmentj` or `xFalseSegmentj` (depending  
 578 on whether the literal  $q$  is negated or not). From the definition of  $\phi$  and the fact that one of  
 579 this segment is in  $\mathcal{R}$ , we know that  $\phi(j)$  has the value that evaluates  $q$  to be `true`. Therefore,  
 580 clause  $c_i$  is satisfied.

581 Consequently,  $\phi$  satisfies all but at most  $a$  clauses in  $S$ .

582 To conclude, given a solution  $\mathcal{R}$  to  $(\mathcal{C}, \mathcal{P})$  we constructed a variable assignment  $\phi$  that  
 583 satisfies at least  $n - a$  clauses of  $S$ . Finally, note that

$$|\mathcal{R}| \geq 3n + 11(n - a) + 12a = 3n + 11n + a = 14n + a,$$

hence

$$15n - |\mathcal{R}| \leq 15n - 14n - a = n - a.$$

584 Therefore,  $\phi$  satisfies at least  $15n - |\mathcal{R}|$  clauses of  $S$ . □



## Chapter 4

# Fixed-parameter tractable algorithm for geometric set cover problem

In this chapter we show fixed-parameter tractable algorithms for the geometric set cover problem in two different settings. Section 4.1 shows a fixed-parameter tractable algorithm for geometric set cover with unweighted segments. The remainder of the chapter presents a fixed-parameter tractable algorithm for geometric set cover with weighted segments with  $\delta$ -extension. We show an algorithm for the setting with  $\delta$ -extension, because the original problem with weights is W[1]-hard, as we show in Chapter 5.

We start with a shared definition for this problem. We define *extreme points* for a set of collinear points.

**Definition 4.1.** For a set of collinear points  $C$  in the plane, **extreme points** of  $C$  are the endpoints of the smallest segment that covers all points from set  $C$ .

If  $C$  consists of one point or is empty, then there are 1 or 0 extreme points respectively.

### 4.1. Fixed-parameter tractable algorithm for unweighted segments

In this section we consider fixed-parameter tractable algorithms for unweighted geometric set cover with segments. The setting where segments are required to be axis-parallel (or limited to a constant number of directions) has a trivial FPT algorithm. We present an FPT algorithm for geometric set cover with unweighted segments, where segments are in arbitrary directions.

#### 4.1.1. Axis-parallel segments

**Theorem 4.1.** (*FPT for segment cover with axis-parallel segments*). There exists an algorithm that given a family  $\mathcal{P}$  of axis-parallel segments, a set of points  $C$  and a parameter  $k$ , runs in time  $\mathcal{O}(2^k)$ , and outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $C$ , or determines that such a set  $\mathcal{R}$  does not exist.

*Proof.* We show an  $\mathcal{O}(2^k)$ -time branching algorithm. In each step, the algorithm selects a point  $a$  which is not yet covered, branches to choose one of the two directions, and greedily chooses a segment  $a$  in that direction to cover. This proceeds until either all points are covered or  $k$  segments are chosen.

Let us take the point  $a = (x_a, y_a)$  which is the smallest among points that are not yet covered in the lexicographic ordering of points in  $\mathbb{R}^2$ . We need to cover  $a$  with some of the remaining segments.

Branch over the choice of one of the coordinates ( $x$  or  $y$ ); without loss of generality, let us assume we chose  $x$ . Among the segments lying on line  $x = x_a$ , we greedily add to the solution the one that covers the most points. As  $a$  was the smallest in the lexicographical order, all points on the line  $x = x_a$  have the  $y$ -coordinate larger than  $y_a$ . Therefore, if we denote the greedily chosen segment as  $s$ , then any other segment on the line  $x = x_a$  that covers  $a$  can only cover a subset of points covered by  $s$ . Thus, greedily choosing  $s$  is optimal.

In each step of the algorithm we add one segment to the solution, thus the recursion can be stopped at depth  $k$ . If no branch finds a solution, then this means that a solution of size at most  $k$  does not exist.  $\square$

Note that the same algorithm can be used for segments in  $d$  directions, where we branch over  $d$  choices of directions, and it runs in complexity  $\mathcal{O}(d^k)$ .

#### 4.1.2. Segments in arbitrary directions

In this section we consider the setting where segments are not constrained to a constant number of directions. We present a fixed-parameter tractable algorithm, parameterized by the size of the solution.

**Theorem 1.2. (FPT for SEGMENT SET COVER).** *There exists an algorithm that given a family  $\mathcal{P}$  of segments (in any direction), a set of points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $k^{\mathcal{O}(k)}(|\mathcal{C}| \cdot |\mathcal{P}|)^2$ , and outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.*

We will need the following lemmas proving properties of any instance of the problem.

**Lemma 4.1.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, without loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any distinct  $A, B \in \mathcal{P}$ , we have  $A \cap \mathcal{C} \not\subseteq B \cap \mathcal{C}$  and  $A \cap \mathcal{C} \not\supseteq B \cap \mathcal{C}$ .*

*Proof.* Assume towards a contradiction that there is an instance  $(\mathcal{P}, \mathcal{C})$ , and two distinct subsets of  $\mathcal{P}$ ,  $A, B$ , such that  $A \cap \mathcal{C} \subseteq B \cap \mathcal{C}$ .

We construct a set  $\mathcal{P}' := \mathcal{P} - \{A\}$ . We prove that for any solution  $\mathcal{R}$  of  $(\mathcal{P}, \mathcal{C})$ , we can construct a solution  $\mathcal{R}' \subseteq \mathcal{P}'$ , such that  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Let us take any solution  $\mathcal{R}$  of  $(\mathcal{P}, \mathcal{C})$ . If  $A \in \mathcal{R}$ , then  $\mathcal{R}' := \mathcal{R} \cup \{B\} - \{A\}$ , otherwise  $\mathcal{R}' := \mathcal{R}$ . Let us consider the case when  $A \in \mathcal{R}$ , because the other case is trivial. Since  $A \cap \mathcal{C} \subseteq B \cap \mathcal{C}$ , then  $\mathcal{R} \cup \{B\} - \{A\}$  covers any point from  $\mathcal{C}$  that was covered by  $\mathcal{R}$ . Also,  $|\mathcal{R} \cup \{B\} - \{A\}| \leq |\mathcal{R}|$ .  $\square$

**Lemma 4.2.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem transformed by Lemma 4.1, if there exists a line  $L$  with at least  $k + 1$  points on it, then there exists a subset  $A \subseteq \mathcal{P}$ , of size at most  $k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|A \cap \mathcal{R}| \geq 1$ . Moreover, such a subset can be found in polynomial time.*

*Proof.* Let us enumerate the points from  $\mathcal{C}$  that lie on  $L$  as  $x_1, x_2, \dots, x_t$  in the order in which they appear on  $L$ . Our proposed set is defined as:

$$A := \{\text{segment collinear with } L \text{ that covers } x_i \text{ and does not cover } x_{i-1} : i \in \{1, \dots, k\}\},$$

where for  $i = 1$  we just take a segment that covers  $x_1$ . If such a segment does not exist for any point  $x$  as above, then  $x$  does not give rise to any segment in  $A$ .

653 We prove the lemma by contradiction. Let us assume that there exists a solution  $\mathcal{R}$  of  
654 size at most  $k$  such that  $\mathcal{R} \cap A = \emptyset$ .

655 Let  $\mathcal{R}_L$  be the set of segments from  $\mathcal{R}$  that are collinear with  $L$ .

656 Every segment that is not collinear with  $L$  can cover at most one of the points that lie  
657 on this line. Hence, if  $\mathcal{R}_L$  was empty, then  $\mathcal{R}$  would cover at most  $k$  points on line  $L$ , but  $L$   
658 had at least  $k + 1$  different points from  $\mathcal{C}$  on it.

659 Therefore, we know that  $\mathcal{R}_L$  is not empty and  $|\mathcal{R} - \mathcal{R}_L| \leq k - 1$ . Segments from  $\mathcal{R} - \mathcal{R}_L$   
660 can cover at most  $k - 1$  points among  $\{x_1, x_2, \dots, x_k\}$ , therefore at least one of these points  
661 must be covered by segments from  $\mathcal{R}_L$ . We take the leftmost point from  $\{x_1, x_2, \dots, x_k\}$  that  
662 is covered in  $\mathcal{R}_L$  and name it  $a$ . After the transformation from Lemma 4.1, in  $\mathcal{R}$  there is only  
663 one segment that starts in  $a$  and is collinear with  $L$ , therefore this segment must be in both  
664  $\mathcal{R}$  and  $A$ . This contradiction concludes the proof that  $|A \cap \mathcal{R}| \geq 1$  for any solution  $\mathcal{R}$  of size  
665 at most  $k$ .  $\square$

666 We are now ready to prove Theorem 1.2.

667 *Proof of Theorem 1.2.* We will prove this theorem by presenting a branching algorithm that  
668 works in desired complexity. It first branches over the choice of segments to cover the lines  
669 with *many* points and then solves a small instance (where every line has at most  $k$  points) by  
670 checking all possible solutions.

671 **Algorithm.** We present a recursive algorithm. Given an instance of the problem:

- 672 (1) Use Lemma 4.1 to remove some redundant segments from our instance.
- 673 (2) If there exists a line with at least  $k + 1$  points from  $\mathcal{C}$ , we branch over the choice of  
674 adding to the solution one of the at most  $k$  possible segments provided by Lemma 4.2;  
675 name this segment  $s$  and name the set of points from  $\mathcal{C}$  that lie on  $s$  as  $S$ . By recursion,  
676 we find a solution  $\mathcal{R}$  for the instance  $(\mathcal{C} - S, \mathcal{P} - \{s\})$ , and parameter  $k - 1$ . We return  
677  $\mathcal{R} \cup \{s\}$ . Note that if Lemma 4.2 returned  $\emptyset$ , then we respond NO.
- 678 (3) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer NO.
- 679 (4) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force: check all subsets of  $\mathcal{P}$  of size at most  $k$ .

680 **Correctness.** Lemma 4.2 proves that at least one segment that we branch over in (1)  
681 must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ . Therefore, the recursive call can find  
682 a solution, provided there exists one.

683 In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which  
684 implies the same about every segment from  $\mathcal{P}$ . Under this assumption we can cover only  $k^2$   
685 points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

686 Checking all possible solutions in (3) is trivially correct.

687 **Complexity.** In the leaves of the recursion we have  $|\mathcal{C}| \leq k^2$ , so  $|\mathcal{P}| \leq k^4$ , because  
688 every segment can be uniquely identified by the two extreme points it covers (by Lemma 4.1).  
689 Therefore, there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $\mathcal{O}(k|\mathcal{C}|)$ .  
690 Thus, (3) takes time  $k^{\mathcal{O}(k)}$ .

691 In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we  
692 have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch  
693 over can be found on each level in time  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)})$ .

Reduction from Lemma 4.1 can be implemented in time  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)})$ .

It follows that the overall complexity is  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)} \cdot k^{\mathcal{O}(k)})$   $\square$

## 4.2. Fixed-parameter tractable algorithm for weighted segments with $\delta$ -extension

In this section we consider the geometric set cover problem for weighted segments relaxed with  $\delta$ -extension. We show that this problem admits an FPT algorithm when parameterized by the size of the solution and  $\delta$ . In the next chapter we show that the assumption about the problem being relaxed with  $\delta$ -extension is necessary: we prove that geometric set cover problem for weighted segments (without extension) is W[1]-hard, which means there does not exist any FPT algorithm parameterized by solution size for it, assuming  $\text{FPT} \neq \text{W}[1]$ .

**Theorem 1.3.** (*FPT for WEIGHTED SEGMENT SET COVER with  $\delta$ -extension*). *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$  points  $\mathcal{C}$ , and parameters  $k$  and  $\delta > 0$ , runs in time  $f(k, \delta) \cdot (nm)^c$  for some computable function  $f$  and a constant  $c$  and outputs a set  $\mathcal{R}$  such that:*

- $\mathcal{R} \subseteq \mathcal{P}$ ,
- $|\mathcal{R}| \leq k$ ,
- $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ ,
- the weight of  $\mathcal{R}$  is not greater than the weight of an optimum solution of size at most  $k$  for this problem without  $\delta$ -extension,

or determines that there is no set  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  such that  $\mathcal{R}$  covers all points in  $\mathcal{C}$ .

To solve this problem we will introduce a lemma about choosing a *dense* subset of points. A dense subset of points for a set of collinear points  $C$  and parameters  $k$  and  $\delta$  is a subset of  $C$  such that if we cover it with at most  $k$  segments, these segments after  $\delta$ -extension will cover all of the points from  $C$ . We will prove that such set of size bounded by some function  $f(k, \delta)$  always exists (Lemma 4.3). Later, Lemma 4.3 will allow us to find a kernel for our original problem.

**Definition 4.2.** For a set of collinear points  $C$ , a subset  $A \subseteq C$  is  $(k, \delta)$ -**dense** if for any set of segments  $R$  that covers  $A$  and such that  $|R| \leq k$ , it holds that  $R^{+\delta}$  covers  $C$ .

**Lemma 4.3.** *For any set of collinear points  $C$ ,  $\delta > 0$  and  $k \geq 1$ , there exists a  $(k, \delta)$ -dense set  $A \subseteq C$  of size at most  $(2 + \frac{2}{\delta})^k$ . Moreover, there exists an algorithm that computes the  $(k, \delta)$ -dense set in time  $\mathcal{O}(|C| \cdot (2 + \frac{2}{\delta})^k)$ .*

*Proof.* We prove this for a fixed  $\delta$  by induction on  $k$ .

**Inductive hypothesis.** For any set of collinear points  $C$ , there exists a set  $A$  such that:

- $A$  is subset of  $C$ ,
- $A$  is  $(\ell, \delta)$ -dense for every  $1 \leq \ell \leq k$ ,
- $|A| \leq (2 + \frac{2}{\delta})^k$ ,
- the extreme points of  $C$  are in  $A$ .



731 **Base case for  $k = 1$ .** It is sufficient that  $A$  consists of the extreme points of  $C$ .  
732 If they are covered with one segment, it must be a segment that includes the extreme  
733 points from  $C$ , so it covers the whole set  $C$ .  
734 There are at most 2 extreme points in  $C$  and  $2 < 2 + \frac{2}{\delta}$ .

735 **Inductive step.** Assuming inductive hypothesis for any set of collinear points  $C$  and  
736 for parameter  $k$ , we will prove it for  $k + 1$ .  
737 Let  $s$  be the minimal segment that includes all points from  $C$ . That is, the extreme points  
738 of  $C$  are endpoints of  $s$ .  
739 We define  $M = \lceil 1 + \frac{2}{\delta} \rceil$  subsegments of  $s$  by splitting  $s$  into  $M$  closed segments of equal  
740 length. We name these segments  $v_i$ , note that  $|v_i| = \frac{|s|}{M}$  for each  $1 \leq i \leq M$ .  
741 Let  $C_i$  be the subset of  $C$  consisting of points lying on  $v_i$ .  
742 Let  $t_i$  be the segment with endpoints being the extreme points of  $C_i$ . It might be a  
743 degenerate segment if  $C_i$  consists of one point, or  $t_i$  might be empty if  $C_i$  is empty.  
744 Figure 4.1 presents an example of such segments  $v_i$  and  $t_i$ .

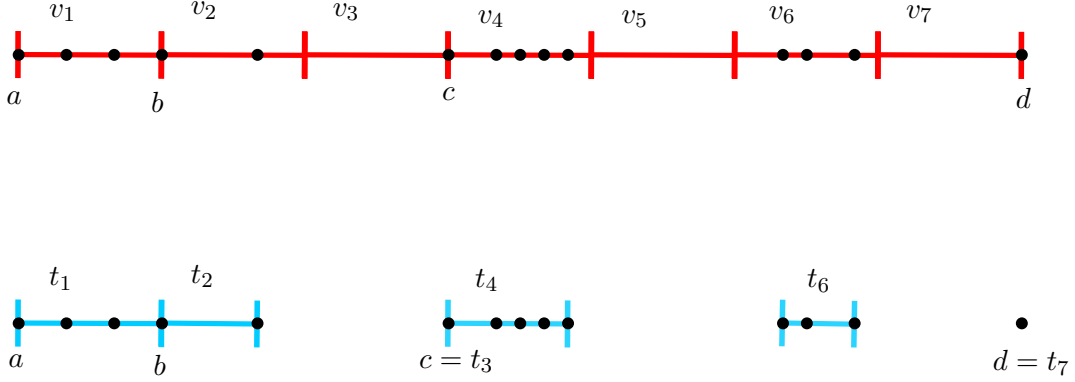


Figure 4.1: **Example of segments  $v_i$  and  $t_i$ .**

Example for  $M = 7$  and some set of points (marked with black circles). The top panel shows segments  $v_i$  and the bottom panel shows segments  $t_i$  on the same set of points.  $a$  and  $b$  are the extreme points and therefore segment  $s$  ends at  $a$  and  $b$ . Red segments depict the split into  $M$  segments of equal length  $v_i$ . Blue segments depict the segments  $t_i$ .  $t_5$  is an empty segment, because there are no points that lie on segment  $v_5$ . Segments  $t_3$  and  $t_7$  are degenerated to one point –  $c$  and  $d$ , respectively. Segments  $t_1$  and  $t_2$  share one point  $b$ .

745 We use the inductive hypothesis to choose  $(k, \delta)$ -dense sets  $A_i$  for sets  $C_i$ . Note that if  
746  $|C_i| \leq 1$ , then  $A_i = C_i$  and it is still a  $(k, \delta)$ -dense set for  $C_i$ .  
747 Then we define  $A = \bigcup_{i=1}^M A_i$ . Thus  $A$  includes the extreme points of  $C$ , because they are  
748 included in the sets  $A_1$  and  $A_M$ .

The size of each  $A_i$  is at most  $(2 + \frac{2}{\delta})^k$  from the inductive hypothesis, therefore size of  $A$  is at most:

$$M \left(2 + \frac{2}{\delta}\right)^k = \left\lceil 1 + \frac{2}{\delta} \right\rceil \cdot \left(2 + \frac{2}{\delta}\right)^k \leq \left(2 + \frac{2}{\delta}\right)^{k+1}.$$

749 **Proof that  $A$  is  $(k+1, \delta)$ -dense for  $C$ .** Let us take any cover of  $A$  with  $k+1$  segments  
750 and call it  $\mathcal{R}$ .

751 For every segment  $t_i$ , if there exists a segment  $x$  in  $\mathcal{R}$  that is disjoint with  $t_i$ , then we have  
752 a cover of  $A_i$  with at most  $k$  segments using  $\mathcal{R} - \{x\}$ . Since  $A_i$  is  $(k, \delta)$ -dense for  $t_i$  and  $C_i$ ,  
753  $(\mathcal{R} - \{x\})^{+\delta}$  covers  $C_i$ . So  $\mathcal{R}^{+\delta}$  covers  $C_i$  as well.  
754 If there exists a segment  $t_i$  for which a segment  $x$  as defined above does not exist, then  
755 all  $k + 1$  segments that cover  $A_i$  intersect  $t_i$ . An example of such segments is depicted in  
756 Figure 4.2. Let us consider any such  $t_i$ . By the inductive hypothesis, the endpoints of  $s$   
757 are in  $A_1$  and  $A_M$  respectively, so  $\mathcal{R}$  must cover them. For each endpoint of  $s$ , there exists  
758 a segment that contains this endpoint and intersects  $t_i$ . Let us call these two segments  $y$   
759 and  $z$ . It follows that:  $|y| + |z| + |t_i| \geq |s|$ . Since  $|t_i| \leq |v_i| = \frac{|s|}{M} \leq \frac{|s|}{1+\frac{2}{\delta}} = \frac{|s|\delta}{\delta+2}$ , we have  
760  $\max(|y|, |z|) \geq |s|(1 - \frac{\delta}{\delta+2})/2 = \frac{|s|}{\delta+2}$ .



Figure 4.2: **Example of all  $k + 1$  segments intersecting one segment  $t_i$ .**  
Both panels show the same set  $\mathcal{C}$  (black circles), the same as in Figure 4.1. The top panel shows blue segments  $t_i$  for  $M = 7$ . The bottom panel shows green segments – solution  $\mathcal{R}$  of size 4. All segments from  $\mathcal{R}$  intersect  $t_4$ . Segments  $z$  and  $y$  are named in the figure.

After  $\delta$ -extension, the longer of these segments will expand at both ends by at least:

$$\max(|y|, |z|)\delta \geq \frac{|s|\delta}{\delta+2} = \frac{|s|}{1+\frac{2}{\delta}} \geq \frac{|s|}{M} = |v_i| \geq |t_i|.$$

761 Therefore, the longer of segments  $y$  and  $z$  will cover the whole segment  $t_i$  after  $\delta$ -extension.  
762 We conclude that  $\mathcal{R}^{+\delta}$  covers  $C_i$ .  
763 Since  $C = \bigcup_{i=1}^M C_i$ , it follows that  $\mathcal{R}^{+\delta}$  covers  $C$ .

**Algorithm.** We can simulate the inductive proof presented above by a recursive algorithm with the following complexity:

$$O\left(|C| + \frac{1}{\delta}\right) + O\left(|C| \cdot \left(2 + \frac{2}{\delta}\right)^k\right).$$

764

□

765 Let us now formulate some claims about the properties for the problem parameterized  
766 by the solution size. These properties provide bounds for different objects in the problem  
767 instance, which help us to find a small kernel for the problem or conclude that the optimum  
768 solution to this instance must be, in terms of size, above some threshold.

769 **Definition 4.3.** A line in the plane is **long** if there are at least  $k + 1$  points from  $\mathcal{C}$  on it.

770 **Claim 4.1.** *If there are more than  $k$  different long lines, then  $\mathcal{C}$  can not be covered with  $k$*   
 771 *segments.*

772 *Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k+1$  different  
 773 long lines in our instance of the problem and there is a solution  $\mathcal{R}$  of size at most  $k$  covering  
 774 points  $\mathcal{C}$ .

775 Choose any long line  $L$ . Every segment from  $\mathcal{R}$  which is not collinear with  $L$ , covers at  
 776 most one point that lies on  $L$ .  $L$  is long, so there are at least  $k+1$  points from  $\mathcal{C}$  that lie on  
 777  $L$ . This implies that there must be a segment in  $\mathcal{R}$  that is collinear with  $L$ .

778 Since we have at least  $k+1$  different long lines, there are at least  $k+1$  segments in  $\mathcal{R}$   
 779 collinear with different lines. This contradicts with the assumption that  $|\mathcal{R}| \leq k$ .  $\square$

780 **Claim 4.2.** *If there are more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then  $\mathcal{C}$*   
 781 *can not be covered with  $k$  segments.*

782 *Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k^2+1$  points  
 783 from  $\mathcal{C}$  that do not lie on any long line, call this set  $A$ , and a solution  $\mathcal{R}$  of size at most  $k$   
 784 covering all points in  $\mathcal{C}$ .

785 Every segment  $s$  from  $\mathcal{R}$  covers at most  $k$  points from  $A$ . This is because if  $s$  covered at  
 786 least  $k+1$  points from  $A$ , then the line in the direction of  $s$  would be a long line and that  
 787 contradicts the definition of  $A$ .

788 If every segment from  $\mathcal{R}$  covers at most  $k$  points from  $A$  and  $|\mathcal{R}| \leq k$ , then at most  $k^2$   
 789 points from  $A$  are covered by  $\mathcal{R}$  and that contradicts the fact that  $\mathcal{R}$  is a solution to the given  
 790 geometric set cover instance.  $\square$

791 We are now ready to give a proof of Theorem 1.3.

792 *Proof of Theorem 1.3.* Our goal is to either answer NO or to find a kernel  $(\mathcal{C}', \mathcal{P}')$  of size  
 793 bounded by  $f(k)$  for some function  $f$ , such that:

- 794 • (*Property 1*) for every solution  $\mathcal{R}$  to  $(\mathcal{C}, \mathcal{P})$  of size at most  $k$ , there exists a set  $\mathcal{R}_1 \subseteq \mathcal{P}'$   
 795 such that  $|\mathcal{R}_1| \leq k$ , the weight of  $\mathcal{R}_1$  is not greater than the weight of  $\mathcal{R}$ , and  $\mathcal{R}_1$  covers  
 796  $\mathcal{C}'$ ;
- 797 • (*Property 2*) for every set  $\mathcal{R}_2 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_2| \leq k$  and  $\mathcal{R}_2$  covers all points in  $\mathcal{C}'$ ,  
 798  $\mathcal{R}_2^{+\delta}$  covers all points in the original set  $\mathcal{C}$ .

799 If we found such sets  $(\mathcal{C}', \mathcal{P}')$ , using *Property 1* we know that an optimum solution of size  
 800 at most  $k$  to  $(\mathcal{C}', \mathcal{P}')$  has no greater weight than an optimum solution of size at most  $k$  to  
 801  $(\mathcal{C}, \mathcal{P})$ . Using *Property 2* we know that any solution to  $(\mathcal{C}', \mathcal{P}')$  after  $\delta$ -extension covers  $\mathcal{C}$ .

802 Therefore, finding such sets and solving the instance  $(\mathcal{C}', \mathcal{P}')$  by iterating over all of the  
 803 subsets of  $\mathcal{P}'$  of size at most  $k$  in desired complexity is sufficient to prove Theorem 1.3.

804 **Definition of  $\mathcal{C}'$  and  $\mathcal{P}'$ .** Let us name the number of different long lines as  $l$ . Applying  
 805 Claims 4.1 and 4.2, if we have more than  $k$  different long lines or more than  $k^2$  points from  
 806  $\mathcal{C}$  that do not lie on any long line, then we answer NO, because these lemmas prove that there  
 807 is no solution of size at most  $k$  to this instance.

808 Otherwise, we can split  $\mathcal{C}$  into at most  $k+1$  sets:

- 809 •  $D$ : points that do not lie on any long line,  $|D| \leq k^2$ ;
- 810 •  $C_i$  for  $1 \leq i \leq l$ : points that lie on the  $i$ -th long line,  $|C_i| > k$ .

811 Note that sets  $C_i$  do not need to be disjoint.

812 Then, for every set  $C_i$  we can use Lemma 4.3 to obtain a  $(k, \delta)$ -dense set  $A_i$  for  $C_i$  with  
813  $|A_i| \leq (2 + \frac{2}{\delta})^k$ .

814 We define  $\mathcal{C}' := D \cup (\bigcup A_i)$ .  $\mathcal{C}'$  has size at most  $k^2 + k(2 + \frac{2}{\delta})^k$ . We define  $\mathcal{P}'$  as  
815 follows: for every pair of points  $\mathcal{C}'$ , we choose one segment from  $\mathcal{P}$  that has the lowest weight  
816 among segments that cover these points or decide that there is no segment that covers them.  
817 There are at most  $|\mathcal{C}'|^2$  different segments in  $\mathcal{P}'$ , therefore both  $\mathcal{P}'$  and  $\mathcal{C}'$  have size bounded  
818 by  $\mathcal{O}((k^2 + k(2 + \frac{2}{\delta}))^2)$ .

819 **Proof of Property 2.** Firstly, we prove that for every set  $\mathcal{R}_2 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_2| \leq k$  and  
820  $\mathcal{R}_2$  covers points in  $\mathcal{C}'$ ,  $\mathcal{R}_2^{+\delta}$  covers points in the original instance  $\mathcal{C}$ .

821 Let us take such a set  $\mathcal{R}_2$ .

822  $\mathcal{C}$  is partitioned into several parts – sets  $D$  and  $C_i$ . Points from  $D$  are covered by  $\mathcal{R}_2$ ,  
823 because  $D$  is part of  $\mathcal{C}'$ . Each point from any  $A_i$  is covered, because  $A_i$  is a part of  $\mathcal{C}'$ ;  $A_i$  is a  
824  $(k, \delta)$ -dense set for  $C_i$ , therefore  $\mathcal{R}_2^{+\delta}$  covers all points in  $C_i$ . Therefore,  $\mathcal{R}_2^{+\delta}$  covers all points  
825 in  $\mathcal{C}$ .

826 **Proof of Property 1.** Secondly, we prove that for every solution  $\mathcal{R}$  to  $(\mathcal{C}, \mathcal{P})$  of size at  
827 most  $k$ , there exists a set  $\mathcal{R}_1 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_1| \leq k$ , the weight of  $\mathcal{R}_1$  is not greater than  
828 the weight of  $\mathcal{R}$  and  $\mathcal{R}_1$  covers  $\mathcal{C}'$ .

829 For every segment in  $\mathcal{R}$ , say  $s$ , let us look at the points from  $\mathcal{C}'$  that lie on  $s$  and call this  
830 set of points  $F$ .  $F$  is of course a set of collinear points. We can cover  $F$  with any segment  
831 that covers extreme points of  $F$ , because all other points lie on the segment between these  
832 points. Therefore, we can replace  $s$  with a segment  $s'$  that has lowest weight among the points  
833 that cover the extreme points of  $F$ . Such a segment belongs to  $\mathcal{P}'$ , because this is how it was  
834 defined. Segment  $s'$  has weight no greater than the weight of  $s$ , because  $s$  also covers  $F$ .

835 Therefore, we produced the set  $\mathcal{R}_1$  that has size not greater than the size of  $\mathcal{R}$  (because  
836 some segments  $s$  can map to the same segment  $s'$ ), weight not greater than  $\mathcal{R}$ , and it covers  
837  $\mathcal{C}'$ .

838 **Complexity** We find a solution of  $(\mathcal{C}', \mathcal{P}')$  by iterating over all the possible subsets of  
839  $\mathcal{P}'$ . Finding sets  $\mathcal{P}'$  and  $\mathcal{C}'$  and then solving problem for kernel has overall complexity  
840  $(|\mathcal{P}| + |\mathcal{C}|)^{\mathcal{O}(1)} \mathcal{O}((2 + \frac{2}{\delta})^k) + \mathcal{O}((k^2 + k(2 + \frac{2}{\delta})^k)^k)$ .  $\square$

## Chapter 5

# W[1]-hardness for axis-parallel weighted segments

In this chapter we consider the geometric set cover problem with axis-parallel or right-diagonal weighted segments. In Theorem 1.4 below, we prove that this problem is W[1]-hard when parameterized by the size of the solution.

We believe that the below construction can be improved to only utilize the axis-parallel segments.

**Theorem 1.4.** *Consider the problem of covering a set  $\mathcal{C}$  of points by selecting at most  $k$  segments from a set of segments  $\mathcal{P}$  with non-negative weights  $w : \mathcal{P} \rightarrow \mathbb{R}^+$  so that the weight of the cover is minimal. Then this problem is W[1]-hard when parameterized by  $k$  and assuming ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ . Moreover, this holds even if all segments in  $\mathcal{P}$  are axis-parallel or right-diagonal.*

### 5.1. Grid Tiling

In order to prove Theorem 1.4 we will show a reduction from a W[1]-hard problem: grid tiling. This problem was introduced in [Marx, 2007] (the author called it matrix tiling instead). It was originally described as an approximation problem, but W[1]-hardness follows directly from the theorems stated there. For a more contemporary description of this problem and a proof of W[1]-hardness, see Chapter 14 of [Cygan et al., 2015].

**Definition 5.1.** We define the **powerset** of a set  $A$ , denoted as  $\text{Pow}(A)$ , as the set of all subsets of  $A$ , i.e.  $\text{Pow}(A) = \{B : B \subseteq A\}$ .

**Definition 5.2.** In the **grid tiling** problem we are given integers  $n$  and  $k$ , and a function  $f : \{1, \dots, k\} \times \{1, \dots, k\} \rightarrow \text{Pow}(\{1, \dots, n\} \times \{1, \dots, n\})$  specifying the set of allowed tiles for each cell of a  $k \times k$  grid. The task is to decide whether there exist functions  $x, y : \{1, \dots, k\} \rightarrow \{1, \dots, n\}$  that assign colors from  $\{1, \dots, n\}$  to respectively columns and rows of the grid, so that  $(x(i), y(j)) \in f(i, j)$  for all  $i, j \in \{1, \dots, k\}$ .

In short, in the grid tiling problem one needs to assign numbers to rows and columns in such a way that for every pair of a row and a column, the pair of colors assigned to the row and column belongs to the allowed set of tiles for this pair. The next theorem describes the complexity of this problem, which is W[1]-hard when parameterized by the size of the grid.

	$x(1) = 3$	$x(2) = 1$	$x(3) = 3$	$x(4) = 7$
$y(4) = 1$	$(\mathbf{2}, \mathbf{1}); (2, 2);$ $(\mathbf{3}, \mathbf{1}); (3, 9)$	$(1, 1); (3, 1)$	$(\mathbf{3}, \mathbf{1}); (7, 2)$	$(\mathbf{2}, \mathbf{1}); (\mathbf{7}, \mathbf{1})$
$y(3) = 1$	$(\mathbf{2}, \mathbf{1}); (\mathbf{3}, \mathbf{1});$ $(4, 2); (8, 2)$	$(1, 1); (1, 3)$	$(\mathbf{3}, \mathbf{1}); (4, 3)$	$(\mathbf{2}, \mathbf{2}); (\mathbf{7}, \mathbf{1})$
$y(2) = 6$	$(\mathbf{2}, \mathbf{6}); (\mathbf{3}, \mathbf{6})$	$(1, 2); (\mathbf{1}, \mathbf{6});$ $(2, 6)$	$(2, 6); (\mathbf{3}, \mathbf{6})$	$(\mathbf{2}, \mathbf{6}); (\mathbf{7}, \mathbf{6})$
$y(1) = 4$	$(\mathbf{2}, \mathbf{4}); (2, 6);$ $(\mathbf{3}, \mathbf{4}); (\mathbf{3}, \mathbf{9})$	$(1, 4); (\mathbf{1}, \mathbf{9})$	$(\mathbf{3}, \mathbf{4}); (\mathbf{3}, \mathbf{9})$	$(\mathbf{2}, \mathbf{9}); (\mathbf{7}, \mathbf{4})$

Figure 5.1: **Example of a grid tiling instance and its solution.**

In the first row and column of the table you can see the solution: functions  $x$  and  $y$ . The tiles used in this solution are marked in **bold**. If we instead chose the tiles marked in **blue** (whenever there is one, taking the tile marked in **bold** otherwise), then that corresponds to setting  $x(1) = 2$ , and would also form a correct solution. On the other hand, if we instead chose the tiles marked in **red** (as before), then this corresponds to setting  $y(1) = 9$  and  $x(4) = 2$  and that would **not** form a correct solution. Even though the first row is correct, the cell with coordinates  $(3, 4)$  requires tile  $(2, 1)$ , not  $(2, 2)$  (marked in **bold red**).

**Theorem 5.1.** [Marx, 2007] *Grid tiling is  $W[1]$ -hard when parameterized by  $k$  and assuming ETH, there is no  $f(k) \cdot n^{o(k)}$ -time algorithm solving the grid tiling problem for any computable function  $f$ .*

The remainder of this section is devoted to proving Theorem 1.4 by a reduction from a grid tiling problem instance with parameter  $k$  (number of rows in the grid) to a geometric set cover instance with parameter  $k^2$  (size of solution). This reduction is described in Lemma 5.1. This proves the  $W[1]$ -hardness of the geometric set cover problem, because if we could solve it with an FPT algorithm, then we could also solve the grid tiling problem (which we reduced to the geometric set cover). Therefore, geometric set cover with setting described in Theorem 1.4 is at least as hard as the grid tiling problem.

## 5.2. Statement of reduction

Let us denote an instance of grid tiling problem as  $(n, k, f)$  consisting of:

- the number of colors  $n$ ,
- the size of the grid  $k$ ,
- the function specifying the allowed tiles  $f : \{1, \dots, k\} \times \{1, \dots, k\} \rightarrow \text{Pow}(\{1, \dots, n\} \times \{1, \dots, n\})$ .

Let us also define constants:

$$\begin{aligned} \epsilon &:= \frac{1}{2k^2} \\ \delta &:= \frac{1}{4k^4} \\ W_{\text{hv}} &:= 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) \end{aligned}$$

which are going to be used when defining the weight of the constructed instance of geometric set cover with weighted segments.

**Lemma 5.1.** *Given an instance  $(n, k, f)$  of the grid tiling problem, we can construct an instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of geometric set cover with weighted segments such that:*

- (1) if the answer to  $(n, k, f)$  is YES, then there exists a solution to  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of weight at most  $W_{\text{hv}} + k^2\delta$ ;*
- (2) if there exists a solution to  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of weight at most  $W_{\text{hv}} + k^2\delta$ , then the answer to  $(n, k, f)$  is YES.*

First, let us prove Theorem 1.4 using Lemma 5.1.

*Proof of Theorem 1.4.* Let us take any instance  $(n, l, f)$  of the grid tiling problem. We prove the theorem by contradiction, therefore we assume that geometric set cover with weighted segments parameterized by solution size  $k$  admits a  $g(k) \cdot n^{o(\sqrt{k})}$ -time algorithm for some computable function  $g$ .

Using Lemma 5.1 let us construct an instance  $I$  for  $(n, l, f)$ . Let us assume that the optimum solution of size at most  $k$  to the instance  $I$  has weight  $u$ . Using (2) we know that if  $u \leq W_{\text{hv}} + k^2\delta$ , then the answer to  $(n, l, f)$  is YES. If  $u > W_{\text{hv}} + k^2\delta$ , then using (1) we know that the answer to  $(n, l, f)$  must be NO.

Therefore if we could find the solution in time  $g(k) \cdot n^{o(\sqrt{k})}$ , then we could solve the grid tiling problem in time  $g(l) \cdot n^{o(l)}$  by constructing an instance of the set cover with weighted segments, solving it for parameter  $k = 3l^2 + 2l$  in time  $n^{o(\sqrt{3l^2+2l})}$  and then answering based on the weight of the optimum solution. As  $\mathcal{O}(n^{o(l)}) \subseteq \mathcal{O}(n^{o(\sqrt{3l^2+2l})})$ , the existence of this algorithm contradicts Theorem 5.1. Hence such an algorithm can not exist.  $\square$

We prove Lemma 5.1 in subsequent sections. First, we define a constructed instance  $I$ , later property (1) is proved by Lemma 5.2 and property (2) is proved by Lemma 5.6.

In the proof of Lemma 5.6 we do not use the assumption that the solution is bounded by the size, which the problem is parameterized by,  $3k^2 + 2k$ . If we had a permissive FPT algorithm that finds a solution of any size that still has weight no more than  $W_{\text{hv}} + k^2\delta$ , then we still would have a contradiction with grid tiling being W[1]-hard in proof of Theorem 1.4. Thus, this reduction proves that the problem is not only W[1]-hard, but assuming ETH there also does not exist permissive FPT algorithm for this problem. Formally we state this in the Theorem 5.2.

**Theorem 5.2. (Permissive FPT does not exist).** *Consider the problem of covering a set  $\mathcal{C}$  of points using segments from a set  $\mathcal{P}$  with non-negative weights  $w : \mathcal{P} \rightarrow \mathbb{R}^+$  so that the weight of the cover is minimal. Let  $\mathcal{R}^k$  be the optimum solution to this problem of size at most  $k$ . The task is to find a solution  $\mathcal{R}$  of any size such that weight of  $\mathcal{R}$  is not greater than the weight of  $\mathcal{R}^k$ .*

*Assuming ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ . Moreover, this holds even if all segments in  $\mathcal{P}$  are axis-parallel or right-diagonal.*

### 5.3. Construction of the Geometric Set Cover instance

We construct an instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of geometric set cover as follows.

First, let us choose any bijection  $\text{order} : \{1, \dots, n^2\} \rightarrow \{1, \dots, n\} \times \{1, \dots, n\}$ .

Define  $\text{match}_v(i, j)$  and  $\text{match}_h(i, j)$  as boolean functions denoting whether two points share x or y coordinate:

$\text{match}_v(i, j)$  is **true**  $\iff$   $\text{order}(i)$  and  $\text{order}(j)$  have the same x coordinate,

$\text{match}_h(i, j)$  is **true**  $\iff$   $\text{order}(i)$  and  $\text{order}(j)$  have the same y coordinate.

### 931 5.3.1. Points

For  $1 \leq i, j \leq k$  and  $1 \leq t \leq n^2$  define points:

$$h_{i,j,t} := (i \cdot (n^2 + 1) + t, j \cdot (n^2 + 1)),$$

$$v_{i,j,t} := (i \cdot (n^2 + 1), j \cdot (n^2 + 1) + t).$$

Let us define sets  $H$  and  $V$  as:

$$H := \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\},$$

$$V := \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}.$$

Let us recall that  $\epsilon = \frac{1}{2k^2}$ . For a point  $p = (x, y)$  we define points:

$$p^L := (x - \epsilon, y),$$

$$p^R := (x + \epsilon, y),$$

$$p^U := (x, y + \epsilon),$$

$$p^D := (x, y - \epsilon).$$

Then we define the point set as follows:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}.$$

932 **Definition 5.3.** For every point  $p \in H$ , we name point  $p^L$  its **left guard** and point  $p^R$  its  
933 **right guard**.

934 Similarly for every points  $p \in V$ , we name point  $p^D$  its **lower guard** and point  $p^U$  its  
935 **upper guard**.

### 936 5.3.2. Segments

937 For  $1 \leq i, j \leq k$  and  $1 \leq t, t_1, t_2 \leq n^2$  define segments:

$$\text{hor}_{i,j,t_1,t_2} := (h_{i,j,t_1}^R, h_{i+1,j,t_2}^L),$$

$$\text{ver}_{i,j,t_1,t_2} := (v_{i,j,t_1}^U, v_{i,j+1,t_2}^D),$$

$$\text{horBeg}_{i,t} := (h_{1,i,1}^L, h_{1,i,t}^L),$$

$$\text{horEnd}_{i,t} := (h_{k,i,t}^R, h_{k,i,n^2}^R),$$

$$\text{verBeg}_{i,t} := (v_{i,1,1}^D, v_{i,1,t}^D),$$

$$\text{verEnd}_{i,t} := (v_{i,k,t}^U, v_{i,k,n^2}^U).$$



938 Next, we define sets of vertical and horizontal segments:

$$\begin{aligned} \text{HOR} &:= \{ \text{hor}_{i,j,t_1,t_2} : 1 \leq i < k, 1 \leq j \leq k, 1 \leq t_1, t_2 \leq n^2, \text{match}_h(t_1, t_2) \text{ holds} \} \\ &\cup \{ \text{horBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \} \\ &\cup \{ \text{horEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \}, \end{aligned}$$

939

$$\begin{aligned} \text{VER} &:= \{ \text{ver}_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, \text{match}_v(t_1, t_2) \text{ holds} \} \\ &\cup \{ \text{verBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \} \\ &\cup \{ \text{verEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \}. \end{aligned}$$

940 An example is depicted in Figure 5.3.

Finally, we also define a set of right-diagonal segments:

$$\text{DIAG} := \{ (h_{i,j,t}, v_{i,j,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, \text{order}(t) \in f(i, j) \}.$$

941 An example of such segments is depicted in Figure 5.2.

942 Every segment in **DIAG** connects points  $(i(n^2+1)+t, j \cdot (n^2+1))$  and  $(i \cdot (n^2+1), j(n^2+1) + t)$   
 943 for some  $1 \leq i, j \leq k, 1 \leq t \leq n^2$ . The line on which it lies can be described by linear equation  
 944  $x + y = t + (i + j)(n^2 + 1)$ , thus these segments are in fact right-diagonal.

945 The constructed segment set is defined as:

$$\mathcal{P} := \text{HOR} \cup \text{VER} \cup \text{DIAG}.$$

946 The weight of each segment in  $\text{HOR} \cup \text{VER}$  is equal to its length, while every segment in  
 947 **DIAG** has weight  $\delta$ .

$$w(s) = \begin{cases} \text{length}(s) & \text{if } s \in \text{HOR} \cup \text{VER} \\ \delta & \text{if } s \in \text{DIAG} \end{cases}$$

## 948 5.4. Proof that reduction is correct

949 Now, we prove that the constructed instance of geometric set cover with weighted segments  
 950 indeed gives a correct and sound reduction of the grid tiling problem. Lemma 5.2 proves that  
 951 if a solution to the instance of the grid tiling instance exists, then there exists a solution with  
 952 suitably bounded size and weight of the constructed instance of geometric set cover. Then  
 953 Lemma 5.6 proves that if there is a solution to the geometric set cover instance with bounded  
 954 weight, then there exists a solution to the original grid tiling instance.

955 **Lemma 5.2.** *If there exists a solution to the grid tiling instance  $(f_{i,j})$ , then there exists*  
 956 *a solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of geometric set cover with weight  $W_{\text{hv}} + k^2\delta$ .*

957 *Proof.* Suppose there exists a solution  $x, y$  of the instance  $(f_{i,j})$  of the grid tiling problem.

958 We define the proposed solution  $\mathcal{R} \subseteq \mathcal{P}$  of the instance of geometric set cover in three



Figure 5.2: **Vertices and segments in DIAG.**

This is an example of constructed points any  $1 \leq i, j \leq k$ . Points from  $H$  and  $V$  are marked in black, their guards are marked in blue. You can also see segments from DIAG with their weights (equal to  $\delta$ ).



Figure 5.3: **Vertices and segments in HOR.**

This is an example for  $n = 2$  and any  $1 \leq j \leq k$ . Points from  $H$  are marked in black, their guards are marked in light blue.  $t_{i,j}$  is a notation that we use for  $\text{order}^{-1}(i, j)$ . Segments are represented as arcs between endpoints. You can see  $\text{horBeg}_{j,t}$  segments in red.  $\text{horBeg}_{j,1}$  is degenerated to a single point at  $h_{1,1,t_{1,1}}^L$ . Segments  $\text{hor}_{i,j,t_{x_1,y},t_{x_2,y}}$  are marked in blue and green. Blue segments connect  $t_{x_1,y}$  and  $t_{x_2,y}$  such that they share y-coordinate equal to 1, for green segments it is equal to 2.

959 parts:  $D \subseteq \text{DIAG}$ ,  $A \subseteq \text{HOR}$  and  $B \subseteq \text{VER}$ :

$$\begin{aligned}
 D &:= \{(v_{i,j,t}, h_{i,j,t}) : 1 \leq i, j \leq k, t = \text{order}^{-1}(x(i), y(j))\}, \\
 A &:= \{\text{horBeg}_{i, \text{order}^{-1}(x(1), y(i))} : 1 \leq i \leq k\} \\
 &\quad \cup \{\text{horEnd}_{i, \text{order}^{-1}(x(k), y(i))} : 1 \leq i \leq k\} \\
 &\quad \cup \{\text{hor}_{i,j, \text{order}^{-1}(x(i), y(j)), \text{order}^{-1}(x(i+1), y(j))} : 1 \leq i < k, 1 \leq j \leq k\}, \\
 B &:= \{\text{verBeg}_{i, \text{order}^{-1}(x(i), y(1))} : 1 \leq i \leq k\} \\
 &\quad \cup \{\text{verEnd}_{i, \text{order}^{-1}(x(i), y(k))} : 1 \leq i \leq k\} \\
 &\quad \cup \{\text{ver}_{i,j, \text{order}^{-1}(x(i), y(j)), \text{order}^{-1}(x(i), y(j+1))} : 1 \leq i \leq k, 1 \leq j < k\}, \\
 \mathcal{R} &:= D \cup A \cup B.
 \end{aligned}$$

960 Since  $\mathcal{C} = H \cup V$ , we show that  $\mathcal{R}$  covers the whole set  $H$ ; the proof for  $V$  is analogous.

961 Fix any  $1 \leq j \leq k$  and define  $t_i := \text{order}^{-1}(x(i), y(j))$ . The two leftmost segments in  $A$   
 962 for this  $j$  are  $\text{horBeg}_{j,t_1} = (h_{1,j,1}^L, h_{1,j,t_1}^L)$  and  $\text{hor}_{1,j,t_1,t_2} = (h_{1,j,t_1}^R, h_{2,j,t_2}^L)$ . Therefore, points  
 963  $h_{1,j,x}, h_{1,j,x}^L$  and  $h_{1,j,x}^R$  for all  $1 \leq x \leq n^2$  are covered by  $\text{horBeg}_{j,t_1}$  and  $\text{hor}_{1,j,t_1,t_2}$ , excluding  
 964 point  $h_{1,j,t_1}$ .

965 Analogously for  $2 \leq i \leq k-1$ , the two consecutive segments  $\text{hor}_{i-1,j,t_{i-1},t_i}$  and  $\text{hor}_{i,j,t_i,t_{i+1}}$   
 966 cover points  $h_{i,j,x}, h_{i,j,x}^L$  and  $h_{i,j,x}^R$  for all  $1 \leq x \leq n^2$ , excluding point  $h_{i,j,t_i}$ .

967 Finally  $\text{hor}_{k-1,j,t_{k-1},t_k}$  and  $\text{horEnd}_{j,t_k}$  cover all points  $h_{k,j,x}, h_{k,j,x}^L$  and  $h_{k,j,x}^R$  for  $1 \leq x \leq n^2$ ,  
 968 excluding point  $h_{k,j,t_k}$ .

969  $D$  covers all points  $h_{i,j,t_i}$  and  $v_{i,j,t_i}$ . As  $j$  was chosen arbitrarily, all points in  $H$  are covered.  
 The size of this proposed solution is:

$$|\mathcal{R}| = |D| + |A| + |B| = k^2 + (k+1)k + (k+1)k = 3k^2 + 2k.$$

970 Then, we need to compute the total weight of the solution  $\mathcal{R}$ . First, we compute the sum  
 971 of weights of segments in  $A$ . Fix  $1 \leq j \leq k$  and consider segments collinear with the  $j$ -th  
 972 horizontal line. All points  $h_{i,j,t}, h_{i,j,t}^L$  and  $h_{i,j,t}^R$  for every  $1 \leq i \leq k$  and  $1 \leq t \leq n^2$  are covered  
 973 by  $A$  excluding points  $h_{i,j, \text{order}^{-1}(x(i), y(j))}$ . Every such point leaves a gap of length  $2\epsilon$  between  
 974  $h_{i,j, \text{order}^{-1}(x(i), y(j))}^L$  and  $h_{i,j, \text{order}^{-1}(x(i), y(j))}^R$ . Therefore, the total weight of segments in  $A$  that  
 975 lie on the line in question equals the length of the segment  $(h_{1,1,1}^L, h_{i,k,n^2}^R)$  minus  $2\epsilon k$ , which is

976  $k(n^2 + 1) - 2(1 - \epsilon) - 2k\epsilon$ . We need to multiply that by  $k$ , as we consider all possible values  
 977 of  $j$ .

978 Computation for vertical segments is analogous and yields the same result. Every segment  
 979 in  $D$  has weight  $\delta$ , therefore the sum of all weights is equal to:

$$2k(k(n^2 + 1) - 2(1 - \epsilon) - 2k\epsilon) + k^2\delta = W_{\text{hv}} + k^2\delta. \quad \square$$

980 Now we present a few additional properties of the constructed instance of the geometric  
 981 set cover that help us to prove Lemma 5.6.

982 **Claim 5.1.** *In any solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$ :*

- 983 • *the left and right guards of points in  $H$  (points in  $\{p^L : p \in H\} \cup \{p^R : p \in H\}$ ) have*  
 984 *to be covered with segments from HOR,*
- 985 • *the lower and upper guards of points in  $V$  (points in  $\{p^D : p \in V\} \cup \{p^U : p \in V\}$ ) have*  
 986 *to be covered with segments from VER.*

987 *Proof.* We prove the claim for the points from  $H$  as the proof for points from  $V$  is analogous.  
 988 Every segment in VER is vertical and has x-coordinate equal to  $i(n^2 + 1)$  for some  $1 \leq i \leq k$ ,  
 989 so they all have different x-coordinate than any left or right guard of points in  $H$ .

990 For every point  $x$  which is a left or right guard of a point in  $H$ , there are  $kn^2$  segments  
 991 from DIAG that intersect with the horizontal line that goes through  $x$ . All of these segments  
 992 intersect with this line in points from set  $H$ , therefore none of them covers any of the guards.

993 Therefore none of the segments from VER or DIAG covers any of the guards of the points  
 994 in  $H$ .  $\square$

995 **Claim 5.2.** *For any  $1 \leq i, j \leq n$  and any solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$ , all*  
 996 *but at most one point  $h_{i,j,t}$  and at most one point  $v_{i,j,t}$  for  $1 \leq t \leq n^2$  must be covered with*  
 997 *segments from HOR or VER.*

998 *Proof.* We prove the claim for horizontal segments, as the proof for vertical segments is anal-  
 999 ogous.

1000 We prove this by contradiction. Assume that we have two points  $h_{i,j,t_1}, h_{i,j,t_2}, 1 \leq t_1 <$   
 1001  $t_2 \leq n^2$ , such that they are not covered with segments from HOR.

1002 Point  $h_{i,j,t_1}^R$  has to be covered with a segment from HOR by Claim 5.1. Every segment in  
 1003 HOR covering  $h_{i,j,t_1}^R$ , but not  $h_{i,j,t_1}$  must start at  $h_{i,j,t_1}^R$  and all such segments cover also  $h_{i,j,t_2}$ .  
 1004 This contradicts the assumption, which concludes the proof.  $\square$

1005 **Lemma 5.3.** *For every solution  $\mathcal{R}$  to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$ , the sum of weights of*  
 1006 *segments chosen from sets HOR and VER is at least  $W_{\text{hv}}$ .*

1007 *Proof.* Let us fix  $1 \leq i \leq k$ .

1008 We provide a lower bound for the sum of lengths of vertical segments from  $\mathcal{R} \cap \text{VER}$ . This  
 1009 bound is the same for each  $i$  and is the same for horizontal lines, thus we need to multiply  
 1010 such a bound by  $2k$ .

(1) The total length between  $v_{i,1,1}^D$  and  $v_{i,k,n^2}^U$  is:

$$(k(n^2 + 1) + n^2 + \epsilon) - ((n^2 + 1) + 1 - \epsilon) = k(n^2 + 1) - 2(1 - \epsilon).$$

1011 (2) For every  $1 \leq j \leq k$  there exists at most one  $1 \leq t \leq n^2$  such that  $v_{i,j,t}$  is not covered  
 1012 by segments from **VER** (Claim 5.2). Its guards (see Definition 5.3)  $v_{i,j,t}^U$  and  $v_{i,j,t}^D$  have  
 1013 to be covered in **VER** (Claim 5.1). Therefore, at most  $k$  spaces of length  $2\epsilon$  can be left  
 1014 not covered by segments from **VER** between  $v_{i,1,1}^D$  and  $v_{i,k,n^2}^U$ .

The sum of these lower bounds for vertical and horizontal lines is:

$$2k(k(n^2 + 1) - 2k\epsilon - 2(1 - \epsilon)) = 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) = W_{\text{hv}}. \quad \square$$

1015 **Lemma 5.4.** *Let  $\mathcal{R}$  be a solution to a constructed instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight at*  
 1016 *most  $W_{\text{hv}} + k^2\delta$ . Then for every  $1 \leq i, j \leq k$  there exists  $1 \leq t \leq n^2$  such that:*

- 1017 (1)  $v_{i,j,t}, h_{i,j,t}$  are not covered by segments from **VER** or **HOR**;
- 1018 (2) segment  $(v_{i,j,t}, h_{i,j,t})$  is in solution  $\mathcal{R}$ ;
- 1019 (3)  $\text{order}(t) \in f(i, j)$ , that is,  $\text{order}(t)$  is an allowed tile for  $(i, j)$ ;
- 1020 (4) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $v_{i,j,s}$  is covered in **VER**;
- 1021 (5) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $h_{i,j,s}$  is covered in **HOR**.

1022 *Proof.* At most one of the points  $\{h_{i,j,t_x} : 1 \leq t_x \leq n^2\}$  and one of the points  $\{v_{i,j,t_y} : 1 \leq$   
 1023  $t_y \leq n^2\}$  is covered with **DIAG** (Claim 5.2).

1024 Moreover, exactly one such point  $h_{i,j,t_x}$  and one such point  $v_{i,j,t_y}$  is covered with **DIAG**,  
 1025 because if none of them were covered, then the solution would have to have weight at least  
 1026  $W_{\text{hv}} + 2\epsilon$  (see the proof of Lemma 5.3), which is more than  $W_{\text{hv}} + k^2\delta$ .

1027 We observe that points  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  have to be covered with the same segment from  
 1028 **DIAG**. Indeed we need to use at least  $k^2$  of them to use exactly one **DIAG** segment for every  
 1029 pair of  $1 \leq i, j \leq k$ , if we used 2 segments from **DIAG** for one pair  $(i, j)$ , then we would have  
 1030 used total weight at least  $W_{\text{hv}} + k^2\delta + \delta$  (Lemma 5.3), which is more than  $W_{\text{hv}} + k^2\delta$ . Since  
 1031 points  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  are covered by a single segment from **DIAG**, we have  $t_x = t_y$ .

1032 Therefore  $t_x = t_y$  and  $\text{order}(t_x)$  is an allowed tile for  $(i, j)$  because the corresponding  
 1033 segment is in **DIAG**.  $\square$

1034 We refer to the function mapping  $1 \leq x \leq k$  to  $t_x$  from Lemma 5.4 as **diagonal** :  $\{1, \dots, k\} \times$   
 1035  $\{1, \dots, k\} \rightarrow \{1, \dots, n^2\}$ .

1036 **Lemma 5.5.** *Let  $\mathcal{R}$  be any solution of a constructed instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight*  
 1037 *at most  $W_{\text{hv}} + k^2\delta$ . Then:*

- 1038 1. for any  $1 \leq i < k, 1 \leq j \leq k$ ,  $\text{match}_h(\text{diagonal}(i, j), \text{diagonal}(i + 1, j))$  is **true**;
- 1039 2. for any  $1 \leq i \leq k, 1 \leq j < k$ ,  $\text{match}_v(\text{diagonal}(i, j), \text{diagonal}(i, j + 1))$  is **true**.

1040 *Proof.* We prove (1) by contradiction, the proof of (2) is analogous.

1041 Let us take any  $1 \leq i < k, 1 \leq j \leq k$  and name  $t_1 = \text{diagonal}(i, j)$  and  $t_2 = \text{diagonal}(i +$   
 1042  $1, j)$ . We also assume that  $\text{match}_h(t_1, t_2)$  is **false**, which is equivalent to the fact that segment  
 1043  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  is not in set **HOR**.

1044 Therefore  $h_{i,j,t_1}$  and  $h_{i+1,j,t_2}$  are not covered by segments from **HOR** (Lemma 5.4), while  
 1045  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  have to be covered by segments from **HOR** (Claim 5.1).

1046 Every segment from **HOR** either:

- 1047 • starts at point  $h_{x,y,z_1}^R$  and ends at point  $h_{x+1,y,z_2}^L$  for some  $1 \leq x < k, 1 \leq y \leq k$  and  
 1048  $1 \leq z_1, z_2 \leq n^2$ ; or
- 1049 • is  $\text{horBeg}_{y,z}$  and starts at  $h_{1,y,1}^L$  and ends at  $h_{1,y,z}^L$  for some  $1 \leq y \leq k$  and  $1 \leq z \leq n^2$ ;  
 1050 or
- 1051 • is  $\text{horEnd}_{y,z}$  and starts at  $h_{k,y,z}^R$  and ends at  $h_{k,y,n^2}^R$  for some  $1 \leq y \leq k$  and  $1 \leq z \leq n^2$ .

1052 All of the points between  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  are covered by segments in HOR and there is no  
 1053 segment  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  in HOR. Hence, there are at least two different segments covering  
 1054 them. If both of these segments are neither  $\text{horBeg}_{y,z}$  nor  $\text{horEnd}_{y,z}$ , then one of them must  
 1055 begin at  $h_{i,j,t_1}^R$  and end at  $h_{i+1,j,z_2}^L$  and there must be other one that begins at  $h_{i,j,z_1}^R$  and ends  
 1056 at  $h_{i+1,j,t_2}^L$  for some  $1 \leq z_1, z_2 \leq n^2$ .

1057 Thus, the space between  $h_{i,j,z_1}^R$  and  $h_{i,j+1,z_2}^L$  would be covered twice and is longer than  $\epsilon$ .  
 1058 The case when one of them is  $\text{horBeg}_{y,z}$  or  $\text{horEnd}_{y,z}$  is analogous. Note that they cannot be  
 1059 both  $\text{horBeg}_{y,z}$  or  $\text{horEnd}_{y,z}$ .

1060 By the proof of Lemma 5.3, the lower bound for weight of such a solution is  $W_{\text{hv}} + \epsilon$  which  
 1061 is more than  $W_{\text{hv}} + k^2\delta$ .

1062 Therefore  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  must be covered by one segment from HOR, namely  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$ .  
 1063 Hence  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  is a segment in HOR and  $\text{match}_h(t_1, t_2)$  is **true**.  $\square$

1064 **Lemma 5.6.** *If there exists a solution to instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight at most*  
 1065  *$W_{\text{hv}} + k^2\delta$ , then there exists a solution to the grid tiling instance  $(f_{i,j})$ .*

1066 *Proof.* Take **diagonal** function from Lemma 5.4.

1067 To define the  $x$  function for every  $1 \leq i \leq k$  set  $x(i) := x_i$  where  $(x_i, a) = \text{order}(v_{i,1})$ .  
 1068 Similarly, to define the  $y$  function, for every  $1 \leq i \leq k$  set  $y(i) := y_i$  where  $(b, y_i) = \text{order}(h_{1,i})$

1069 To prove that this is a correct solution to grid tiling, we need to prove that for every  
 1070  $1 \leq i, j \leq k$ ,  $(x(i), y(j))$  is in the allowed tiles set  $f(i, j)$ .

1071 Let us take any  $1 \leq i, j \leq k$ . By Lemma 5.5 and simple induction, we know that  
 1072  $\text{match}_h(\text{diagonal}(1, j), \text{diagonal}(i, j))$  and  $\text{match}_v(\text{diagonal}(i, 1), \text{diagonal}(i, j))$  are **true**. There-  
 1073 fore  $\text{order}(\text{diagonal}(i, j)) = (x(i), y(j))$ . By Lemma 5.4 we know that  $\text{order}(\text{diagonal}(i, j))$  is in  
 1074  $f(i, j)$ . Therefore  $(x(i), y(j))$  is in  $f(i, j)$ .  $\square$

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