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# Approximation and Parameterized Algorithms for Segment Set Cover

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Master's thesis

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

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## **Tytuł pracy w języku polskim**

36 Algorytmy aproksymacyjne i parametryzowane dla problemu pokrywania punktów  
37 odcinkami na płaszczyźnie



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# Chapter 1

## Introduction

Some problems in Computer Science are known to be NP-complete, meaning that assuming  $P \neq NP$  there is no polynomial time algorithm that can solve these problems. Even so, they still can have a variety of interesting properties. Set Cover is one classical example of an NP-complete problem, which has been proven in literature to be APX-hard and W[2]-hard [Cygan et al., 2015]. We can also restrict the problem to various specialized settings, which can yield more complexity and approximation results. In this paper we take a closer look at the Geometric Set Cover problem on a plane, where points to cover are points on a plane and sets to cover them with are geometric objects.

**Approximation** Over the years there has been a lot of work related to approximation of Geometric Set Cover. Notably, Geometric Set Cover with unweighted unit disks admits PTAS (see Corollary 1.1 in [Mustafa and Ray, 2010]). When we consider the same problem with weighted unit disks (or unit squares), the problem admits QPTAS [Mustafa et al., 2014], later improved in Theorem 2 in [Pilipczuk et al., 2020]. On the other hand, [Chan and Grant, 2014] proves that cover with unweighted axis-parallel rectangles is APX-hard, which they also show for set cover with many other standard geometric objects.

**Parametrization** We consider Geometric Set Cover parameterized by the size of solution. Geometric Set Cover with unit squares was first proven to be W[1]-hard in [Marx, 2005] (Theorem 5), later follow-up [Marx and Pilipczuk, 2015] shows that there is an algorithm running in time  $\mathcal{O}(n^{\sqrt{k}})$  that solves Geometric Set Cover with unit squares or disks and that there is no algorithm running in time  $f(k) \cdot n^{\sqrt{k}-\epsilon}$  for any  $\epsilon > 0$ , so this is a tight bound for this problem.

We also consider parametrization in the weighted setting. There does not seem to be a consensus of what parametrization in weighted setting is exactly; there was an attempt to introduce quite complicated general framework of weighted parameterized setting in [Shachnai and Zehavi, 2017]. Kernels for several well known weighted problems such as Subset Sum or Knapsack are presented in [Etscheid et al., 2017]. Another work [Kim et al., 2021] presents weighted parametrization of Weighted Directed Feedback Set and Weighted *st*-Cut.

**$\delta$ -extensions** In this paper, we focus on Geometric Set Cover with segments with  $\delta$ -extensions.  $\delta$ -extensions is a problem relaxation method based on the  $\delta$ -shrinking model which was introduced in [Adamaszek et al., 2015] and later used in [Pilipczuk et al., 2016] and [Wiese, 2018].

Similar model is used to prove that Geometric Set Cover with fat polygons relaxed with  $\delta$ -extensions admits EPTAS [Har-Peled and Lee, 2009].

## Our contribution

In this paper we make the following contributions.

We show that approximation of unweighted Geometric Set Cover with axis-parallel segments (even if we relax it with  $\frac{1}{2}$ -extensions) is APX-hard (Theorem 3.1). This expands the previous result of Geometric Set Cover with unweighted axis-parallel rectangles being APX-hard in [Chan and Grant, 2014]. That also proves that the assumption in [Har-Peled and Lee, 2009] for EPTAS about polygons being fat is necessary, because cover with arbitrary polygons with  $\delta$ -extensions is APX-hard.

We also provide two FPT algorithms for parameterized Geometric Set Cover with unweighted segments (Theorem 4.2) and weighted segments relaxed with  $\delta$ -extensions (Theorem 4.3). But Geometric Set Cover with weighted axis-parallel segments is W[1]-hard (Theorem 5.1) and assuming ETH there does not exist algorithm for this problem that runs in time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$ . See Figure 1.1 for a summary of parameterized results for the weighted setting.

	exact	$\delta$ -extensions
axis-parallel	W[1]-hard	FPT*
any direction	W[1]-hard*	FPT

Figure 1.1: **Our results for Geometric Set Cover problem with weighted segments parameterized by the size of solution.**

Results marked with \* directly follow from more or less restricted settings.



## Chapter 2

## Definitions

In this chapter we present some definitions that are later used across the different chapters.

### 2.1. Geometric set cover

Every time we refer to geometric set cover, we consider a geometric set cover problem on a 2-dimensional plane.

In the geometric set cover problem we are given  $\mathcal{P}$  – a set of objects, which are connected subsets of the plane and  $\mathcal{C}$  – a set of points in the plane. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that every point in  $\mathcal{C}$  is inside some object from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized.

In the parameterized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$  or decide that there is no such set  $\mathcal{R}$ .

In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$  and we would like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

### 2.2. Approximation

Let us recall some definitions related to optimization problems that are used in the following sections.

**Definition 2.1.** A **polynomial-time approximation scheme (PTAS)** for a minimization problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$  and in polynomial time finds a solution that is within a factor of  $(1 + \epsilon)$  of being optimal. That means the reported solution has weight at most  $(1 + \epsilon)opt(I)$ , where  $opt(I)$  is the weight of an optimal solution to  $I$ .

**Definition 2.2.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

### 2.3. Problem modification with $\delta$ -extension

Another idea presented here, much less versatile than the previous concepts, is  $\delta$ -extension. We define it specifically for the geometric set cover problem.

It is based on the similar idea of  $\delta$ -shrinking for the geometric independent set problem, which was introduced in [Adamaszek et al., 2015] and later utilized in [?] and [Wiese, 2018].

Intuitively, we consider a problem with slightly larger objects, which makes the instance more permissive. However, we aim to find a solution that is not larger than the optimum

solution to the original problem, so this is substantially easier than just solving the problem for the larger objects. It may even be the case that we are able to find the solution of size smaller than the optimum solution to the original problem.

First, we formally define  $\delta$ -extended objects.

**Definition 2.3.** For any  $\delta > 0$  and a center-symmetric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  $\delta$ -**extension** of  $L$  is the object  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ , that is,  $L^{+\delta}$  is the image of  $L$  under homothety centered at  $S$  with scale  $(1 + \delta)$ .

The geometric set cover problem with  $\delta$ -extension is a modified version of geometric set cover with the following modifications.

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extension).
- We look for a solution that is not larger than the optimum solution to the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 2.4.** The **geometric set cover problem with  $\delta$ -extension** is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is not larger than the optimal solution to the problem without extension, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

At last, we formulate a definition of the polynomial-time approximation scheme (PTAS) of the problem with  $\delta$ -extension.

**Definition 2.5.** We define a **PTAS for geometric set cover with  $\delta$ -extension** as a family of algorithms  $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$ , and in polynomial-time outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$  factor of the optimal solution to this problem without extension, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## Chapter 3

# APX-hardness of geometric set cover problem

In this section we analyze whether there exists a PTAS for geometric set cover for rectangles. We show that we can restrict this problem to a very simple setting: segments parallel to axes and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just degenerated rectangles with one side being very narrow.

Our results can be summarized in the following theorem and this section aims to prove it.

**Theorem 3.1.** (*axis-parallel segment set cover with  $1/2$ -extension is APX-hard*). *Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension) is APX-hard. That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

Theorem 3.1 implies the following.

**Corollary 3.1.** (*rectangle set cover is APX-hard*). *Unweighted geometric set cover with axis-parallel rectangles (even with  $1/2$ -extension) is APX-hard.*

We prove Theorem 3.1 by taking a problem that is APX-hard and showing a reduction. For this problem we choose MAX-(3,3)-SAT which we define below.

### 3.1. MAX-(3,3)-SAT and statement of reduction

**Definition 3.1.** MAX-3SAT is the following maximization problem. We are given a 3-CNF formula, and need to find an assignment of variables that satisfies the most clauses.

**Definition 3.2.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction that every variable appears in exactly 3 clauses and every clause contains exactly 3 literals of 3 different variables. Note that thus, the number of clauses is equal to the number of variables.

In our proof of Theorem 3.1 we use hardness of approximation of MAX-(3,3)-SAT proved in [Håstad, 2001] and described in Theorem 3.2 below.

**Definition 3.3** ( $\alpha$ -satisfiable MAX-3SAT formula). MAX-3SAT formula with  $m$  clauses is at most  $\alpha$ -satisfiable, if every assignment of variables satisfies no more than  $\alpha m$  clauses.

**Theorem 3.2.** [Håstad, 2001] *For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most  $(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel segment set cover problem such that for a sufficiently small  $\epsilon > 0$ , a polynomial time  $(1 + \epsilon)$ -approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-(3,3)-SAT is fully satisfiable or is at most  $(7/8 + \epsilon)$ -satisfiable. However, according to Theorem 3.2 the latter problem is NP-hard. This would imply  $P = NP$ , contradicting the assumption.

The following lemma encapsulates the properties of the reduction described in this section, and it allows us to prove Theorem 3.1.

**Lemma 3.1.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value  $opt(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in 2D such that:*

(1) *For every solution  $X$  of instance  $I$ , there exists a solution to  $S$  that satisfies at least  $15n - |X|$  clauses.*

(2) *For every solution to instance  $S$  that satisfies  $w$  clauses, there exists a solution to  $I$  of size  $15n - w$ .*

(3) *Every solution with  $1/2$ -extension of  $I$  is also a solution to the original instance  $I$ .*

Therefore, the optimum size of a solution to  $I$  is  $opt(I) = 15n - opt(S)$ .

We prove Lemma 3.1 in subsequent sections, but meanwhile let us prove Theorem 3.1 using Lemma 3.1 and Theorem 3.2.

*Proof of Theorem 3.1.* Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in 2D with  $(1/2)$ -extension. We construct an algorithm that solves the problem stated in Theorem 3.2, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $I$  using Lemma 3.1. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ . Denote the size of the solution returned by this algorithm as  $approx(I)$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $approx(I) \geq 15n - (\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $approx(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$opt(S) = n.$$

From Lemma 3.1 we have:

$$opt(I) = 14n.$$

Therefore,

$$\begin{aligned} approx(I) &\leq (1 + \epsilon)opt(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n. \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)$  satisfiable, we have:

$$opt(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3.1 we have:

$$\text{opt}(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

221 Since a solution to  $I$  with  $\frac{1}{2}$ -extension is also a solution without any extension, by Lemma  
 222 3.1 (3), we have:

$$\text{approx}(I) \geq \text{opt}(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

223 Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to distin-  
 224 guish the case when  $S$  is satisfiable: from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable,  
 225 it suffices to compare  $\text{approx}(I)$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation  
 226 algorithm cannot exist, unless  $P = NP$ .  $\square$

## 227 3.2. Reduction

228 We proceed to the proof of Lemma 3.1. That is, we show a reduction from the MAX-(3,3)-  
 229 SAT problem to geometric set cover with segments parallel to axis. Moreover, the obtained  
 230 instance of geometric set cover will be robust to 1/2-extension (have the same optimal solution  
 231 after 1/2-extension).

232 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and  
 233 **CLAUSE-gadgets**. **CLAUSE-gadgets** will be constructed using two **OR-gadgets** connected  
 234 together.

### 235 3.2.1. VARIABLE-gadget

236 **VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It  
 237 allows two minimum solutions of size 3 each. These two choices correspond to the two Boolean  
 238 values of the variable corresponding to this gadget.

239 **Points.** Define points  $a, b, c, d, e, f, g, h$  as follows, where  $L = 22n$ :



Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $\text{pointsVariable}_i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $\text{chooseVariable}_i^{\text{false}}$  and the set of blue segments as  $\text{chooseVariable}_i^{\text{true}}$ .

240

$$\begin{array}{llll} a = (-3L, 0) & b = (-2L, 0) & c = (-L, 0) & d = (-3L, 1) \\ e = (-2L, 1) & f = (-2L, 2) & g = (L, 0) & h = (L, 2) \end{array}$$

Let us define:

$$\text{pointsVariable} = \{a, b, c, d, e, f\}$$

and, for any  $1 \leq i \leq n$ ,

$$\text{pointsVariable}_i = \text{pointsVariable} + (0, 4i).$$

241 We denote  $a_i := a + (0, 4i)$  etc.

242 **Segments.** Let us define:

$$\text{chooseVariable}_i^{\text{true}} := \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\},$$

$$\text{chooseVariable}_i^{\text{false}} := \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\},$$

$$\text{segmentsVariable}_i := \text{chooseVariable}_i^{\text{true}} \cup \text{chooseVariable}_i^{\text{false}}.$$

243 We also name two of these segment for future reference:  $\text{xTrueSegment}_i := (c_i, g_i)$ ,  
244  $\text{xFalseSegment}_i := (f_i, h_i)$ .

245 **Lemma 3.2.** *For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can be covered using 3 segments*  
246 *from  $\text{segmentsVariable}_i$ .*

247 *Proof.* We can use either set  $\text{chooseVariable}_i^{\text{true}}$  or  $\text{chooseVariable}_i^{\text{false}}$ . □

248 **Lemma 3.3.** *For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can not be covered with fewer than*  
249 *3 segments from  $\text{segmentsVariable}_i$ .*

250 *Proof.* No segment of  $\text{segmentsVariable}_i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  
251  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments. □

252 **Lemma 3.4.** *For every set  $A \subseteq \text{segmentsVariable}_i$  such that  $A$  covers  $\text{pointsVariable}_i$  and*  
253  *$\text{xTrueSegment}_i, \text{xFalseSegment}_i \in A$ , it holds that  $|A| \geq 4$ .*

254 *Proof.* No segment from  $\text{segmentsVariable}_i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  
255  $\text{pointsVariable}_i - \{c_i, f_i, g_i, h_i\}$  can not be covered with fewer than 2 segments. □

### 256 3.2.2. OR-gadget

257 OR-gadget connects input and output segments (see Figure 3.2) in a way that is supposed to  
258 simulate a binary *or* function.

259 Input segments are the only segments that cover points outside of the gadget, as their left  
260 ends lie outside of it. Point  $v_{i,j}$  is the only one that can be covered by segments that do not  
261 belong to the gadget.

262 The OR-gadget has the property that every set of segments that covers all the points in  
263 the gadget uses at least 3 segments from it.. Moreover, the output segment belongs to the  
264 solution of size 3 only if at least one of the input segments belong to the solution. Therefore,  
265 optimum solutions restricted to the OR-gadget behave like a binary *or* function for the input  
266 segments.



Figure 3.2: **OR-gadget**. Segments from  $\text{chooseOr}_{i,j}^{\text{false}}$  are **red**, segments from  $\text{chooseOr}_{i,j}^{\text{true}}$  are blue (both **light blue** and **dark blue**), segments from  $\text{orMoveVariable}_{i,j}$  are **green** and **yellow**. **Dark blue** segment is the *output* segment. Grey segments  $\text{input}_x$  and  $\text{input}_y$  are input segments that are not part of  $\text{segmentsOr}_{i,j}$ .

267 **Points.**

$$\begin{array}{llll}
 l_0 := (0, 0) & m_0 := (0, 1) & n_0 := (0, 2) & o_0 := (0, 3) \\
 p_0 := (0, 4) & q_0 := (1, 1) & r_0 := (1, 3) & s_0 := (2, 1) \\
 t_0 := (2, 2) & u_0 := (2, 3) & v_0 := (3, 2) & 
 \end{array}$$

$$\text{vec}_{i,j} := (20i + 3 + 3j, 4(n + 1) + 2j)$$

269 For integers  $i, j$ , define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $\text{vec}_{i,j}$ , i.e.  $l_{i,j} =$   
 270  $l_0 + \text{vec}_{i,j}$  etc.

271 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$\text{pointsOr}_{i,j} := \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

272 Note that  $\text{pointsOr}_{i,j}$  does not include the point  $v_{i,j}$

273 **Segments.** We define set of segments in several parts:

$$\begin{aligned}
 \text{chooseOr}_{i,j}^{\text{false}} &:= \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}, \\
 \text{chooseOr}_{i,j}^{\text{true}} &:= \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\},
 \end{aligned}$$

$$\text{orMoveVariable}_{i,j} := \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}.$$

274 Finally all segments in OR-gadget are defined as:

$$\text{segmentsOr}_{i,j} := \text{chooseOr}_{i,j}^{\text{false}} \cup \text{chooseOr}_{i,j}^{\text{true}} \cup \text{orMoveVariable}_{i,j}$$

275 **Lemma 3.5.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$ , points in  $\text{pointsOr}_{i,j} - \{x\} \cup$   
 276  $\{v_{i,j}\}$  can be covered with 4 segments from  $\text{segmentsOr}_{i,j}$ .

277 *Proof.* We can do that using one segment from  $\text{orMoveVariable}_{i,j}$ , the one that does not cover  
 278  $x$ , and all segments from  $\text{chooseOr}_{i,j}^{\text{true}}$ .  $\square$

279 **Lemma 3.6.** *For any  $1 \leq i \leq n, j \in \{0,1\}$ , points in  $\text{pointsOr}_{i,j}$  can be covered with 4*  
 280 *segments from  $\text{segmentsOr}_{i,j}$ .*

281 *Proof.* We can do that using segments from  $\text{orMoveVariable}_{i,j} \cup \text{chooseOr}_{i,j}^{\text{false}}$ .  $\square$

### 282 3.2.3. CLAUSE-gadget

283 A CLAUSE-gadget is responsible for determining whether variable values assigned in variable  
 284 gadgets satisfy the corresponding clause in the input formula  $\phi$ . It has a minimum solution  
 285 to weight  $w$  if and only if the clause is satisfied, i.e. at least one of the respective variables is  
 286 assigned the correct value. Otherwise, its minimum solution has weight  $w + 1$ . In this way,  
 287 by analyzing the cost of the minimum solution to the entire constructed instance, we will be  
 288 able to tell how many clauses it was possible to satisfy in the optimum solution to  $\phi$ .



Figure 3.3: **CLAUSE-gadget for a clause  $a \vee b \vee \neg c$ .** Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}, y_{i,0}$  and  $z_{i,0}$  depend on the variables in the  $i$ -th clause. Grey segments corresponds to the values of variables satisfying the  $i$ -th clause.

289 **Points.** First, we define auxiliary functions for literals. For a literal  $w$ , let  $\text{idx}(w)$  be the  
 290 index of the variable in  $w$ , and  $\text{neg}(w)$  be the Boolean value whether the variable is negated  
 291 in  $w$  or not.

292 Let us assume that clause  $C_i = a \vee b \vee c$  for any literals  $a, b, c$ . Then, we define points in  
 293 the gadget as:



$$\begin{aligned}
x_{i,0} &:= (20i, 4 \cdot \text{id}x(a) + 2 \cdot \text{neg}(c)), & x_{i,1} &:= (20i, 4(n+1)), \\
y_{i,0} &:= (20i+1, 4 \cdot \text{id}x(b) + 2 \cdot \text{neg}(b)), & y_{i,1} &:= (20i+1, 4(n+1)+4), \\
z_{i,0} &:= (20i+2, 4 \cdot \text{id}x(c) + 2 \cdot \text{neg}(c)), & z_{i,1} &:= (20i+2, 4(n+1)+6).
\end{aligned}$$

We are now ready to define set of points:

$$\text{moveVariable}_i := \{x_{i,j} : j \in \{0,1\}\} \cup \{y_{i,j} : j \in \{0,1\}\} \cup \{z_{i,j} : j \in \{0,1\}\},$$

$$\text{pointsClause}_i := \text{moveVariable}_i \cup \text{pointsOr}_{i,0} \cup \text{pointsOr}_{i,1} \cup \{v_{i,1}\}.$$

Note that these two points are equal:  $v_{i,0} = l_{i,1}$ . This translates to the fact, that output of the one OR-gadget is an input to the other OR-gadget to create *or* of 3 segments.

**Segments.** We also define segments for the clause gadget as below:

$$\begin{aligned}
\text{segmentsClause}_i &:= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\
&\cup \text{segmentsOr}_{i,0} \cup \text{segmentsOr}_{i,1}.
\end{aligned}$$

The CLAUSE-gadgets consist of two OR-gadgets. Ideally, we would place the  $i$ -th CLAUSE-gadget close to the  $\text{xTrueSegment}_{j_1}$  or  $\text{xFalseSegment}_{j_1}$  segments corresponding to the literals that occur in the  $i$ -th clause. It would be inconvenient to position them there, because between these segments there may be additional  $\text{xTrueSegment}_{j_2}$  or  $\text{xFalseSegment}_{j_2}$  segments corresponding to the other literals.

Instead, we use simple auxiliary gadgets to *transfer* whether the segment is in a solution, i.e. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$  in this gadget. Each gadget consists of two segments  $(x_{i,0}, x_{i,1}), (x_{i,1}, a)$ . These are the only segments that can cover  $x_{i,1}$ . We place  $x_{i,0}$  on a segment that we want to transfer (i.e. segment responsible for choosing the variable value satisfying the corresponding literal). If in some solution  $x_{i,0}$  is already covered by this segment, then we can cover  $x_{i,1}$  by  $(x_{i,1}, a)$ , thus also covering  $a$ . If  $x_{i,0}$  is not covered by this segment, then the only way to cover  $x_{i,0}$  is to use segment  $(x_{i,0}, x_{i,1})$ . Intuitively, in any optimal solution the two segments *transfer* the state of whether  $x_{i,0}$  is covered onto whether  $a$  is covered. Therefore, the number of segments in the optimal solution is increased by one, and we get a point  $a$  that was effectively placed on some segment  $s$ , but it can be placed anywhere on the plane instead, consequently simplifying the construction.

**Lemma 3.7.** *For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , there is a set  $\text{solClause}_i^{\text{true},a} \subseteq \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{true},a}| = 11$  that covers all points in  $\text{pointsClause}_i - \{a\}$ .*

*Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 3.5 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments in  $\text{chooseOr}_{i,0}^{\text{true}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{i,0}$ : Using Lemma 3.6 and Lemma 3.5 with  $x = p_{i,1}$ , we obtain 8 segments in  $\text{chooseOr}_{i,0}^{\text{false}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 3.8.** *For any  $1 \leq i \leq n$  there is a set  $\text{solClause}_i^{\text{false}} \subseteq \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{false}}| = 12$  that covers all points in  $\text{pointsClause}_i$ .*

326 *Proof.* Using Lemma 3.6 twice we can cover  $\text{pointsOr}_{i,0}$  and  $\text{pointsOr}_{i,1}$  with 8 segments. To  
 327 cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$   
 328  $\square$

329 **Lemma 3.9.** *For any  $1 \leq i \leq n$ :*

- 330 (1) *points in  $\text{pointsClause}_i$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$*   
 331 *of size smaller than 12;*
- 332 (2) *points in  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using any subset of segments*  
 333 *from  $\text{segmentsClause}_i$  of size smaller than 11.*

*Proof of (1).* No segment in  $\text{segmentsClause}_i$  covers more than 1 point from

$$\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0} = l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}.$$

334 Therefore we need to use at least 12 segments.  $\square$

335 *Proof of (2).* We can define disjoint sets  $X, Y, Z$  such that  $X \cup Y \cup Z \subseteq \text{pointsClause}_i -$   
 336  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  such that there are no segments in  $\text{segmentsClause}_i$  covering points from dif-  
 337 ferent sets. And we prove a lower bound for each of these sets. First, let:

$$X := \{x_{i,1}, y_{i,1}, z_{i,1}\}.$$

338 No two points in  $X$  can be covered with one segment of  $\text{segmentsClause}_i$ , so it must be  
 339 covered with 3 different segments. Next we define other sets:

$$Y := \text{pointsOr}_{i,0} - \{l_{i,0}, p_{i,0}\},$$

$$Z := \text{pointsOr}_{i,1} - \{l_{i,1}, p_{i,1}\}.$$

340 For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments of  $\text{segmentsClause}_i$  to  
 341 conclude that none of them cover the considered, so both  $Y$  and  $Z$  have to be covered with  
 342 disjoint sets of 4 segments each.

343 Therefore,  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  must be covered with at least  $3 + 4 + 4 = 11$   
 344 segments from  $\text{segmentsClause}_i$ .  $\square$

### 345 3.2.4. Summary

346 Add some smart lemmas that sets will be exclusive to each other.

347 **Lemma 3.10. Robustness to 1/2-extension.** *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+\frac{1}{2}}$  cover*  
 348 *the same points from  $\mathcal{C}$ .*

349 *Proof.* We can just check every segment. Most of the segments  $s$  are collinear only with points  
 350 that lie on  $s$ , so trivially  $s^{+\frac{1}{2}}$  cannot cover more points than  $s$  does.

351 Within VARIABLE-gadget for any  $1 \leq i \leq n$  after  $\frac{1}{2}$ -extension:  $(c_i, g_i)$  does not cover  $b_i$ .

352 Within OR-gadget some of the segments are collinear and share one point; specifically, for  
 353 any  $1 \leq i \leq n$  and  $j \in \{0, 1\}$ , after  $\frac{1}{2}$ -extension:

- 354 •  $(l_{i,j}, n_{i,j})$  does not cover  $o_{i,j}$ ,
- 355 •  $(n_{i,j}, p_{i,j})$  does not cover  $m_{i,j}$ ,
- 356 •  $(t_{i,j}, v_{i,j})$  does not cover  $n_{i,j}$ .



Figure 3.4: **Schema of the whole construction.**

General layout of VARIABLE-gadgets and CLAUSE-gadgets and how they interact with each other.

357 Within CLAUSE-gadget, for any  $1 \leq i \leq n$  after  $\frac{1}{2}$ -extension:

- 358 •  $(o_{i,0}, u_{i,0})$  does not cover  $m_{i,1}$ ,
- 359 •  $(m_{i,1}, s_{i,1})$  does not cover  $u_{i,0}$ ,
- 360 •  $(y_{i,1}, p_{i,0})$  does not cover  $n_{i,1}$ .

361 For two consecutive VARIABLE-gadgets, for any  $1 \leq i < n$  after  $\frac{1}{2}$ -extension:  $(b_i, f_i)$  does  
 362 not cover  $b_{i+1}$  (nor  $f_{i-1}$  for  $i > 1$ ). Similarly  $(a_i, d_i)$  does not cover  $a_{i+1}$  (nor  $d_{i-1}$  for  $i > 1$ ),  
 363 because this segment is shorter than the previous one and  $a_i$  and  $b_i$  share y-coordinate.

364 For two consecutive CLAUSE-gadgets, segments from one do not cover anything from the  
 365 other, as the gadgets have width 9 and every leftmost x-coordinate is divisible by 20. Hence  
 366 two different gadgets do not interact with each other after  $\frac{1}{2}$ -extension.

367 Next we need to check whether VARIABLE-gadget's segments do not cover any points  
 368  $x_{i,0}, y_{i,0}$  or  $z_{i,0}$  from CLAUSE-gadget. For any  $1 \leq i \leq n$  and  $1 \leq j \leq n$ , all points  $x_{j,0}, y_{j,0}$   
 369 and  $z_{j,0}$  have x-coordinate strictly positive. Segment  $(a_i, c_i)$  have length  $2L$  and  $c_i$  has x-  
 370 coordinate equal to  $-L$ , so after  $\frac{1}{2}$ -extension this segment does not cover any points with a  
 371 positive x-coordinate.

372 □

### 373 3.2.5. Summary of construction

Finally we define set of points and segments for the constructed instance:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} \text{pointsVariable}_i \cup \text{pointsClause}_i,$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} \text{segmentsVariable}_i \cup \text{segmentsClause}_i.$$



Figure 3.5: **Schema of the whole construction.**

General layout of VARIABLE-gadgets and CLAUSE-gadgets and how they interact with each other.

### 3.3. Construction lemmas and proof of Lemma 3.1

In order to prove Lemma 3.1 we introduce several auxiliary lemmas proving properties of the construction described in the previous section.

Consider an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution satisfying  $k$  clauses. Let us construct an instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover as described in Section 3.2 for the instance  $S$  of MAX-(3,3)-SAT.

**Lemma 3.11.** *Instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover admits a solution of size  $15n - k$ .*

*Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Let the variable assignment in the optimum solution to  $S$  be  $\phi : \{x_1, x_2 \dots x_n\} \rightarrow \{\mathbf{true}, \mathbf{false}\}$ .

We cover every VARIABLE-gadget with solution described in Lemma 3.2, where in the  $i$ -th gadget we choose the set of segments corresponding to the value of  $\phi(x_i)$ .

For every clause that is satisfied, say  $c_i$ , let us name the variable that is **true** in it as  $x_i$  and point corresponding to  $x_i$  in  $\mathbf{pointsClause}_i$  as  $a$ . Points in  $\mathbf{pointsClause}_i$  are covered with set  $\mathbf{solClause}_i^{\mathbf{true}, a}$  described in Lemma 3.7. For every clause that is not satisfied, say  $c_j$ , points in  $\mathbf{pointsClause}_j$  are covered with set  $\mathbf{solClause}_j^{\mathbf{false}}$  described in Lemma 3.8.

Formally we define sets responsible for choosing variable assignment and satisfying clauses,  $R_i$  and  $C_i$  respectively, as following:

$$\begin{aligned}
R_i &:= \begin{cases} \text{chooseVariable}_i^{\text{true}} & \text{if } \phi(x_i) = \text{true} \\ \text{chooseVariable}_i^{\text{false}} & \text{if } \phi(x_i) = \text{false} \end{cases} \\
C_i &:= \begin{cases} \text{solClause}_i^{\text{true},a} & \text{if } c_i \text{ satisfied by literal corresponding to point } a \\ \text{solClause}_i^{\text{false}} & \text{if } c_i \text{ not satisfied} \end{cases} \\
\mathcal{R} &:= \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}.
\end{aligned}$$

391 This set covers all the points from  $\mathcal{C}$ , because the sets  $R_i$ ,  $C_i$  individually cover their  
392 corresponding gadgets, as proved in the respective lemmas.

393 All of these sets are disjoint, so the size of the obtained solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k. \quad \square$$

394 **Lemma 3.12.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover.*  
395 *Then there exists a solution  $\mathcal{R}'$ , such that  $|\mathcal{R}'| \leq |\mathcal{R}|$ , and  $\mathcal{R}'$  contains at most one of the*  
396 *segments  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$  from each VARIABLE-gadget.*

397 *Proof.* Assume that we have  $\{\text{xTrueSegment}_i, \text{xFalseSegment}_i\} \subseteq \mathcal{R}$  for some  $i$ . We will show  
398 how to modify  $\mathcal{R}$  into  $\mathcal{R}'$ , such that the number of such  $i$  decreases, while  $\mathcal{R}'$  is still a valid  
399 solution to  $(\mathcal{C}, \mathcal{P})$ , and  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Then, by repeating this procedure, we can eventually  
400 construct a solution satisfying the property from the Lemma.

401 To construct  $\mathcal{R}'$ , we first remove from  $\mathcal{R}$  all segments belonging to  $\text{segmentsVariable}_i$ .  
402 Recall that the  $i$ -th VARIABLE-gadget corresponds to variable  $x_i$  in  $S$ . As every variable in  
403  $S$  is used in exactly 3 clauses, then one literal  $x_i$  or  $\neg x_i$  must appear in at least 2 clauses. If  
404 that literal is  $x_i$ , then we add to the constructed solution all segments from  $\text{chooseVariable}_i^{\text{true}}$ ,  
405 otherwise we add all segments from  $\text{chooseVariable}_i^{\text{false}}$ .

406 Now, there exists at most one CLAUSE-gadget which needs adjustment to make  $\mathcal{R}'$  valid;  
407 assuming it is the  $j$ -th clause, then one of the points  $x_{j,0}, y_{j,0}$  or  $z_{j,0}$  for this CLAUSE-gadget  
408 might be not covered, say  $y_{j,0}$ . We amend the solution by adding  $(y_{j,0}, y_{j,1})$  to  $\mathcal{R}'$ .

409 By Lemma 3.4 we know that  $\mathcal{R}$  used at least 4 segments from  $\text{segmentsVariable}_i$ . Therefore,  
410 we removed at least 4 segments and added at most 4 segments, so  $|\mathcal{R}'| \leq |\mathcal{R}|$ .  $\square$

411 **Lemma 3.13.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover*  
412 *that is of size  $w$ . Then there exists a solution to  $S$  that satisfies at least  $15n - w$  clauses.*

413 *Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Given a solution  
414  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover, we use Lemma 3.12 to modify  $\mathcal{R}$  such that  
415 for any  $i$  it contains at most one of  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$ ; this may decrease the  
416 cost of  $\mathcal{R}$ , but that does not matter in the subsequent construction. To simplify notation, in  
417 the remainder of this proof we use  $\mathcal{R}$  to refer to the modified solution.

Given  $\mathcal{R}$ , we construct a solution to  $S$  by defining an assignment of variables:

$$\phi : \{x_1, x_2 \dots x_n\} \rightarrow \{\text{true}, \text{false}\}$$

418 that satisfies at least  $15n - w$  clauses in  $S$ .

**Definition of  $\phi$ .** Recall that due to Lemma 3.12,  $\mathcal{R}$  contains at most one of  $\text{xTrueSegment}_i$  and  $\text{xFalseSegment}_i$ .

We define the value  $\phi(x_i)$  for the variable  $x_i$  as follows:

$$\begin{cases} \phi(x_i) = \text{true} & \text{if } \text{xTrueSegment}_i \in \mathcal{R} \\ \phi(x_i) = \text{false} & \text{otherwise} \end{cases}$$

Moreover, from Lemma 3.3 we get  $|\text{segmentsVariable}_i \cap \mathcal{R}| \geq 3$  for every  $i$ .

**Clauses satisfied with the chosen variable assignment.** For a clause  $c_i$ ,  $\mathcal{R}$  needs to use at least 11 segments to cover  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  in the  $i$ -th CLAUSE-gadget (Lemma 3.9).

Moreover, if none of the points  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  are covered by the segments from  $\mathcal{R} \cap \text{segmentsVariable}_i$ , then  $\mathcal{R}$  needs to cover  $\text{pointsClause}_i$  with at least 12 segments by Lemma 3.9.

Let us denote  $a$  as the amount of such clauses  $c_i$  for which none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  were covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$  for any  $1 \leq j \leq n$ .

Consider a clause  $c_i$  for which at least one of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  were covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$  for some  $1 \leq j \leq n$ , then denote this point as  $t$  and say it corresponds to literal  $q$  and variable  $x_j$ . Point  $t$  can be only covered in  $\text{segmentsVariable}_j$  by a corresponding segment  $\text{xTrueSegment}_j$  or  $\text{xFalseSegment}_j$  (depending on whether the literal  $q$  is negated or not). From the definition of  $\phi$  and the fact that one of this segment is in  $\mathcal{R}$ , we know that  $\phi(j)$  has the value that evaluates  $w$  to be true. Therefore, clause  $c_i$  is satisfied.

Consequently,  $\phi$  satisfies all but at most  $a$  clauses in  $S$ .

To conclude, given a solution to  $(\mathcal{C}, \mathcal{P})$  of size  $w$  we constructed a variable assignment  $\phi$  that satisfies at least  $n - a$  clauses of  $S$ . Finally, note that

$$w \geq 3n + 11(n - a) + 12a = 3n + 11n + a = 14n + a,$$

hence

$$15n - w \leq 15n - 14n - a = n - a.$$

Therefore  $\phi$  satisfies at least  $15n - w$  clauses of  $S$ . □

We are ready to conclude the proof of Lemma 3.1.

*Proof of Lemma 3.1.* By Lemma 3.11, we know that there exists a solution to  $(\mathcal{C}, \mathcal{P})$  of size  $15n - k$ , so:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \leq 15n - k.$$

Since the optimum solution to  $S$  satisfies  $k$  clauses, then according to Lemma 3.13:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \geq 15n - k.$$

Therefore, the solution given by Lemma 3.11 of size  $15n - k$  is an optimum solution to the instance  $(\mathcal{C}, \mathcal{P})$ . □

## Chapter 4

# Fixed-parameter tractable algorithm for geometric set cover problem

In this chapter we show fixed-parameter tractable algorithms for the geometric set cover problem in two different settings. Section 4.1 shows a fixed-parameter tractable algorithm for geometric set cover with unweighted segments. The remainder of the chapter presents a fixed-parameter tractable algorithm for geometric set cover with weighted segments with  $\delta$ -extensions. We show an algorithm for the setting with  $\delta$ -extensions, because the original problem with weights is W[1]-hard, as we show in Chapter 5.

We start with a shared definition for this problem. We define *extreme points* for a set of collinear points.

**Definition 4.1.** For a set of collinear points  $C$  in the plane, **extreme points** of  $C$  are the endpoints of the smallest segment that covers all points from set  $C$ .

If  $C$  consists of one point or is empty, then there are 1 or 0 extreme points respectively.

### 4.1. Fixed-parameter tractable algorithm for unweighted segments

In this section we consider fixed-parameter tractable algorithms for unweighted geometric set cover with segments. The setting where segments are required to be axis-parallel (or limited to a constant number of directions) has an FPT algorithm already present in literature in the Parametrized Algorithms book [Cygan et al., 2015]. We present an FPT algorithm for geometric set cover with unweighted segments, where segments are in arbitrary directions.

#### 4.1.1. Axis-parallel segments

**Theorem 4.1.** (*FPT for segment cover with axis-parallel segments*). There exists an algorithm that given a family  $\mathcal{P}$  of axis-parallel segments, a set of points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $\mathcal{O}(2^k)$ , and outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.

We present here a simple algorithm from [Cygan et al., 2015] for completeness.

*Proof.* We show an  $\mathcal{O}(2^k)$ -time branching algorithm. In each step, the algorithm selects a point  $a$  which is not yet covered, branches to choose one of the two directions, and greedily

chooses a segment  $a$  in that direction to cover. This proceeds until either all points are covered or  $k$  segments are chosen.

Let us take the point  $a = (x_a, y_a)$  which is the smallest among points that are not yet covered in the lexicographic ordering of points in  $\mathbb{R}^2$ . We need to cover  $a$  with some of the remaining segments.

Branch over the choice of one of the coordinates ( $x$  or  $y$ ); without loss of generality, let us assume we chose  $x$ . Among the segments lying on line  $x = x_a$ , we greedily add to the solution the one that covers the most points. As  $a$  was the smallest in the lexicographical order, all points on the line  $x = x_a$  have the  $y$ -coordinate larger than  $y_a$ . Therefore, if we denote the greedily chosen segment as  $s$ , then any other segment on the line  $x = x_a$  that covers  $a$  can only cover a subset of points covered by  $s$ . Thus, greedily choosing  $s$  is optimal.

In each step of the algorithm we add one segment to the solution, thus the recursion can be stopped at depth  $k$ . If no branch finds a solution, then this means that a solution of size at most  $k$  does not exist.  $\square$

Note that the same algorithm can be used for segments in  $d$  directions, where we branch over  $d$  choices of directions, and it runs in complexity  $\mathcal{O}(d^k)$ .

#### 4.1.2. Segments in arbitrary directions

In this section we consider the setting where segments are not constrained to a constant number of directions. We present a fixed-parameter tractable algorithm, parameterized by the size of the solution.

**Theorem 4.2. (FPT for segment cover).** *There exists an algorithm that given a family  $\mathcal{P}$  of segments (in any direction), a set of points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $k^{\mathcal{O}(k)} \cdot (|\mathcal{C}| \cdot |\mathcal{P}|)^2$ , and outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.*

We will need the following lemmas proving properties of any instance of the problem.

**Lemma 4.1.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, without loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any distinct  $A, B \in \mathcal{P}$ , we have  $A \cap \mathcal{C} \not\subseteq B \cap \mathcal{C}$  and  $A \cap \mathcal{C} \not\supseteq B \cap \mathcal{C}$ .*

*Proof.* Assume towards a contradiction that there is an instance  $(\mathcal{P}, \mathcal{C})$ , and two distinct subsets of  $\mathcal{P}$ ,  $A, B$ , such that  $A \cap \mathcal{C} \subseteq B \cap \mathcal{C}$ .

We construct a set  $\mathcal{P}' := \mathcal{P} - \{A\}$ . We prove that for any solution  $\mathcal{R}$  of  $(\mathcal{P}, \mathcal{C})$ , we can construct a solution  $\mathcal{R}' \subseteq \mathcal{P}'$ , such that  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Let us take any solution  $\mathcal{R}$  of  $(\mathcal{P}, \mathcal{C})$ . If  $A \in \mathcal{R}$ , then  $\mathcal{R}' := \mathcal{R} \cup \{B\} - \{A\}$ , otherwise  $\mathcal{R}' := \mathcal{R}$ . Let us consider the case when  $A \in \mathcal{R}$ , because the other case is trivial. Since  $A \cap \mathcal{C} \subseteq B \cap \mathcal{C}$ , then  $\mathcal{R} \cup \{B\} - \{A\}$  covers any point from  $\mathcal{C}$  that was covered by  $\mathcal{R}$ . Also,  $|\mathcal{R} \cup \{B\} - \{A\}| \leq |\mathcal{R}|$ .  $\square$

**Lemma 4.2.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem transformed by Lemma 4.1, if there exists a line  $L$  with at least  $k + 1$  points on it, then there exists a subset  $A \subseteq \mathcal{P}$ , of size at most  $k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|A \cap \mathcal{R}| \geq 1$ . Moreover, such a subset can be found in polynomial time.*

*Proof.* Let us enumerate the points from  $\mathcal{C}$  that lie on  $L$  as  $x_1, x_2, \dots, x_t$  in the order in which they appear on  $L$ . Our proposed set is defined as:

$$A := \{\text{segment collinear with } L \text{ that covers } x_i \text{ and does not cover } x_{i-1} : i \in \{1, \dots, k\}\}.$$



511 Where for  $i = 1$  we just take a segment that covers  $x_1$ .

512 If such a segment does not exist for any point  $x$  as above, then  $x$  does not give rise to  
 513 any segment in  $A$ . We prove the lemma by contradiction. Let us assume that there exists  
 514 a solution  $\mathcal{R}$  of size at most  $k$  such that  $\mathcal{R} \cap A = \emptyset$ .

515 Let us define a set  $\mathcal{R}_L$ , which is defined as segments from  $\mathcal{R}$  that are collinear with  $L$ .

516 Every segment that is not collinear with  $L$  can cover at most one of the points that lie  
 517 on this line. Hence, if  $\mathcal{R}_L$  was empty, then  $\mathcal{R}$  would cover at most  $k$  points on line  $L$ , but  $L$   
 518 had at least  $k + 1$  different points from  $\mathcal{C}$  on it.

519 Therefore, we know that  $\mathcal{R}_L$  is not empty and  $|\mathcal{R} - \mathcal{R}_L| \leq k - 1$ . Segments from  $\mathcal{R} - \mathcal{R}_L$   
 520 can cover at most  $k - 1$  points among  $\{x_1, x_2, \dots, x_k\}$ , therefore at least one of these points  
 521 must be covered by segments from  $\mathcal{R}_L$ . We take the leftmost point from  $\{x_1, x_2, \dots, x_k\}$  that  
 522 is covered in  $\mathcal{R}_L$  and name it  $a$ . After the transformation from Lemma 4.1, in  $\mathcal{R}$  there is only  
 523 one segment that starts in  $a$  and is collinear with  $L$ , therefore this segment must be in both  
 524  $\mathcal{R}$  and  $A$ . This contradiction concludes the proof that  $|A \cap \mathcal{R}| \geq 1$  for any solution  $\mathcal{R}$  of size  
 525 at most  $k$ .  $\square$

526 We are now ready to prove Theorem 4.2.

527 *Proof of Theorem 4.2.* We will prove this theorem by presenting a branching algorithm that  
 528 works in desired complexity. It first branches over the choice of segments to cover the lines  
 529 with *many* points and then solves a small instance (where every line has at most  $k$  points) by  
 530 checking all possible solutions.

531 **Algorithm.** We present a recursive algorithm. Given an instance of the problem:

- 532 (1) Use Lemma 4.1 to remove some redundant segments from our instance.
- 533 (2) If there exists a line with at least  $k + 1$  points from  $\mathcal{C}$ , we branch over the choice of  
 534 adding to the solution one of the at most  $k$  possible segments provided by Lemma 4.2;  
 535 name this segment  $s$  and name the set of points from  $\mathcal{C}$  that lie on  $s$  as  $S$ . By recursion,  
 536 we find a solution  $\mathcal{R}$  for the instance  $(\mathcal{C} - S, \mathcal{P} - \{s\})$ , and parameter  $k - 1$ . We return  
 537  $\mathcal{R} \cup \{s\}$ . Note that if Lemma 4.2 returned  $\emptyset$ , then we respond NO.
- 538 (3) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer NO.
- 539 (4) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force: check all subsets of  $\mathcal{P}$  of size at most  $k$ .

540 **Correctness.** Lemma 4.2 proves that at least one segment that we branch over in (1)  
 541 must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ . Therefore, the recursive call can find  
 542 a solution, provided there exists one.

543 In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which  
 544 implies the same about every segment from  $\mathcal{P}$ . Under this assumption we can cover only  $k^2$   
 545 points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

546 Checking all possible solutions in (3) is trivially correct.

547 **Complexity.** In the leaves of the recursion we have  $|\mathcal{C}| \leq k^2$ , so  $|\mathcal{P}| \leq k^4$ , because  
 548 every segment can be uniquely identified by the two extreme points it covers (by Lemma 4.1).  
 549 Therefore, there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $\mathcal{O}(k|\mathcal{C}|)$ .  
 550 Thus, (3) takes time  $k^{\mathcal{O}(k)}$ .

In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch over can be found on each level in time  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)})$ .

Reduction from Lemma 4.1 can be implemented in time  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)})$ .

It follows that the overall complexity is  $\mathcal{O}((|\mathcal{C}| \cdot |\mathcal{P}|)^{\mathcal{O}(1)} \cdot k^{\mathcal{O}(k)})$   $\square$

## 4.2. Fixed-parameter tractable algorithm for weighted segments with $\delta$ -extensions

In this section we consider the geometric set cover problem for weighted segments relaxed with  $\delta$ -extensions. We show that this problem admits an FPT algorithm when parameterized by the size of the solution and  $\delta$ . In the next chapter we show that the assumption about the problem being relaxed with  $\delta$ -extensions is necessary: we prove that geometric set cover problem for weighted segments (without extensions) is W[1]-hard, which means there does not exist any FPT algorithm parameterized by solution size for it, assuming  $\text{FPT} \neq \text{W}[1]$ .

**Theorem 4.3** (FPT for weighted segment cover with  $\delta$ -extensions). *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$  points  $\mathcal{C}$ , and parameters  $k$  and  $\delta > 0$ , such that it runs in time  $f(k, \delta) \cdot (nm)^c$  for some computable function  $f$  and a constant  $c$  and outputs a set  $\mathcal{R}$  such that:*

- $\mathcal{R} \subseteq \mathcal{P}$ ,
- $|\mathcal{R}| \leq k$ ,
- $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ ,
- the weight of  $\mathcal{R}$  is not greater than the weight of an optimum solution of size at most  $k$  for this problem without  $\delta$ -extensions

or determines that there is no set  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  such that  $\mathcal{R}$  covers all points in  $\mathcal{C}$ .

To solve this problem we will introduce a lemma about choosing a *dense* subset of points. A dense subset of points for a set of collinear points  $C$  and parameters  $k$  and  $\delta$  is a subset of  $C$  such that if we cover it with at most  $k$  segments, these segments after  $\delta$ -extensions will cover all of the points from  $C$ . We will prove that such set of size bounded by some function  $f(k, \delta)$  always exists (Lemma 4.3). Later, Lemma 4.3 will allow us to find a kernel for our original problem.

**Definition 4.2.** For a set of collinear points  $C$ , a subset  $A \subseteq C$  is  $(k, \delta)$ -**dense** if for any set of segments  $R$  that covers  $A$  and such that  $|R| \leq k$ , it holds that  $R^{+\delta}$  covers  $C$ .

**Lemma 4.3.** *For any set of collinear points  $C$ ,  $\delta > 0$  and  $k \geq 1$ , there exists a  $(k, \delta)$ -dense set  $A \subseteq C$  of size at most  $(2 + \frac{2}{\delta})^k$ . Moreover, there exists an algorithm that computes the  $(k, \delta)$ -dense set in time  $\mathcal{O}(|C| \cdot (2 + \frac{2}{\delta})^k)$ .*

*Proof.* We prove this for a fixed  $\delta$  by induction on  $k$ .

586 **Inductive hypothesis.** For any set of collinear points  $C$ , there exists a set  $A$  such that:

- 587 •  $A$  is subset of  $C$ ,
- 588 •  $A$  is  $(\ell, \delta)$ -dense for every  $1 \leq \ell \leq k$ ,
- 589 •  $|A| \leq (2 + \frac{2}{\delta})^k$ ,
- 590 • the extreme points of  $C$  are in  $A$ .

591 **Base case for  $k = 1$ .** It is sufficient that  $A$  consists of the extreme points of  $C$ .

592 If they are covered with one segment, it must be a segment that includes the extreme

593 points from  $C$ , so it covers the whole set  $C$ .

594 There are at most 2 extreme points in  $C$  and  $2 < 2 + \frac{2}{\delta}$ .

595 **Inductive step.** Assuming inductive hypothesis for any set of collinear points  $C$  and

596 for parameter  $k$ , we will prove it for  $k + 1$ .

597 Let  $s$  be the minimal segment that includes all points from  $C$ . That is, the extreme points

598 of  $C$  are endpoints of  $s$ .

599 We define  $M = \lceil 1 + \frac{2}{\delta} \rceil$  subsegments of  $s$  by splitting  $s$  into  $M$  closed segments of equal

600 length. We name these segments  $v_i$ , note that  $|v_i| = \frac{|s|}{M}$  for each  $1 \leq i \leq M$ .

601 Let  $C_i$  be the subset of  $C$  consisting of points lying on  $v_i$ .

602 Let  $t_i$  be the segment with endpoints being the extreme points of  $C_i$ . It might be a

603 degenerate segment if  $C_i$  consists of one point, or  $t_i$  might be empty if  $C_i$  is empty.

604 Figure 4.1 presents an example of such segments  $v_i$  and  $t_i$ .

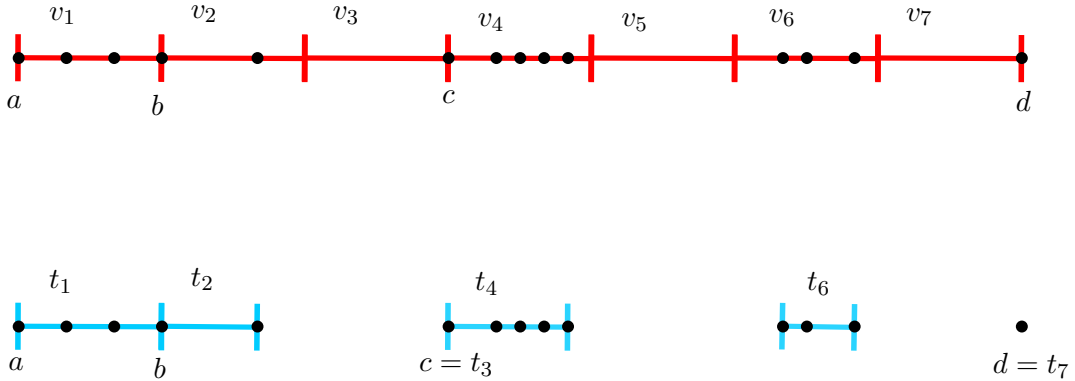


Figure 4.1: **Example of segments  $v_i$  and  $t_i$ .**

Example for  $M = 7$  and some set of points (marked with black circles). The top panel shows segments  $v_i$  and the bottom panel shows segments  $t_i$  on the same set of points.  $a$  and  $b$  are the extreme points and therefore segment  $s$  ends at  $a$  and  $b$ . Red segments depict the split into  $M$  segments of equal length  $v_i$ . Blue segments depict the segments  $t_i$ .  $t_5$  is an empty segment, because there are no points that lie on segment  $v_5$ . Segments  $t_3$  and  $t_7$  are degenerated to one point –  $c$  and  $d$ , respectively. Segments  $t_1$  and  $t_2$  share one point  $b$ .

605 We use the inductive hypothesis to choose  $(k, \delta)$ -dense sets  $A_i$  for sets  $C_i$ . Note that if

606  $|C_i| \leq 1$ , then  $A_i = C_i$  and it is still a  $(k, \delta)$ -dense set for  $C_i$ .

607 Then we define  $A = \bigcup_{i=1}^M A_i$ . Thus  $A$  includes the extreme points of  $C$ , because they are  
 608 included in the sets  $A_1$  and  $A_M$ .

The size of each  $A_i$  is at most  $(2 + \frac{2}{\delta})^k$  from the inductive hypothesis, therefore size of  $A$  is at most:

$$M \left(2 + \frac{2}{\delta}\right)^k = \left\lceil 1 + \frac{2}{\delta} \right\rceil \cdot \left(2 + \frac{2}{\delta}\right)^k \leq \left(2 + \frac{2}{\delta}\right)^{k+1}.$$

609 **Proof that  $A$  is  $(k, \delta)$ -dense for  $C$ .** Let us take any cover of  $A$  with  $k + 1$  segments  
 610 and call it  $\mathcal{R}$ .

611 For every segment  $t_i$ , if there exists a segment  $x$  in  $\mathcal{R}$  that is disjoint with  $t_i$ , then we have  
 612 a cover of  $A_i$  with at most  $k$  segments using  $\mathcal{R} - \{x\}$ . Since  $A_i$  is  $(k, \delta)$ -dense for  $t_i$  and  $C_i$ ,  
 613  $(\mathcal{R} - \{x\})^{+\delta}$  covers  $C_i$ . So  $\mathcal{R}^{+\delta}$  covers  $C_i$  as well.

614 If there exists a segment  $t_i$  for which a segment  $x$  as defined above does not exist, then  
 615 all  $k + 1$  segments that cover  $A_i$  intersect  $t_i$ . An example of such segments is depicted in  
 616 Figure 4.2. Let us consider any such  $t_i$ . By the inductive hypothesis, the endpoints of  $s$   
 617 are in  $A_1$  and  $A_M$  respectively, so  $\mathcal{R}$  must cover them. For each endpoint of  $s$ , there exists  
 618 a segment that contains this endpoint and intersects  $t_i$ . Let us call these two segments  $y$   
 619 and  $z$ . It follows that:  $|y| + |z| + |t_i| \geq |s|$ . Since  $|t_i| \leq |v_i| = \frac{|s|}{M} \leq \frac{|s|}{1 + \frac{2}{\delta}} = \frac{|s|\delta}{\delta + 2}$ , we have  
 620  $\max(|y|, |z|) \geq |s|(1 - \frac{\delta}{\delta + 2})/2 = \frac{|s|}{\delta + 2}$ .

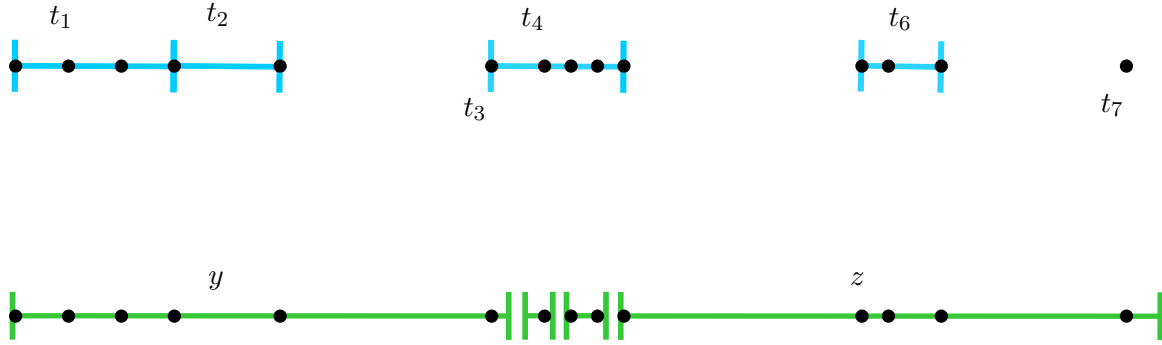


Figure 4.2: **Example of all  $k + 1$  segments intersecting one segment  $t_i$ .**

Both panels show the same set  $\mathcal{C}$  (black circles), the same as in Figure 4.1. The top panel shows blue segments  $t_i$  for  $M = 7$ . The bottom panel shows green segments – solution  $\mathcal{R}$  of size 4. All segments from  $\mathcal{R}$  intersect  $t_4$ . Segments  $z$  and  $y$  are named in the figure.

After  $\delta$ -extension, the longer of these segments will expand at both ends by at least:

$$\max(|y|, |z|)\delta \geq \frac{|s|\delta}{\delta + 2} = \frac{|s|}{1 + \frac{2}{\delta}} \geq \frac{|s|}{M} = |v_i| \geq |t_i|.$$

621 Therefore, the longer of segments  $y$  and  $z$  will cover the whole segment  $t_i$  after  $\delta$ -extension.  
 622 We conclude that  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

623 Since  $C = \bigcup_{i=1}^M C_i$ , it follows that  $\mathcal{R}^{+\delta}$  covers  $C$ .

**Algorithm.** We can simulate the inductive proof presented above by a recursive algorithm with the following complexity:

$$O\left(|C| + \frac{1}{\delta}\right) + O\left(|C| \cdot \left(2 + \frac{2}{\delta}\right)^k\right).$$

Let us now formulate some claims about the properties for the problem parameterized by the solution size. These properties provide bounds for different objects in the problem instance, which help us to find a small kernel for the problem or conclude that the optimum solution to this instance must be, in terms of size, above some threshold.

**Definition 4.3.** A line in the plane is **long** if there are at least  $k + 1$  points from  $\mathcal{C}$  on it.

**Claim 4.1.** *If there are more than  $k$  different long lines, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

*Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k + 1$  different long lines in our instance of the problem and there is a solution  $\mathcal{R}$  of size at most  $k$  covering points  $\mathcal{C}$ .

Choose any long line  $L$ . Every segment from  $\mathcal{R}$  which is not collinear with  $L$ , covers at most one point that lies on  $L$ .  $L$  is long, so there are at least  $k + 1$  points from  $\mathcal{C}$  that lie on  $L$ . This implies that there must be a segment in  $\mathcal{R}$  that is collinear with  $L$ .

Since we have at least  $k + 1$  different long lines, there are at least  $k + 1$  segments in  $\mathcal{R}$  collinear with different lines. This contradicts with the assumption that  $|\mathcal{R}| \leq k$ . □

**Claim 4.2.** *If there are more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

*Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k^2 + 1$  points from  $\mathcal{C}$  that do not lie on any long line, call this set  $A$ , and a solution  $\mathcal{R}$  of size at most  $k$  covering all points in  $\mathcal{C}$ .

Every segment  $s$  from  $\mathcal{R}$  covers at most  $k$  points from  $A$ . This is because if  $s$  covered at least  $k + 1$  points from  $A$ , then the line in the direction of  $s$  would be a long line and that contradicts the definition of  $A$ .

If every segment from  $\mathcal{R}$  covers at most  $k$  points from  $A$  and  $|\mathcal{R}| \leq k$ , then at most  $k^2$  points from  $A$  are covered by  $\mathcal{R}$  and that contradicts the fact that  $\mathcal{R}$  is a solution to the given geometric set cover instance. □

We are now ready to give a proof of Theorem 4.3.

*Proof of Theorem 4.3.* Our goal is to either answer NO or to find a kernel  $(\mathcal{C}', \mathcal{P}')$  of size bounded by  $f(k)$  for some function  $f$ , such that:

- (*Property 1*) for every solution  $\mathcal{R}$  to  $(\mathcal{C}, \mathcal{P})$  of size at most  $k$ , there exists a set  $\mathcal{R}_1 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_1| \leq k$ , weight of  $\mathcal{R}_1$  is not greater than weight of  $\mathcal{R}$  and  $\mathcal{R}_1$  covers  $\mathcal{C}'$ ;
- (*Property 2*) for every set  $\mathcal{R}_2 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_2| \leq k$  and  $\mathcal{R}_2$  covers points in  $\mathcal{C}'$ ,  $\mathcal{R}_2^{+\delta}$  covers points in original instance  $\mathcal{C}$ .

If we found such sets  $(\mathcal{C}', \mathcal{P}')$ , using *Property 1* we know that optimum solution of size at most  $k$  to  $(\mathcal{C}', \mathcal{P}')$  has no greater weight than optimum solution of size at most  $k$  to  $(\mathcal{C}, \mathcal{P})$ . Using *Property 2* we know that any solution to  $(\mathcal{C}', \mathcal{P}')$  after  $\delta$ -extensions covers  $\mathcal{C}$ .

Therefore finding such sets and solving the instance  $(\mathcal{C}', \mathcal{P}')$  by iterating over all of the subsets of  $\mathcal{P}'$  of size at most  $k$  in desired complexity is sufficient to prove Theorem 4.3.

**Definition of  $\mathcal{C}'$  and  $\mathcal{P}'$ .** Let us name the number of different long lines as  $l$ . Applying Claims 4.1 and 4.2, if we have more than  $k$  different long lines or more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then we answer NO, because these lemmas prove that there is no solution of size at most  $k$  to this instance.

Otherwise, we can split  $\mathcal{C}$  into at most  $k + 1$  sets:

- $D$ : points that do not lie on any long line,  $|D| \leq k^2$ ;
- $C_i$  for  $1 \leq i \leq l$ : points that lie on the  $i$ -th long line,  $|C_i| > k$ .

Note that sets  $C_i$  do not need to be disjoint.

Then, for every set  $C_i$  we can use Lemma 4.3 to obtain a  $(k, \delta)$ -dense set  $A_i$  for  $C_i$  with  $|A_i| \leq (2 + \frac{2}{\delta})^k$ .

We define  $\mathcal{C}' := D \cup (\bigcup A_i)$ .  $\mathcal{C}'$  has size at most  $k^2 + k(2 + \frac{2}{\delta})^k$ . We define  $\mathcal{P}'$  as follows: for every pair of points  $\mathcal{C}'$ , we choose one segment from  $\mathcal{P}$  that has the lowest weight among segments that cover these points or decide that there is no segment that covers them. There are at most  $|\mathcal{C}'|^2$  different segments in  $\mathcal{P}'$ , therefore both  $\mathcal{P}'$  and  $\mathcal{C}'$  have size bounded by  $\mathcal{O}((k^2 + k(2 + \frac{2}{\delta})^k)^2)$ .

**Proof of Property 2.** First, we prove that for every set  $\mathcal{R}_2 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_2| \leq k$  and  $\mathcal{R}_2$  covers points in  $\mathcal{C}'$ ,  $\mathcal{R}_2^{+\delta}$  covers points in the original instance  $\mathcal{C}$ .

Let us take such a set  $\mathcal{R}_2$ .

$\mathcal{C}$  is separated into several parts – sets  $D$  and  $C_i$ . Points from  $D$  are covered by  $\mathcal{R}_2$ , because  $D$  is part of  $\mathcal{C}'$ . Each point from any  $A_i$  is covered, because  $A_i$  is a part of  $\mathcal{C}'$ ;  $A_i$  is a  $(k, \delta)$ -dense set for  $C_i$ , therefore  $\mathcal{R}_2^{+\delta}$  covers all points in  $C_i$ . Therefore,  $\mathcal{R}_2^{+\delta}$  covers all points in  $\mathcal{C}$ .

**Proof of Property 1.** Secondly, we prove that for every solution  $\mathcal{R}$  to  $(\mathcal{C}, \mathcal{P})$  of size at most  $k$ , there exists a set  $\mathcal{R}_1 \subseteq \mathcal{P}'$  such that  $|\mathcal{R}_1| \leq k$  and the weight of  $\mathcal{R}_1$  is not greater than the weight of  $\mathcal{R}$ .

For every segment in  $\mathcal{R}$ , say  $s$ , let us look at the points from  $\mathcal{C}'$  that lie on  $s$  and call this set of points  $F$ .  $F$  is of course a set of collinear points. We can cover  $F$  with any segment that covers extreme points of  $F$ , because all other points lie on the segment between these points. Therefore, we can replace  $s$  with a segment  $s'$  that has lowest weight among the points that cover the extreme points of  $F$ . Such a segment belongs to  $\mathcal{P}'$ , because this is how it was defined. Segment  $s'$  has weight no greater than the weight of  $s$ , because  $s$  also covers  $F$ .

Therefore, we produced the set  $\mathcal{R}_1$  that has size not greater than size of  $\mathcal{R}$  (because some segments  $s$  can map to the same segment  $s'$ ), weight not greater than  $\mathcal{R}$ , and it covers  $\mathcal{C}'$ .

**Complexity** We find a solution of  $(\mathcal{C}', \mathcal{P}')$  by iterating over all the possible subsets of  $\mathcal{P}'$ . Finding sets  $\mathcal{P}'$  and  $\mathcal{C}'$  and then solving problem for kernel has overall complexity  $(|\mathcal{P}| + |\mathcal{C}|)^{\mathcal{O}(1)} \mathcal{O}((2 + \frac{2}{\delta})^k) + \mathcal{O}((k^2 + k(2 + \frac{2}{\delta})^k)^k)$ .  $\square$

## Chapter 5

# W[1]-hardness for axis-parallel weighted segments

In this chapter we consider geometric set cover problem with axis-parallel weighted segments. In Theorem 5.1 below, we prove that this problem is W[1]-hard when parameterized by the size of the solution. For simplicity we present here solution that uses segments in 3 directions, but there exists isomorphism that transposes these points in such a way, that all segments become axis-parallel.

**Definition 5.1.** A line is **right-diagonal** if it is described by linear function  $x + y = d$  for some  $d \in \mathbb{R}$ . Segment is **right-diagonal** if its direction is a right-diagonal line.

**Theorem 5.1.** *Consider the problem of covering a set  $\mathcal{C}$  of points by selecting at most  $k$  segments from a set of segments  $\mathcal{P}$  with non-negative weights  $w : \mathcal{P} \rightarrow \mathbb{R}^+$  so that the weight of the cover is minimal. Then this problem is W[1]-hard when parameterized by  $k$  and assuming ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ . Moreover, this holds even if all segments in  $\mathcal{P}$  are axis-parallel or right-diagonal.*

In order to prove Theorem 5.1 we will show reduction from a W[1]-hard problem. We introduce the grid tiling problem, which was introduced in [Marx, 2007] (authors call it matrix tiling instead). It was originally described as parametrization problem, but W[1]-hardness follows directly from the theorems states there. For more contemporary description of this artificial problem and proof of W[1]-hardness see Chapter 14 of [?].

**Definition 5.2.** We define the **powerset** of a set  $A$ , denoted as  $\text{Pow}(A)$ , as the set of all subsets of  $A$ , ie.  $\text{Pow}(A) = \{B : B \subseteq A\}$ .

**Definition 5.3.** In the **grid tiling** problem we are given integers  $n$  and  $k$ , and a function  $f : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \text{Pow}(\{1 \dots n\} \times \{1 \dots n\})$  specifying the set of allowed tiles for each cell of a  $k \times k$  grid. The task is to decide whether there exists such functions  $x, y : \{1 \dots k\} \rightarrow \{1 \dots n\}$  that assign colors from  $\{1 \dots n\}$  to respectively columns and rows of the grid, so that  $(x(i), y(j)) \in f(i, j)$  for all  $i, j \in \{1 \dots k\}$ .

In short, in grid tiling problem one needs to assign numbers to rows and columns in such a way that for every pair of a row and a column, the pair of colors assigned to the row and column belongs to the allowed set of tiles for this pair. The next theorem describes the complexity of this problem, which is W[1]-hard when parameterized by the size of the grid.

	$x(1) = 3$	$x(2) = 1$	$x(3) = 3$	$x(4) = 7$
$y(4) = 1$	$(\mathbf{2}, \mathbf{1}); (2, 2);$ $(\mathbf{3}, \mathbf{1}); (3, 9)$	$(1, 1); (3, 1)$	$(\mathbf{3}, \mathbf{1}); (7, 2)$	$(\mathbf{2}, \mathbf{1}); (\mathbf{7}, \mathbf{1})$
$y(3) = 1$	$(\mathbf{2}, \mathbf{1}); (\mathbf{3}, \mathbf{1});$ $(4, 2); (8, 2)$	$(1, 1); (1, 3)$	$(\mathbf{3}, \mathbf{1}); (4, 3)$	$(\mathbf{2}, \mathbf{2}); (\mathbf{7}, \mathbf{1})$
$y(2) = 6$	$(\mathbf{2}, \mathbf{6}); (\mathbf{3}, \mathbf{6})$	$(1, 2); (1, \mathbf{6});$ $(2, 6)$	$(2, 6); (\mathbf{3}, \mathbf{6})$	$(\mathbf{2}, \mathbf{6}); (\mathbf{7}, \mathbf{6})$
$y(1) = 4$	$(\mathbf{2}, \mathbf{4}); (2, 6);$ $(\mathbf{3}, \mathbf{4}); (\mathbf{3}, \mathbf{9})$	$(1, 4); (\mathbf{1}, \mathbf{9})$	$(\mathbf{3}, \mathbf{4}); (\mathbf{3}, \mathbf{9})$	$(\mathbf{2}, \mathbf{9}); (\mathbf{7}, \mathbf{4})$

Figure 5.1: **Example of a grid tiling instance and its solution.**

In the first row and column of the table you can see the solution: functions  $x$  and  $y$ . The tiles used in this solution are marked in **bold**. If we instead chose the tiles marked in **blue** (whenever there is one, taking the tile marked in **bold** otherwise), then that corresponds to setting  $x(1) = 2$ , and would also form a correct solution. On the other hand, if we instead chose the tiles marked in **red** (as before), then this corresponds to setting  $y(1) = 9$  and  $x(4) = 2$  and that would **not** form a correct solution. Even though the first row is correct, the cell with coordinates  $(3, 4)$  requires tile  $(2, 1)$ , not  $(2, 2)$  (marked in **bold red**).

**Theorem 5.2.** [Marx, 2007] *Grid tiling is  $W[1]$ -hard when parameterized by  $k$  and assuming ETH, there is no  $f(k) \cdot n^{o(k)}$ -time algorithm solving the grid tiling problem for any computable function  $f$ .*

The remainder of this section is devoted to proving Theorem 5.1 by a reduction from a grid tiling problem instance with parameter (number of rows in the grid)  $k$  to a geometric set cover instance with parameter (size of solution)  $k^2$ . This reduction is described in Lemma 5.1. This proves the  $W[1]$ -hardness of the geometric set cover problem, because if we could solve it with an FPT algorithm, then we could also solve the grid tiling problem (which we reduced to the geometric set cover). Therefore geometric set cover with setting described in Theorem 5.1 is at least as hard as the grid tiling problem.

Let us denote instance of grid tiling problem as  $(n, k, f)$ , it consists of:

- the size of the grid  $k$ ,
- the number of colors  $n$ ,
- the function of allowed tiles  $f : \{1, \dots, k\} \times \{1, \dots, k\} \rightarrow \text{Pow}(\{1, \dots, n\} \times \{1, \dots, n\})$ .

Let us also define a constant  $W_{hv} := 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$ , that is going to be used when defining the weight of the constructed instance of geometric set cover with weighted segments.

**Lemma 5.1.** *Given an instance  $(n, k, f)$  of the grid tiling problem, we can construct an instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of the geometric set cover with weighted segments such that:*

- (1) *if the answer to  $(n, k, f)$  is YES, then there exists a solution to  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of weight at most  $W_{hv} + k^2\delta$ ;*
- (2) *if there exists a solution to  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of weight at most  $W_{hv} + k^2\delta$ , then the answer to  $(n, k, f)$  is YES.*



754 First, let us prove Theorem 5.1 using Lemma 5.1.

755 *Proof.* Let us take any instance  $(n, k, f)$  of the grid tiling problem of the grid size  $l$ . We prove  
 756 the theorem by contradiction, therefore we assume that geometric set cover with weighted  
 757 segments with solution of size at most  $k$  segments admits a  $n^{o(\sqrt{k})}$  algorithm.

758 Using Lemma 5.1 let us construct an instance  $I$  for  $(n, k, f)$ . Let us assume that the  
 759 optimum solution of size at most  $k$  to the instance  $I$  has weight  $w$ . Using (2) we know that  
 760 if  $w \leq W_{hv} + k^2\delta$ , then answer to  $(n, k, f)$  is YES. If  $w > W_{hv} + k^2\delta$ , then using (1) we know  
 761 that the answer to  $(n, k, f)$  must be NO.

762 Therefore if we could find it in time  $n^{o(\sqrt{k})}$ , then we could solve the grid tiling problem  
 763 in  $n^{o(l)}$  by constructing an instance of the set cover with weighted segments, solving it for  
 764 parameter  $k = 3l^2 + 2l$  in time  $n^{o(\sqrt{3l^2+2l})} \geq n^{o(l)}$  and then answering based on the weight  
 765 of the optimum solution. Existence of this algorithm contradicts Theorem 5.2.

766 Hence such an algorithm can not exist.  $\square$

767 We prove Lemma 5.1 in subsequent sections. First, we define a constructed instance  $I$ ,  
 768 later property (1) is proved by Lemma 5.2 and property (2) is proved by Lemma 5.6.

769 TODO: Do the same summary of Proofs in chapter 3.

770 In proof of Lemma 5.6 we do not use the assumption that the solution is bounded by the  
 771 size, which the problem is parametrized by,  $3k^2 + 2k$ . Thus this reduction proves that the  
 772 problem is not only W[1]-hard, but assuming ETH there also does not exist permissive FPT  
 773 algorithm for this problem.

774 TODO: rewrite this Permissive FPT is a stronger result, where we need to find an optimal  
 775 solution of any size, but it is still parametrized by the size of the. Similar idea for local search  
 776 was presented in [Marx and Schlotter, 2011], [Gaspers et al., 2012].

777 **Theorem 5.3. Permissive FPT does not exist.**

778 TODO: Here state stronger theorem

779 **Construction.** We construct an instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of geometric set cover as follows.

780 First, let us choose any bijection  $\text{order} : \{1, \dots, n^2\} \rightarrow \{1, \dots, n\} \times \{1, \dots, n\}$ .

Define  $\text{match}_v(i, j)$  and  $\text{match}_h(i, j)$  as boolean functions denoting whether two points  
 share x or y coordinate:

$\text{match}_v(i, j)$  is true  $\iff$   $\text{order}(i)$  and  $\text{order}(j)$  have the same x coordinate,

$\text{match}_h(i, j)$  is true  $\iff$   $\text{order}(i)$  and  $\text{order}(j)$  have the same y coordinate.

**Points.** For  $1 \leq i, j \leq k$  and  $1 \leq t \leq n^2$  define points:

$$h_{i,j,t} := (i \cdot (n^2 + 1) + t, j \cdot (n^2 + 1)),$$

$$v_{i,j,t} := (i \cdot (n^2 + 1), j \cdot (n^2 + 1) + t).$$

Let us define sets  $H$  and  $V$  as:

$$H := \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\},$$

$$V := \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}.$$

Let  $\epsilon = \frac{1}{2k^2}$ . For a point  $p = (x, y)$  we define points:

$$p^L := (x - \epsilon, y),$$

$$\begin{aligned}
p^R &:= (x + \epsilon, y), \\
p^U &:= (x, y + \epsilon), \\
p^D &:= (x, y - \epsilon).
\end{aligned}$$

Then we define the point set as follows:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}.$$

**Definition 5.4.** For every point  $p \in H$ , we name point  $p^L$  its **left guard** and point  $p^R$  its **right guard**.

Similarly for every points  $p \in V$ , we name point  $p^D$  its **lower guard** and point  $p^U$  its **upper guard**.

**Segments.** For  $1 \leq i, j \leq k$  and  $1 \leq t_1, t_2 \leq n^2$  define segments:

$$\begin{aligned}
\text{hor}_{i,j,t_1,t_2} &:= (h_{i,j,t_1}^R, h_{i+1,j,t_2}^L), \\
\text{ver}_{i,j,t_1,t_2} &:= (v_{i,j,t_1}^U, v_{i,j+1,t_2}^D), \\
\text{horBeg}_{i,t} &:= (h_{1,i,1}^L, h_{1,i,t}^L), \\
\text{horEnd}_{i,t} &:= (h_{k,i,t}^R, h_{k,i,n^2}^R), \\
\text{verBeg}_{i,t} &:= (v_{i,1,1}^D, v_{i,1,t}^D), \\
\text{verEnd}_{i,t} &:= (v_{i,k,t}^U, v_{i,k,n^2}^U).
\end{aligned}$$

Next, we define sets of vertical and horizontal segments:

$$\begin{aligned}
\text{HOR} &:= \{ \text{hor}_{i,j,t_1,t_2} : 1 \leq i < k, 1 \leq j \leq k, 1 \leq t_1, t_2 \leq n^2, \text{match}_h(t_1, t_2) \text{ holds} \} \\
&\cup \{ \text{horBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \} \\
&\cup \{ \text{horEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \},
\end{aligned}$$

$$\begin{aligned}
\text{VER} &:= \{ \text{ver}_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, \text{match}_v(t_1, t_2) \text{ holds} \} \\
&\cup \{ \text{verBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \} \\
&\cup \{ \text{verEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2 \}.
\end{aligned}$$

You can see an example of these segments in Figure 5.3.

Finally, we also define a set of right-diagonal segments:

$$\text{DIAG} := \{ (h_{i,j,t}, v_{i,j,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, \text{order}(t) \in f(i, j) \}.$$

You can see an example of such segments in Figure 5.2.

Every segment in **DIAG** connects points  $(i(n^2+1)+t, j \cdot (n^2+1))$  and  $(i \cdot (n^2+1), j(n^2+1) + t)$  for some  $1 \leq i, j \leq k, 1 \leq t \leq n^2$ . The line on which it lies can be described by linear equation  $y = -x + (t + (i+j)(n^2+1))$ , thus these segments are in fact right-diagonal.

The constructed segment set is defined as:

$$\mathcal{P} := \text{HOR} \cup \text{VER} \cup \text{DIAG}.$$

The weight of each segment in  $\text{HOR} \cup \text{VER}$  is equal to its length, while every segment in **DIAG** has weight  $\delta := \frac{1}{4k^4}$ .

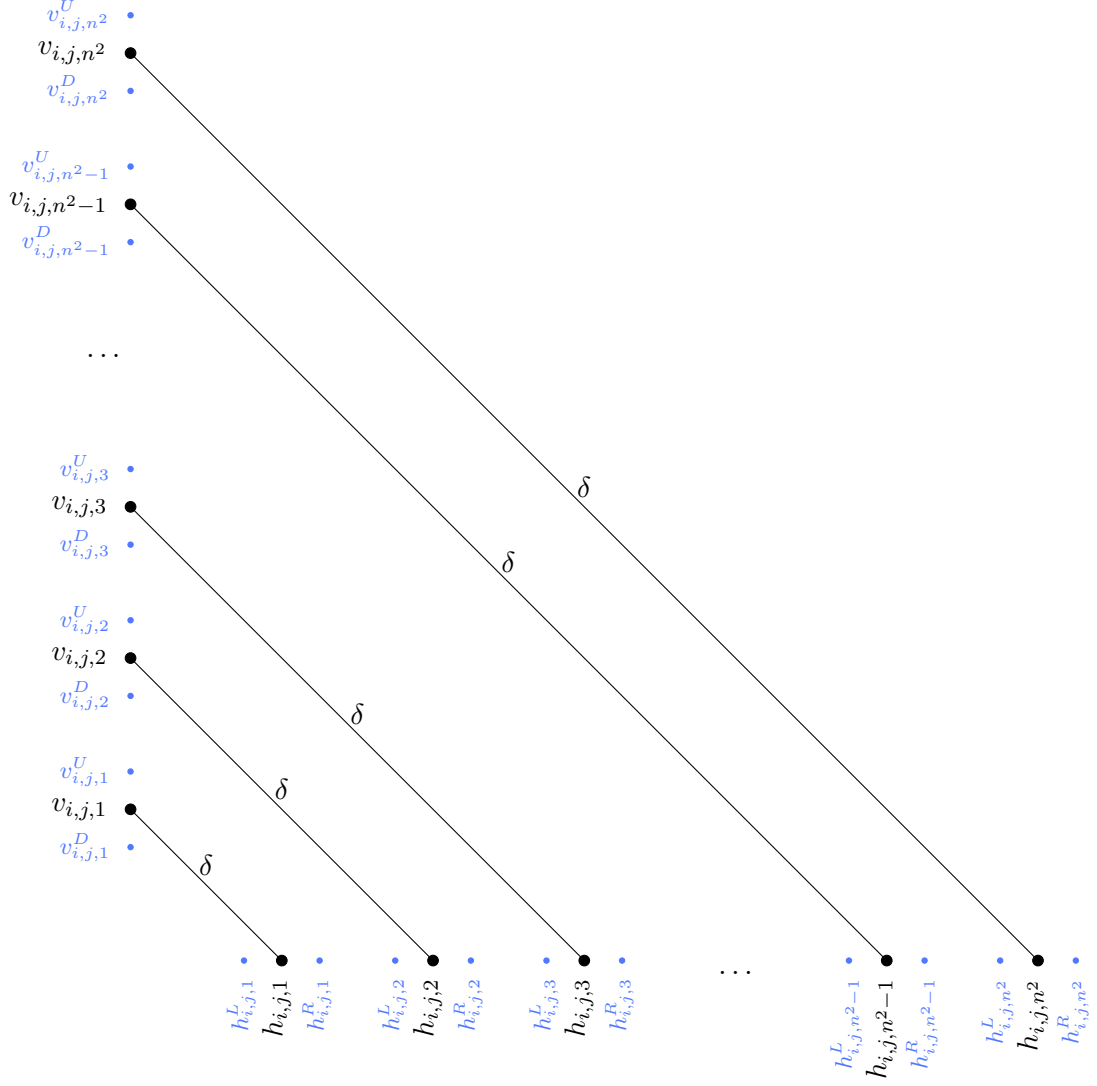


Figure 5.2: **Vertices and segments in DIAG.**

This is an example of constructed points any  $1 \leq i, j \leq k$ . Points from  $H$  and  $V$  are marked in black, their guards are marked in blue. You can also see segments from DIAG with their weights (equal to  $\delta$ ).

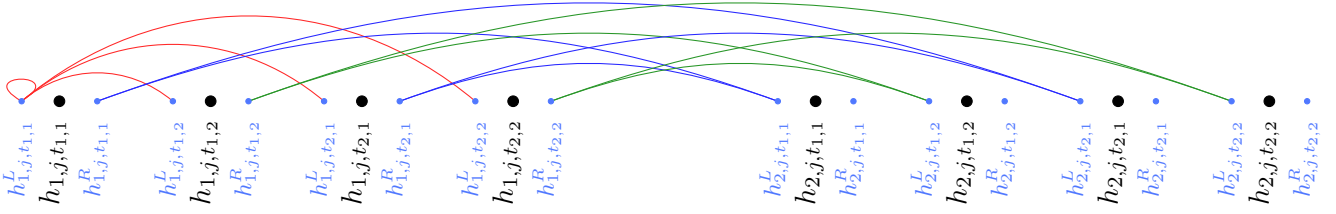


Figure 5.3: Vertices and segments in HOR.

This is an example for  $n = 2$  and any  $1 \leq j \leq k$ . Points from  $H$  are marked in black, their guards are marked in light blue.  $t_{i,j}$  is a notation that we use for  $\text{order}^{-1}(i, j)$ . Segments are represented as arcs between endpoints. You can see  $\text{horBeg}_{j,t}$  segments in red.  $\text{horBeg}_{j,1}$  is degenerated to a single point at  $h_{1,1,t_{1,1}}^L$ . Segments  $\text{hor}_{i,j,t_{x_1,y},t_{x_2,y}}$  are marked in blue and green. Blue segments connect  $t_{x_1,y}$  and  $t_{x_2,y}$  such that they share y-coordinate equal to 1, for green segments it is equal to 2.

$$w(s) = \begin{cases} \text{length}(s) & \text{if } s \in \text{HOR} \cup \text{VER} \\ \delta & \text{if } s \in \text{DIAG} \end{cases}$$

Now, we prove that the constructed instance of geometric set cover with weighted segments is indeed a correct and sound reduction of the grid tiling problem. Lemma 5.2 proves that if the solution to the instance of the grid tiling instance exists, then there exists a solution with bounded size and weight of the constructed instance of geometric set cover problem.

Then Lemma 5.6 proves that if the solution to the geometric set cover instance with bounded weight exists, then there exists a solution to the original grid tiling instance.

**Lemma 5.2.** *If there exists a solution to the grid tiling instance  $(f_{i,j})$ , then there exists a solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  of geometric set cover with weight  $W_{hv} + k^2\delta$ .*

*Proof.* Suppose there exists a solution  $x, y$  of the instance  $(f_{i,j})$  of the grid tiling problem.

We define the proposed solution  $\mathcal{R} \subset \mathcal{P}$  of the instance of geometric set cover in three parts  $D \subset \text{DIAG}$ ,  $A \subset \text{HOR}$  and  $B \subset \text{VER}$ :

$$\begin{aligned} D &:= \{(v_{i,j,t}, h_{i,j,t}) : 1 \leq i, j \leq k, t = \text{order}^{-1}(x(i), y(j))\}, \\ A &:= \{\text{horBeg}_{i, \text{order}^{-1}(x(1), y(i))} : 1 \leq i \leq k\} \\ &\quad \cup \{\text{horEnd}_{i, \text{order}^{-1}(x(k), y(i))} : 1 \leq i \leq k\} \\ &\quad \cup \{\text{hor}_{i,j, \text{order}^{-1}(x(i), y(j)), \text{order}^{-1}(x(i+1), y(j))} : 1 \leq i < k, 1 \leq j \leq k\}, \\ B &:= \{\text{verBeg}_{i, \text{order}^{-1}(x(i), y(1))} : 1 \leq i \leq k\} \\ &\quad \cup \{\text{verEnd}_{i, \text{order}^{-1}(x(i), y(k))} : 1 \leq i \leq k\} \\ &\quad \cup \{\text{ver}_{i,j, \text{order}^{-1}(x(i), y(j)), \text{order}^{-1}(x(i), y(j+1))} : 1 \leq i \leq k, 1 \leq j < k\}, \end{aligned}$$

$$\mathcal{R} := D \cup A \cup B.$$

Since  $\mathcal{C} = H \cup V$ , we show that  $\mathcal{R}$  covers the whole set  $H$ , proof for  $V$  is analogous.

Take any  $1 \leq j \leq k$  and define  $t_i := \text{order}^{-1}(x(i), y(j))$ . The two leftmost segments in  $A$  for this  $j$  are  $\text{horBeg}_{j,t_1} = (h_{1,j,t_1}^L, h_{1,j,t_1}^L)$  and  $\text{hor}_{1,j,t_1,t_2} = (h_{1,j,t_1}^R, h_{2,j,t_2}^L)$ . Therefore points  $h_{1,j,x}, h_{1,j,x}^L$  and  $h_{1,j,x}^R$  for all  $1 \leq x \leq n^2$  are covered by  $\text{horBeg}_{j,t_1}$  and  $\text{hor}_{1,j,t_1,t_2}$ , excluding point  $h_{1,j,t_1}$ .

812 Analogously for  $2 \leq i \leq k-1$  for two consecutive segments  $\text{hor}_{i-1,j,t_{i-1},t_i}$  and  $\text{hor}_{i,j,t_i,t_{i+1}}$   
 813 we prove that all points  $h_{i,j,x}$ ,  $h_{i,j,x}^L$  and  $h_{i,j,x}^R$  for all  $1 \leq x \leq n^2$  are covered by these segments  
 814 excluding point  $h_{i,j,t_i}$ .

815 Finally  $\text{hor}_{k-1,j,t_{k-1},t_k}$  and  $\text{horEnd}_{j,t_k}$  cover all points  $h_{k,j,x}$ ,  $h_{k,j,x}^L$  and  $h_{k,j,x}^R$  for all  $1 \leq x \leq n^2$   
 816 excluding point  $h_{k,j,t_k}$ .

817  $D$  covers all points  $h_{i,j,t_i}$  and  $v_{i,j,t_i}$ , therefore all points in  $H$  are covered.

Size of this proposed solution is:

$$|\mathcal{R}| = |D| + |A| + |B| = k^2 + (k+1)k + (k+1)k = 3k^2 + 2k.$$

818 Then, we need to compute the total weight of the solution  $\mathcal{R}$ . First, we compute the sum  
 819 of weights of segments in  $A$ . Fix  $1 \leq j \leq k$  and compute segments collinear with the  $j$ -th  
 820 line. All points  $h_{i,j,t}$ ,  $h_{i,j,t}^L$  and  $h_{i,j,t}^R$  for every  $1 \leq i \leq k$  and  $1 \leq t \leq n^2$  are covered by  
 821  $A$  excluding points  $h_{i,j,\text{order}^{-1}(x(i),y(j))}$ . Every such point leaves a gap of length  $2\epsilon$  between  
 822  $h_{i,j,\text{order}^{-1}(x(i),y(j))}^L$  and  $h_{i,j,\text{order}^{-1}(x(i),y(j))}^R$ . Therefore, the total weight of segments in  $A$  that  
 823 lie on the line in question equals the length of the segment  $(h_{i,1,1}^L, h_{i,k,n^2}^R)$  minus  $2\epsilon k$ , which is  
 824  $k(n^2 + 1) - 2(1 - \epsilon) - 2k\epsilon$ . We need to multiply that by  $k$ , as we consider all possible values  
 825 of  $j$ .

826 Calculation for vertical segments is analogous and has the same result. Every segment  
 827 in  $D$  has weight  $\delta$ , therefore the sum of all weights is equal to:

$$2k(k(n^2 + 1) - 2(1 - \epsilon) - 2k\epsilon) + k^2\delta = W_{hv} + k^2\delta$$

828 □

829 **Claim 5.1.** *In any solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$ :*

- 830 • *left and right guards of points in  $H$  (points in  $\{p^L : p \in H\} \cup \{p^R : p \in H\}$ ) have*  
 831 *to be covered with segments from HOR,*
- 832 • *lower and upper guards of points in  $V$  (points in  $\{p^D : p \in V\} \cup \{p^U : p \in V\}$ ) have*  
 833 *to be covered with segments from VER.*

834 *Proof.* We prove the claim for the points from  $H$  as the proof for points from  $V$  is analogous.

835 Every segment in VER is vertical and has x-coordinate equal to  $i(n^2 + 1)$  for some  $1 \leq i \leq k$ ,  
 836 so they all have different x-coordinate than any left or right guard of points in  $H$ .

837 Every point  $x$ , which is a left or right guard of points in  $H$  have  $kn^2$  segments from DIAG  
 838 that intersect with the horizontal line that goes through  $x$ . All of these segments intersect  
 839 with this line in points from set  $H$ , therefore none of them cover any of the guards.

840 Therefore none of the segments from VER or DIAG cover any of the guards of the points  
 841 in  $H$ . □

842 Now we present a few additional properties of the constructed instance of the geometric  
 843 set cover that help us to prove Lemma 5.6.

844 **Claim 5.2.** *For any  $1 \leq i, j \leq n$  and any solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  all,*  
 845 *but at most one point  $h_{i,j,t}$  and at most one point  $v_{i,j,t}$  for  $1 \leq t \leq n^2$  must be covered with*  
 846 *segments from HOR or VER.*

847 *Proof.* We prove the claim for horizontal segments, as the proof for vertical segments is ana-  
 848 loguous.

We prove this by contradiction. Assume that we have two points  $h_{i,j,t_1}, h_{i,j,t_2}$  such that they are not covered with segments from **HOR** for any  $1 \leq t_1 < t_2 \leq n^2$ .

Point  $h_{i,j,t_1}^R$  has to be covered with **HOR** by Claim 5.1. Every segment in **HOR** covering  $h_{i,j,t_1}^R$ , but not  $h_{i,j,t_1}$  must start at  $h_{i,j,t_1}^R$  and all such segments cover also  $h_{i,j,t_2}$ . This contradicts the assumption, which concludes the proof.  $\square$

**Lemma 5.3.** *For every solution to the instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$ , the sum of weights of segments chosen from sets **HOR** and **VER** is at least  $2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$ .*

*Proof.* We prove the lemma for vertical lines, as the proof for horizontal segments is analogous.

Let us fix  $1 \leq i \leq k$ .

We provide a lower bound for the sum of lengths of vertical segments from  $\mathcal{R} \cap \mathbf{VER}$ . This bound is the same for each  $i$  and is the same for horizontal lines, thus we need to multiply such bound by  $2k$ .

(1) The total length between  $v_{i,1,1}^D$  and  $v_{i,k,n^2}^U$  is:

$$(k(n^2 + 1) + n^2 + \epsilon) - ((n^2 + 1) + 1 - \epsilon) = k(n^2 + 1) - 2(1 - \epsilon).$$

(2) For every  $1 \leq j \leq k$  there exists at most one  $1 \leq t \leq n^2$  such that  $v_{i,j,t}$  is not covered by segments from **VER** (Claim 5.2). Its guards (see Definition 5.4)  $v_{i,j,t}^U$  and  $v_{i,j,t}^D$  have to be covered in **VER** (Claim 5.1). Therefore, at most  $k$  spaces of length  $2\epsilon$  can be left not covered by segments from **VER** between  $v_{i,1,1}^D$  and  $v_{i,k,n^2}^U$ .

The sum of these lower bounds for vertical and horizontal lines is:

$$2k(k(n^2 + 1) - 2k\epsilon - 2(1 - \epsilon)) = 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$$

$\square$

Let us name the bound from the previous lemma as  $W_{hv} := 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$  for future reference.

**Lemma 5.4.** *Let  $\mathcal{R}$  be a solution to a constructed instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight at most  $W_{hv} + k^2\delta$ . Then for every  $1 \leq i, j \leq k$  there exists such  $1 \leq t \leq n^2$  that:*

- (1)  $v_{i,j,t}, h_{i,j,t}$  are not covered by segments from **VER** or **HOR**;
- (2) segment  $(v_{i,j,t}, h_{i,j,t})$  is in solution  $\mathcal{R}$ ;
- (3)  $\text{order}(t) \in f(i, j)$ , that is,  $\text{order}(t)$  is an allowed tile for  $(i, j)$ ;
- (4) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $v_{i,j,s}$  is covered in **VER**;
- (5) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $h_{i,j,s}$  is covered in **HOR**.

*Proof.* At most one of points  $\{h_{i,j,t_x} : 1 \leq t_x \leq n^2\}$  and one of points  $\{v_{i,j,t_y} : 1 \leq t_y \leq n^2\}$  is covered with **DIAG** (Claim 5.2).

Moreover, exactly one such point  $h_{i,j,t_x}$  and one such point  $v_{i,j,t_y}$  is covered with **DIAG**, because if none of them were covered, then the solution would have to have weight at least  $W_{hv} + 2\epsilon$  (Lemma 5.3), which is more than  $W_{hv} + k^2\delta$ .

We observe that points  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  have to be covered with the same segment from **DIAG**. Indeed we need to use at least  $k^2$  of them to use exactly one **DIAG** segment for every

pair of  $1 \leq i, j \leq k$ , if we used 2 segments from **DIAG** for one pair  $(i, j)$ , then we would have used  $W_{hv} + k^2\delta + \delta$  (Lemma 5.3), which if more than  $W_{hv} + k^2\delta$ . Since points  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  are covered by a single segment from **DIAG**, we have  $t_x = t_y$ .

Therefore  $t_x = t_y$  and  $\text{order}(t_x)$  is an allowed tile for  $(i, j)$  because the corresponding segment is in **DIAG**.  $\square$

We refer to the function mapping  $1 \leq x \leq k$  to  $t_x$  from Lemma 5.4 as  $\text{diagonal} : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \{1 \dots n^2\}$ .

**Lemma 5.5.** *For any solution  $\mathcal{R}$  of a constructed instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight at most  $W_{hv} + k^2\delta$ :*

1. for any  $1 \leq i < k, 1 \leq j \leq k$ ,  $\text{match}_h(\text{diagonal}(i, j), \text{diagonal}(i + 1, j))$  is **true**;
2. for any  $1 \leq i \leq k, 1 \leq j < k$ ,  $\text{match}_v(\text{diagonal}(i, j), \text{diagonal}(i, j + 1))$  is **true**.

*Proof.* We prove (1) by contradiction, the proof of (2) is analogous.

Let us take any  $1 \leq i < k, 1 \leq j \leq k$  and name  $t_1 = \text{diagonal}(i, j)$  and  $t_2 = \text{diagonal}(i + 1, j)$ . We also assume that  $\text{match}_h(t_1, t_2)$  is **false**, which is equivalent to the fact that segment  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  is not in set **HOR**.

Therefore  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  are not covered by segments from **HOR** (Lemma 5.4), while  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  have to be covered by segments from **HOR** (Claim 5.1).

Every segment from **HOR** starts at point  $h_{x,y,z_1}^R$  and ends at point  $h_{x+1,y,z_2}^L$  for some  $1 \leq x < k, 1 \leq y \leq k$  and  $1 \leq z_1, z_2 \leq n^2$ . All of the points between  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  are covered by segments in **HOR** and there is no segment  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  in **HOR**. Hence, there are at least two different segments covering them. One of them must begin at  $h_{i,j,t_1}^R$  and end at  $h_{i+1,j,z_2}^L$  and there must be other one that begins at  $h_{i,j,z_1}^R$  and ends at  $h_{i+1,j,t_2}^L$  for some  $1 \leq z_1, z_2 \leq n^2$ .

Thus, the space between  $h_{i,j,z_1}^R$  and  $h_{i,j+1,z_2}^L$  would be covered twice and is longer than  $\epsilon$ . By Lemma 5.3, the lower bound for weight of such a solution is  $W_{hv} + \epsilon$  which is more than  $W_{hv} + k^2\delta$ .

Therefore  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  must be covered by one segment from **HOR**,  $(h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$  is a segment in **HOR** and  $\text{match}_h(t_1, t_2)$  is **true**.  $\square$

**Lemma 5.6.** *If there exists solution to instance  $(\mathcal{C}, \mathcal{P}, w, 3k^2 + 2k)$  with weight at most  $W_{hv} + k^2\delta$ , then there exists a solution to the grid tiling instance  $(f_{i,j})$ .*

*Proof.* Take  $\text{diagonal}$  function from Lemma 5.4.

To define the  $x$  function for every  $1 \leq i \leq k$  set  $x(i) := x_i$  where  $(x_i, a) = \text{order}(v_{i,1})$ . Similarly, to define the  $y$  function, for every  $1 \leq i \leq k$  set  $y(i) := y_i$  where  $(b, y_i) = \text{order}(h_{1,i})$ .

To prove that it is a correct solution to grid tiling, we need to prove that for every  $1 \leq i, j \leq k$   $(x(i), y(j))$  is in allowed tiles set  $f(i, j)$ .

Let us take any  $1 \leq i, j \leq k$ . By Lemma 5.5 and simple induction, we know that  $\text{match}_h(\text{diagonal}(1, j), \text{diagonal}(i, j))$  and  $\text{match}_v(\text{diagonal}(i, 1), \text{diagonal}(i, j))$  are **true**. Therefore  $\text{order}(\text{diagonal}(i, j)) = (x(i), y(j))$ . By Lemma 5.4 we know that  $\text{order}(\text{diagonal}(i, j))$  is in  $f(i, j)$ . Therefore  $(x(i), y(j))$  is in  $f(i, j)$ .  $\square$





## <sup>921</sup> Chapter 6

## <sup>922</sup> Conclusions

<sup>923</sup> We know FPT for axis-parallel segments without  $\delta$ -extensions.



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