Approximation and parameterization of Segment Set Cover

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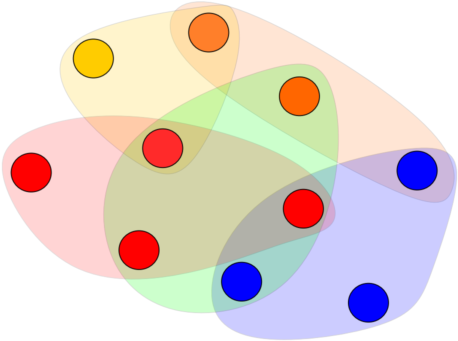
Overview

Problem Statement: SET COVER

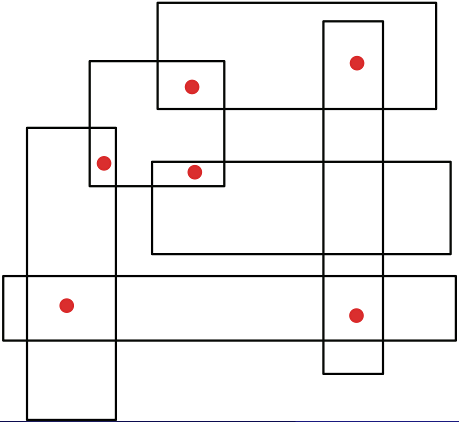
Input: universe $\mathcal{U}$, family S of subsets of $\mathcal{U}$
Output: $C \subseteq S$ such that $|C|$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in $\mathcal{U}$.



Problem Statement: GEOMETRIC SET COVER

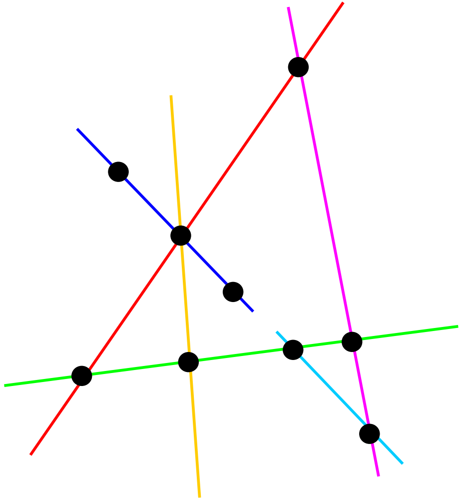
- Universe $\mathcal{U}$ is set of points in the plane
- Sets S are some geometric shapes
- Formally, each set in S is intersection of $\mathcal{U}$ with some geometric shape.

Example for rectangles:



Problem Statement: Segment Set Cover

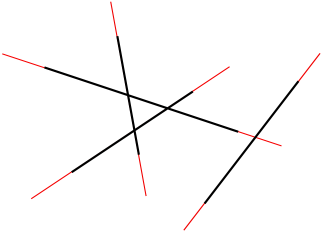
- Universe $\mathcal{U}$ is set of points in the plane
- Sets S are segments.



δ-extension for SEGMENT SET COVER

- δ-extension for a segment is a segment which is longer by δ fraction at both ends
- We accept solution in which segments cover solution after δ-extension
- The solution is compared to optimal solution without extension.

Example:



It is based on similar concept of δ-shrinking, which helped to provide FPT algorithm and EPTAS for (originally W[1]-hard and APX-hard) MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES.

Preliminaries: Approximation

Given:

- instance I of the optimization problem (looking for minimal solution)
- weight of the optimal solution $\mathrm{opt}(I)$

p-approximation is algorithm that yields solution of weight not larger than $p \cdot \mathrm{opt}(I)$

PTAS – polynomial time approximation scheme

For every $\epsilon > 0$, there exists $(1+\epsilon)$-approximation algorithm running in time $n^{f(\epsilon)}$ for some computable function f.
Example: KNAPSACK.

APX-hardness

For sufficiently small $\epsilon > 0$, if $(1+\epsilon)$-approximation exists, then P $=$ NP.
Example: MAX-3-SAT (how many clauses in 3-SAT can be solved)

Approximation results for SET COVER from literature

SET COVER

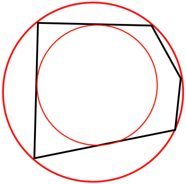
- $\log n$ approximation, no $o(\log n)$ approximation assuming $P \neq NP$

GEOMETRIC SET COVER

- with fat rectangles is APX-hard
- EPTAS for fat polygons with δ-extensions **[Har-Peled and Lee, 2012]**

Definition

Fatness of polygons τ is a ratio between a radius of circle circumscribed on this polygon and inscribed in this polygon.



MAX-(3,3)-SAT

How do we prove that there doesn't exist $(1+\epsilon)$-approximation?

MAX-(3,3)-SAT problem

Given 3-SAT instance with n variables, m clauses and every variable appears in exactly 3 clauses, find an assignment that satisfies the maximum number of clauses (not necessarily all of them).

MAX-(3,3)-SAT is NP-complete, therefore:

NP-completeness

Distinguishing if result of MAX-(3,3)-SAT is n or less than n is NP-complete.

PCP theorem

PCP Theorem

$\mathrm{NP} = \mathrm{PCP}(\log n, \mathcal{O}(1))$.

PCP theorem implication [Håstad, 2001]

Distinguishing whether optimum result of the instance of MAX-(3,3)-SAT is m or at most $\frac{49}{50} \cdot m$ is NP-hard.

Therefore MAX-(3,3)-SAT does not have $(1 + \frac{1}{49})$-approximation

Approximation of SEGMENT SET COVER

Theorem 1

SEGMENT SET COVER is APX-hard, i.e. there exists such $\epsilon > 0$ that $(1 + \epsilon)$-approximation running in time $n^{f(\epsilon)}$ does not exist if P $\neq$ NP.

How can we make this problem *easier*?

- Allow only segments in 2 directions (parallel to the two axes)
- Allow segments to *almost* cover the points, i.e. δ-extensions

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Remark: Set cover with polygons of bounded fatness (at least τ):

1. is APX-hard;
2. with δ-extension has EPTAS.

Theorem 1 (rephrased)

Segment Set Cover is APX-hard, even if segments are axis-parallel and problem is relaxed with $\frac{1}{2}$-extension.

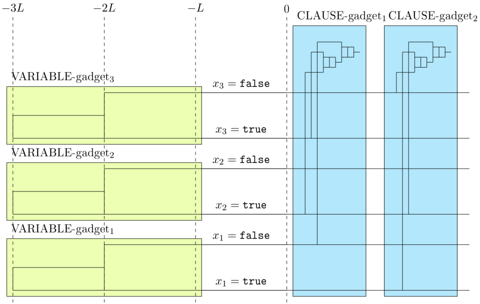

Figure: Scheme of the construction.

Results for approximation of Geometric Set Cover

	exact	δ-extensions
fat polygons	APX-hard [Chan and Grant, 2014]	EPTAS [Har-Peled and Lee, 2012]
segments	APX-hard*	APX-hard (Theorem 1)
any polygons	APX-hard*	APX-hard*

Results marked with * follow from results for more restricted settings.

Prelimiaries: Parameterized algorithms

Instance I of a parameterized problem now has:

- size of the instance n;
- parameter k (usually size of the solution).

Class	**Upper Bound of Complexity**	**Example**
FPT	$f(k) \cdot n^{O(1)}$	VERTEX COVER
W[1]	$n^{O(k)}$	k-CLIQUE, GRID TILING

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Set Cover is:

- W[2]-complete
- no algorithm with running time $n^{k-\epsilon}$ for any $\epsilon > 0$

Parameterization of GEOMETRIC SET COVER

Theoem [Marx, 2005]

GEOMETRIC SET COVER with unit squares parameterized by the solution size k is W[1]-hard.

The followup work [Marx and Pilipczuk, 2022] shows a tight bound for this problem $n^{\mathcal{O}(\sqrt{k})}$.

Parameterization of Segment Set Cover

Theorem 2

Segment Set Cover parameterized by the solution size k can be solved in time $k^k \cdot n^{O(1)}$.

Technique: Branching over at most $k+1$ segments on the lines with more than k points on them.

Parameterization of Segment Set Cover

Theorem 2

Segment Set Cover parameterized by the solution size k can be solved in time $k^k \cdot n^{O(1)}$.

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Now let's make problem **harder**...

Definition of Weighted Segment Set Cover

Input: universe $\mathcal{U}$, subfamily S of subsets of $\mathcal{U}$ and function of weights assigned to sets $w : S \to \mathbb{R}^+$
Output: $C \subseteq S$ such that $\sum_{c \in C} w(c)$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in $\mathcal{U}$.

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FPT results for weighted problems in literature:

- general weighted FPT framework [Shachnai and Zehavi, 2017]
- kernels for Weighted Subset Sum, Weighted Knapsack [Etscheid et al.,2017]
- Weighted *st*-Cut Weighted Directed Feedback Set [Kim et al., 2021]

Parameterization of Weighted Segment Set Cover

Theorem 3

Weighted Segment Set Cover is W[1]-hard and there does not exists an algorithm running in time $n^{o(\sqrt{k})}$.

This result is particularly interesting, because unweighted problem has an FPT algorithm.

How can we make this problem easier?

- allow less directions (maybe just two parallel to axes);
- allow δ-extension.

Parameterization of WEIGHTED SEGMENT SET COVER

Theorem 3 (rephrased)

WEIGHTED SEGMENT SET COVER is W[1]-hard and there does not exists an algorithm running in time $n^{o(\sqrt{k})}$ even when segments are limited to 3 directions.

Theorem 4

WEIGHTED SEGMENT SET COVER relaxed with δ-extension has an FPT algorithm.
It can be solved with algorithm running in time $\mathcal{O}((2 + k/\delta)^k \cdot n^{O(1)})$.

Technique intuition: Provide a kernel where we choose (k, δ)-**good** set of points of size at most $f(k, \delta)$ on each line with more than $k + 1$ points on them.
Subset of a set of collinear points S is (k, δ)-**good** if no matter how we cover it with k segments, then these segments after δ-extension cover all points from the original set of points S.

Results of parameterization of Segment Set Cover

	exact	δ-extensions
unweighted	FPT (Theorem 2)	FPT*
weighted 2 directions	???	FPT*
weighted 3 directions	W[1]-hard (Theorem 3)	FPT*
weighted any directions	W[1]-hard*	FPT (Theorem 4)

Theorem 3 (reminder)

WEIGHTED SEGMENT SET COVER is W[1]-hard and there does not exists an algorithm running in time $n^{o(\sqrt{k})}$ even when segments are limited to 3 directions.

Future work:

- prove Theorem 3 for axis-parallel segments (2 directions);
- remove the gap between the lower bound of complexity $n^{o(\sqrt{k})}$ and a simple algorithm running in time $n^{\mathcal{O}(k)}$.

The end.