

Approximation and parametrization of Segment Set Cover

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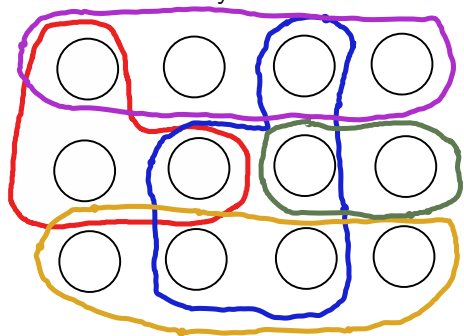
- 1 SET COVER
 - GEOMETRIC SET COVER
 - SEGMENT SET COVER
- 2 δ -extension
- 3 Approximation
- 4 Results for set cover with weighted segments

Problem Statement: SET COVER

Input: universe \mathcal{U} , family S of subsets of \mathcal{U}

Output: $C \subseteq S$ such that $|C|$ is minimal and $\bigcup C = \mathcal{U}$

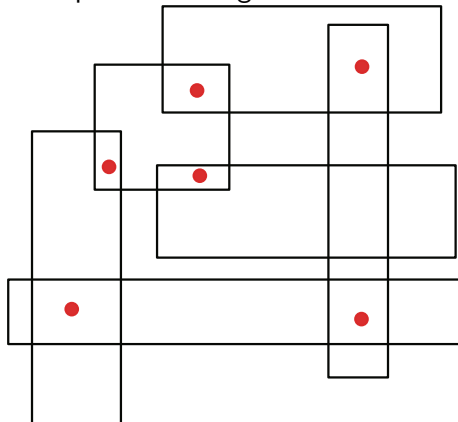
Minimal subfamily that covers all elements in \mathcal{U} .



Problem Statement: GEOMETRIC SET COVER

- Universe \mathcal{U} is set of points in the plane
- Sets S are some geometric shapes
- Formally, each set in S is intersection of \mathcal{U} with some geometric shape.

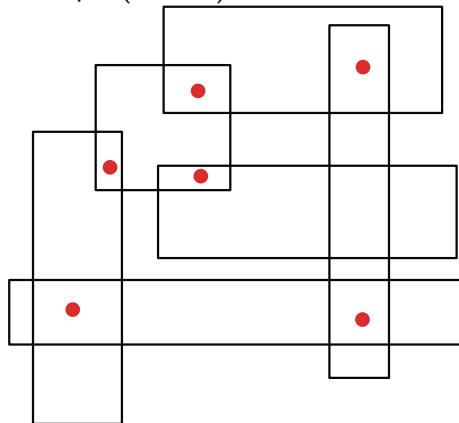
Example for rectangles:



Problem Statement: SEGMENT SET COVER

- Universe \mathcal{U} is set of points in the plane
- Sets S are segments.

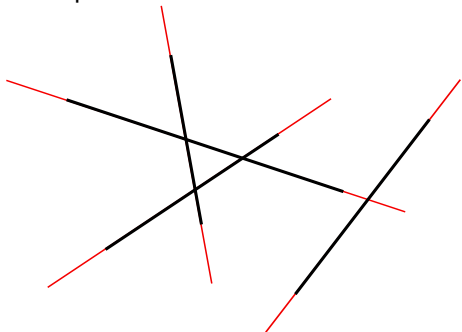
Example (TODO):



δ -extension for SEGMENT SET COVER

- δ -extension for a segment is a segment which is longer by δ fraction at both ends
- We accept solution in which segments cover solution after δ -extension
- The solution is compared to optimal solution without extension.

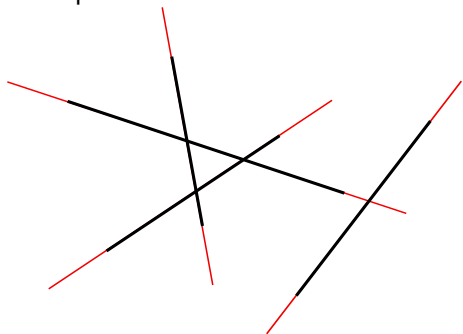
Example:



δ -extensions for SEGMENT SET COVER

- We are given universe \mathcal{U} and set of segments S
- We construct set $S^{+\delta}$ consisting of segments longer by δ fraction at both ends
- We look for $C \subseteq S^{+\delta}$ that $\bigcup C = \mathcal{U}$
- solution C is compared with optimal solution for S without extensions for both parametrized and approximation setting

Example:



Preliminaries: Approximation

Given:

- instance I of the optimization problem (looking for minimal solution)
- weight of the optimal solution $\text{opt}(I)$

p -approximation is algorithm that yields solution of weight not larger than $p \cdot \text{opt}(I)$

PTAS – polynomial time approximation scheme

For every $\epsilon > 0$, there exists $(1 + \epsilon)$ -approximation algorithm running in time $n^{f(\epsilon)}$ for any computable function f .

Example: KNAPSACK.

APX-hardness

For sufficiently small $\epsilon > 0$, if $(1 + \epsilon)$ -approximation exists, then $P = NP$.

Example: MAX-3-SAT (how many clauses in 3-SAT can be solved)

Approximation results for SET COVER from literature

SET COVER

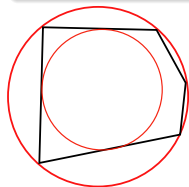
- $\log n$ -approximation, no $o(\log n)$ approximation assuming $P \neq NP$

GEOMETRIC SET COVER

- with fat rectangles is APX-hard
- (E?Q?)PTAS for FAT polygons with δ -ext (TODO reference)

Definition

Fatness of polygons τ is a ratio between a radius of circle circumscribed on this polygon and inscribed in this polygon.



PCP theorem

How do we prove that there doesn't exist $(1 + \epsilon)$ -approximation?

NP-completeness

Distinguishing if result of MAX-(3,3)-SAT is n or less than n is NP-complete.

Remark

Result of MAX-(3,3)-SAT is not less than $\frac{7}{8}n$

PCP theorem

Distinguishing if result of MAX-(3,3)-SAT is n or at most $(\frac{7}{8} + \epsilon)n$ cannot be done in polynomial time assuming $NP \not\subseteq DTIME(2^{O(\log n \log \log n)})$

Therefore MAX-(3,3)-SAT does not have $(\frac{7}{8} + \epsilon)$ -approximation

Approximation of SEGMENT SET COVER

Theorem 2

SEGMENT SET COVER is APX-hard, ie.

for small $\epsilon = 0.001$, doesn't have $(1 + \epsilon)$ -approximation unless $P = NP$.

How can we make this problem *easier*?

- Allow only segments in 2 directions (parallel to the two axes)
- Allow segments to *almost* cover the points, ie. be very close to them

Theorem 2 (rephrased)

SEGMENT SET COVER is APX-hard, even if segments are axis-parallel and problem is relaxed with $\frac{1}{2}$ -extension.

Parametrized algorithms vs. $W[1]$ -hardness

Given:

- parameter k (usually size of the solution)
- size of the instance n

Class	Complexity	Example
$W[1]$ -hard	$n^{O(k)}$	k -CLIQUE, GRID TILING
FPT	$f(k) \cdot n^{O(1)}$	VERTEX COVER

Parametrization of SET COVER

W[2]-complete (no algorithm with running time $n^{k-\epsilon}$)

Theorem 1

SEGMENT SET COVER parametrized by the solution size k can be solved in time $k^k \cdot n^{O(1)}$.

Technique: Branching over at most $k + 1$ segments on the lines with more than k points on them.

Approximation of set cover with segments

How do we prove that there doesn't exist $(1 + \epsilon)$ -approximation?

MAX-(3,3)-SAT problem

Given 3-SAT instance with n variables, n clauses and every variable appears in exactly 3 clauses, find an assignment that satisfies the maximum number of clauses (not necessarily all of them).

Approximation of set cover with segments

I initially started with rectangles problem. . .

Theorem 2.5 (Har-Peled also did that in 2012)

Set cover with polygons of bounded fatness (at least τ) with δ -extension has $(1 + \epsilon)$ -approximation for any $\epsilon > 0$.

Definition of set cover with weighted segments

Input: universe \mathcal{U} , subfamily S of subsets of \mathcal{U} and function of weights assigned to sets $w : S \rightarrow \mathbb{N}$

Output: $C \subseteq S$ such that $\sum_{c \in C} w(c)$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in \mathcal{U} .

In parametrized setting: We look for the *best* solution with restricted size $|C| \leq k$.

In approximation setting: It is still APX-hard, nothing interesting

Theorem 3

Weighted set cover with segments is $W[1]$ -hard.

Doesn't have FPT algorithm

How can we make this problem easier?

- Allow less directions (maybe just two parallel to axes)
- Allow δ -extension

Theorem 4

Weighted set cover with segments with δ -extensions is in FPT.
It can be solved with complexity $O((k/\delta)^k \cdot n^{O(1)})$.

Technique intuition: Provide kernel in problem where you need to cover both points and some segments (these are required by high density of points to cover on them)

Parametrization for set cover with weighted segments

Theorem 3 (rephrased)

Weighted set cover with segments in 3 directions is $W[1]$ -hard.
These directions are: parallel to axis and diagonal 45 degrees.
Doesn't have FPT algorithm

Nobody¹ knows if this still holds with 2 directions or in 3D with segments parallel to axis.

This is particularly interesting, because unweighted problem is in FPT.

Summary of results for parametrization of set cover with segments

Unweighted

	2 directions	arbitrarily many (at least 3)
exact	FPT	FPT (Theorem 1)
δ -extensions	FPT	FPT

Weighted

	2 directions	arbitrarily many (at least 3)
exact	???	W[1]-hard (Theorem 3)
δ -extensions	FPT	FPT (Theorem 4)

Summary of results for approximation of set cover with segments

Unweighted

	2 directions	any directions	Fat Polygons
exact	APX-hard	APX-hard	APX-hard
δ -extensions	APX-hard (Th. 2)	APX-hard	PTAS (Har-Peled)

Approximation of set cover with lines

We can reduce set cover with lines to Vertex Cover, so it's NP-hard and APX-hard.

Nobody² knows if there exist 2-approximation.