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# Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

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in COMPUTER SCIENCE

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
12 that it fulfils the requirements for the degree of Master of Computer Science.

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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

28

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## **Tytuł pracy w języku polskim**

36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów  
37 odcinkami na płaszczyźnie



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# Chapter 1

## Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja] We are given a family of sets and have to choose the smallest subfamily of these sets that cover all their elements. This problem naturally extends to settings where we put different weights on the sets and look for the subfamily of the minimal weight. This problem is NP-complete even without weights and if we put restrictions on what the sets can be. One of such variants is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric shapes and only some points of the plane have to be covered. When these shapes are rectangles with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of size  $k$  cannot be found in  $n^o(k)$  time), APX-complete (for sufficiently small  $\epsilon > 0$ , the problem does not admit  $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can be solved in polynomial time.

There is a notion of  $\delta$ -expansions, which loosen the restrictions on geometric set cover. We allow the objects to cover the points after  $\delta$ -expansion and compare the result to the original setting. This way we can produce both FPT and EPTAS for the rectangle set cover with  $\delta$ -extensions [referencje].

**Our contribution.** In this work, we prove that unweighted geometric set cover with segments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted axis-parallel segments, even with  $1/2$ -extensions. So the problem for very thin rectangles also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme (EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow  $\delta$ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.





## Chapter 2

## Definitions

### 2.1. Geometric Set Cover

In the geometric set cover problem we are given  $\mathcal{P}$  – a set of objects, which are connected subsets of the plane,  $\mathcal{C}$  – a set of points in the plane. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that every point in  $\mathcal{C}$  is inside some element from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized.

In the parametrized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$ .

In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$ , and we would like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

### 2.2. Approximation

Let us recall some definitions related to optimization problems that are used in the following sections.

**Definition 2.1.** A **polynomial-time approximation scheme (PTAS)** for a minimization problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$  and in polynomial time finds a solution that is within a factor  $(1 + \epsilon)$  of being optimal. That means the reported solution has weight at most  $(1 + \epsilon)\text{opt}(I)$ , where  $\text{opt}(I)$  is the weight of an optimal solution for  $I$ .

**Definition 2.2.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

### 2.3. $\delta$ -extensions

TODO PLACEHOLDER for introductory text

$\delta$ -extensions is one of the modifications to a problem, that makes geometric set cover problem easier, it has been already used in literature (place some refrence here).

**Definition 2.3** ( $\delta$ -extensions for center-symmetric objects). For any  $\delta > 0$  and a center-symmetric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  **$\delta$ -extension** of  $L$  is the object  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ , that is,  $L^{+\delta}$  is the image of  $L$  under homothety centered at  $S$  with scale  $(1 + \delta)$

The geometric set cover problem with  $\delta$ -extensions is a modified version of geometric set cover where:

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extensions);

- We look for a solution that is no larger than the optimum solution for the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 2.4** (Geometric set cover problem with  $\delta$ -extensions). The geometric set cover problem with  $\delta$ -extensions is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is no larger than the optimal solution for the problem without extensions, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

TODO: Some text

**Definition 2.5** (Geometric set cover PTAS with  $\delta$ -extensions). We define a PTAS for geometric set cover with  $\delta$ -extensions as a family of algorithms  $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$ , and in polynomial-time outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$  factor of the optimal solution for this problem without extensions, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## Chapter 3

# APX-completeness Geometric Set Cover

### 3.1. APX-completeness for segments parallel to axes

In this section we analyze whether there exists PTAS for geometric set cover for rectangles. We show that we can restrict this problem to a very simple setting: segments parallel to axes and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just degenerated rectangles with one side being very narrow.

Our results can be summarized in the following theorem and this section aims to prove it.

**Theorem 3.1.** *(axis-parallel segment set cover with  $1/2$ -extension is APX-hard). Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension) is APX-hard. That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

Theorem 3.1 implies the following.

**Corollary 3.1.** *(rectangle set cover is APX-hard). Unweighted geometric set cover with rectangles (even with  $1/2$ -extension) is APX-hard.*

We prove Theorem 3.1 by taking a problem that is APX-hard and showing a reduction. For this problem we choose MAX-(3,3)-SAT which we define below.

#### 3.1.1. MAX-(3,3)-SAT and statement of reduction

**Definition 3.1.** MAX-3SAT is the following maximization problem. We are given a 3-CNF formula, and need to find an assignment of variables that satisfies the most clauses.

**Definition 3.2.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction that every variable appears in exactly 3 clauses. Note that thus, the number of clauses is equal to the number of variables.

In our proof of Theorem 3.1 we use hardness of approximation of MAX-(3,3)-SAT proved in [Håstad, 2001] and described in Theorem 3.2 below.

**Definition 3.3** ( $\alpha$ -satisfiable MAX-3SAT formula). MAX-3SAT formula of size  $n$  is at most  $\alpha$ -satisfiable, if every assignment of variables satisfies no more than  $\alpha n$  clauses.

**Theorem 3.2.** [Håstad, 2001]

For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most  $(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.

Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel segment set cover problem, such that for a sufficiently small  $\epsilon > 0$ , a polynomial time  $(1 + \epsilon)$ -approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-(3,3)-SAT is fully satisfiable or is at most  $(7/8 + \epsilon)$ -satisfiable. However, according to (Theorem 3.2) the latter problem is NP-hard. This would imply  $P = NP$ , contradicting the assumption.

The following lemma encapsulates the properties of the reduction described in this section, and it allows us to prove Theorem 3.1.

**Lemma 3.1.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value  $opt(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in  $2D$ , such that:*

(1) *For every solution  $X$  of instance  $I$ , there exists a solution of  $S$  that satisfies at least  $15n - |X|$  clauses.*

(2) *For every solution of instance  $S$  that satisfies  $w$  clauses, there exists a solution of  $I$  of size  $15n - w$ .*

(3) *Every solution with  $1/2$ -extensions of  $I$  is also a solution to the original instance  $I$ .*

*Therefore, the optimum size of a solution of  $I$  is  $opt(I) = 15n - opt(S)$ .*

We prove Lemma 3.1 in subsequent sections, but meanwhile let us prove Theorem 3.1 using Lemma 3.1 and Theorem 3.2.

*Proof of Theorem 3.1.*

Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in  $2D$  with  $(1/2)$ -extensions. We construct an algorithm that solves the problem stated in Theorem 3.2, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $I$  using Lemma 3.1. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ . Denote the size of the solution returned by this algorithm as  $approx(I)$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $approx(I) \geq 15n - (\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $approx(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$opt(S) = n.$$

From Lemma 3.1 we have:

$$opt(I) = 14n.$$

Therefore,

$$\begin{aligned} approx(I) &\leq (1 + \epsilon)opt(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$\text{opt}(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3.1 we have:

$$\text{opt}(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

205 Since a solution to  $I$  with  $\frac{1}{2}$ -extensions is also a solution without extensions, by Lemma  
206 3.1 (3.), we have:

$$\text{approx}(I) \geq \text{opt}(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

207 Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to dis-  
208 tinguish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable,  
209 it suffices to compute  $\text{approx}(I)$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation  
210 algorithm cannot exist, unless  $P = NP$ .  $\square$

### 211 3.1.2. Reduction

212 We proceed to the proof of Lemma 3.1. That is, we show a reduction from MAX-(3,3)-SAT  
213 problem to geometric set cover with segments parallel to axis. Moreover, the obtained instance  
214 of geometric set cover will be robust to  $1/2$ -extensions (have the same optimal solution after  
215  $1/2$ -extension).

216 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and  
217 **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** con-  
218 nected together.

#### 219 3.1.2.1. VARIABLE-gadget

220 **VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It  
221 allows two minimum solutions of size 3 each. These two choices correspond to the two Boolean  
222 values of the variable corresponding to this gadget.

223 **Points.** Define points  $a, b, c, d, e, f, g, h$  as follows, where  $L = 12n$ :

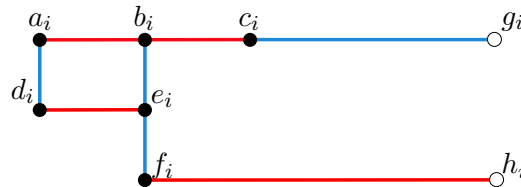


Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $\text{pointsVariable}_i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $\text{chooseVariable}_i^{\text{false}}$  and the set of blue segments as  $\text{chooseVariable}_i^{\text{true}}$ .

$$\begin{array}{llll}
a = (-L, 0) & b = (-\frac{2}{3}L, 0) & c = (-\frac{1}{3}L, 0) & d = (-L, 1) \\
e = (-\frac{2}{3}L, 1) & f = (-\frac{2}{3}L, 2) & g = (L, 0) & h = (L, 2)
\end{array}$$

Let us define:

$$\text{pointsVariable} = \{a, b, c, d, e, f\}$$

and

$$\text{pointsVariable}_i = \text{pointsVariable} + (0, 4i)$$

We denote  $a_i = a + (0, 4i)$  etc.

**Segments.** Let us define:

$$\text{chooseVariable}_i^{\text{true}} = \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\}$$

$$\text{chooseVariable}_i^{\text{false}} = \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\}$$

$$\text{segmentsVariable}_i = \text{chooseVariable}_i^{\text{true}} \cup \text{chooseVariable}_i^{\text{false}}$$

**Lemma 3.2.** For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can be covered using 3 segments from  $\text{segmentsVariable}_i$ .

*Proof.* We can use either set  $\text{chooseVariable}_i^{\text{true}}$  or  $\text{chooseVariable}_i^{\text{false}}$ .  $\square$

**Lemma 3.3.** For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments from  $\text{segmentsVariable}_i$ .

*Proof.* No segment of  $\text{segmentsVariable}_i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments.  $\square$

**Lemma 3.4.** For every set  $A \subseteq \text{segmentsVariable}_i$  such that  $A$  covers  $\text{pointsVariable}_i$  and  $(c_i, g_i), (f_i, h_i) \in A$ , it holds that  $|A| \geq 4$ .

*Proof.* No segment from  $\text{segmentsVariable}_i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  $\text{pointsVariable}_i - \{c_i, f_i, g_i, h_i\}$  can not be covered with fewer than 2 segments.  $\square$

### 3.1.2.2. OR-gadget

OR-segment connects input and output segments that are connected to other parts of constructions.

Output segment is part of OR-segment, but input is not.

For every solution  $\mathcal{R}$  of the whole construction. Define  $\mathcal{R}'$  as intersection of  $\mathcal{R}$  and the gadget segments. Minimum solution of OR-gadget has size  $w$ , i.e.  $|\mathcal{R}'| \leq w$ . *output* segments can be part of  $\mathcal{R}'$  only if *input<sub>x</sub>* or *input<sub>y</sub>* are part of the chosen solution  $\mathcal{R}$ . If none of them are chosen, then solution containing *output* segment has weight at least  $w + 1$ . Therefore the following formula holds:

$$\text{output} \in \mathcal{R}' \wedge |\mathcal{R}'| = w \Rightarrow (x \in \mathcal{R}) \vee (y \in \mathcal{R})$$

Only 3 points that belong to this segment:  $l_{i,j}, p_{i,j}, v_{i,j}$  can be covered by segment not from the OR-gadget.



Figure 3.2: **OR-gadget**. Figure presenting OR-gadget: segments from  $\text{chooseOr}_{i,j}^{false}$  are red, segments from  $\text{chooseOr}_{i,j}^{true}$  are blue, segments from  $\text{orMoveVariable}_{i,j}$  are yellow and green. Dark blue segment is an *output* segment. Grey segments  $input_x$  and  $input_y$  are input segments that are not part of  $\text{segmentsOr}_{i,j}$ .

249 **Points.**

$$\begin{array}{llll}
 l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\
 p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\
 t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) & 
 \end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

251 Define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $vec_{i,j}$

252 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$\text{pointsOr}_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

253 Note that  $\text{pointsOr}_{i,j}$  does not include point  $v_{i,j}$

254 **Segments.** We define names subsets of segments, to refer to them in lemmas.

$$\text{chooseOr}_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$\text{chooseOr}_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$\text{orMoveVariable}_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

255 Segments in OR-gadget:

$$\text{segmentsOr}_{i,j} = \text{chooseOr}_{i,j}^{false} \cup \text{chooseOr}_{i,j}^{true} \cup \text{orMoveVariable}_{i,j}$$

256 **Lemma 3.5.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$ , points in  $\text{pointsOr}_{i,j} - \{x\} \cup$   
 257  $\{v_{i,j}\}$  can be covered with 4 segments from  $\text{segmentsOr}_{i,j}$ .

258 *Proof.* We can do that using one segment from  $\text{orMoveVariable}_{i,j}$ , the one that does not cover  
 259  $x$ , and all segments from  $\text{chooseOr}_{i,j}^{true}$ .  $\square$

260 **Lemma 3.6.** For any  $1 \leq i \leq n, j \in \{0, 1\}$ , points in  $\text{pointsOr}_{i,j}$  can be covered with 4  
 261 segments from  $\text{segmentsOr}_{i,j}$ .

262 *Proof.* We can do that using segments from  $\text{orMoveVariable}_{i,j}$  and  $\text{chooseOr}_{i,j}^{\text{false}}$ .  $\square$

### 263 3.1.2.3. CLAUSE-gadget

264 CLAUSE-gadget is responsible for calculating if variables values assigned in variable gadgets  
 265 satisfy the respective clause in CNF. It has minimum solution of weight  $w$  if and only if the  
 266 clause is satisfied, i.e. at least one of the respective variables is assigned a correct value.  
 267 Otherwise it has minimum solution of weight  $w + 1$ . This way, by analyzing the minimum  
 268 solution for the whole problem, we can tell how many clauses were possible to satisfy in the  
 269 optimum solution of CNF.

270 The CLAUSE-gadgets consist of two OR-gadgets. It would be inconvenient to posi-  
 271 tion the CLAUSE-gadgets in between the very long variable segments. Instead, we use  
 272 a simple auxiliary gadget to *transfer* whether the segment is in a solution, i.e. segments  
 273  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ . Each gadget consists of two segments  $(x_{i,0}, x_{i,1}), (x_{i,1}, a)$ .  
 274 These are the only segments that can cover  $x_{i,1}$ . If  $x_{i,0}$  is already covered by some other  
 275 gadget, we can cover  $x_{i,1}$  by the other segment covering another point from the gadget, say  $a$ .  
 276 If  $x_{i,0}$  is not covered, then the only way to cover  $x_{i,0}$  is to use segment  $(x_{i,0}, x_{i,1})$ . Intuitively,  
 277 the two segments *transfer* the state of  $x_{i,0}$  onto  $a$ , but there are less restrictions on where  $a$   
 278 can be placed, simplifying the construction.



Figure 3.3: **CLAUSE-gadget.** This figure presents CLAUSE-gadget. Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the variables in the  $i$ -th clause. Grey segments corresponds to the values of variables satisfying the  $i$ -th clause.

279 **Points.** TODO: Rephrase it

280 Assuming clause  $C_i = a \vee b \vee c$ , function  $\text{idx}(w)$  returns index of the variable  $w$ , function

281  $\text{neg}(w)$  returns whether variable  $w$  is negated in a clause.



$$\begin{aligned}
x_{i,0} &= (10i + 1, 4 \cdot \text{idx}(a) + 2 \cdot \text{neg}(c)) & x_{i,1} &= (10i + 1, 4n) \\
y_{i,0} &= (10i + 2, 4 \cdot \text{idx}(b) + 2 \cdot \text{neg}(b)) & y_{i,1} &= (10i + 2, 4n + 4) \\
z_{i,0} &= (10i + 3, 4 \cdot \text{idx}(c) + 2 \cdot \text{neg}(c)) & z_{i,1} &= (10i + 3, 4n + 6)
\end{aligned}$$

$$\text{moveVariable}_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$\text{pointsClause}_i = \text{moveVariable}_i \cup \text{pointsOr}_{i,0} \cup \text{pointsOr}_{i,1} \cup \{v_{i,1}\}$$

**Segments.**

$$\begin{aligned}
\text{segmentsClause}_i &= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\
&\cup \text{segmentsOr}_{i,0} \cup \text{segmentsOr}_{i,1}
\end{aligned}$$

**Lemma 3.7.** For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , there is a  $\text{solClause}_i^{\text{true},a} \subset \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{true},a}| = 11$  that covers points in  $\text{pointsClause}_i - \{a\}$ .

*Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 3.5 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments  $\text{chooseOr}_{i,0}^{\text{true}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{i,0}$ : Using Lemma 3.6 and Lemma 3.5 with  $x = p_{i,1}$ , resulting with 8 segments  $\text{chooseOr}_{i,0}^{\text{false}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 3.8.** For any  $1 \leq i \leq n$  there is  $\text{solClause}_i^{\text{false}} \subset \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{false}}| = 12$  that covers points in  $\text{pointsClause}_i$ .

*Proof.* Using Lemma 3.6 twice we can cover  $\text{pointsOr}_{i,0}$  and  $\text{pointsOr}_{i,1}$  with 8 segments.

To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$   $\square$

**Lemma 3.9.** For any  $1 \leq i \leq n$ :

(1) points in  $\text{pointsClause}_i$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 12;

(2) points in  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 11.

*Proof of (1).* No segment in  $\text{segmentsClause}_i$  covers more than 2 points from  $\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0}, l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$ .

Therefore we need to use at least 12 segments.  $\square$

*Proof of (2).* We can choose disjoint sets  $X, Y, Z$  such that  $X \cup Y \cup Z \subseteq \text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  and there are no segments covering points from different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

No two points in  $X$  are covered with one segment of  $\text{segmentsClause}_i$ , so it must be covered with 3 different segments.

$$Y = \text{pointsOr}_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = \text{pointsOr}_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments of  $\text{segmentsClause}_i$  with brutforce that none of them cover the set of points, so both  $Y$  and  $Z$  have to be covered with disjointed sets of 4 segments.

TODO: Funny fact, neither  $Y$  nor  $Z$  doesn't have independent set of size 4.

Therefore  $\text{pointsClause}_i$  must be covered with at least  $3 + 4 + 4 = 11$  segments.  $\square$

#### 3.1.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

**Lemma 3.10. Robustness to 1/2-extensions.** *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+1/2}$  cover the same points from  $\mathcal{C}$ .*

*Proof.* We can just check every segment. Most of the segments  $s$  are collinear only with points that lay on  $s$ , so trivially  $s^{+1/2}$  cannot cover more points than  $s$  does.

TODO: list problematic segments here

In the same gadget:  $(n_{i,j}, p_{i,j})$  does not cover  $m_{i,j}$  and symmetrically.  $(t_{i,j}, v_{i,j})$  does not cover  $n_{i,j}$ .  $(o_{i,0}, u_{i,0})$  does not cover  $m_{i,1}$  and symmetrically.  $(y_{i,1}, p_{i,0})$  does not cover  $n_{i,j}$ .

From different gadgets:  $(b_i, f_i)$  after  $\frac{1}{2}$ -extensions does not cover  $b_{i+1}$  point.

VARIABLE-gadget's  $(a_i, c_i)$  after  $\frac{1}{2}$ -extensions does not cover any points  $x_{i,0}, y_{i,0}$  or  $z_{i,0}$  from CLAUSE-gadget.

$\square$

#### 3.1.2.5. Summary of construction

We define:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} \text{pointsVariable}_i \cup \text{pointsClause}_i$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} \text{segmentsVariable}_i \cup \text{segmentsClause}_i$$

The subsequent sections define these sets.

We prove some properties of different gadgets. Every segment for a gadget will only cover points in this gadget (won't interact with any different gadget), so we can prove lemmas *locally*.

TODO:  $y$  axis is increasing values downward on figures (not upwards like in normal).

#### 3.1.3. Construction lemmas and proof of Lemma 3.1

In order to prove Lemma 3.1 we introduce several auxiliary lemmas proving properties of the construction described in the previous section.

Consider an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution satisfying  $k$  clauses. Let us construct an instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover as described in Section 3.1.2 for instance  $S$  of MAX-(3,3)-SAT.

**Lemma 3.11.** *Instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover admits a solution of size  $15n - k$ .*



Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

*Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Let the assignment of the variables in the optimum solution to  $S$  be  $\phi : \{x_1, x_2 \dots x_n\} \rightarrow \{\text{true}, \text{false}\}$ .

We cover every VARIABLE-gadget with solution described in Lemma 3.2, in the  $i$ -th gadget choosing the set of segments corresponding to the value of  $\phi(x_i)$ .

For every clause that is satisfied, say  $c_i$ , let us name the variable that is **true** in it as  $x_i$  and point corresponding to  $x_i$  in  $\text{pointsClause}_i$  as  $a$ . Points in  $\text{pointsClause}_i$  are covered with set  $\text{solClause}_i^{\text{true}, a}$  described in Lemma 3.7. For every clause that is not satisfied, say  $c_j$ , points in  $\text{pointsClause}_j$  are covered with set  $\text{solClause}_j^{\text{false}}$  described in Lemma 3.8.

Formally we define sets responsible for choosing variable and satisfying the variable,  $R_i$  and  $C_i$  respectively, as following:

$$\begin{aligned} R_i &= \begin{cases} \text{chooseVariable}_i^{\text{true}} & \text{if } \phi(x_i) = \text{true} \\ \text{chooseVariable}_i^{\text{false}} & \text{if } \phi(x_i) = \text{false} \end{cases} \\ C_i &= \begin{cases} \text{solClause}_i^{\text{true}, a} & \text{if } c_i \text{ satisfied} \\ \text{solClause}_i^{\text{false}} & \text{if } c_i \text{ not satisfied} \end{cases} \\ \mathcal{R} &= \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}. \end{aligned}$$

This set covers all the points from  $\mathcal{C}$ , because the sets  $R_i, C_i$  individually cover their corresponding gadgets, as proved in the respective lemmas.

All of these sets are disjoint, so the size of the obtained solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k.$$

□

**Lemma 3.12.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover. Then there exists a solution  $\mathcal{R}'$ , such that  $|\mathcal{R}'| \leq |\mathcal{R}|$ , and for each VARIABLE-gadget  $\mathcal{R}'$  contains at most one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$ .*

*Proof.* Assume that we have  $\{(c_i, g_i), (f_i, h_i)\} \subseteq \mathcal{R}$  for some  $i$ . We will show how to modify  $\mathcal{R}$  into  $\mathcal{R}'$ , such that the number of such  $i$  decreases, while  $\mathcal{R}'$  is still a valid solution of  $(\mathcal{C}, \mathcal{P})$ , and  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Then, by repeating this procedure, we can eventually construct a solution satisfying the property from the Lemma.

To construct  $\mathcal{R}'$ , we remove either  $(c_i, g_i)$  or  $(f_i, h_i)$  from  $\mathcal{R}$ , and then add one extra segment to make  $\mathcal{R}'$  valid. Recall that the  $i$ -th VARIABLE-gadget corresponds to variable  $x_i$  in  $S$ . As every variable in  $S$  is used in exactly 3 clauses, one of the ways of setting  $x_i$  (to either **true** or **false**) must satisfy at least 2 clauses. If that setting is  $x_i = \text{true}$ , then we remove  $(f_i, h_i)$ , otherwise we remove  $(c_i, g_i)$ . Now, there exists at most one CLAUSE-gadget which needs adjustment to make  $\mathcal{R}'$  valid; we do that by adding  $(t_{j,1}, v_{j,1})$  to  $\mathcal{R}'$ .

TODO: Can we really just remove one segment and add another one? I'd think we need to "restructure"  $\mathcal{R}$  around  $\text{pointsVariable}_i$  (saying one segment due to Lemma 3.3 and Lemma 3.4) and then again restructure  $\mathcal{R}$  around the clause that we need to fix? □

**Lemma 3.13.** *Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover that is of size  $w$ . Then there exists a solution of  $S$  that satisfies at least  $15n - w$  clauses.*

372 *Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Given a solution  
 373  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover, we use Lemma 3.12 to modify  $\mathcal{R}$  such that  
 374 for any  $i$  it contains at most one of  $(c_i, g_i)$  and  $(f_i, h_i)$ ; this may decrease the cost of  $\mathcal{R}$ , but  
 375 that does not matter in the subsequent construction. To simplify notation, in the remainder  
 376 of this proof we use  $\mathcal{R}$  to refer to the modified solution.

377 Given  $\mathcal{R}$ , we construct a solution of  $S$  by constructing an assignment of variables  $\phi :$   
 378  $\{x_1, x_2 \dots x_n\} \rightarrow \{\text{true}, \text{false}\}$  that satisfies at least  $15n - w$  clauses in  $S$ .

379 **Variables** Recall that due to Lemma 3.12,  $\mathcal{R}$  contains at most one of  $(c_i, g_i)$  and  $(f_i, h_i)$ .  
 We define the value  $\phi(x_i)$  for the variable  $x_i$  as follows:

$$\begin{cases} \phi(x_i) = \text{true} & \text{if } (c_i, g_i) \in \mathcal{R} \\ \phi(x_i) = \text{false} & \text{if } (f_i, h_i) \in \mathcal{R} \\ \phi(x_i) = \text{false} & \text{otherwise} \end{cases} \quad (3.1)$$

380 Moreover, from Lemma 3.3 we get  $|\text{pointsVariable}_i \cap \mathcal{R}| \geq 3$  for every  $i$ .

381 **Clauses** For a clause  $C_i = x \vee y \vee z$ ,  $\mathcal{R}$  needs to use at least 11 segments to cover  
 382  $\text{pointsClause}_i - \{x, y, z\}$  in CLAUSE-gadget (Lemma 3.9).

383 TODO: maybe put something with cases and names of sets as above

384 Moreover, if all of the points  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  are not covered by the segments from  $\mathcal{R} \cap \text{pointsVariable}_i$ ,  
 385 then  $\mathcal{R}$  needs to cover  $\text{pointsClause}_i$  with at least 12 segments by Lemma 3.9.

TODO: Maybe remove section below, because we do this calculation at the end anyway  
 We covered CLAUSE-gadget with at least 11 or at least 12 segments:

$$|\bigcup_{i=1}^n \text{segmentsClause}_i \cap \mathcal{R}| \geq 11n + a$$

386 where  $a$  is the number of clauses where none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  were covered by  
 387  $\mathcal{R} \cap \text{segmentsVariable}_j$  for their respective variable  $x_j$ .

388 **Satisfied clauses with chosen variable assignment.** Consider a clause, say  $c_i$ . If  
 389 none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  were covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$ ,  
 390 this clause is not satisfied by assignment  $\phi$ .

391 If one of these points is covered by segments from VARIBALE-gadget (TODO better this  
 392 or  $\mathcal{R} \cap \text{segmentsVariable}_j$ ), then denote this point as  $t$  and say it corresponds to variable  $x_j$ .  
 393 Consider the cases of choosing value of  $\phi(x_j)$  in equation (3.1).

394 If  $\mathcal{R}$  contains exactly one of the segments  $(c_j, g_j)$  and  $(f_j, h_j)$ , then the value  $\phi(x_j)$  satisfies  
 395  $c_i$ .

396 If  $\mathcal{R}$  contains neither  $(c_j, g_j)$  nor  $(f_j, h_j)$ , then it is impossible that  $t$  is covered by segments  
 397 in  $\mathcal{R} \cap \text{segmentsVariable}_j$ .

398 This means that  $\phi$  satisfies all but at most  $a$  clauses in  $S$ .

399 To conclude, we proved that given a solution of  $(\mathcal{C}, \mathcal{P})$  of size  $w$ , we have constructed a  
 400 variables assignment  $\phi$  that satisfies at least  $n - a$  clauses of  $S$ . Finally, note that

$$w \geq 3n + 11(n - a) + 12a = 3n + 11n + a = 14n + a,$$

hence

$$15n - w \leq 15n - 14n - a = n - a.$$

401 So  $\phi$  satisfies at least  $15n - w$  clauses of  $S$ . □

402 We are ready to conclude the proof of Lemma 3.1.

*Proof of Lemma 3.1.* By Lemma 3.11, we know that there exists a solution to  $(\mathcal{C}, \mathcal{P})$  of size  $15n - k$ , so:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \leq 15n - k.$$

Since the optimum solution of  $S$  satisfies  $k$  clauses, then according to Lemma 3.13:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \geq 15n - k.$$

403 Therefore, the solution given by Lemma 3.11 of size  $15n - k$  is an optimum solution to the  
404 instance  $(\mathcal{C}, \mathcal{P})$ . □

## Chapter 4

# FPT for Geometric Set Cover for segments with $\delta$ -extensions

### 4.1. FPT for segments

In this section we consider the fixed-parameter tractable algorithms for unweighted geometric set cover with segments. Setting where segments are limited to be axis-parallel (or limited to constant number of directions) has an FPT algorithm already present in literature. We present an FPT algorithm for unweighted geometric set cover with segments, where segments are in arbitrary directions.

#### 4.1.1. Axis-parallel segments

You can find this in Platypus book. (TODO add referece)

We show an  $\mathcal{O}(2^k)$ -time branching algorithm. In each step, the algorithm selects a point  $a$  which is not yet covered, branches to choose one of the two directions, and greedily chooses a segment in that direction to cover  $a$ . This proceeds until either all points are covered or  $k$  segments are chosen.

Let us take the point  $a = (x_a, y_a)$  which is the smallest among points that are not yet covered in the lexicographic ordering of points in  $\mathbb{R}^2$ . We need to cover  $a$  with some of the remaining segments.

Branch over the choice of one of the coordinates ( $x$  or  $y$ ); without loss of generality, let us assume we chose  $x$ . Among the segments lying on line  $x = x_a$ , we greedily add to the solution the one that covers the most points. As  $a$  was the smallest in the lexicographical order, then all points on line  $x = x_a$  have the  $y$ -coordinate larger than  $y_a$ . Therefore, if we denote the greedily chosen segment as  $s$ , then any other segment on  $x = x_a$  that covers  $a$  can only cover a (possibly improper) subset of points covered by  $s$ . Thus, greedily choosing  $s$  is optimal.

In each step of the algorithm we add one segment to the solution, thus each branch can stop at depth  $k$ . If no branch finds a solution, then that means a solution of size at most  $k$  does not exist.

TODO: Maybe split it into theorem + algorithm + explanation like in section 4.1.2

**Remark 4.1.** *The same algorithm can be used for segments in  $d$  directions, where we branch over  $d$  directions and it runs in complexity  $\mathcal{O}(d^k)$ .*

### 4.1.2. Segments in arbitrary directions

In this section we consider setting where segments are not constrained to only  $d$  directions. We present a fixed-parameter tractable algorithm, where parameter is the size of the solution.

**Theorem 4.1. (*FPT for segment cover*).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $k^{O(k)} \cdot (nm)^2$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.*

We will need the following lemmas.

**Lemma 4.1.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, without a loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any distinct  $A, B \in \mathcal{P}$ , we have  $A \cap \mathcal{C} \not\subseteq B \cap \mathcal{C}$  and  $A \cap \mathcal{C} \not\supseteq B \cap \mathcal{C}$ .*

*Proof.* Trivial. □

**Lemma 4.2.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, if there exists a line  $L$  with at least  $k + 1$  points on it, then there exists a subset  $\mathcal{A} \subseteq \mathcal{P}$ ,  $|\mathcal{A}| \leq k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|\mathcal{A} \cap \mathcal{R}| \geq 1$ . Moreover, such a subset can be found in polynomial time.*

*Proof.* First we use Lemma 4.1.

Let us enumerate the points from  $\mathcal{C}$  that lie on  $L$  as  $x_1, x_2, \dots, x_t$  in the order in which they appear on  $L$ . Every segment that is not collinear with  $L$  can cover at most one of these points. Therefore, in any solution of size not larger than  $k$ , among any  $k$  of these points at least one must be covered with segment collinear with  $L$ .

Therefore, every solution needs to take one of the segments collinear with  $L$  that covers any of the points  $x_1, x_2, \dots, x_k$ . After using reduction from Lemma 4.1, there are at most  $k$  such segments that are distinct. □

We are ready to prove Theorem 4.1.

*Proof of Theorem 4.1.*

We will prove this theorem by presenting a branching algorithm that works in desired complexity. It branches over the choice of segments to cover lines with *a lot* of points, then finally solving the small instance, where every line has at most  $k$  points by checking all possible solutions.

**Algorithm.** First we use Lemma 4.1.

Next, we present a recursive algorithm. Given an instance of the problem:

- (1) If there exist a line with at least  $k + 1$  points from  $\mathcal{C}$ , we branch over adding to the solution one of the at most  $k$  possible segments provided by Lemma 4.2; name this segment  $S$ . Then we find a solution  $\mathcal{R}$  for the problem for points  $\mathcal{C} - S$ , segments  $\mathcal{P} - \{S\}$ , and parameter  $k - 1$ . We return  $\mathcal{R} \cup \{S\}$ .
- (2) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer NO.
- (3) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force: check all subsets of  $\mathcal{P}$  of size at most  $k$ .



473 **Correctness.** Lemma 4.2 proves that at least one segment that we branch over in (1)  
 474 must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ . Therefore, the recursive call can find a  
 475 solution, provided there exists one.

476 In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which  
 477 implies the same about every segment from  $\mathcal{P}$ . Under this assumption we can cover only  $k^2$   
 478 points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

479 Checking all possible solutions in (3) is trivially correct.

480 **Complexity.** In the leaves of recursion we have  $|\mathcal{C}| \leq k^2$ , so  $|\mathcal{P}| \leq k^4$ , because every  
 481 segments can be uniquely identified by the two extreme points it covers (by Lemma 4.1).  
 482 Therefore, there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $O(k|\mathcal{C}|)$ .  
 483 Therefore, (3) takes time  $k^{O(k)}$ .

484 In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we  
 485 have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch  
 486 over can be found on each level in time  $O((nm)^2)$ .

487 Reduction from Lemma 4.1 can be implemented in time  $O(n^2m)$ .

488 It follows that the overall complexity is  $O((nm)^2 \cdot k^{O(k)})$  □

## 489 4.2. FPT for weighted segments with $\delta$ -extensions

490 In this section we consider a geometric set cover problem for weighted segments relaxed with  
 491  $\delta$ -extensions. We show that this problem admits an FPT algorithm when parametrized with  
 492 size of solution and  $\delta$ . Assumption about the problem being relaxed with  $\delta$ -extensions is  
 493 relevant. In next chapter we show that geometric set cover problem for weighted segments is  
 494 W[1]-hard, that means there does not exist an FPT algorithm parametrized by solution size  
 495 for it.

496 **Theorem 4.2** (FPT for weighted segment cover with  $\delta$ -extensions). *There exists an algorithm*  
 497 *that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$  points  $\mathcal{C}$ , and*  
 498 *parameters  $k$  and  $\delta > 0$ , runs in time  $f(k, \delta) \cdot (nm)^c$  for some computable function  $f$  and a*  
 499 *constant  $c$ , and outputs a set  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ , or*  
 500 *determines that such a set  $\mathcal{R}$  does not exist.*

501 To solve this problem we will introduce a lemma about choosing a *good* subset of points.  
 502 Good subset of points for a set of collinear points  $C$  and parameters  $k$  and  $\delta$  is a subset of  
 503  $C$ , such that if we cover good subset of  $C$  with at most  $k$  segments, these segments after  
 504  $\delta$ -extensions will cover all of the points from  $C$ .

505 Such set of size bounded by some function  $f(k, \delta)$  always exists. In later part of the section  
 506 this lemma will allow us to find a kernel for our original problem.

507 **Definition 4.1.** For a set of collinear points  $C$ , a subset  $A \subseteq C$  is  $(k, \delta)$ -**good** if for any set  
 508 of segments  $R$  that covers  $A$  and such that  $|R| \leq k$ , it holds that  $R^{+\delta}$  covers  $C$ .

509 **Lemma 4.3.** *There exists an algorithm that for any set of collinear points  $C$ ,  $\delta > 0$  and*  
 510  *$k \geq 1$ , outputs a  $(k, \delta)$ -good set  $A \subseteq C$  of size at most  $(2 + \frac{2}{\delta})^k$ . Moreover it can be computed*  
 511 *in time  $O(|C| \cdot (2 + \frac{2}{\delta}))$ .*

512 To prove this theorem we also define *extreme points* for a set of collinear points.

513 **Definition 4.2.** For a set of collinear points  $C$ , **extreme points** are the ends of the smallest  
 514 segment that covers all points from set  $C$ .

515 If  $C$  consists of one point or is empty, then there exists 1 or 0 extreme points respectively.

516 *Proof of Lemma 4.3.* We prove this for a fixed  $\delta$  by induction over  $k$ .

517 **Inductive hypothesis.** For any set of collinear points  $C$ , there exists a set  $A$  such that:

- 518 •  $A$  is subset of  $C$ ,
- 519 •  $A$  is  $(\ell, \delta)$ -good for every  $1 \leq \ell \leq k$ ,
- 520 •  $|A| \leq (2 + \frac{2}{\delta})^k$ ,
- 521 • extreme points from  $C$  are in  $A$ .

522 **Base case for  $k = 1$ .** It is sufficient that  $A$  consists of extreme points of  $C$ .

523 If they are covered with one segment, it must be a segment that includes the extreme  
524 points from  $C$ , so it covers the whole set  $C$ .

525 There are at most 2 extreme points in  $C$  and  $2 < 2 + \frac{2}{\delta}$ .

526 **Inductive step.** Assuming inductive hypothesis for any set of collinear points  $C$  and  
527 for parameter  $k$ , we will prove hypothesis for  $k + 1$ .

528 Let  $s$  be the minimal segment that includes all points from  $C$ . That is, the extreme points  
529 of  $C$  are endpoints of  $s$ .

530 We define  $M = \lceil 1 + \frac{2}{\delta} \rceil$  subsegments of  $s$  in the following way. We split  $s$  into  $M$  closed  
531 segments of equal length. We name these as segments  $v_i$  and  $|v_i| = \frac{|s|}{M}$  for each  $1 \leq i \leq M$ .

532 Let  $C_i$  be the subset of  $C$  consisting of points laying on  $v_i$ .

533 Let  $t_i$  be the segment with endpoints being the extreme points of  $C_i$ . It might be degen-  
534 erated segment if  $C_i$  consists of one point or empty if  $C_i$  is empty.

535 TODO: Add a picture with  $v_i$  and  $t_i$  here

536 We use the inductive hypothesis to choose  $(k, \delta)$ -good sets  $A_i$  for sets  $C_i$ . Note that if  
537  $|C_i| \leq 1$ , then  $A_i = C_i$  and it's still a  $(k, \delta)$ -good set for  $C_i$ .

538 Then we define  $A = \bigcup_{i=1}^M A_i$ . Thus  $A$  includes the extreme points of  $C$ , because they are  
539 included in the sets  $A_1$  and  $A_M$ .

Size of each  $A_i$  is at most  $(2 + \frac{2}{\delta})(k - 1)$  from inductive hypothesis, therefore size of  $A$  is  
at most:

$$M(2 + \frac{2}{\delta})^{(k-1)} = \lceil 1 + \frac{2}{\delta} \rceil \cdot (2 + \frac{2}{\delta})^{(k-1)} \leq (2 + \frac{2}{\delta})^k$$

540 **Proof that  $A$  is  $(k, \delta)$ -good for  $C$ .** Let us take any cover of  $A$  with  $k + 1$  segments  
541 and call it  $\mathcal{R}$ .

542 For every segment  $t_i$ , if there exists a segment  $x$  in  $\mathcal{R}$  that is disjoint with  $t_i$ , then we have  
543 a cover of  $A_i$  with at most  $k$  segments using  $\mathcal{R} - \{x\}$ . Since  $A_i$  is  $(k, \delta)$ -good for  $t_i$  and  $C_i$ ,  
544 then  $(\mathcal{R} - \{x\})^{+\delta}$  covers  $C_i$ . So  $\mathcal{R}^{+\delta}$  covers  $C_i$  as well.

545 If there exists a segment  $t_i$  for which a segment  $x$  as defined above does not exist, then all  
546  $k + 1$  segments that cover  $A_i$  intersect with  $t_i$ . Note that there may exist only one such segment  
547  $t_i$ . From the inductive hypothesis endpoints of  $s$  are in  $A_1$  and  $A_M$  respectively, so  $\mathcal{R}$  must  
548 cover them. For each endpoint of  $s$ , there exists a segment that starts in this endpoint and ends  
549 somewhere in  $t_i$ . Let us call these two segments  $y$  and  $z$ . It follows that:  $|y| + |z| + |t_i| \geq |s|$ .  
550 Since  $|t_i| \leq |v_i| = \frac{|s|}{M} \leq \frac{|s|}{1 + \frac{2}{\delta}} = \frac{|s|\delta}{\delta + 2}$ , we have  $\max(|y|, |z|) \geq |s|(1 - \frac{\delta}{\delta + 2})/2 = \frac{|s|}{\delta + 2}$ .

551 TODO: Add a picture with such segments here

After  $\delta$ -extension, the longer of these segments will expand at both ends by at least:

$$\frac{|s|\delta}{\delta + 2} = \frac{|s|}{1 + \frac{2}{\delta}} \geq \frac{|s|}{M} = v_i \geq t_i.$$

Therefore the longer of segments  $y$  and  $z$  will cover the whole segment  $t_i$  after  $\delta$ -extension.

We conclude that  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

Since  $C = \bigcup_{i=1}^M C_i$ , it follows that  $\mathcal{R}^{+\delta}$  covers  $C$ .

**Algorithm.** We can simulate the inductive proof by a recursive algorithm.

TODO: Maybe just say "it's easy to implement"?

We apply recursion for each subset  $C_i$ . Every point from  $C$  belongs to at most 2 sets  $C_i$ .

Apart from recursive algorithm we perform operations linear in  $|C| + M$  to calculate the sets  $C_i$ .

Therefore the algorithm has complexity:

$$O(|C| + M) + \sum_i^M O(|C_i|k(1 + \frac{1}{\delta})) = O(|C| + (1 + \frac{1}{\delta})) + O((\sum_i^M |C_i|)k(1 + \frac{1}{\delta})) \leq O(|C|k(1 + \frac{1}{\delta})).$$

□

Let us now formulate some claims about the properties for the problem parametrized by the solution size. These properties provide bounds for different objects in the problem instance, that help us to find small kernel of the problem or claim that the minimal solution of this instance must be above some threshold.

**Definition 4.3.** A line in  $\mathbb{R}$  is **long** if there are at least  $k + 1$  points from  $\mathcal{C}$  on it.

**Claim 4.1.** *If there are more than  $k$  different long lines, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

*Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k + 1$  different long lines in our instance of the problem and solution  $\mathcal{R}$  of size at most  $k$  covering points  $\mathcal{C}$ .

Choose any long line  $L$ . Every segment from  $\mathcal{R}$ , which is not collinear with  $L$ , covers at most one point that lies on  $L$ .

$L$  is long, so there are at least  $k + 1$  points from  $\mathcal{C}$  that lie on  $L$ . That implies that there must be a segment in  $\mathcal{R}$  that is collinear with  $L$ .

Since we have at least  $k + 1$  different long lines, then there are at least  $k + 1$  segments in  $\mathcal{R}$  collinear with different lines. It contradicts with the assumption that  $|\mathcal{R}| \leq k$ . □

**Claim 4.2.** *If there are more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

*Proof.* We prove the claim by contradiction. Let us assume that we have at least  $k^2 + 1$  points from  $\mathcal{C}$  that do not lie on any long line, call this set  $A$ , and a solution  $\mathcal{R}$  of size at most  $k$  covering points  $\mathcal{C}$ .

For every segment  $s$  from  $\mathcal{R}$  it covers at most  $k$  points from  $A$ . It holds because if  $s$  covered at least  $k + 1$  points from  $A$ , then the line in the direction of  $s$  would be a long line and that contradicts of definition of  $A$ .

If every segment from  $\mathcal{R}$  covers at most  $k$  points from  $A$  and  $|\mathcal{R}| \leq k$ , then at most  $k^2$  points from  $A$  are covered by  $\mathcal{R}$  and that contradicts the fact that  $\mathcal{R}$  is a solution of given geometric set cover instance. □

We are now ready to give a proof of Theorem 4.2 algorithm.

*Proof of Theorem 4.2.* Applying the claims 4.1 and 4.2, if we have more than  $k$  different long lines or more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then we answer that there is no solution of size at most  $k$ .

Otherwise, we can split  $\mathcal{C}$  into at most  $k + 1$  sets:

- $D$ , at most  $k^2$  points that do not lie on any long line;
- $C_i$ , points that lie on  $i$ -th long line.

Sets  $C_i$  do not need to be disjoint.

Then for every set  $C_i$  we can use Lemma 4.3 to obtain a  $(k, \delta)$ -good set  $A_i$  for  $C_i$  with  $|A_i| \leq (2 + \frac{2}{\delta})^k$ .

Then we have a set  $\mathcal{C}' = D \cup \bigcup A_i$  of size at most  $f(k, \delta)$  for some computable function  $f$ . Observe that if we have a solution  $\mathcal{R}$  of size at most  $k$  that covers  $\mathcal{C}'$ , then  $\mathcal{R}^{+\delta}$  covers  $\mathcal{C}$ .

$\mathcal{C}$  is separated into several parts – sets  $D$  and  $C_i$ . Points from  $D$  are covered by  $\mathcal{R}$ , because  $D$  is part of  $\mathcal{C}'$ . Each  $A_i$  is covered, because  $A_i$  is part of  $\mathcal{C}'$ ;  $A_i$  is a  $(k, \delta)$ -good set for  $C_i$ , therefore  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

TODO: Rewrite After that we shrunk down  $\mathcal{C}$  to  $\mathcal{C}'$  of size  $f(k, \delta)$  for some computable function  $f$ . Then we would like to shrink down  $\mathcal{P}$  to some set of interesting segments of bounded size as well.

For every pair of points  $\mathcal{C}'$ , we can choose one segment from  $\mathcal{P}$  that have the lowest weight among segments that cover these points or decide there is no segment that cover them. Call this set  $\mathcal{P}'$  and name these segments **interesting**. There are at most  $|\mathcal{C}'|^2$  different segments in  $\mathcal{P}'$ .

We need to show that when we cover  $\mathcal{C}'$  with segments from  $\mathcal{P}'$  we achieve the same minimal solution as when we cover them with segments from  $\mathcal{P}$ . In order to prove this, consider a minimal solution  $\mathcal{R}$  that covers  $\mathcal{C}'$  with segments from  $\mathcal{P}'$  and take any segment  $s$  from  $\mathcal{R}$ . Let us look at the points from  $\mathcal{C}'$  that lie on  $s$  and call this set of points  $F$ .  $F$  is a set of collinear points for course. We can cover  $F$  with any segment that covers extreme points of  $F$ , because all other points lay on the segment between these points. Therefore we can change  $s$  to an interesting segment  $s'$  and interesting segments are defined in such a way, that  $s'$  has weight no larger than weight of  $s$ .

This has complexity  $O(|\mathcal{C}'|^2|\mathcal{P}|)$  and produce shrunk down set of segments  $\mathcal{P}'$  of size  $f(k, \delta)$  for some computable function  $f$ .

Then we can iterate over all subsets of  $\mathcal{P}'$  and choose the set with the lowest sum of weights that cover  $\mathcal{C}'$ . This solution would have weight not larger than optimal solution for the problem without extension, because we iterate over all possibilities of covering the subset of  $\mathcal{C}'$ .  $\square$

## Chapter 5

# W[1]-completeness for weighted segments in 3 directions

TODO: some introduction

**Definition 5.1.** Line is **right-diagonal** if it is described by linear function  $y = x + d$  for any  $d \in \mathbb{R}$ . Segment is **right-diagonal** if its direction is a right-diagonal line.

**Theorem 5.1.** *Consider the problem of covering a set  $\mathcal{C}$  of points by selecting at most  $k$  segments from a set of segments  $\mathcal{P}$  with non-negative weights  $w : \mathcal{P} \rightarrow \mathbb{R}^*$  so that the weight of the cover is minimal. Then this problem is W[1]-hard parameterized by  $k$  and assuming ETH, there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ . Moreover, this holds even if all segments in  $\mathcal{P}$  are axis-parallel or right-diagonal.*

Theorem 5.1 is also true for less restricted problem where segments have any direction. We prove more tight setting in this section.

In order to prove Theorem 5.1 we will show reduction from an W[1]-hard problem. We introduce grid tiling problem, which is proven to be W[1]-complete in literature.

**Definition 5.2.** In the **grid tiling** problem we are given integers  $n$  and  $k$ , and a function  $f : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \wp(\{1 \dots n\} \times \{1 \dots n\})$  specifying the set of allowed tiles for each cell of a  $k \times k$  grid. The task is to find functions  $x, y : \{1 \dots k\} \rightarrow \{1 \dots n\}$  that assign numbers from  $\{1 \dots n\}$  to respectively columns and rows of the grid, so that  $(x(i), y(j)) \in f(i, j)$  for all valid  $i$  and  $j$ , or conclude that such an assignment does not exist.

The next theorem is describing complexity of grid tiling problem.

**Theorem 5.2.** *Grid tiling is W[1]-hard parametrized by  $k$  and assuming ETH, there is no  $f(k) \cdot n^{o(\sqrt{k})}$ -time algorithm solving this problem for any computable function  $f$ .*

TODO: proof from reference in literature (platus book page 490)

The reminder of this section is proving Theorem 5.1 by reduction of grid tiling problem to geometric set cover. That would prove that geometric set cover is W[1]-hard, because if we could solve it with an FPT algorithm, then we could also solve grid tiling problem (which we reduced to geometric set cover). Therefore geometric set cover with setting described in Theorem 5.1 is at least as hard as grid tiling problem.

We start with an instance of the grid tiling problem  $(n, k, f)$  size of the grid  $k$ , number of colors  $n$  and function of allowed tiles  $f : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \wp(\{1 \dots n\} \times \{1 \dots n\})$ .

TODO: nice picture of instance of grid tiling with solution

656 **Construction.** We construct an instance  $(\mathcal{C}, \mathcal{P}, w)$  of geometric set cover as follows.  
 657 First let us choose any bijection  $order : \{1, \dots, n^2\} \rightarrow \{1, \dots, n\} \times \{1, \dots, n\}$ .  
 658 Define  $match_v(i, j)$  and  $match_h(i, j)$  as functions denoting whether two points share x or  
 659 y coordinate.

$$match_v(i, j) \iff order(i) = \{x_i, y_i\} \wedge order(j) = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff order(i) = \{x_i, y_i\} \wedge order(j) = \{x_j, y_j\} \wedge y_i = y_j$$

**Points.** Define points:

$$h_{i,j,t} = (i \cdot (n^2 + 1) + t, j \cdot (n^2 + 1))$$

$$v_{i,j,t} = (i \cdot (n^2 + 1), j \cdot (n^2 + 1) + t)$$

Let's define sets  $H$  and  $V$  as:

$$H = \{h_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j \leq k, 1 \leq t \leq n^2\}$$

Let us define  $\epsilon = \frac{1}{2k^2}$ . For a point  $p = \{x, y\}$  we define points:

$$p^L = \{x - \epsilon, y\},$$

$$p^R = \{x + \epsilon, y\},$$

$$p^U = \{x, y + \epsilon\},$$

$$p^D = \{x, y - \epsilon\}.$$

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

660 **Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^U, v_{i,j+1,t_2}^D)$$

$$horBeg_{i,t} = (h_{1,i,1}^L, h_{1,i,t}^L)$$

$$horEnd_{i,t} = (h_{k,i,t}^R, h_{k,i,n^2}^R)$$

$$verBeg_{i,t} = (v_{i,1,1}^D, v_{i,1,t}^D)$$

$$verEnd_{i,t} = (v_{i,k,t}^U, v_{i,k,n^2}^U)$$

661 Next we define sets of vertical and horizontal segments:

$$\begin{aligned} \text{HOR} &= \{hor_{i,j,t_1,t_2} : 1 \leq i < k, 1 \leq j \leq k, 1 \leq t_1, t_2 \leq n^2, match_h(t_1, t_2) \text{ holds}\} \\ &\cup \{horBeg_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{horEnd_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \end{aligned}$$

$$\begin{aligned}
\text{VER} &= \{\text{ver}_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, \text{match}_v(t_1, t_2)\} \\
&\cup \{\text{verBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\
&\cup \{\text{verEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}
\end{aligned}$$

Finally we also define set of right-diagonal segments:

$$\text{DIAG} := \{(h_{i,j,t}, v_{i,j,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, \text{order}(t) \in f(i, j)\}$$

662 TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := \text{HOR} \cup \text{VER} \cup \text{DIAG}$$

663 The weight of each segment in  $\text{HOR} \cup \text{VER}$ , while every segment in  $\text{DIAG}$  has weight  
664  $\delta = \frac{1}{4k^4}$ .

665 TODO: Put a picture of small instance like 3x3 with  $n=2$

$$w(s) = \begin{cases} \text{length}(s) & \text{if } s \in \text{HOR} \cup \text{VER} \\ \delta & \text{if } s \in \text{DIAG} \end{cases}$$

666 **Lemma 5.1.** *If there exists a solution for the grid tiling instance  $f_{i,j}$ , then there exists a*  
667 *solution for the instance  $(\mathcal{C}, \mathcal{P}, w)$  of geometric set cover with weight  $2k^2(n^2 + 1) - 4k^2\epsilon -$*   
668  *$4k(1 - \epsilon) + k^2\delta$ .*

*Proof.* Suppose there exists a solution  $x, y$  to the grid tiling problem. Then consider a solution that covers all points

$$\{h_{i,j,t} : 1 \leq i, j \leq k, \text{order}(t) = (x(i), y(j))\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, \text{order}(t) = (x(i), y(j))\}$$

669 with  $k^2$  segments from  $\text{DIAG}$  and the rest in  $\text{VER}$  or  $\text{HOR}$ . This solution has weight  
670  $2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$ .  $\square$

671 **Claim 5.1.** *Points in  $\{p^L : p \in H\} \cup \{p^R : p \in H\}$  have to be covered with segments from*  
672  *$\text{HOR}$ .*

673 *Points in  $\{p^U : p \in V\} \cup \{p^D : p \in V\}$  have to be covered with segments from  $\text{VER}$ .*

674 **Claim 5.2.** *For given  $1 \leq i, j \leq n$  and any solution of an instance  $(\mathcal{C}, \mathcal{P}, w)$  no two points*  
675  *$h_{i,j,t_1}, h_{i,j,t_2}$  ( $v_{i,j,t_1}, v_{i,j,t_2}$ ) for  $1 \leq t_1 < t_2 \leq n^2$  can be not covered with segments from  $\text{HOR}$*   
676 *( $\text{VER}$ ).*

677 *Proof.* Proof for horizontal segments. Proof for vertical is analogous.

678 Assume point  $h_{i,j,t_1}$  is not covered with segments from  $\text{HOR}$ . Point  $h_{i,j,t_1}^R$  has to be covered  
679 with  $\text{HOR}$  from Claim 5.1. And every segment in  $\text{HOR}$  covering  $h_{i,j,t_1}^R$ , but not  $h_{i,j,t_1}$  covers  
680 also  $h_{i,j,t_2}$ .  $\square$

681 **Lemma 5.2.** *For a constructed instance  $(\mathcal{C}, \mathcal{P}, w)$  any solution has to have sum of weights*  
682 *from sets  $\text{HOR}$  and  $\text{VER}$  at least  $W_{hv} = 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$ .*

683 *Proof.* We know that for every  $1 \leq i, j \leq k$  only one there exists only one  $t_x$  and  $t_y$  such that  
684  $v_{i,j,t_x}$  and  $h_{i,j,t_y}$  can be not covered by segments from  $\text{HOR}$  and  $\text{VER}$  (Claim 5.2),

685 We sum the lower bound for sum of length for horizontal/vertical lines for a single vertical  
686 line (the bound is the same for every horizontal line).

687 Let us fix  $1 \leq i \leq k$ .

(1) Length between  $v_{i,1,1}^D$  and  $v_{i,k,n^2}^U$  is:

$$(k(n^2 + 1) + n^2 + \epsilon) - ((n^2 + 1) + 1 - \epsilon) = k(n^2 + 1) - 2(1 - \epsilon).$$

(2) For every  $1 \leq j \leq k$  there exists at most one  $1 \leq t \leq n^2$  such that  $v_{i,j,t}$  is not covered by segments from **VER** (Claim 5.2). Its guards  $v_{i,j,t}^U$  and  $v_{i,j,t}^D$  have to be covered in **VER** (Claim 5.1). Therefore at most  $k$  spaces of length  $2\epsilon$  can be left not covered by segments from **VER**.

Therefore sum of these lower bounds for vertical and horizontal lines are:

$$2k(k(n^2 + 1) - 2k\epsilon - 2(1 - \epsilon)) = 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon)$$

692

□

**Lemma 5.3.** For a constructed instance  $(\mathcal{C}, \mathcal{P}, w)$  for any solution  $\mathcal{R}$  with weight equal to  $2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$ , for every  $1 \leq i, j \leq k$  there exists such  $1 \leq t \leq n^2$  that:

(1)  $v_{i,j,t}, h_{i,j,t}$  are not covered by segments from **VER** or **HOR**;

(2) segment  $(v_{i,j,t}, h_{i,j,t})$  is in solution  $\mathcal{R}$ ;

(3)  $order(t) \in f(i, j)$ , ie. it is an allowed tile for  $(i, j)$ ;

(4) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $v_{i,j,s}$  is covered in **VER**;

(5) for every  $1 \leq s \leq n^2$ ,  $s \neq t$ ,  $h_{i,j,s}$  is covered in **HOR**.

Name the function of this  $t$  as diagonal :  $\{1 \dots k\} \times \{1 \dots k\} \rightarrow \{1 \dots n^2\}$ .

*Proof.* At most one  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  points are covered with **DIAG** (Claim 5.2).

Exactly one  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  points are covered with **DIAG**, because if one of them were not, then we would use too much weight

$$W_{hv} + 2\epsilon > 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$$

This points are covered with the same segment from **DIAG**, because we need to use at least  $k^2$  of them to use exactly one **DIAG** segment for every pair of  $1 \leq i, j \leq k$ , if we used 2 segments from **DIAG** for one pair  $(i, j)$ , then we would have used too much weight by  $\delta$ . Since these points  $h_{i,j,t_x}$  and  $v_{i,j,t_y}$  are covered by segment from **DIAG**, therefore  $t_x = t_y$ .

Therefore  $diagonal(i, j) = t_x = t_y$  and  $order(t_x)$  is an allowed tile for  $(i, j)$  because the respective segment is in **DIAG**.

708

□

**Lemma 5.4.** For a constructed instance  $(\mathcal{C}, \mathcal{P}, w)$  for any solution of weight  $2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$  it holds that diagonal function from Lemma 5.3:

1. for any  $1 \leq i < k, 1 \leq j \leq k$ ,  $match_h(diagonal(i, j), diagonal(i + 1, j))$  must be true;

2. for any  $1 \leq i \leq k, 1 \leq j < k$ ,  $match_v(diagonal(i, j), diagonal(i, j + 1))$  must be true.



713 *Proof.* Every space between points in  $H$  and  $V$  covered by HOR/VER has to be covered by  
 714 only one segment, because otherwise it would use  $W_{hv} + \epsilon > 2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$ ,  
 715 therefore such solution would be too costly.

716 Proof for vertical (2), proof for horizontal is analogous.

717 Let us take any  $1 \leq i < k, 1 \leq j \leq k$  and name  $t_1 = \text{diagonal}(i, j)$  and  $t_2 = \text{diagonal}(i +$   
 718  $1, j)$ . Therefore  $h_{i,j,t_1}$  and  $h_{i+1,j,t_2}$  are not covered by segments from HOR,  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$   
 719 have to be covered by segments from HOR (Claim 5.1). Every segment from HOR starts  
 720 at  $h_{x,y,z_1}^R$  segment and finishes at  $h_{x,y+1,z_2}^L$  segment for some  $1 \leq x \leq k, 1 \leq z < k$  and  
 721  $1 \leq z_1, z_2 \leq n^2$ . Since all of the points between  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  are covered by segments  
 722 in HOR, and there are two different segments covering these points, one of them must begin  
 723 at  $h_{i,j,t_1}^R$  and end at  $h_{i,j+1,z_2}^L$  and the other one begin at  $h_{i,j,z_1}^R$  and end at  $h_{i+1,j,t_2}^L$  for some  
 724  $1 \leq z_1, z_2 \leq n^2$ . Therefore space between  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  is covered twice and is longer  
 725 than  $\epsilon$ . By Lemma 5.2 lower bound for weight of such solution is  $W_{hv} + \epsilon$  which contradicts  
 726 that the given solution has size  $2k^2(n^2 + 1) - 4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta < W_{hv} + \epsilon$  Therefore  
 727  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  must be covered by one segment from HOR and these points are ends of  
 728 this segment.

729  $h_{i,j,t_1}^R$  and  $h_{i+1,j,t_2}^L$  are ends of a segment from HOR, therefore  $\text{match}_h(t_1, t_2)$  must be  
 730 true.  $\square$

731 **Corollary 5.1.** *For a constructed instance  $(\mathcal{C}, \mathcal{P}, w)$  for any solution of weight  $2k^2(n^2 + 1) -$*   
 732  *$4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$  it holds that diagonal function from Lemma 5.3:*

733 1. *for any  $1 \leq i, j \leq k$ ,  $\text{match}_h(\text{diagonal}(1, j), \text{diagonal}(i, j))$  must be true;*

734 2. *for any  $1 \leq i, j \leq k$ ,  $\text{match}_v(\text{diagonal}(i, 1), \text{diagonal}(i, j))$  must be true.*

735 *Proof.* Simple inductive proof based on Lemma 5.4.  $\square$

736 **Lemma 5.5.** *If there exists solution of instance  $(\mathcal{C}, \mathcal{P}, w)$  with weight at most  $2k^2(n^2 + 1) -$*   
 737  *$4k^2\epsilon - 4k(1 - \epsilon) + k^2\delta$ , then there exists a solution for grid tiling instance.*

738 *Proof.* Take *diagonal* function from Lemma 5.3.

739 To define  $x$  function for every  $1 \leq i \leq k$  as  $x(i) = x_i$  where  $(x_i, a) = \text{order}(v_{i,1})$  and  $y$   
 740 function for every  $1 \leq i \leq k$  as  $y(i) = y_i$  where  $(b, y_i) = \text{order}(h_{1,i})$

741 To prove that it is a correct solution for grid tiling, we need to prove that for every  
 742  $1 \leq i, j \leq k$   $(x(i), y(j))$  is in allowed tiles set  $f(i, j)$ .

743 Let us take any  $1 \leq i, j \leq k$ , By Corollary 5.1 we know that  $\text{match}_h(\text{diagonal}(1, j), \text{diagonal}(i, j))$   
 744 and  $\text{match}_v(\text{diagonal}(i, 1), \text{diagonal}(i, j))$  are true. Therefore  $\text{order}(\text{diagonal}(i, j)) = (x(i), y(i))$ .  
 745 By Lemma 5.3 we know that  $\text{order}(\text{diagonal}(i, j))$  is in  $f(i, j)$ . Therefore  $(x(i), y(i))$  is in  
 746  $f(i, j)$ .  $\square$

748 *Proof of Theorem 5.1.* Based on Lemmas 5.1 and 5.5 this is true.  $\square$



## 749 Chapter 6

# 750 Geometric Set Cover with lines

### 751 6.1. Lines parallel to one of the axis

752 When  $\mathcal{R}$  consists only of lines parallel to one of the axis, the problem can be solved in  
753 polynomial time.

754 We create bipartial graph  $G$  with node for every line on the input split into sets:  $H$  –  
755 horizontal lines and  $V$  – vertical lines. If any two lines cover the same point from  $\mathcal{C}$ , then we  
756 add edge between them.

757 Of course there will be no edges between nodes inside  $H$ , because all of them are pararell  
758 and if they share one point, they are the same lines. Similar argument for  $V$ . So the graph is  
759 bipartial.

760 Now Geometric Set Cover can be solved with Vertex Cover on graph  $G$ . Since Vertex  
761 Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

762 Short note for myself just to remember how to this in polynomial time:

763 Non-weighted setting - Konig theorem + max matching

764 Weighted setting - Min cut in graph of  $\neg A$  or  $\neg B$  (edges directed from  $V$  to  $H$ )

### 765 6.2. FPT for arbitrary lines

766 You can find this is Platypus book. We will show FPT kernel of size at most  $k^2$ .

767 (Maybe we need to reduce lines with one point/points with one line).

768 For every line if there is more than  $k$  points on it, you have to take it. At the end, if there  
769 is more than  $k^2$  points, return NO. Otherwise there is no more than  $k^4$  lines.

770 In weighted settings among the same lines with different weights you leave the cheapest  
771 one and use the same algorithm.

### 772 6.3. APX-completeness for arbitrary lines

773 We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex  
774 Cover problem for graph  $G$ . We will create a set of  $|V(G)|$  pairwise non-pararell lines, such  
775 that no three of them share a common point.

776 Then for every edge in  $(v, w) \in E(G)$  we put a point on crossing of lines for vertices  $v$   
777 and  $w$ . They are not pararell, so there exists exactly one such point and any other line don't  
778 cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph  $G$ . For every point (edge) we need to choose at least one of lines (vertices)  $v$  or  $w$  to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

## 6.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do  $d$ -approximation, where  $d$  is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least  $k$  points and all lines have at least  $k$  points on them. It can be created by casting  $k$ -grid in  $k$ -D space on 2D space.

Greedy algorithm yields  $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than  $k$ ) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

## 6.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from  $\mathcal{C}$ , line from  $\mathcal{P}$ ).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.

## 803 Chapter 7

# 804 Geometric Set Cover with polygons

### 805 7.1. State of the art

806 Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons  
807 with  $\delta$ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

808 Although with thin objects, even if we allow  $\delta$ -expansion, the Set Cover with rectangles is  
809 APX-complete (for  $\delta = 1/2$ ), it follows from APX-completeness for segments with  $\delta$ -expansion  
810 in Section 3.1.

811 Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming  
812 *SETH*, there is no  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$  time algorithm for any computable function  $f$  and  
813  $\epsilon > 0$  that decides if there are  $k$  polygons in  $\mathcal{P}$  that together cover  $\mathcal{C}$ , *Theorem 1.9* in [Marx  
814 and Pilipczuk, 2015].



## 815 Chapter 8

## 816 Conclusions

817 We do not know FPT for axis-parallel segments without  $\delta$ -extensions.





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