

Approximation and parameterization of Segment Set Cover

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21.06.2022

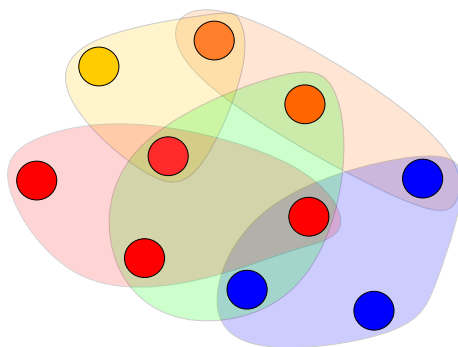
- 1 SET COVER
 - GEOMETRIC SET COVER
 - SEGMENT SET COVER
- 2 δ -extension
- 3 Approximation
- 4 Parameterization
- 5 WEIGHTED SEGMENT SET COVER

Problem Statement: SET COVER

Input: universe \mathcal{U} , family S of subsets of \mathcal{U}

Output: $C \subseteq S$ such that $|C|$ is minimal and $\bigcup C = \mathcal{U}$

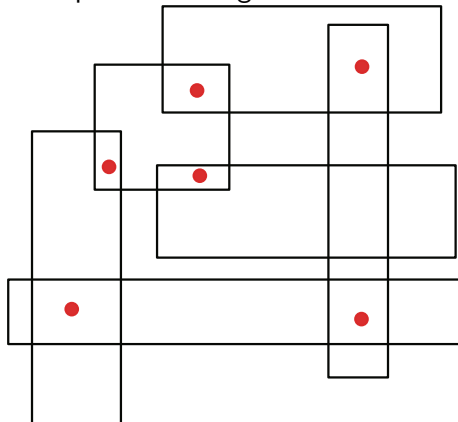
Minimal subfamily that covers all elements in \mathcal{U} .



Problem Statement: GEOMETRIC SET COVER

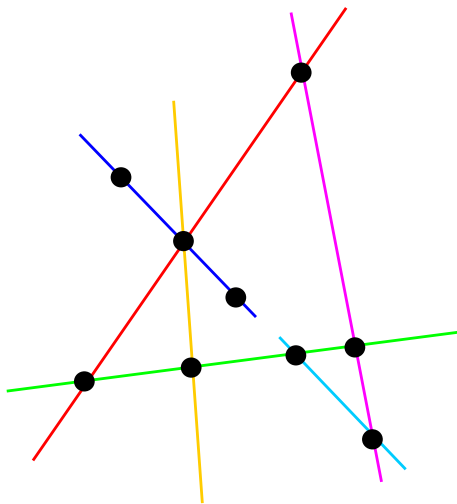
- Universe \mathcal{U} is set of points in the plane
- Sets S are some geometric shapes
- Formally, each set in S is intersection of \mathcal{U} with some geometric shape.

Example for rectangles:



Problem Statement: SEGMENT SET COVER

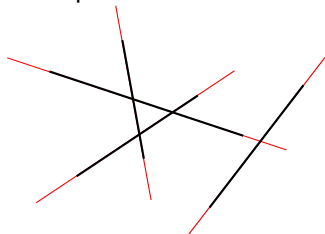
- Universe \mathcal{U} is a set of points in the plane.
- Sets S are line segments.



δ -extension for SEGMENT SET COVER

- δ -extension for a segment is a segment which is longer by δ fraction at both ends
- We accept solution in which segments cover solution after δ -extension
- The solution is compared to optimal solution without extension.

Example:



It is based on similar concept of δ -shrinking, which helped to provide FPT algorithm and EPTAS for (originally W[1]-hard and APX-hard) MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES.

Preliminaries: Approximation

Given:

- instance I of the optimization problem (looking for minimal solution)
- weight of the optimal solution $\text{opt}(I)$

p -approximation is algorithm that yields solution of weight not larger than $p \cdot \text{opt}(I)$

PTAS – polynomial time approximation scheme

For every $\epsilon > 0$, there exists $(1 + \epsilon)$ -approximation algorithm running in time $n^{f(\epsilon)}$ for some computable function f .

Example: KNAPSACK.

APX-hardness

For sufficiently small $\epsilon > 0$, if $(1 + \epsilon)$ -approximation exists, then $P = NP$.
Example: MAX-3-SAT (how many clauses in 3-SAT can be solved)

Approximation results for SET COVER from literature

SET COVER

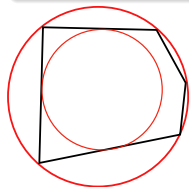
- $\log n$ approximation, no $o(\log n)$ approximation assuming $P \neq NP$

GEOMETRIC SET COVER

- with fat rectangles is APX-hard
- EPTAS for fat polygons with δ -extension [**Har-Peled and Lee, 2012**]

Definition

Fatness of polygons τ is a ratio between a radius of circle circumscribed on this polygon and inscribed in this polygon.



MAX-(3,3)-SAT

How do we prove that there doesn't exist $(1 + \epsilon)$ -approximation?

MAX-(3,3)-SAT problem

Given 3-SAT instance with n variables, m clauses and every variable appears in exactly 3 clauses, find an assignment that satisfies the maximum number of clauses (not necessarily all of them).

MAX-(3,3)-SAT is NP-complete, therefore:

NP-completeness

Distinguishing if result of MAX-(3,3)-SAT is n or less than n is NP-complete.

PCP theorem

PCP Theorem

$\text{NP} = \text{PCP}(\log n, \mathcal{O}(1))$.

PCP theorem implication [Håstad, 2001]

Distinguishing whether optimum result of the instance of MAX-(3,3)-SAT is m or at most $\frac{49}{50} \cdot m$ is NP-hard.

Therefore MAX-(3,3)-SAT does not have $(1 + \frac{1}{49})$ -approximation

Approximation of SEGMENT SET COVER

Theorem 1

SEGMENT SET COVER is APX-hard, i.e. there exists such $\epsilon > 0$ that $(1 + \epsilon)$ -approximation running in time $n^{f(\epsilon)}$ does not exist if $P \neq NP$.

How can we make this problem *easier*?

- Allow only segments in 2 directions (parallel to the two axes)
- Allow segments to *almost* cover the points, i.e. δ -extension

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Remark: Set cover with polygons of bounded fatness (at least τ):

- 1 is APX-hard;
- 2 with δ -extension has EPTAS.

Theorem 1 (rephrased)

SEGMENT SET COVER is APX-hard, even if segments are axis-parallel and problem is relaxed with $\frac{1}{2}$ -extension.

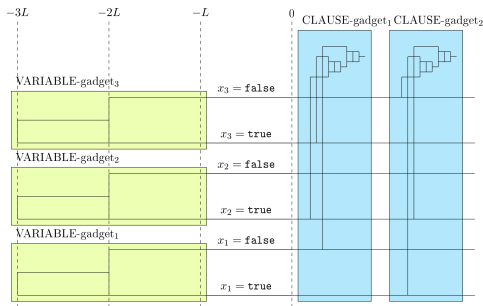


Figure: Scheme of the construction.

Results for approximation of GEOMETRIC SET COVER

	exact	δ -extension
fat polygons	APX-hard [Chan and Grant, 2014]	EPTAS [Har-Peled and Lee, 2012]
segments	APX-hard*	APX-hard (Theorem 1)
any polygons	APX-hard*	APX-hard*

Results marked with * follow from results for more restricted settings.

Prelimiaries: Parameterized algorithms

Instance I of a parameterized problem now has:

- size of the instance n ;
- parameter k (usually size of the solution).

Class	Upper Bound of Complexity	Example
FPT	$f(k) \cdot n^{O(1)}$	VERTEX COVER
W[1]	$n^{O(k)}$	k -CLIQUE, GRID TILING

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SET COVER is:

- W[2]-complete
- no algorithm with running time $n^{k-\epsilon}$ for any $\epsilon > 0$

Theorem [Marx, 2005]

GEOMETRIC SET COVER with unit squares parameterized by the solution size k is $W[1]$ -hard.

The follow-up work [Marx and Pilipczuk, 2022] shows a tight bound for this problem $n^{\Theta(\sqrt{k})}$.

Theorem 2

SEGMENT SET COVER parameterized by the solution size k can be solved in time $k^k \cdot n^{O(1)}$.

Technique: Branching over at most $k + 1$ segments on the lines with more than k points on them.

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Now let's make problem **harder**...

Definition of WEIGHTED SEGMENT SET COVER

Input: universe \mathcal{U} , subfamily S of subsets of \mathcal{U} and function of weights assigned to sets $w : S \rightarrow \mathbb{R}^+$

Output: $C \subseteq S$ such that $\sum_{c \in C} w(c)$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in \mathcal{U} .

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FPT results for weighted problems in literature:

- general weighted FPT framework [Shachnai and Zehavi, 2017]
- kernels for WEIGHTED SUBSET SUM, WEIGHTED KNAPSACK [Etscheid et al., 2017]
- WEIGHTED *st*-CUT WEIGHTED DIRECTED FEEDBACK SET [Kim et al., 2021]

Theorem 3

WEIGHTED SEGMENT SET COVER is $W[1]$ -hard and there does not exist an algorithm running in time $n^{o(\sqrt{k})}$.

This result is particularly interesting, because unweighted problem has an FPT algorithm.

How can we make this problem easier?

- allow less directions (maybe just two parallel to axes);
- allow δ -extension.

Theorem 3 (rephrased)

WEIGHTED SEGMENT SET COVER is $W[1]$ -hard and there does not exist an algorithm running in time $n^{o(\sqrt{k})}$ even when segments are limited to 3 directions.

Theorem 4

WEIGHTED SEGMENT SET COVER relaxed with δ -extension has an FPT algorithm.

It can be solved with algorithm running in time $\mathcal{O}((2 + k/\delta)^k \cdot n^{O(1)})$.

Technique intuition: Provide a kernel where we choose (k, δ) -**good** set of points of size at most $f(k, \delta)$ on each line with more than $k + 1$ points on them.

Subset of a set of collinear points S is (k, δ) -**good** if no matter how we cover it with k segments, then these segments after δ -extension cover all points from the original set of points S .

Results of parameterization of GEOMETRIC SET COVER

GEOMETRIC SET COVER

	exact	δ -extension
fat polygons	W[1]-hard [Marx, 2005]	???
segments	FPT (Theorem 2)	FPT*

SEGMENT SET COVER

	exact	δ -extension
unweighted	FPT (Theorem 2)	FPT*
weighted 2 directions	???	FPT*
weighted 3 directions	W[1]-hard (Theorem 3)	FPT*
weighted any directions	W[1]-hard*	FPT (Theorem 4)

Theorem 3 (reminder)

WEIGHTED SEGMENT SET COVER is $W[1]$ -hard and there does not exist an algorithm running in time $n^{o(\sqrt{k})}$ even when segments are limited to 3 directions.

Future work:

- prove Theorem 3 for axis-parallel segments (2 directions);
- remove the gap between the lower bound of complexity $n^{o(\sqrt{k})}$ and a simple algorithm running in time $n^{O(k)}$;
- investigate whether GEOMETRIC SET COVER with fat polygons relaxed with δ -extension has FPT algorithm.

The end.