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# Approximation and Parametrized Algorithms for Segment Set Cover

6

Master's thesis

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in COMPUTER SCIENCE

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9

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10 **Supervisor's statement**

11 Hereby I confirm that the presented thesis was prepared under my supervision and  
12 that it fulfils the requirements for the degree of Master of Computer Science.

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## **Abstract**

23 The work presents a study of different geometric set cover problems. It mostly focuses on  
24 segment set cover and its connection to the polygon set cover.

25

## **Keywords**

26 set cover, geometric set cover, FPT,  $W[1]$ -completeness, APX-completeness, PCP theorem,  
27 NP-completeness

28

## **Thesis domain (Socrates-Erasmus subject area codes)**

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30

31

## **Subject classification**

32 D. Software

33 D.127. Blabalgorithms

34 D.127.6. Numerical blabalysis

35

## **Tytuł pracy w języku polskim**

36 Algorytmy parametryzowania i trudność aproksymacji problemu pokrywania zbiorów  
37 odcinkami na płaszczyźnie



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# Chapter 1

## Introduction

The Set Cover problem is one of the most common NP-complete problems. [tutaj referencja] We are given a family of sets and have to choose the smallest subfamily of these sets that cover all their elements. This problem naturally extends to settings where we put different weights on the sets and look for the subfamily of the minimal weight. This problem is NP-complete even without weights and if we put restrictions on what the sets can be. One of such variants is Vertex Cover problem, where sets have size 2 (they are edges in a graph).

In this work we focus on another such variant where the sets correspond to some geometric shapes and only some points of the plane have to be covered. When these shapes are rectangles with edges parallel to the axis, the problem can be proven to be W[1]-complete (solution of size  $k$  cannot be found in  $n^o(k)$  time), APX-complete (for sufficiently small  $\epsilon > 0$ , the problem does not admit  $1 + \epsilon$ -approximation scheme) [referencje].

Some of these settings are very easy. Set cover with lines parallel to one of the axis can be solved in polynomial time.

There is a notion of  $\delta$ -expansions, which loosen the restrictions on geometric set cover. We allow the objects to cover the points after  $\delta$ -expansion and compare the result to the original setting. This way we can produce both FPT and EPTAS for the rectangle set cover with  $\delta$ -extensions [referencje].

**Our contribution.** In this work, we prove that unweighted geometric set cover with segments is fixed parameter tractable (FPT).

Moreover, we show that geometric set cover with segments is APX-complete for unweighted axis-parallel segments, even with  $1/2$ -extensions. So the problem for very thin rectangles also can't admit PTAS. Therefore, in the efficient polynomial-time approximation scheme (EPTAS) for *fat polygons* by [Har-Peled and Lee, 2009], the assumption about polygons being fat is necessary.

Finally, we show that geometric set cover with weighted segments in 3 directions is W[1]-complete. However, geometric set cover with weighted segments is FPT if we allow  $\delta$ -extension.

This result is especially interesting, since it's counter-intuitive that the unweighted setting is FPT and the weighted setting is W[1]-complete. Most of such problems (like vertex cover or [wiecej przykladow]) are equally hard in both weighted and unweighted settings.





## 101 Chapter 2

## 102 Definitions

### 103 2.1. Geometric Set Cover

104 In the geometric set cover problem we are given  $\mathcal{P}$  – a set of objects, which are connected  
105 subsets of the plane,  $\mathcal{C}$  – a set of points in the plane. The task is to choose  $\mathcal{R} \subseteq \mathcal{P}$  such that  
106 every point in  $\mathcal{C}$  is inside some element from  $\mathcal{R}$  and  $|\mathcal{R}|$  is minimized.

107 In the parametrized setting for a given  $k$ , we only look for a solution  $\mathcal{R}$  such that  $|\mathcal{R}| \leq k$ .

108 In the weighted setting, there is some given weight function  $f : \mathcal{P} \rightarrow \mathbb{R}^+$ , and we would  
109 like to find a solution  $\mathcal{R}$  that minimizes  $\sum_{R \in \mathcal{R}} f(R)$ .

### 110 2.2. Approximation

111 Let us recall some definitions related to optimization problems that are used in the following  
112 sections.

113 **Definition 1.** A **polynomial-time approximation scheme (PTAS)** for a minimization  
114 problem  $\Pi$  is a family of algorithms  $\mathcal{A}_\epsilon$  for every  $\epsilon > 0$  such that  $\mathcal{A}_\epsilon$  takes an instance  $I$  of  $\Pi$   
115 and in polynomial time finds a solution that is within a factor  $(1 + \epsilon)$  of being optimal. That  
116 means the reported solution has weight at most  $(1 + \epsilon)\text{opt}(I)$ , where  $\text{opt}(I)$  is the weight of  
117 an optimal solution for  $I$ .

118 **Definition 2.** A problem  $\Pi$  is **APX-hard** if assuming  $P \neq NP$ , there exists  $\epsilon > 0$  such that  
119 there is no polynomial-time  $(1 + \epsilon)$ -approximation algorithm for  $\Pi$ .

### 120 2.3. $\delta$ -extensions

121 TODO PLACEHOLDER for introductory text

122  $\delta$ -extensions is one of the modifications to a problem, that makes geometric set cover  
123 problem easier, it has been already used in literature (place some refrence here).

124 **Definition 3** ( $\delta$ -extensions for center-symmetric objects). For any  $\delta > 0$  and a center-  
125 symmetric object  $L$  with centre of symmetry  $S = (x_s, y_s)$ , the  **$\delta$ -extension** of  $L$  is the  
126 object  $L^{+\delta} = \{(1 + \delta) \cdot (x - x_s, y - y_s) + (x_s, y_s) : (x, y) \in L\}$ , that is,  $L^{+\delta}$  is the image of  $L$   
127 under homothety centered at  $S$  with scale  $(1 + \delta)$

128 The geometric set cover problem with  $\delta$ -extensions is a modified version of geometric set  
129 cover where:

- We need to cover all the points in  $\mathcal{C}$  with objects from  $\{P^{+\delta} : P \in \mathcal{P}\}$  (which always include no fewer points than the objects before  $\delta$ -extensions);

- We look for a solution that is no larger than the optimum solution for the original problem. Note that it does not need to be an optimal solution in the modified problem.

Formally, we have the following.

**Definition 4** (Geometric set cover problem with  $\delta$ -extensions). The geometric set cover problem with  $\delta$ -extensions is the problem where for an input instance  $I = (\mathcal{P}, \mathcal{C})$ , the task is to output a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is no larger than the optimal solution for the problem without extensions, i.e.  $|\mathcal{R}| \leq |\text{opt}(I)|$ .

TODO: Some text

**Definition 5** (Geometric set cover PTAS with  $\delta$ -extensions). We define a PTAS for geometric set cover with  $\delta$ -extensions as a family of algorithms  $\{\mathcal{A}_{\delta, \epsilon}\}_{\delta, \epsilon > 0}$  that each takes as an input instance  $I = (\mathcal{P}, \mathcal{C})$ , and in polynomial-time outputs a solution  $\mathcal{R} \subseteq \mathcal{P}$  such that the  $\delta$ -extended set  $\{R^{+\delta} : R \in \mathcal{R}\}$  covers  $\mathcal{C}$  and is within a  $(1 + \epsilon)$  factor of the optimal solution for this problem without extensions, i.e.  $(1 + \epsilon)|\mathcal{R}| \leq |\text{opt}(I)|$ .

## Chapter 3

# Geometric Set Cover with segments

### 3.1. FPT for segments

In this section we consider the fixed-parameter tractable algorithms for unweighted geometric set cover with segments. Setting where segments are limited to be axis-parallel (or limited to constant number of directions) has an FPT algorithm already present in literature. We present an FPT algorithm for unweighted geometric set cover with segments, where segments are in arbitrary directions.

#### 3.1.1. Axis-parallel segments

You can find this in Platypus book. (TODO add referece)

We show an  $\mathcal{O}(2^k)$ -time branching algorithm. In each step, the algorithm selects a point  $a$  which is not yet covered, branches to choose one of the two directions, and greedily chooses a segment in that direction to cover  $a$ . This proceeds until either all points are covered or  $k$  segments are chosen.

Let us take the point  $a = (x_a, y_a)$  which is the smallest among points that are not yet covered in the lexicographic ordering of points in  $\mathbb{R}^2$ . We need to cover  $a$  with some of the remaining segments.

Branch over the choice of one of the coordinates ( $x$  or  $y$ ); without loss of generality, let us assume we chose  $x$ . Among the segments lying on line  $x = x_a$ , we greedily add to the solution the one that covers the most points. As  $a$  was the smallest in the lexicographical order, then all points on line  $x = x_a$  have the  $y$ -coordinate larger than  $y_a$ . Therefore, if we denote the greedily chosen segment as  $s$ , then any other segment on  $x = x_a$  that covers  $a$  can only cover a (possibly improper) subset of points covered by  $s$ . Thus, greedily choosing  $s$  is optimal.

In each step of the algorithm we add one segment to the solution, thus each branch can stop at depth  $k$ . If no branch finds a solution, then that means a solution of size at most  $k$  does not exist.

TODO: Maybe split it into theorem + algorithm + explanation like in section 3.1.2

**Remark 1.** *The same algorithm can be used for segments in  $d$  directions, where we branch over  $d$  directions and it runs in complexity  $\mathcal{O}(d^k)$ .*

#### 3.1.2. Segments in arbitrary directions

In this section we consider setting where segments are not constrained to only  $d$  directions. We present a fixed-parameter tractable algorithm, where parameter is the size of the solution.

**Theorem 1. (FPT for segment cover).** *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  segments (in any direction), a set of  $m$  points  $\mathcal{C}$  and a parameter  $k$ , runs in time  $k^{O(k)} \cdot (nm)^2$ , and outputs a subfamily  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.*

We will need the following lemmas.

**Lemma 1.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, without a loss of generality we can assume that no segment covers a superset of what another segment covers. That is, for any distinct  $A, B \in \mathcal{P}$ , we have  $A \cap C \not\subseteq B \cap C$  and  $A \cap C \not\supseteq B \cap C$ .*

*Proof.* Trivial. □

**Lemma 2.** *Given an instance  $(\mathcal{P}, \mathcal{C})$  of the segment cover problem, if there exists a line  $L$  with at least  $k + 1$  points on it, then there exists a subset  $\mathcal{A} \subseteq \mathcal{P}$ ,  $|\mathcal{A}| \leq k$ , such that every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$  satisfies  $|\mathcal{A} \cap \mathcal{R}| \geq 1$ . Moreover, such a subset can be found in polynomial time.*

*Proof.* First we use Lemma 1.

Let us enumerate the points from  $\mathcal{C}$  that lie on  $L$  as  $x_1, x_2, \dots, x_t$  in the order in which they appear on  $L$ . Every segment that is not collinear with  $L$  can cover at most one of these points. Therefore, in any solution of size not larger than  $k$ , among any  $k$  of these points at least one must be covered with segment collinear with  $L$ .

Therefore, every solution needs to take one of the segments collinear with  $L$  that covers any of the points  $x_1, x_2, \dots, x_k$ . After using reduction from Lemma 1, there are at most  $k$  such segments that are distinct. □

We are ready to prove Theorem 1.

*Proof of Theorem 1.*

We will prove this theorem by presenting a branching algorithm that works in desired complexity. It branches over the choice of segments to cover lines with a lot of points, then finally solving the small instance, where every line has at most  $k$  points by checking all possible solutions.

**Algorithm.** First we use Lemma 1.

Next, we present a recursive algorithm. Given an instance of the problem:

- (1) If there exist a line with at least  $k + 1$  points from  $\mathcal{C}$ , we branch over adding to the solution one of the at most  $k$  possible segments provided by Lemma 2; name this segment  $S$ . Then we find a solution  $\mathcal{R}$  for the problem for points  $\mathcal{C} - S$ , segments  $\mathcal{P} - \{S\}$ , and parameter  $k - 1$ . We return  $\mathcal{R} \cup \{S\}$ .
- (2) If every line has at most  $k$  points on it and  $|\mathcal{C}| > k^2$ , then answer NO.
- (3) If  $|\mathcal{C}| \leq k^2$ , solve the problem by brute force: check all subsets of  $\mathcal{P}$  of size at most  $k$ .

**Correctness.** Lemma 2 proves that at least one segment that we branch over in (1) must be present in every solution  $\mathcal{R}$  with  $|\mathcal{R}| \leq k$ . Therefore, the recursive call can find a solution, provided there exists one.

In (2) the answer is no, because every line covers no more than  $k$  points from  $\mathcal{C}$ , which implies the same about every segment from  $\mathcal{P}$ . Under this assumption we can cover only  $k^2$  points with a solution of size  $k$ , which is less than  $|\mathcal{C}|$ .

Checking all possible solutions in (3) is trivially correct.

219 **Complexity.** In the leaves of recursion we have  $|\mathcal{C}| \leq k^2$ , so  $|\mathcal{P}| \leq k^4$ , because every  
 220 segments can be uniquely identified by the two extreme points it covers (by Lemma 1).  
 221 Therefore, there are  $\binom{k^4}{k}$  possible solutions to check, each can be checked in time  $O(k|\mathcal{C}|)$ .  
 222 Therefore, (3) takes time  $k^{O(k)}$ .

223 In this branching algorithm our parameter  $k$  is decreased with every recursive call, so we  
 224 have at most  $k$  levels of recursion with branching over  $k$  possibilities. Candidates to branch  
 225 over can be found on each level in time  $O((nm)^2)$ .

226 Reduction from Lemma 1 can be implemented in time  $O(n^2m)$ .

227 It follows that the overall complexity is  $O((nm)^2 \cdot k^{O(k)})$  □

## 228 3.2. APX-completeness for segments parallel to axes

229 In this section we analyze whether there exists PTAS for geometric set cover for rectangles.  
 230 We show that we can restrict this problem to a very simple setting: segments parallel to axes  
 231 and allow  $(1/2)$ -extension, and the problem is still APX-hard. Note that segments are just  
 232 degenerated rectangles with one side being very narrow.

233 Our results can be summarized in the following theorem and this section aims to prove it.

234 **Theorem 2.** *(axis-parallel segment set cover with  $1/2$ -extension is APX-hard).*  
 235 *Unweighted geometric set cover with axis-parallel segments in 2D (even with  $1/2$ -extension)*  
 236 *is APX-hard. That is, assuming  $P \neq NP$ , there does not exist a PTAS for this problem.*

237 Theorem 2 implies the following.

238 **Corollary 1.** *(rectangle set cover is APX-hard).* *Unweighted geometric set cover with*  
 239 *rectangles (even with  $1/2$ -extension) is APX-hard.*

240 We prove Theorem 2 by taking a problem that is APX-hard and showing a reduction. For  
 241 this problem we choose MAX-(3,3)-SAT which we define below.

### 242 3.2.1. MAX-(3,3)-SAT and statement of reduction

243 **Definition 6.** MAX-3SAT is the following maximization problem. We are given a 3-CNF  
 244 formula, and need to find an assignment of variables that satisfies the most clauses.

245 **Definition 7.** MAX-(3,3)-SAT is a variant of MAX-3SAT with an additional restriction  
 246 that every variable appears in exactly 3 clauses. Note that thus, the number of clauses is  
 247 equal to the number of variables.

248 In our proof of Theorem 2 we use hardness of approximation of MAX-(3,3)-SAT proved  
 249 in [Håstad, 2001] and described in Theorem 3 below.

250 **Definition 8** ( $\alpha$ -satisfiable MAX-3SAT formula). MAX-3SAT formula of size  $n$  is at most  
 251  $\alpha$ -satisfiable, if every assignment of variables satisfies no more than  $\alpha n$  clauses.

252 **Theorem 3.** [Håstad, 2001]

253 *For any  $\epsilon > 0$ , it is NP-hard to distinguish satisfiable (3,3)-SAT formulas from at most*  
 254  *$(7/8 + \epsilon)$ -satisfiable (3,3)-SAT formulas.*

Given an instance  $I$  of MAX-(3,3)-SAT, we construct an instance  $J$  of axis-parallel segment set cover problem, such that for a sufficiently small  $\epsilon > 0$ , a polynomial time  $(1 + \epsilon)$ -approximation algorithm for  $J$  would be able to distinguish whether an instance  $I$  of MAX-(3,3)-SAT is fully satisfiable or is at most  $(7/8 + \epsilon)$ -satisfiable. However, according to (Theorem 3) the latter problem is NP-hard. This would imply  $P = NP$ , contradicting the assumption.

The following lemma encapsulates the properties of the reduction described in this section, and it allows us to prove Theorem 2.

**Lemma 3.** *Given an instance  $S$  of MAX-(3,3)-SAT with  $n$  variables and optimum value  $opt(S)$ , we can construct an instance  $I$  of geometric set cover with axis-parallel segments in  $2D$ , such that:*

(1) *For every solution  $X$  of instance  $I$ , there exists a solution of  $S$  that satisfies at least  $15n - |X|$  clauses.*

(2) *For every solution of instance  $S$  that satisfies  $w$  clauses, there exists a solution of  $I$  of size  $15n - w$ .*

(3) *Every solution with 1/2-extensions of  $I$  is also a solution to the original instance  $I$ .*

*Therefore, the optimum size of a solution of  $I$  is  $opt(I) = 15n - opt(S)$ .*

We prove Lemma 3 in subsequent sections, but meanwhile let us prove Theorem 2 using Lemma 3 and Theorem 3.

*Proof of Theorem 2.*

Consider any  $0 < \epsilon < 1/(15 \cdot 8)$ .

Let us assume that there exists a polynomial-time  $(1 + \epsilon)$ -approximation algorithm for unweighted geometric set cover with axis-parallel segments in  $2D$  with  $(1/2)$ -extensions. We construct an algorithm that solves the problem stated in Theorem 3, thereby proving that  $P = NP$ .

Take an instance  $S$  of MAX-(3,3)-SAT to be distinguished and construct an instance of geometric set cover  $I$  using Lemma 3. We now use the  $(1 + \epsilon)$ -approximation algorithm for geometric set cover on  $I$ . Denote the size of the solution returned by this algorithm as  $approx(I)$ . We prove that if in  $S$  one can satisfy at most  $(\frac{7}{8} + \epsilon)n$  clauses, then  $approx(I) \geq 15n - (\frac{7}{8} + \epsilon)n$  and if  $S$  is satisfiable, then  $approx(I) < 15n - (\frac{7}{8} + \epsilon)n$ .

**Assume  $S$  satisfiable.** From the definition of  $S$  being satisfiable, we have:

$$opt(S) = n.$$

From Lemma 3 we have:

$$opt(I) = 14n.$$

Therefore,

$$\begin{aligned} approx(I) &\leq (1 + \epsilon)opt(I) = 14n(1 + \epsilon) = 14n + 14\epsilon \cdot n = \\ &= 14n + (15\epsilon - \epsilon)n < 14n + \left(\frac{1}{8} - \epsilon\right)n = 15n - \left(\frac{7}{8} + \epsilon\right)n \end{aligned}$$

**Assume  $S$  is at most  $(\frac{7}{8} + \epsilon)$  satisfiable.** From the definition of  $S$  being at most  $(\frac{7}{8} + \epsilon)n$  satisfiable, we have:

$$opt(S) \leq \left(\frac{7}{8} + \epsilon\right)n$$

From Lemma 3 we have:

$$opt(I) \geq 15n - \left(\frac{7}{8} + \epsilon\right)n$$

285 Since a solution to  $I$  with  $\frac{1}{2}$ -extensions is also a solution without extensions, by Lemma 3  
286 (3.), we have:

$$approx(I) \geq opt(I) = 15n - \left(\frac{7}{8} + \epsilon\right)n$$

287 Therefore, by using the assumed  $(1 + \epsilon)$ -approximation algorithm, it is possible to dis-  
288 tinguish the case when  $S$  is satisfiable from the case when it is at most  $(\frac{7}{8} + \epsilon)n$  satisfiable,  
289 it suffices to compute  $approx(I)$  with  $15n - (\frac{7}{8} + \epsilon)n$ . Hence, the assumed approximation  
290 algorithm cannot exist, unless  $P = NP$ .  $\square$

### 291 3.2.2. Reduction

292 We proceed to the proof of Lemma 3. That is, we show a reduction from MAX-(3,3)-SAT  
293 problem to geometric set cover with segments parallel to axis. Moreover, the obtained instance  
294 of geometric set cover will be robust to 1/2-extensions (have the same optimal solution after  
295 1/2-extension).

296 The construction will be composed of 2 types of gadgets: **VARIABLE-gadgets** and  
297 **CLAUSE-gadgets**. **CLAUSE-gadgets** would be constructed using two **OR-gadgets** con-  
298 nected together.

#### 299 3.2.2.1. VARIABLE-gadget

300 **VARIABLE-gadget** is responsible for choosing the value of a variable in a CNF formula. It  
301 allows two minimum solutions of size 3 each. These two choices correspond to the two Boolean  
302 values of the variable corresponding to this gadget.

303 **Points.** Define points  $a, b, c, d, e, f, g, h$  as follows, where  $L = 12n$ :



Figure 3.1: **VARIABLE-gadget**. We denote the set of points marked with black circles as  $\text{pointsVariable}_i$ , and they need to be covered (are part of the set  $\mathcal{C}$ ). Note that some of the points are not marked as black dots and exists only to name segments for further reference. We denote the set of red segments as  $\text{chooseVariable}_i^{\text{false}}$  and the set of blue segments as  $\text{chooseVariable}_i^{\text{true}}$ .

$$\begin{array}{llll}
a = (-L, 0) & b = (-\frac{2}{3}L, 0) & c = (-\frac{1}{3}L, 0) & d = (-L, 1) \\
e = (-\frac{2}{3}L, 1) & f = (-\frac{2}{3}L, 2) & g = (L, 0) & h = (L, 2)
\end{array}$$

Let us define:

$$\text{pointsVariable} = \{a, b, c, d, e, f\}$$

and

$$\text{pointsVariable}_i = \text{pointsVariable} + (0, 4i)$$

We denote  $a_i = a + (0, 4i)$  etc.

**Segments.** Let us define:

$$\text{chooseVariable}_i^{\text{true}} = \{(a_i, d_i), (b_i, f_i), (c_i, g_i)\}$$

$$\text{chooseVariable}_i^{\text{false}} = \{(a_i, c_i), (d_i, e_i), (f_i, h_i)\}$$

$$\text{segmentsVariable}_i = \text{chooseVariable}_i^{\text{true}} \cup \text{chooseVariable}_i^{\text{false}}$$

**Lemma 4.** For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can be covered using 3 segments from  $\text{segmentsVariable}_i$ .

*Proof.* We can use either set  $\text{chooseVariable}_i^{\text{true}}$  or  $\text{chooseVariable}_i^{\text{false}}$ .  $\square$

**Lemma 5.** For any  $1 \leq i \leq n$ , points in  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments from  $\text{segmentsVariable}_i$ .

*Proof.* No segment of  $\text{segmentsVariable}_i$  covers more than one point from  $\{d_i, f_i, c_i\}$ , therefore  $\text{pointsVariable}_i$  can not be covered with fewer than 3 segments.  $\square$

**Lemma 6.** For every set  $A \subseteq \text{segmentsVariable}_i$  such that  $A$  covers  $\text{pointsVariable}_i$  and  $(c_i, g_i), (f_i, h_i) \in A$ , it holds that  $|A| \geq 4$ .

*Proof.* No segment from  $\text{segmentsVariable}_i$  covers more than one point from  $\{a_i, e_i\}$ , therefore  $\text{pointsVariable}_i - \{c_i, f_i, g_i, h_i\}$  can not be covered with fewer than 2 segments.  $\square$

### 3.2.2.2. OR-gadget

OR-segment connects input and output segments that are connected to other parts of constructions.

Output segment is part of OR-segment, but input is not.

For every solution  $\mathcal{R}$  of the whole construction. Define  $\mathcal{R}'$  as intersection of  $\mathcal{R}$  and the gadget segments. Minimum solution of OR-gadget has size  $w$ , i.e.  $|\mathcal{R}'| \leq w$ . *output* segments can be part of  $\mathcal{R}'$  only if *input<sub>x</sub>* or *input<sub>y</sub>* are part of the chosen solution  $\mathcal{R}$ . If none of them are chosen, then solution containing *output* segment has weight at least  $w + 1$ . Therefore the following formula holds:

$$\text{output} \in \mathcal{R}' \wedge |\mathcal{R}'| = w \Rightarrow (x \in \mathcal{R}) \vee (y \in \mathcal{R})$$

Only 3 points that belong to this segment:  $l_{i,j}, p_{i,j}, v_{i,j}$  can be covered by segment not from the OR-gadget.





Figure 3.2: **OR-gadget**. Figure presenting OR-gadget: segments from  $\text{chooseOr}_{i,j}^{false}$  are red, segments from  $\text{chooseOr}_{i,j}^{true}$  are blue, segments from  $\text{orMoveVariable}_{i,j}$  are yellow and green. Dark blue segment is an *output* segment. Grey segments  $input_x$  and  $input_y$  are input segments that are not part of  $\text{segmentsOr}_{i,j}$ .

329 **Points.**

$$\begin{array}{llll}
 l_0 = (0, 0) & m_0 = (0, 1) & n_0 = (0, 2) & o_0 = (0, 3) \\
 p_0 = (0, 4) & q_0 = (1, 1) & r_0 = (1, 3) & s_0 = (2, 1) \\
 t_0 = (2, 2) & u_0 = (2, 3) & v_0 = (3, 2) & 
 \end{array}$$

$$vec_{i,j} = (10i + 3 + 3j, 4n + 2j)$$

331 Define  $\{l_{i,j}, m_{i,j} \dots v_{i,j}\}$  as  $\{l_0, m_0 \dots v_0\}$  shifted by  $vec_{i,j}$

332 Note that  $v_{i,0} = l_{i,1}$  (see Figure 3.3)

$$\text{pointsOr}_{i,j} = \{l_{i,j}, m_{i,j}, n_{i,j}, o_{i,j}, p_{i,j}, q_{i,j}, r_{i,j}, s_{i,j}, t_{i,j}, u_{i,j}\}$$

333 Note that  $\text{pointsOr}_{i,j}$  does not include point  $v_{i,j}$

334 **Segments.** We define names subsets of segments, to refer to them in lemmas.

$$\text{chooseOr}_{i,j}^{false} = \{(q_{i,j}, r_{i,j}), (s_{i,j}, u_{i,j})\}$$

$$\text{chooseOr}_{i,j}^{true} = \{(m_{i,j}, s_{i,j}), (o_{i,j}, u_{i,j}), (t_{i,j}, v_{i,j})\}$$

$$\text{orMoveVariable}_{i,j} = \{(l_{i,j}, n_{i,j}), (n_{i,j}, p_{i,j})\}$$

335 Segments in OR-gadget:

$$\text{segmentsOr}_{i,j} = \text{chooseOr}_{i,j}^{false} \cup \text{chooseOr}_{i,j}^{true} \cup \text{orMoveVariable}_{i,j}$$

336 **Lemma 7.** For any  $1 \leq i \leq n, j \in \{0, 1\}$  and  $x \in \{l_{i,j}, p_{i,j}\}$ , points in  $\text{pointsOr}_{i,j} - \{x\} \cup \{v_{i,j}\}$   
 337 can be covered with 4 segments from  $\text{segmentsOr}_{i,j}$ .

338 *Proof.* We can do that using one segment from  $\text{orMoveVariable}_{i,j}$ , the one that does not cover  
 339  $x$ , and all segments from  $\text{chooseOr}_{i,j}^{true}$ .  $\square$

**Lemma 8.** For any  $1 \leq i \leq n, j \in \{0, 1\}$ , points in  $\text{pointsOr}_{i,j}$  can be covered with 4 segments from  $\text{segmentsOr}_{i,j}$ .

*Proof.* We can do that using segments from  $\text{orMoveVariable}_{i,j}$  and  $\text{chooseOr}_{i,j}^{\text{false}}$ .  $\square$

### 3.2.2.3. CLAUSE-gadget

CLAUSE-gadget is responsible for calculating if variables values assigned in variable gadgets satisfy the respective clause in CNF. It has minimum solution of weight  $w$  if and only if the clause is satisfied, i.e. at least one of the respective variables is assigned a correct value. Otherwise it has minimum solution of weight  $w + 1$ . This way, by analyzing the minimum solution for the whole problem, we can tell how many clauses were possible to satisfy in the optimum solution of CNF.

The CLAUSE-gadgets consist of two OR-gadgets. It would be inconvenient to position the CLAUSE-gadgets in between the very long variable segments. Instead, we use a simple auxiliary gadget to *transfer* whether the segment is in a solution, i.e. segments  $(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})$ . Each gadget consists of two segments  $(x_{i,0}, x_{i,1}), (x_{i,1}, a)$ . These are the only segments that can cover  $x_{i,1}$ . If  $x_{i,0}$  is already covered by some other gadget, we can cover  $x_{i,1}$  by the other segment covering another point from the gadget, say  $a$ . If  $x_{i,0}$  is not covered, then the only way to cover  $x_{i,0}$  is to use segment  $(x_{i,0}, x_{i,1})$ . Intuitively, the two segments *transfer* the state of  $x_{i,0}$  onto  $a$ , but there are less restrictions on where  $a$  can be placed, simplifying the construction.



Figure 3.3: **CLAUSE-gadget.** This figure presents CLAUSE-gadget. Every green rectangle is an OR-gadget.  $y$ -coordinates of  $x_{i,0}$ ,  $y_{i,0}$  and  $z_{i,0}$  depend on the variables in the  $i$ -th clause. Grey segments corresponds to the values of variables satisfying the  $i$ -th clause.

**Points.** TODO: Rephrase it

Assuming clause  $C_i = a \vee b \vee c$ , function  $\text{idx}(w)$  returns index of the variable  $w$ , function  $\text{neg}(w)$  returns whether variable  $w$  is negated in a clause.

$$\begin{aligned}
x_{i,0} &= (10i + 1, 4 \cdot idx(a) + 2 \cdot neg(c)) & x_{i,1} &= (10i + 1, 4n) \\
y_{i,0} &= (10i + 2, 4 \cdot idx(b) + 2 \cdot neg(b)) & y_{i,1} &= (10i + 2, 4n + 4) \\
z_{i,0} &= (10i + 3, 4 \cdot idx(c) + 2 \cdot neg(c)) & z_{i,1} &= (10i + 3, 4n + 6)
\end{aligned}$$

$$\text{moveVariable}_i = \{x_{i,j} : j \in \{0, 1\}\} \cup \{y_{i,j} : j \in \{0, 1\}\} \cup \{z_{i,j} : j \in \{0, 1\}\}$$

$$\text{pointsClause}_i = \text{moveVariable}_i \cup \text{pointsOr}_{i,0} \cup \text{pointsOr}_{i,1} \cup \{v_{i,1}\}$$

**Segments.**

$$\begin{aligned}
\text{segmentsClause}_i &= \{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (x_{i,1}, l_{i,0}), (y_{i,1}, p_{i,0}), (z_{i,1}, p_{i,1}), \} \cup \\
&\cup \text{segmentsOr}_{i,0} \cup \text{segmentsOr}_{i,1}
\end{aligned}$$

**Lemma 9.** For any  $1 \leq i \leq n$  and  $a \in \{x_{i,0}, y_{i,0}, z_{i,0}\}$ , there is a  $\text{solClause}_i^{\text{true},a} \subset \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{true},a}| = 11$  that covers points in  $\text{pointsClause}_i - \{a\}$ .

*Proof.* For  $a = x_{i,0}$  (analogous proof for  $y_{i,0}$ ): First we use Lemma 7 twice with excluded  $x = l_{i,0}$  and  $x = l_{i,1} = v_{i,0}$ , resulting with 8 segments  $\text{chooseOr}_{i,0}^{\text{true}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,1}, y_{i,0}, y_{i,1}, z_{i,0}, z_{i,1}, l_{i,0}$ . We cover those using additional 3 segments:  $\{(x_{i,1}, l_{i,0}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1})\}$

For  $a = z_{i,0}$ : Using Lemma 8 and Lemma 7 with  $x = p_{i,1}$ , resulting with 8 segments  $\text{chooseOr}_{i,0}^{\text{false}} \cup \text{chooseOr}_{i,1}^{\text{true}}$  which cover all required points apart from  $x_{i,0}, x_{i,1}, y_{i,0}, y_{i,1}, z_{i,1}, p_{i,1}$ . We cover those using additional 3 segments:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,1}, p_{i,1})\}$ .  $\square$

**Lemma 10.** For any  $1 \leq i \leq n$  there is  $\text{solClause}_i^{\text{false}} \subset \text{segmentsClause}_i$  with  $|\text{solClause}_i^{\text{false}}| = 12$  that covers points in  $\text{pointsClause}_i$ .

*Proof.* Using Lemma 8 twice we can cover  $\text{pointsOr}_{i,0}$  and  $\text{pointsOr}_{i,1}$  with 8 segments.

To cover the remaining points we additionally use:  $\{(x_{i,0}, x_{i,1}), (y_{i,0}, y_{i,1}), (z_{i,0}, z_{i,1}), (t_{i,1}, v_{i,1})\}$   $\square$

**Lemma 11.** For any  $1 \leq i \leq n$ :

(1) points in  $\text{pointsClause}_i$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 12;

(2) points in  $\text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  can not be covered using any subset of segments from  $\text{segmentsClause}_i$  of size smaller than 11.

*Proof of (1).* No segment in  $\text{segmentsClause}_i$  covers more than 2 points from  $\{x_{i,0}, y_{i,0}, z_{i,0}, l_{i,0}, p_{i,0}, q_{i,0}, u_{i,0}, v_{i,0}, l_{i,1}, p_{i,1}, q_{i,1}, u_{i,1}, v_{i,1}\}$ .

Therefore we need to use at least 12 segments.  $\square$

*Proof of (2).* We can choose disjoint sets  $X, Y, Z$  such that  $X \cup Y \cup Z \subseteq \text{pointsClause}_i - \{x_{i,0}, y_{i,0}, z_{i,0}\}$  and there are no segments covering points from different sets. And we prove lower bounds for each of these sets.

$$X = \{x_{i,1}, y_{i,1}, z_{i,1}\}$$

No two points in  $X$  are covered with one segment of  $\text{segmentsClause}_i$ , so it must be covered with 3 different segments.

$$Y = \text{pointsOr}_{i,0} - \{l_{i,0}, p_{i,0}\}$$

$$Z = \text{pointsOr}_{i,1} - \{l_{i,1}, p_{i,1}\}$$

For both  $Y$  and  $Z$  we can check all of the subsets of 3 segments of  $\text{segmentsClause}_i$  with brutforce that none of them cover the set of points, so both  $Y$  and  $Z$  have to be covered with disjointed sets of 4 segments.

TODO: Funny fact, neither  $Y$  nor  $Z$  doesn't have independent set of size 4.

Therefore  $\text{pointsClause}_i$  must be covered with at least  $3 + 4 + 4 = 11$  segments.  $\square$

### 3.2.2.4. Summary

Add some smart lemmas that sets will be exclusive to each other.

**Lemma 12. Robustness to 1/2-extensions.** *For every segment  $s \in \mathcal{P}$ ,  $s$  and  $s^{+1/2}$  cover the same points from  $\mathcal{C}$ .*

*Proof.* We can just check every segment. Most of the segments  $s$  are collinear only with points that lay on  $s$ , so trivially  $s^{+1/2}$  cannot cover more points than  $s$  does.

TODO: list problematic segments here

In the same gadget:  $(n_{i,j}, p_{i,j})$  does not cover  $m_{i,j}$  and symmetrically.  $(t_{i,j}, v_{i,j})$  does not cover  $n_{i,j}$ .  $(o_{i,0}, u_{i,0})$  does not cover  $m_{i,1}$  and symmetrically.  $(y_{i,1}, p_{i,0})$  does not cover  $n_{i,j}$ .

From different gadgets:  $(b_i, f_i)$  after  $\frac{1}{2}$ -extensions does not cover  $b_{i+1}$  point.

VARIABLE-gadget's  $(a_i, c_i)$  after  $\frac{1}{2}$ -extensions does not cover any points  $x_{i,0}, y_{i,0}$  or  $z_{i,0}$  from CLAUSE-gadget.

$\square$

### 3.2.2.5. Summary of construction

We define:

$$\mathcal{C} := \bigcup_{1 \leq i \leq n} \text{pointsVariable}_i \cup \text{pointsClause}_i$$

$$\mathcal{P} := \bigcup_{1 \leq i \leq n} \text{segmentsVariable}_i \cup \text{segmentsClause}_i$$

The subsequent sections define these sets.

We prove some properties of different gadgets. Every segment for a gadget will only cover points in this gadget (won't interact with any different gadget), so we can prove lemmas *locally*.

TODO:  $y$  axis is increasing values downward on figures (not upwards like in normal).

### 3.2.3. Construction lemmas and proof of Lemma 3

In order to prove Lemma 3 we introduce several auxiliary lemmas proving properties of the construction described in the previous section.

Consider an instance  $S$  of MAX-(3,3)-SAT of size  $n$  with optimum solution satisfying  $k$  clauses. Let us construct an instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover as described in Section 3.2.2 for instance  $S$  of MAX-(3,3)-SAT.

**Lemma 13.** *Instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover admits a solution of size  $15n - k$ .*

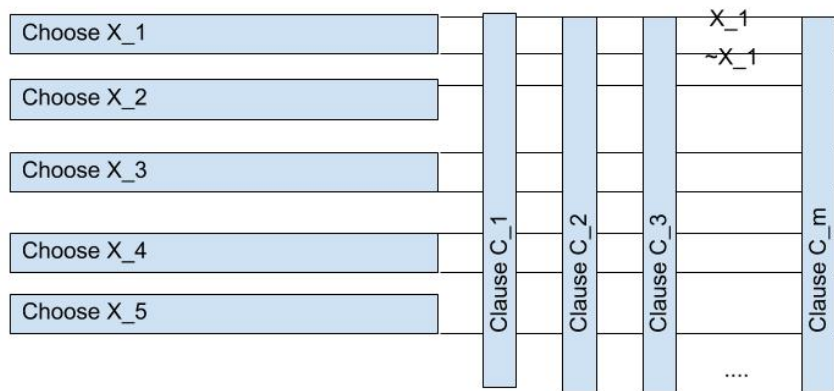


Figure 3.4: **General schema.**

General layout of VARIABLE-gadget and CLAUSE-gadget and how they interact with each other.

TODO: Rename Choose X to VARIABLE-gadget and Clause C to CLAUSE-gadget.

*Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Let the assignment of the variables in the optimum solution to  $S$  be  $\phi : \{x_1, x_2 \dots x_n\} \rightarrow \{\text{true}, \text{false}\}$ .

We cover every VARIABLE-gadget with solution described in Lemma 4, in the  $i$ -th gadget choosing the set of segments corresponding to the value of  $\phi(x_i)$ .

For every clause that is satisfied, say  $c_i$ , let us name the variable that is **true** in it as  $x_i$  and point corresponding to  $x_i$  in  $\text{pointsClause}_i$  as  $a$ . Points in  $\text{pointsClause}_i$  are covered with set  $\text{solClause}_i^{\text{true}, a}$  described in Lemma 9. For every clause that is not satisfied, say  $c_j$ , points in  $\text{pointsClause}_j$  are covered with set  $\text{solClause}_j^{\text{false}}$  described in Lemma 10.

Formally we define sets responsible for choosing variable and satisfying the variable,  $R_i$  and  $C_i$  respectively, as following:

$$\begin{aligned} R_i &= \begin{cases} \text{chooseVariable}_i^{\text{true}} & \text{if } \phi(x_i) = \text{true} \\ \text{chooseVariable}_i^{\text{false}} & \text{if } \phi(x_i) = \text{false} \end{cases} \\ C_i &= \begin{cases} \text{solClause}_i^{\text{true}, a} & \text{if } c_i \text{ satisfied} \\ \text{solClause}_i^{\text{false}} & \text{if } c_i \text{ not satisfied} \end{cases} \\ \mathcal{R} &= \bigcup_{i=1}^n \{R_i \cup C_i : 1 \leq i \leq n\}. \end{aligned}$$

This set covers all the points from  $\mathcal{C}$ , because the sets  $R_i, C_i$  individually cover their corresponding gadgets, as proved in the respective lemmas.

All of these sets are disjoint, so the size of the obtained solution is:

$$|\mathcal{R}| = \sum_{i=1}^n R_i + \sum_{i=1}^n C_i = 3n + 11k + 12(n - k) = 15n - k.$$

□

**Lemma 14.** Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover. Then there exists a solution  $\mathcal{R}'$ , such that  $|\mathcal{R}'| \leq |\mathcal{R}|$ , and for each VARIABLE-gadget  $\mathcal{R}'$  contains at most one of the segments  $(c_i, g_i)$  and  $(f_i, h_i)$ .

*Proof.* Assume that we have  $\{(c_i, g_i), (f_i, h_i)\} \subseteq \mathcal{R}$  for some  $i$ . We will show how to modify  $\mathcal{R}$  into  $\mathcal{R}'$ , such that the number of such  $i$  decreases, while  $\mathcal{R}'$  is still a valid solution of  $(\mathcal{C}, \mathcal{P})$ , and  $|\mathcal{R}'| \leq |\mathcal{R}|$ . Then, by repeating this procedure, we can eventually construct a solution satisfying the property from the Lemma.

To construct  $\mathcal{R}'$ , we remove either  $(c_i, g_i)$  or  $(f_i, h_i)$  from  $\mathcal{R}$ , and then add one extra segment to make  $\mathcal{R}'$  valid. Recall that the  $i$ -th VARIABLE-gadget corresponds to variable  $x_i$  in  $S$ . As every variable in  $S$  is used in exactly 3 clauses, one of the ways of setting  $x_i$  (to either **true** or **false**) must satisfy at least 2 clauses. If that setting is  $x_i = \text{true}$ , then we remove  $(f_i, h_i)$ , otherwise we remove  $(c_i, g_i)$ . Now, there exists at most one CLAUSE-gadget which needs adjustment to make  $\mathcal{R}'$  valid; we do that by adding  $(t_{j,1}, v_{j,1})$  to  $\mathcal{R}'$ .

TODO: Can we really just remove one segment and add another one? I'd think we need to "restructure"  $\mathcal{R}$  around  $\text{pointsVariable}_i$  (saving one segment due to Lemma 5 and Lemma 6) and then again restructure  $\mathcal{R}$  around the clause that we need to fix? □

**Lemma 15.** Suppose we have a solution  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover that is of size  $w$ . Then there exists a solution of  $S$  that satisfies at least  $15n - w$  clauses.

452 *Proof.* Let the clauses in  $S$  be  $c_1, c_2 \dots c_n$  and the variables be  $x_1, x_2 \dots x_n$ . Given a solution  
 453  $\mathcal{R}$  of the instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover, we use Lemma 14 to modify  $\mathcal{R}$  such that for  
 454 any  $i$  it contains at most one of  $(c_i, g_i)$  and  $(f_i, h_i)$ ; this may decrease the cost of  $\mathcal{R}$ , but that  
 455 does not matter in the subsequent construction. To simplify notation, in the remainder of  
 456 this proof we use  $\mathcal{R}$  to refer to the modified solution.

457 Given  $\mathcal{R}$ , we construct a solution of  $S$  by constructing an assignment of variables  $\phi :$   
 458  $\{x_1, x_2 \dots x_n\} \rightarrow \{\text{true}, \text{false}\}$  that satisfies at least  $15n - w$  clauses in  $S$ .

459 **Variables** Recall that due to Lemma 14,  $\mathcal{R}$  contains at most one of  $(c_i, g_i)$  and  $(f_i, h_i)$ .  
 We define the value  $\phi(x_i)$  for the variable  $x_i$  as follows:

$$\begin{cases} \phi(x_i) = \text{true} & \text{if } (c_i, g_i) \in \mathcal{R} \\ \phi(x_i) = \text{false} & \text{if } (f_i, h_i) \in \mathcal{R} \\ \phi(x_i) = \text{false} & \text{otherwise} \end{cases} \quad (3.1)$$

460 Moreover, from Lemma 5 we get  $|\text{pointsVariable}_i \cap \mathcal{R}| \geq 3$  for every  $i$ .

461 **Clauses** For a clause  $C_i = x \vee y \vee z$ ,  $\mathcal{R}$  needs to use at least 11 segments to cover  
 462  $\text{pointsClause}_i - \{x, y, z\}$  in CLAUSE-gadget (Lemma 11).

463 TODO: maybe put something with cases and names of sets as above

464 Moreover, if all of the points  $\{x_{i,0}, y_{i,0}, z_{i,0}\}$  are not covered by the segments from  $\mathcal{R} \cap \text{pointsVariable}_i$ ,  
 465 then  $\mathcal{R}$  needs to cover  $\text{pointsClause}_i$  with at least 12 segments by Lemma 11.

TODO: Maybe remove section below, because we do this calculation at the end anyway  
 We covered CLAUSE-gadget with at least 11 or at least 12 segments:

$$|\bigcup_{i=1}^n \text{segmentsClause}_i \cap \mathcal{R}| \geq 11n + a$$

466 where  $a$  is the number of clauses where none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  were covered by  
 467  $\mathcal{R} \cap \text{segmentsVariable}_j$  for their respective variable  $x_j$ .

468 **Satisfied clauses with chosen variable assignment.** Consider a clause, say  $c_i$ . If  
 469 none of the points  $x_{i,0}, y_{i,0}, z_{i,0}$  in  $\text{pointsClause}_i$  were covered by segments from  $\mathcal{R} \cap \text{segmentsVariable}_j$ ,  
 470 this clause is not satisfied by assignment  $\phi$ .

471 If one of these points is covered by segments from VARIBALE-gadget (TODO better this  
 472 or  $\mathcal{R} \cap \text{segmentsVariable}_j$ ), then denote this point as  $t$  and say it corresponds to variable  $x_j$ .  
 473 Consider the cases of choosing value of  $\phi(x_j)$  in equation (3.1).

474 If  $\mathcal{R}$  contains exactly one of the segments  $(c_j, g_j)$  and  $(f_j, h_j)$ , then the value  $\phi(x_j)$  satisfies  
 475  $c_i$ .

476 If  $\mathcal{R}$  contains neither  $(c_j, g_j)$  nor  $(f_j, h_j)$ , then it is impossible that  $t$  is covered by segments  
 477 in  $\mathcal{R} \cap \text{segmentsVariable}_j$ .

478 This means that  $\phi$  satisfies all but at most  $a$  clauses in  $S$ .

479 To conclude, we proved that given a solution of  $(\mathcal{C}, \mathcal{P})$  of size  $w$ , we have constructed a  
 480 variables assignment  $\phi$  that satisfies at least  $n - a$  clauses of  $S$ . Finally, note that

$$w \geq 3n + 11(n - a) + 12a = 3n + 11n + a = 14n + a,$$

hence

$$15n - w \leq 15n - 14n - a = n - a.$$

481 So  $\phi$  satisfies at least  $15n - w$  clauses of  $S$ . □

We are ready to conclude the proof of Lemma 3.

*Proof of Lemma 3.* By Lemma 13, we know that there exists a solution to  $(\mathcal{C}, \mathcal{P})$  of size  $15n - k$ , so:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \leq 15n - k.$$

Since the optimum solution of  $S$  satisfies  $k$  clauses, then according to Lemma 15:

$$\text{opt}((\mathcal{C}, \mathcal{P})) \geq 15n - k.$$

Therefore, the solution given by Lemma 13 of size  $15n - k$  is an optimum solution to the instance  $(\mathcal{C}, \mathcal{P})$ .  $\square$

### 3.3. FPT for weighted segments with $\delta$ -extensions

TODO: Some intro

**Theorem 4** (FPT for weighted segment cover with  $\delta$ -extensions). *There exists an algorithm that given a family  $\mathcal{P}$  of  $n$  weighted segments (in any direction), a set of  $m$  points  $\mathcal{C}$ , and parameters  $k$  and  $\delta > 0$ , runs in time  $f(k, \delta) \cdot (nm)^c$  for some computable function  $f$  and a constant  $c$ , and outputs a set  $\mathcal{R} \subseteq \mathcal{P}$  such that  $|\mathcal{R}| \leq k$  and  $\mathcal{R}^{+\delta}$  covers all points in  $\mathcal{C}$ , or determines that such a set  $\mathcal{R}$  does not exist.*

To solve this problem we will introduce a lemma about choosing a *good* subset of points.

TODO: Some intuition

**Definition 9.** For a set of collinear points  $C$ , a subset  $A \subseteq C$  is  $(k, \delta)$ -**good** if for any set of segments  $R$  that covers  $A$  and such that  $|R| \leq k$ , it holds that  $R^{+\delta}$  covers  $C$ .

**Lemma 16.** *There exists an algorithm that for any set of collinear points  $C$ ,  $\delta > 0$  and  $k \geq 1$ , outputs a  $(k, \delta)$ -good set  $A \subseteq C$  of size at most  $f(k, \delta)$  for some computable function  $f$ . This algorithm runs in time  $O(|C| \cdot f(k, \delta))$ .*

*Proof.* We prove this for a fixed  $\delta$  by induction over  $k$ .

**Inductive hypothesis.** For any set of collinear points  $C$ , there exists an algorithm that runs in time  $O(|C|k(1 + \frac{1}{\delta}))$  and finds a set  $A$  such that:

- $A$  is  $(\ell, \delta)$ -good for every  $1 \leq \ell \leq k$ ,
- $A$  has size  $|A| < f(\delta, k)$  for some computable function  $f$ ,
- extreme points from  $C$  are in  $A$ .

**Base case for  $k = 1$ .** It is sufficient that  $A$  consists of 2 points: extreme points from  $C$  or a single point if  $|C| = 1$ .

If they are covered with one segment, it must be a segment that includes the extreme points from  $C$ , so it covers the whole set  $C$ .



509 **Inductive step.** Assuming inductive hypothesis for any set of collinear points  $C$  and  
 510 for parameter  $k$ , we will prove hypothesis for  $k + 1$ .

511 Let be  $s$  the minimal segment that includes all points from  $C$ . That is, the extreme points  
 512 of  $C$  are endpoints of  $s$ .

513 We define  $M = \lceil 1 + \frac{2}{\delta} \rceil$  subsegments of  $s$  in the following way. We split  $s$  into  $M$  parts  $v_i$   
 514 of equal length, that is  $|v_i| = \frac{|s|}{M}$  for each  $1 \leq i \leq M$ .

515 Let  $C_i$  be the subset of  $C$  consisting of points laying on  $v_i$ .

516 Let  $t_i$  be the segment with endpoints being the extreme points of  $C_i$  (it might be degen-  
 517 erated segment if  $|C_i| = 1$  or it might be empty if  $C_i$  is empty).

518 TODO: Add a picture with  $v_i$  and  $t_i$  here

519 We use the inductive hypothesis to choose  $(k, \delta)$ -good sets  $A_i$  for sets  $C_i$ . Note that if  
 520  $|C_i| \leq 1$ , then  $A_i = C_i$  and it's still a  $(k, \delta)$ -good set for  $C_i$ .

521 Then we define  $A = \bigcup_{i=1}^M A_i$ . Thus  $A$  includes the extreme points of  $C$ , because they are  
 522 included in the sets  $A_1$  and  $A_M$ .

523 **Proof that  $A$  is  $(k, \delta)$ -good for  $C$ .** Let us take any cover of  $A$  with  $k + 1$  segments  
 524 and call it  $\mathcal{R}$ .

525 For every segment  $t_i$ , if there exists a segment  $x$  in  $\mathcal{R}$  that is disjoint with  $t_i$ , then we have  
 526 a cover of  $A_i$  with at most  $k$  segments using  $\mathcal{R} - \{x\}$ . Since  $A_i$  is  $(k, \delta)$ -good for  $t_i$  and  $C_i$ ,  
 527 then  $(\mathcal{R} - \{x\})^{+\delta}$  covers  $C_i$ . So  $\mathcal{R}^{+\delta}$  covers  $C_i$  as well.

528 If there exists a segment  $t_i$  for which a segment  $x$  as defined above does not exist, then all  
 529  $k + 1$  segments that cover  $A_i$  intersect with  $t_i$ . (Note: There may exist only one such segment  
 530  $t_i$ ). From the inductive hypothesis end points of  $s$  are in  $A_1$  and  $A_M$  respectively, so  $\mathcal{R}$  must  
 531 cover them. Hence there must exist segments starting in the ends of  $s$  and ending somewhere  
 532 in  $t_i$ . Let us call these two segments  $y$  and  $z$ . It follows that:  $|y| + |z| + |t_i| \geq |s|$ . Since  
 533  $|t_i| \leq |v_i| = \frac{|s|}{M} \leq \frac{|s|}{1 + \frac{2}{\delta}} = \frac{|s|\delta}{\delta + 2}$ , we have  $\max(|y|, |z|) \geq |s|(1 - \frac{\delta}{\delta + 2})/2 = \frac{|s|}{\delta + 2}$ .

534 TODO: Add a picture with such segments here

After  $\delta$ -extension, the longer of these segments will lengthen both ways by at least:

$$\frac{|s|\delta}{\delta + 2} = \frac{|s|}{1 + \frac{2}{\delta}} > \frac{|s|}{M} = v_i > t_i.$$

535 Therefore the longer of segments  $y$  and  $z$  will cover the segment  $t_i$  after  $\delta$ -extension,  
 536 therefore  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

537 Since  $C = \bigcup_{i=1}^M C_i$ , then  $\mathcal{R}^{+\delta}$  covers  $C$ .

538 **Complexity** We use the recursive algorithm for subsets  $C_i$ . Every point from  $C$  belongs  
 539 to at most 2 sets  $C_i$ .

540 Apart from recursive algorithm we perform operations linear in size of  $|C| + M$  to calculate  
 541 the sets  $C_i$ .

Therefore it has complexity:

$$O(|C| + M) + \sum_i^M O(|C_i|k(1 + \frac{1}{\delta})) = O(|C| + (1 + \frac{1}{\delta})) + O((\sum_i^M |C_i|)k(1 + \frac{1}{\delta})) \leq O(|C|k(1 + \frac{1}{\delta})).$$

542 □

543 *Proof of Theorem 4.* To construct an algorithm for this problem let us formulate some claims  
 544 about the problem first.

**Definition 10.** Line is **long** if there are at least  $k + 1$  points from  $\mathcal{C}$  on it.

**Claim 1.** *If there are more than  $k$  long lines, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

**Claim 2.** *If there is more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then  $\mathcal{C}$  can not be covered with  $k$  segments.*

Applying the above claims, if we have more than  $k$  long lines or more than  $k^2$  points from  $\mathcal{C}$  that do not lie on any long line, then we answer that there is no solution of size at most  $k$ .

Otherwise, we can split  $\mathcal{C}$  into at most  $k + 1$  sets:  $D$ , at most  $k^2$  points that do not lie on any long line and  $C_i$  – points that lay on  $i$ -th long line. Sets  $C_i$  do not need to be disjoint.

Then for every set  $C_i$ , we can use Lemma 16 to get  $(k, \delta)$ -good set  $A_i$  for  $C_i$ .

Then we have set  $D \cup \bigcup A_i$  of size at most  $f(k, \delta)$  for some computable function  $f$ , that if we have a solution  $\mathcal{R}$  of size at most  $k$  that covers  $D \cup \bigcup A_i$ , then  $\mathcal{R}^{+\delta}$  covers  $\mathcal{C}$ . This is because  $\mathcal{R}$  already covers points  $D$ , they cover  $C_i$ , because they cover  $(k, \delta)$ -good set  $A_i$  with at most  $k$  segments, so  $\mathcal{R}^{+\delta}$  covers  $C_i$ .

After that we shrunk down size of  $\mathcal{C}$  to size of  $f(k, \delta)$  for some computable function  $f$ . Then we would like to shrink down size of  $\mathcal{P}$ . For every collinear subset of  $D$ , we can choose one segment from  $\mathcal{P}$  that covers these points and have the lowest weight or decide there is no segment that cover them. There are at most  $|D|^2$  different segments, because we can distinguish these collinear sets by their extreme points.

This has complexity  $O(|D|^2|\mathcal{P}|)$  and produce shrunk down set  $\mathcal{P}$  of size  $f(k, \delta)$  for some computable function  $f$ .

Then we can iterate over all subsets of shrunk down set  $\mathcal{P}$  and choose the set with the lowest sum of weights that cover  $D$ . This solution would have weight not larger than optimal solution for the problem without extension, because we iterate over all possibilities of covering the subset of  $\mathcal{C}$ .

□

### 3.4. $W[1]$ -completeness for weighted segments in 3 directions

**Theorem 5.** *Weighted geometric set cover with segments in 3 directions is  $W[1]$ -hard. Consider the problem of covering a set  $\mathcal{C}$  of points by selecting  $k$  axis-parallel or right-diagonal weighted segments from a set  $\mathcal{P}$  with weights  $w : \mathcal{P} \rightarrow \mathbb{R}$  with minimal weight. Assuming  $ETH$ , there is no algorithm for this problem with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ .*

**Corollary 2.** *Weighted geometric set cover is  $W[1]$ -hard. Assuming  $ETH$ , there is no algorithm for weighted geometric set cover with running time  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{o(\sqrt{k})}$  for any computable function  $f$ .*

*Proof.* Trivial from Theorem 5.

□

In order to prove theorem 5 we will show reduction from grid tiling problem.

**Definition 11.** In the **grid tiling** problem we are given integers  $n$  and  $k$ , and a function  $f : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \mathcal{P}(\{1 \dots n\} \times \{1 \dots n\})$  specifying the set of allowed tiles for each cell of a  $k \times k$  grid. The task is to find functions  $x, y : \{1 \dots k\} \rightarrow \{1 \dots n\}$  that assign numbers from  $\{1 \dots n\}$  to respectively rows and columns of the grid, so that  $(x(i), y(j)) \in f(i, j)$  for all valid  $i$  and  $j$ , or conclude that such assignment does not exist.

586 **Theorem 6.** Assuming ETH, there is no algorithm for grid tiling problem  $f(k) \cdot n^{o(\sqrt{k})}$  for  
 587 any computable function  $f$ .

588 TODO: proof from reference in literature

589 Let us have an instance of grid tiling problem – size of the grid  $k$ , number of colors  $n$  and  
 590 function of allowed tiles  $f : \{1 \dots k\} \times \{1 \dots k\} \rightarrow \{1 \dots n\} \times \{1 \dots n\}$ .

591 **Construction.** We construct a set  $\mathcal{P}$  of segments and a set  $\mathcal{C}$  of points.

592 First let us choose any ordering of  $n^2$  elements and denote it with bijective function  
 593  $order : \{1 \dots n\} \times \{1 \dots n\} \rightarrow \{1 \dots n^2\}$ .

594 Define  $match_v(i, j)$  and  $match_h(i, j)$  as functions denoting whether two points share x or  
 595 y coordinate.

$$match_v(i, j) \iff order^{-1}(i) = \{x_i, y_i\} \wedge order^{-1}(j) = \{x_j, y_j\} \wedge x_i = x_j$$

$$match_h(i, j) \iff order^{-1}(i) = \{x_i, y_i\} \wedge order^{-1}(j) = \{x_j, y_j\} \wedge y_i = y_j$$

**Points.** Define points:

$$h_{i,j,t} = (i \cdot (n^2 + 1) + t, j \cdot (n^2 + 1))$$

$$v_{i,j,t} = (i \cdot (n^2 + 1), j \cdot (n^2 + 1) + t)$$

Let's define sets  $H$  and  $V$  as:

$$H = \{h_{i,j,t} : 1 \leq i, j, \leq k, 1 \leq t \leq n^2\}$$

$$V = \{v_{i,j,t} : 1 \leq i, j, \leq k, 1 \leq t \leq n^2\}$$

Let us define  $\epsilon = \frac{1}{2k^2}$ . For a point  $p = \{x, y\}$  we define points:

$$p^L = \{x - \epsilon, y\},$$

$$p^R = \{x + \epsilon, y\},$$

$$p^U = \{x, y + \epsilon\},$$

$$p^D = \{x, y - \epsilon\}.$$

Then we define:

$$\mathcal{C} := H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\} \cup V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$$

596 **Segments.** Define horizontal segments.

$$hor_{i,j,t_1,t_2} = (h_{i,j,t_1}^R, h_{i+1,j,t_2}^L)$$

$$ver_{i,j,t_1,t_2} = (v_{i,j,t_1}^D, v_{i,j+1,t_2}^U)$$

$$horBeg_{i,t} = (h_{1,i,1}^L, h_{1,i,t}^L)$$

$$horEnd_{i,t} = (h_{k,i,t}^R, h_{k,i,n^2}^R)$$

$$\begin{aligned}\text{verBeg}_{i,t} &= (v_{i,1,1}^U, v_{i,1,t}^U) \\ \text{verEnd}_{i,t} &= (v_{i,k,t}^D, v_{i,k,n^2}^D)\end{aligned}$$

$$\begin{aligned}HOR &= \{\text{hor}_{i,j,t_1,t_2} : 1 \leq i < k, 1 \leq j \leq k, 1 \leq t_1, t_2 \leq n^2, \text{match}_h(t_1, t_2)\} \\ &\cup \{\text{horBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{\text{horEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}\end{aligned}$$

$$\begin{aligned}VER &= \{\text{ver}_{i,j,t_1,t_2} : 1 \leq i \leq k, 1 \leq j < k, 1 \leq t_1, t_2 \leq n^2, \text{match}_v(t_1, t_2)\} \\ &\cup \{\text{verBeg}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\} \\ &\cup \{\text{verEnd}_{i,t} : 1 \leq i \leq k, 1 \leq t \leq n^2\}\end{aligned}$$

$$DIAG := \{(h_{i,j,t}, v_{i,j,t}) : 1 \leq i, j \leq k, 1 \leq t \leq n^2, a_t \in f(i, j)\}$$

597 TODO: explain that these segments are in fact diagonal

$$\mathcal{P} := HOR \cup VER \cup DIAG$$

598 Weight function is equal to length of the segment for  $HOR$  and  $VER$  and equal to  $\delta = \frac{1}{4k^4}$   
599 for  $DIAG$ .

$$w(s) = \begin{cases} |s| & \text{if } s \in HOR \cup VER \\ \delta & \text{if } s \in DIAG \end{cases}$$

600 **Lemma 17.** *If there exists solution for grid tiling, then there exists solution of instance  $(\mathcal{C}, \mathcal{P})$*   
601 *of geometric set cover with weight  $2k^2(n^2 + 1) - 2k^2\epsilon + k^2\delta - 4(k - \epsilon)$ .*

*Proof.* If there exists a solution to the grid tiling problem  $x, y$ , then there exists a solution that covers all points

$$\{h_{i,j,t} : 1 \leq i, j \leq k, t = \text{order}(x(i), y(j))\} \cup \{v_{i,j,t} : 1 \leq i, j \leq k, t = \text{order}(x(i), y(j))\}$$

602 with  $k^2$  segments from  $DIAG$  and the rest in  $VER$  or  $HOR$  and has weight  $2k^2(n^2 +$   
603  $1) - 2k^2\epsilon + k^2\delta - 4(k - \epsilon)$ .  $\square$

604 **Lemma 18.** *If there exists solution of instance  $(\mathcal{C}, \mathcal{P})$  of geometric set cover with weight at*  
605 *most  $2k \cdot (k(n^2 + 1) - 2 - 2\epsilon(k - 1) + k^2\delta)$ , then there exists a solution for grid tiling*

606 **Proof.** This follows from Lemma ??, because we just take which points are covered with  
607  $DIAG$ .

608 **Claim 3.** *Points  $p^L, p^R, p^U, p^D$  cannot be covered with  $DIAG$ .*

609 **Claim 4.** *Points in  $H \cup \{p^L : p \in H\} \cup \{p^R : p \in H\}$  cannot be covered with  $VER$ .*

610 *Points in  $V \cup \{p^U : p \in V\} \cup \{p^D : p \in V\}$  cannot be covered with  $HOR$ .*

611 **Claim 5.** *For given  $i, j$  if none of the points  $h_{i,j,t}$  ( $v_{i,j,t}$ ) for  $1 \leq t \leq n^2$  are covered with*  
612  *$DIAG$ , then some spaces between neighbouring points were covered twice.*

613 **Claim 6.** *For given  $i, j$  two points  $h_{i,j,t_1}, h_{i,j,t_2}$  ( $v_{i,j,t_1}, v_{i,j,t_2}$ ) for  $1 \leq t_1 < t_2 \leq n^2$  are covered*  
614 *with  $DIAG$ , then one of them had to be also covered with a segment from  $HOR$  ( $VER$ ).*

615 **Proof.** Point  $v_{i,j,t_2}^L$  had to be covered with  $VER$  from Claims 3 and 4. And every segment  
616 in  $VER$  covering  $v_{i,j,t_2}^L$ , covers also  $v_{i,j,t_1}^L$ .

### 617 **3.5. What is missing**

618 We don't know FPT for axis-parallel segments without  $\delta$ -extensions.



## Chapter 4

# Geometric Set Cover with lines

### 4.1. Lines parallel to one of the axis

When  $\mathcal{R}$  consists only of lines parallel to one of the axis, the problem can be solved in polynomial time.

We create bipartial graph  $G$  with node for every line on the input split into sets:  $H$  – horizontal lines and  $V$  – vertical lines. If any two lines cover the same point from  $\mathcal{C}$ , then we add edge between them.

Of course there will be no edges between nodes inside  $H$ , because all of them are parallel and if they share one point, they are the same lines. Similar argument for  $V$ . So the graph is bipartial.

Now Geometric Set Cover can be solved with Vertex Cover on graph  $G$ . Since Vertex Cover (even in weighted setting) on bipartial graphs can be solved in polynomial time.

Short note for myself just to remember how to this in polynomial time:

Non-weighted setting - Konig theorem + max matching

Weighted setting - Min cut in graph of  $\neg A$  or  $\neg B$  (edges directed from  $V$  to  $H$ )

### 4.2. FPT for arbitrary lines

You can find this in Platypus book. We will show FPT kernel of size at most  $k^2$ .

(Maybe we need to reduce lines with one point/points with one line).

For every line if there is more than  $k$  points on it, you have to take it. At the end, if there is more than  $k^2$  points, return NO. Otherwise there is no more than  $k^4$  lines.

In weighted settings among the same lines with different weights you leave the cheapest one and use the same algorithm.

### 4.3. APX-completeness for arbitrary lines

We will show a reduction from Vertex Cover problem. Let's take an instance of the Vertex Cover problem for graph  $G$ . We will create a set of  $|V(G)|$  pairwise non-parallel lines, such that no three of them share a common point.

Then for every edge in  $(v, w) \in E(G)$  we put a point on crossing of lines for vertices  $v$  and  $w$ . They are not parallel, so there exists exactly one such point and any other line doesn't cover this point (any three of them don't cross in the same point).

Solution of Geometric Set Cover for this instance would yield a sound solution of Vertex Cover for graph  $G$ . For every point (edge) we need to choose at least one of lines (vertices)  $v$  or  $w$  to cover this point.

Vertex Cover for arbitrary graph is APX-complete, so this problem is also APX-complete.

#### 4.4. 2-approximation for arbitrary lines

Vertex Cover has an easy 2-approximation algorithm, but here very many lines can cross through the same point, so we can do  $d$ -approximation, where  $d$  is the biggest number of lines crossing through the same point. So for set where any 3 lines don't cross in the same point it yields 2-approximation.

The problematic cases are where through all points cross at least  $k$  points and all lines have at least  $k$  points on them. It can be created by casting  $k$ -grid in  $k$ -D space on 2D space.

Greedy algorithm yields  $\log |\mathcal{R}|$ -approximation, but I have example for this for bipartial graph and reduction with taking all lines crossing through some point (if there are no more than  $k$ ) would solve this case. So maybe it works.

Unfortunately I haven't done this :(

I can link some papers telling it's hard to do.

#### 4.5. Connection with general set cover

Problem with finite set of lines with more dimensions is equivalent to problem in 2D, because we can project lines on the plane which is not perpendicular to any plane created by pairs of (point from  $\mathcal{C}$ , line from  $\mathcal{P}$ ).

Of course every two lines have at most one common point, so is every family of sets that have at most one point in common equivalent to some geometric set cover with lines?

No, because of Desargues's theorem. Have to write down exactly what configuration is banned.



## Chapter 5

# Geometric Set Cover with polygons

### 5.1. State of the art

Covering points with weighted discs admits PTAS [Li and Jin, 2015] and with fat polygons with  $\delta$ -extensions with unit weights admits EPTAS [Har-Peled and Lee, 2009].

Although with thin objects, even if we allow  $\delta$ -expansion, the Set Cover with rectangles is APX-complete (for  $\delta = 1/2$ ), it follows from APX-completeness for segments with  $\delta$ -expansion in Section 3.2.

Covering points with squares is W[1]-hard [Marx, 2005]. It can be proven that assuming *SETH*, there is no  $f(k) \cdot (|\mathcal{C}| + |\mathcal{P}|)^{k-\epsilon}$  time algorithm for any computable function  $f$  and  $\epsilon > 0$  that decides if there are  $k$  polygons in  $\mathcal{P}$  that together cover  $\mathcal{C}$ , *Theorem 1.9* in [Marx and Pilipczuk, 2015].



<sup>685</sup> Chapter 6

<sup>686</sup> Conclusions



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