Approximation and parametrization of Segment Set Cover

Katarzyna Kowalska, Michał Pilipczuk

University of Warsaw, MIMUW

kk371053@students.mimuw.edu.pl

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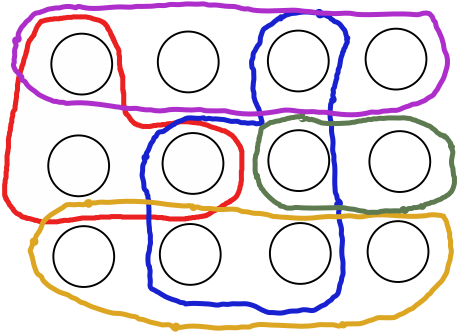
Overview

Problem Statement: Set Cover

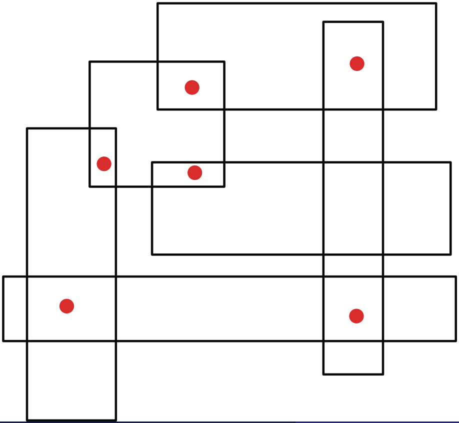
Input: universe $\mathcal{U}$, family S of subsets of $\mathcal{U}$
Output: $C \subseteq S$ such that $|C|$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in $\mathcal{U}$.



Problem Statement: Geometric Set Cover

- Universe $\mathcal{U}$ is set of points in the plane
- Sets S are some geometric shapes
- Formally, each set in S is intersection of $\mathcal{U}$ with some geometric shape.

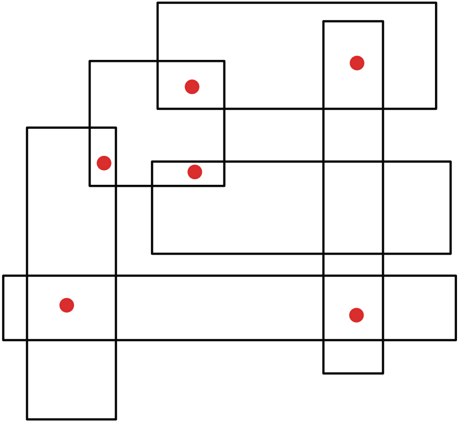
Example for rectangles:



Problem Statement: Segment Set Cover

- Universe $\mathcal{U}$ is set of points in the plane
- Sets S are segments.

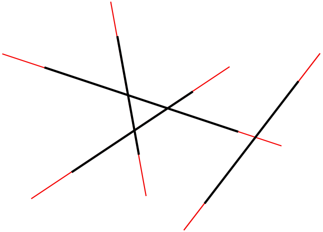
Example (TODO):



δ-extension for SEGMENT SET COVER

- δ-extension for a segment is a segment which is longer by δ fraction at both ends
- We accept solution in which segments cover solution after δ-extension
- The solution is compared to optimal solution without extension.

Example:

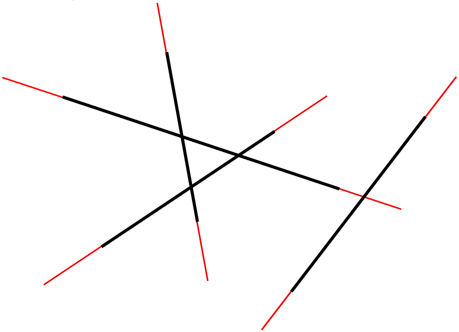


It is based on similar concept of δ-shrinking, which helped to provide FPT algorithm and EPTAS for (originally W[1]-hard and APX-hard) MAXIMUM WEIGHT INDEPENDENT SET OF RECTANGLES.

δ-extensions for Segment Set Cover

- We are given universe $\mathcal{U}$ and set of segments S
- We construct set $S^{+\delta}$ consisting of segments longer by δ fraction at both ends
- We look for $C \subseteq S^{+\delta}$ that $\bigcup C = \mathcal{U}$
- solution C is compared with optimal solution for S without extensions for both parametrized and approximation setting

Example:



Preliminaries: Approximation

Given:

- instance I of the optimization problem (looking for minimal solution)
- weight of the optimal solution $\mathrm{opt}(I)$

p-approximation is algorithm that yields solution of weight not larger than $p \cdot \mathrm{opt}(I)$

PTAS – polynomial time approximation scheme

For every $\epsilon > 0$, there exists $(1+\epsilon)$-approximation algorithm running in time $n^{f(\epsilon)}$ for any computable function f.
Example: $\textsc{Knapsack}$.

APX-hardness

For sufficiently small $\epsilon > 0$, if $(1+\epsilon)$-approximation exists, then $\mathrm{P} = \mathrm{NP}$.
Example: $\mathrm{MAX\text{-}3\text{-}SAT}$ (how many clauses in 3-SAT can be solved)

Approximation results for Set Cover from literature

Set Cover

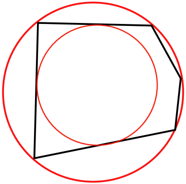
- $\log n$ approximation, no $o(\log n)$ approximation assuing $\mathrm{P} \neq \mathrm{NP}$

Geometric Set Cover

- with fat rectangles is APX-hard
- EPTAS for FAT polygons with δ-extensions **[Har-Peled and Lee, 2012]**

Definition

Fatness of polygons τ is a ratio between a radius of circle circumscribed on this polygon and inscribed in this polygon.



How do we prove that there doesn't exist $(1+\epsilon)$-approximation?

MAX-(3,3)-SAT problem

Given 3-SAT instance with n variables, n clauses and every variable appears in exactly 3 clauses, find an assignment that satisfies the maximum number of clauses (not necessarily all of them).

MAX-(3,3)-SAT is NP-complete, therefore:

NP-completeness

Distinguishing if result of MAX-(3,3)-SAT is n or less than n is NP-complete.

PCP theorem

PCP Theorem

$\mathrm{NP} = \mathrm{PCP}(\log n, \mathcal{O}(1))$.

Remark

Result of MAX-(3,3)-SAT is not less than $\frac{7}{8}n$

PCP theorem implication

Distinguishing if result of MAX-(3,3)-SAT is n or at most $(\frac{7}{8} + \epsilon)n$ is NP-hard.

Therefore MAX-(3,3)-SAT does not have $(\frac{7}{8} + \epsilon)$-approximation

Approxmiation of SEGMENT SET COVER

Theorem APX

SEGMENT SET COVER is APX-hard, ie.
for small $\epsilon = 0.001$, doesn't have $(1+\epsilon)$-approximation unless P = NP.

How can we make this problem *easier*?

- Allow only segments in 2 directions (pararell to the two axes)
- Allow segments to *almost* cover the points, ie. δ-extensions

Theorem APX (rephrased)

SEGMENT SET COVER is APX-hard, even if segments are axis-parallel and problem is relaxed with $\frac{1}{2}$-extension.

Theorem [Har-Peled and Lee, 2012]

Set cover with polygons of bounded fatness (at least τ) with δ-extension has EPTAS.

Results for approximation of Geometric Set Cover

	exact	δ-extensions
fat polygons	APX-hard [Chan and Grant, 2014]	EPTAS [Har-Peled and Lee, 2012]
segments	APX-hard*	APX-hard (Theorem APX)
any polygons	APX-hard*	APX-hard*

Results marked with * follow from results for more restricted settings.

Parametrized algorithms vs. W[1]-hardness

Given:

- parameter k (usually size of the solution)
- size of the instance n

Class	Upper Bound of Complexity	Example
W[1]-hard	$n^{O(k)}$	k-Clique, Grid Tiling
FPT	$f(k) \cdot n^{O(1)}$	Vertex Cover

Set Cover is:

- W[2]-complete
- no algorithm with running time $n^{k-\epsilon}$ for any $\epsilon > 0$

Theorem 1

SEGMENT SET COVER parametrized by the solution size k can be solved in time $k^k \cdot n^{O(1)}$.

Technique: Branching over at most $k+1$ segments on the lines with more than k points on them.

Definition of WEIGHTED SEGMENT SET COVER

Input: universe $\mathcal{U}$, subfamily S of subsets of $\mathcal{U}$ and function of weights assigned to sets $w : S \to \mathbb{N}$
Output: $C \subseteq S$ such that $\sum_{c \in C} w(c)$ is minimal and $\bigcup C = \mathcal{U}$
Minimal subfamily that covers all elements in $\mathcal{U}$.

In parametrized setting: We look for the *best* solution with restricted size $|C| \leqslant k$.
In approximation setting: It is still APX-hard, nothing interesting

Parametrization for set cover with weighted segments

Theorem 3

Weighted set cover with segments is W[1]-hard.
Doesn't have FPT algorithm

How can we make this problem easier?

- Allow less directions (maybe just two parallel to axes)
- Allow δ-extension

Parametrization for set cover with weighted segments

Theorem 4

Weighted set cover with segments with δ-extensions is in FPT.
It can be solved with complexity $O((k/\delta)^k \cdot n^{O(1)})$.

Technique intuition: Provide kernel in problem where you need to cover both points and some segments (these are required by high density of points to cover on them)

Parametrization for set cover with weighted segments

Theorem 3 (rephrased)

Weighted set cover with segments in 3 directions is W[1]-hard.
These directions are: parallel to axis and diagonal 45 degrees.
Doesn't have FPT algorithm

This result is particalarily interesting, because unweighted problem is in FPT.
Nobody[1] knows if this still holds with 2 directions.

Results of parametrization of SEGMENT SET COVER

	exact	δ-extensions
unweighted	FPT (Theorem 1)	FPT*
weighted 2 directions	???	FPT*
weighted 3 directions	W[1]-hard (Theorem 3)	FPT*
weighted any directions	W[1]-hard*	FPT (Theorem 4)