Notes

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Contents

Chapter 1

Microeconomics

1.1 Consumer

1.1.1 Budget

Definition 1 (budget constraint)**.** *How to define "afford"?*

1. *A consumer's consumption bundle is* (x_1, x_2)*.*
2. *The affordable consumption bundle is* $p_1x_1 + p_2x_2 \leq m$
3. *The affordable consumption bundle and the income* m *are called the budget set of the consumer.*
4. *The budget line is the set of* (x_1, x_2) *that* $p_1x_1 + p_2x_2 = m$*.*
5. *The slope of the budget line is the opportunity cost of consuming good 1.*
6. *If* $p_2 = 1$*, good 2 is the composite good that stands for everything else the consumer might want to consume other than good 1.*

How the budget line changes?

1. If m increase, the budget line parallel shift outward (or inward).
2. If p_1 increase, the budget line become steeper (or flatter).

We could change the budget line from $p_1x_1 + p_2x_2 = m$ to $\frac{p_1}{p_2}x_1 + x_2 = \frac{m}{p_2}$. Here the price of x_2 is 1, which is the numeraire price. All prices and income will be measured against it.

How to change the budget constraint?

1. quantity tax: $p_1 \Rightarrow p_1 + t$
2. value tax (ad valorem tax): $p_1 \Rightarrow (1 + \tau)p_1$
3. lump-sum tax: $m \Rightarrow m - t$
4. subsidy:
   (a) quantity subsidy: $p_1 \Rightarrow p_1 - t$
   (b) value subsidy (ad valorem subsidy): $p_1 \Rightarrow (1 - \tau)p_1$
   (c) lump-sum subsidy: $m \Rightarrow m + t$
5. rationing: $x_1 \leq \hat{x}_1$

Sometimes tax, subsidy and rationing are combined.

1.1.2 Consumer Preference

The preference relation between two consumption bundle (x_1, x_2) and (y_1, y_2):

1. $(x_1, x_2) \succ (y_1, y_2)$ (strictly preferred): the consumer definitely wants x-bundle rather than y-bundle.
2. $(x_1, x_2) \sim (y_1, y_2)$ (indifferent): the consumer is equality satisfied by the two bundle.
3. $(x_1, x_2) \succeq (y_1, y_2)$ (weakly preferred): $\succ$ or $\sim$.

Assumption about preference:

- Complete: $(x_1, x_2) \succeq (y_1, y_2)$ or $(y_1, y_2) \succeq (x_1, x_2)$. Quite reasonable.
- Reflexive: $(x_1, x_2) \succeq (x_1, x_2)$. May not work for child.
- Transitive: If $(x_1, x_2) \succeq (y_1, y_2)$ and $(y_1, y_2) \succeq (z_1, z_2)$, we have $(x_1, x_2) \succeq (z_1, z_2)$. This one is controversial. It is a hypothesis, not a statement of logic.

The weakly preferred set for (x_1, x_2) are $\{(y_1, y_2) : (y_1, y_2) \succeq (x_1, x_2)\}$.

The indifference curve is $\{(y_1, y_2) : (y_1, y_2) \sim (x_1, x_2)\}$.

Theorem 1. *The indifference curves cannot cross.*

Classification of preferences:

1. perfect substitute: the slope of indifference curve are all -1.
2. perfect complement: the slope is L-shape.
3. bad: the slope points to top right. The direction of increasing preference is down and to the right.
4. neutral: the consumer doesn't care about it. the slope is vertical line.
5. satiation: circles of preference. The better preference is in the center.

The well-behaved preference:

1. monotonicity of preference:
   - If $y_1 > x_1$ and $y_2 > x_2$, then $(y_1, y_2) \succ (x_1, x_2)$.
   - It implies the slope is negative. If we start at (x_1, x_2) and move around. The up and right will increase preference ,the bottom and left will decrease preference. The indifference position must be top left and bottom right, so the slope is negative.
2. We prefer average to extreme:
   - So the weakly preferred set is convex set:
   $$\forall t \in (0,1], \big(x_1, x_2\big) \sim (y_1, y_2) \Rightarrow \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2\big) \succeq (x_1, x_2) \tag{1.1}$$
   - Why is the preference convex?
     – We often consumes goods togethwer, rather than only one of them.
     – In longer time horizon such as months, we consume goods together.
   - One extension is strict convexity, which replace $\succeq$ by $\succ$.

1.1.3 Marginal Rate of Substitution

1. The marginal rate of substitution (MRS) is the slope of indifference $\dfrac{\mathrm{d}x_2}{\mathrm{d}x_1}$, which is called the marginal rate of substitution of good 2 for good 1.
2. The MRS is typically negative.
3. THe MRS measure the exchange rate. The consumer will trade one for another if the exchange rate is not MRS.
4. If x_2 is composite good ($p_2 = 1$), the MRS is the price of x_1.
5. For strictly convex indifference curve, the MRS exhibits a diminishing marginal rate of substitution.

1.1.4 Utility

The utility function measures the order of preference. It assigns a number to every indifference curve that respect preference order. Only the order matters, and the magnitude does not matter. So it is ordinal utility.

There is a cardinal utility, but cardinality did not add anything to the description of choices.

Theorem 2. *If $f(x)$ is a monotonic function ($f'(x) > 0$), then for a utility function $u(x_1, x_2)$, $f\big(u(x_1, x_2)\big)$ represents the same preference.*

Given a utility function, we could calculate the indifference curve by calculating its level set.

The utility functions for common preferences:

1. perfect substitute: $u(x_1, x_2) = ax_1 + bx_2$.
2. perfect complement: $u(x_1, x_2) = \mathbf{min}\,\{ax_1, bx_2\}$.
3. quasilinear preference: $u(x_1, x_2) = k = v(x_1) + x_2$. k is the height of the curve. It is not realistic, but easy to work with.
4. Cobb-Douglas: $u(x_1, x_2) = x_1^c x_2^d$. Other forms are $x_1^a x_2^{1-a}$ and $c \ln x_1 + d \ln x_2$.

The implication of curvature of preference:

- Perfect substitute: don't care about swap between two goods.
- Perfect complements: never allow swap. Remain constant ratio.

So the utility curve is often convex, which means the consumer would like to trade some for another, but not all, which is a balance between never swap and don't care.

The marginal utility of good 1 measures the rate of change by good 1. Marginal in economics means derivative. So

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} \tag{1.2}$$

The marginal utility depends on the magnitude of utility function. So it will change after applying monotonic transformation.

For x_1 and x_2 in indifference curve $u(x_1, x_2) = k$, take the differentials and we have

$$\mathrm{d}u = \frac{\partial u}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial u}{\partial x_2}\,\mathrm{d}x_2 = 0 \tag{1.3}$$

So

$$\text{MRS} = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{\dfrac{\partial u}{\partial x_1}}{\dfrac{\partial u}{\partial x_2}} \tag{1.4}$$

If there is monotonic transformation $v(x_1, x_2) = f(u(x_1, x_2))$, we have

$$\text{MRS} = -\frac{\dfrac{\partial v}{\partial x_1}}{\dfrac{\partial v}{\partial x_2}} = -\frac{\dfrac{\mathrm{d}f}{\mathrm{d}u}\dfrac{\partial u}{\partial x_1}}{\dfrac{\mathrm{d}f}{\mathrm{d}u}\dfrac{\partial u}{\partial x_2}} = -\frac{\dfrac{\mathrm{d}f}{\mathrm{d}u}}{\dfrac{\mathrm{d}f}{\mathrm{d}u}} \times \frac{\dfrac{\partial u}{\partial x_1}}{\dfrac{\partial u}{\partial x_2}} = -\frac{\dfrac{\partial u}{\partial x_1}}{\dfrac{\partial u}{\partial x_2}} \tag{1.5}$$

So the monotonic transformation did not change MRS.

1.1.5 Choice

The optimal choice is a affordable consumption bundle with highest utility. It has several exceptions:

1. The choice is often the tangent point between budget line and the indifference curve.
2. Sometimes it is not the tangent point. So we have boundary optimum and interior optimum.
3. If the preference is convex, all the tangent points are optimal points.
4. If the preference is strictly convex, there is only one optimal choice.

The optimal choice is called demanded bundle.

In optimal choice, the exchange rate is $\dfrac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\dfrac{p_1}{p_2}$ of budget line, which equals MRS. So we need to calculate (x_1, x_2) that

$$\text{MRS} = -\frac{p_1}{p_2} \tag{1.6}$$

So we need to solve the Lagrangian with auxiliary funtion and λ:

$$L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m) \tag{1.7}$$

with solution

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} - \lambda p_1 &= 0 \\ \frac{\partial L}{\partial x_2} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} - \lambda p_2 &= 0 \\ \frac{\partial L}{\partial \lambda} &= p_1 x_1^* + p_2 x_2^* - m &= 0 \end{aligned} \tag{1.8}$$

Example 1. *The result for Cobb-Douglas preference is*

$$\begin{aligned} x_1^* &= \frac{c}{c+d} \times \frac{m}{p_1} \\ x_2^* &= \frac{d}{c+d} \times \frac{m}{p_2} \end{aligned} \tag{1.9}$$

So

$$\frac{p_1 x_1^*}{m} = \frac{c}{c+d} \tag{1.10}$$

Thus in Cobb-Douglas consumer always spends a fixed fraction of his income on each good. This could be used to check whether the preference is Cobb-Douglas.

Example 2. *The quantity tax is the tax on quantity, and the income tax is the tax on income. The original budget line is:*

$$p_1 x_1 + p_2 x_2 = m \tag{1.11}$$

The optimal choice after quantity tax is:

$$(p_1 + t) q_1^* + p_2 q_2^* = m \tag{1.12}$$

Assuming the government always receive the same tax revenue in both tax form, the optimal choice after income tax is:

$$p_1 i_1^* + p_2 i_2^* = m - t q_1^* \tag{1.13}$$

Because (q_1^*, q_2^*) also satisfy (1.13) and (i_1^*, i_2^*) is chosen, we have $(i_1^*, i_2^*) \succeq (q_1^*, q_2^*)$, which means income tax is not worse than quantity tax for consumer, and the government is indifferent.

So the conclusion is that taxing the income is better than taxing the expense.

The limitation of the conclusion is that this applies to one consumer and the consumer preference will not change after the tax is applied.

1.1.6 Demand

The demand function gives the optimal amount of the good as a function of price and income: $x_i = x_i(p_1, p_2, m)$.

1.1.6.1 Income Offer Curve

- The normal good is the one that $\dfrac{\mathrm{d}x_1}{\mathrm{d}m} > 0$.
- The inferior good is $\dfrac{\mathrm{d}x_1}{\mathrm{d}m} < 0$.
- The luxury good is $\dfrac{\mathrm{d}x_1}{\mathrm{d}m} \times \dfrac{m}{x_1} > 1$.
- The necessary good is $0 < \dfrac{\mathrm{d}x_1}{\mathrm{d}m} \times \dfrac{m}{x_1} < 1$.
- The implicit function between x_1 and x_2 when m changes is called income offer curve, also known as income expansion path.
- The curve of $x_1 \sim m$ is called Engel curve.

Example 3. *The income offset curve for quisilinear utility is very special:*

$$x_2 = \begin{cases} 0 & \text{when } m \le p_2 \\ \dfrac{m}{p_2} - 1 & \text{when } m > p_2 \end{cases} \tag{1.14}$$

So $x_2 = 0$ for a while, and then x_1 become constant.

Theorem 3. *If the consumer preference is homothetic preference, that is*

$$\forall t > 0, u(x_1, x_2) > u(y_1, y_2) \Rightarrow u(tx_1, tx_2) > u(ty_1, ty_2) \tag{1.15}$$

the income offer curve are all straight line through the origin. Perfect substitute, perfect complement and Cobb-Douglas are all homothetic preference. homothetic preference is not realistic for large t.

1.1.6.2 Price Offer Curve

- The implicit function between x_1 and x_2 when p_1 changes is called price offer curve. It rotates against $(0, x_1)$.
- The curve of $x_1 \sim p_1$ is called demand curve.
- Normally we have $\dfrac{\mathrm{d}x_1}{\mathrm{d}p_1} < 0$, so the demand curve goes downward.
- The Giffen good is $\dfrac{\mathrm{d}x_1}{\mathrm{d}p_1} > 0$.

1.1.6.3 Discrete Good

Definition 2 (reservation price)**.** *The reservation price is the price at which the consumer is just indifferent to consuming or not consuming the good.*

Suppose at reservation price r_n the consumer is indifferent between n and $n-1$ x_1, we have

$$u(n, m - r_n \times n) = u(n-1, m - r_n \times (n-1)) \tag{1.16}$$

If we assume the utility function is quisilinear, that is $u(x_1, x_2) = v(x_1) + x_2$, for r_n we have

$$v(n) + m - r_n \times n = v(n-1) + m - r_n \times (n-1) \tag{1.17}$$

So

$$r(n) = v(n) - v(n-1) = \Delta v(n) \tag{1.18}$$

It means the reservation price is the increase in utility.

1.1.6.4 Substitutes and Complements

- The substitute (gross substitute) is $\frac{\mathrm{d}x_1}{\mathrm{d}x_2} > 0$.
- The complement (gross complement) is $\frac{\mathrm{d}x_1}{\mathrm{d}x_2} < 0$.

If there are more than 2 goods, it is possible that a is a substitute for b, and b is a complement for a.

1.1.7 Revealed Preference

Assumption:

- The preference is strictly convex.
- observation does not change preference.

When (x_1, x_2) is chosen when (y_1, y_2) is affordable, that is $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$, we say (x_1, x_2) is directly revealed preferred to (y_1, y_2). So it is a relation between actual demand and the could have been demand.

Theorem 4. *If (x_1, x_2) is directly revealed preferred to (y_1, y_2), then it is preferred.*

It is natural to define indirectly revealed preferred relation. Because that two comparision may happen on different prices and budget liens. The directly and indirectly revealed prefeered relation is called revealed preferred.

If we assume preference is:

- convex.
- monotonic.

Then the revealed preference area is the top right section.

Theorem 5 (weak axiom of revealed preference, WARP)**.** *If (x_1, x_2) is directly revealed preferred to (y_1, y_2), and two bundles are not the same, then it cannot happen that (y_1, y_2) is directly revealed preferred to (x_1, x_2). So if we choose x-bundle when y-bundle is affordable, then when y-bundle is choosen, the x-bundle must be unaffordable.*

To check whether the preference satisfies WRAP, we calculate a square matrix using prices from i and bundle from j:

$$\begin{bmatrix} \ddots & & & & \\ & m_{ii} & \cdots & m_{ij}^* & \\ & & \ddots & & \\ & m_{ji}^* & \cdots & m_{jj} & \\ & & & & \ddots \end{bmatrix} \tag{1.19}$$

where $m_{ij} = \vec{p_i} \times \vec{x_j}$. We then compare m_{ij} and m_{ii} (same i, different j, not m_{jj}) and mark m_{ij} as m_{ij}^* if $m_{ij} < m_{ii}$. A violation of WARP is found if there exists $i \neq j$ that m_{ij}^* and m_{ji}^* both have a star.

Theorem 6 (strong axiom of revealed preference, SARP)**.** *If (x_1, x_2) is revealed preferred to (y_1, y_2), and two bundles are not the same, then it cannot happen that (y_1, y_2) is revealed preferred to (x_1, x_2).*

SARP is both a necessary and sufficient condition for observed choices to be compatible with the economic model of consumer choice.

The algorithm of detecting SARP is almost the same as WARP. We need to add the chain of stars so if $m_{ij}^* < m_{ii}$ and $m_{jk}^* < m_{jj}$, we mark m_{ik} with star too:

$$\begin{bmatrix} \ddots & & & & & \\ & m_{ii} & \cdots & m_{ij}^* & \cdots & m_{ik}^* \\ & & \ddots & & & \vdots \\ & & & m_{jj} & \cdots & m_{jk}^* \\ & & & & \ddots & & \ddots \end{bmatrix} \tag{1.20}$$

The validation of SARP is the same as WARP.

Example 4 (index)**.** *The index is an average of consumption in year t compare to base year b. There are many choices:*

1. *Laspeyres quantity index*: $L_q = \frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b}$.

2. *Paasche quantity index*: $P_q = \dfrac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b}$.
3. *Laspeyres price index*: $L_p = \dfrac{p_1^t x_1^b + p_2^t x_2^b}{p_1^b x_1^b + p_2^b x_2^b}$.
4. *Paasche price index*: $P_p = \dfrac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^t + p_2^b x_2^t}$.
5. *expenditure index*: $E = \dfrac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^b + p_2^b x_2^b}$.

The differences are:

- Quantity index will change quantity.
- Price index will change price.
- Laspeyres chooses base year.
- Paasche chooses target year.

The conclusion from 4 indexes are:

- If $P_q > 1$, we have $p_1^t x_1^t + p_2^t x_2^t > p_1^t x_1^b + p_2^t x_2^b$, so $(x_1^t, x_2^t) \succeq (x_1^b, x_2^b)$ and the target year is better off.
- If $P_q < 1$, we have $p_1^t x_1^t + p_2^t x_2^t < p_1^t x_1^b + p_2^t x_2^b$. We cannot say anything because (x_1^b, x_2^b) is not chosen at t.
- If $L_q < 1$, we have $(x_1^b, x_2^b) \succeq (x_1^t, x_2^t)$.
- If $P_p > E$, we have $p_1^b x_1^b + p_2^b x_2^b > p_1^b x_1^t + p_2^b x_2^t$, so the base year is better off.
- If $L_p < E$, we have $p_1^t x_1^b + p_2^t x_2^b < p_1^t x_1^t + p_2^t x_2^t$, so the target year is better off.

Example 5. *The indexing of social security payment is done so the average senior citizen could buy the same bundle in year t as in year b. Because the bundle in year b is affordable in year t, the optimal bundle in year t is not worse, so they will be better off.*

1.1.8 Slutsky Equation

When the price of good 1 change:

- Slutsky substitution effect: the exchange rate between two goods changed.
- income effect: if we maintain the same ratio of two goods, the same income will have different purchasing power.

So the price move could be broken into two steps:

- Change the relative price so the purchasing power remain the same.
- Change the income and holding the ratio constant.

Assume the (x_1, x_2) are the optimal choice in m. When the price of x_1 raise from p_1 to $p_1{}'$, we need to adjusted income to m' to make the same (x_1, x_2) affordable. So

$$\begin{aligned} p_1 x_1 + p_2 x_2 &= m \\ p_1{}' x_1 + p_2 x_2 &= m' \end{aligned} \tag{1.21}$$

So $m' - m = (p_1{}' - p_1)x$, which is

$$\Delta m = \Delta p_1 \times x_1 \tag{1.22}$$

Here is the change of optimal choice:

1. $x_1(p_1, m)$: the original optimal choice.
2. $x_1(p_1^{'}, m')$: the optimal choice after substitution effect.
3. $x_1(p_1^{'}, m)$: the optimal choice after income effect.

So the total change in demand is:

$$\begin{aligned} x_1(p_1^{'}, m) - x_1(p_1, m) = \quad & x_1(p_1^{'}, m') \quad - x_1(p_1, m) \quad \text{(Slutsky effect)} \\ & + x_1(p_1^{'}, m) \quad - x_1(p_1^{'}, m') \quad \text{(income effect)} \end{aligned} \tag{1.23}$$

Assume the original demand bundle is $(\overline{x}_1, \overline{x}_2, \overline{p}_1, \overline{p}_1, \overline{m})$. The Slutsky demand is

$$x_1^s(p_1, p_2, \overline{x_1}, \overline{x_2}) = x_1(p_1, p_2, p_1\overline{x}_1 + p_2\overline{x}_2) \tag{1.24}$$

Take the derivative against p_1, we have:

$$\frac{\partial x_1^s}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial m} \times \overline{x}_1 \tag{1.25}$$

Rearranging the items we have:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1}{\partial m} \times \overline{x}_1 \tag{1.26}$$

The substitution effect $\frac{\partial x_1^s}{\partial p_1}$ is always negative. The income effect $\frac{\partial x_1}{\partial m} \times \overline{x}_1$ is positive for normal good and negative for inferior. So the end result is negative for normal good, and not sure for inferior good. If it is Giffen good, the income effect has to be very large.

Example 6. *The perfect complement has no substitution effect. The perfect substitute and quisilinear have no income effect.*

Definition 3 (Hicks substitution effect)**.** *The substitution effect that keeps the utility constant rather than keeping purchasing power constant. The Hicks substitution effect is always negative. And when the change to price is small, the Hicks is the same as Slutsky.*

So there are 3 ways to deal with price change:

1. Hold income fixed, the demand curve
2. Hold purchasing power fixed (Slutsky), the Slutsky demand curve.
3. Hold utility fixed (Hicks), the compensated demand curve.

1.1.9 Buying and Selling

1.1.9.1 Definition

In previous sections the income m was given. Now the (x_1, x_2) are given and we calculate the response to price change:

- The given bundle is called endowment and denoted by (ω_1, ω_2).
- The endowment is always on the budget line.
- The gross demand (x_1, x_2) is the final bundle.
- The net demand $(x_1 - \omega_1, x_2 - \omega_2)$ is the difference.
- If $(x_1 - \omega_1) > 0$, the consumer is net buyer of good 1, or it is net seller.

1.1.9.2 Changes caused by Endowment and Price

If we want to know how the gross demand changes when the endowment changes, treat it like income change.

If the price of good 1 change:

- If the consumer is a seller of x_1
  - If $p_1 \downarrow$:
    - If it remains a seller, worse off.
    - Switch to buyer: not sure.
  - If $p_1 \uparrow$: remain a seller.
- If the consumer is a buyer of x_1:
  - If $p_1 \uparrow$:
    - If it remains a buyer, worse off.
    - Switch to seller: not sure.
  - If $p_1 \downarrow$: remain a buyer.

1.1.9.3 Price Offer Curve and Demand Curve

For the price offer curve :

- The price offer curve must pass the endowment:
  1. One utility curve will pass the endowment.
  2. All budget line will pass through the endowment.
  3. There must exists one budget line that is tangent to the utility curve.
- The price offer curve is higher than the utility curve just mentioned.

For the demand curve: There is a point $(\omega_1, p_1(\omega_1))$. For $p_1 > p_1(\omega_1)$, the consumer is a seller. Or he is a buyer.

1.1.9.4 Slutsky Equation with Endowment

The Slutsky equation for endowment will change because m is now a function of p_1. let $x_1(p_1, m(p_1))$ be the demand function of good 1. Differentiate it and we have:

$$\frac{\partial x_1(p_1, m(p_1))}{\partial p_1} = \frac{\partial x_1(p_1, m)}{\partial p_1} + \frac{\partial x_1(p_1, m)}{\partial m} \times \frac{\mathrm{d}m(p_1)}{\mathrm{d}p_1} \tag{1.27}$$

Because we have $m(p_1) = p_1\omega_1 + p_2\omega_2$, we have

$$\frac{\mathrm{d}m(p_1)}{\mathrm{d}p_1} = \omega_1 \tag{1.28}$$

From Slutsky equation we have

$$\frac{\partial x_1(p_1, m)}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1(p_1, m)}{\partial m} \times x_1 \tag{1.29}$$

Put (1.28) and (1.29) into (1.27), we have

$$\frac{\partial x_1(p_1, m(p_1))}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} + \frac{\partial x_1(p_1, m)}{\partial m} \times (\omega_1 - x_1) \tag{1.30}$$

Now the purchasing power change due to price change has 3 components:

1. $\frac{\partial x_1^s}{\partial p_1}$: the Slutsky effect.
2. $-\frac{\partial x_1(p_1, m)}{\partial m} \times x_1$: the ordinary income effect . It ignores the endowment and adjust the total income so x_2 of good 2 could be satisfied.
3. $\frac{\partial x_1(p_1, m)}{\partial m} \times \omega_1$: the endowment income effect . It adjust the ordinary income effect so the endowment is met.

1.1.9.5 Labor Supply

Here is the assumption of labor supply:

- M: money endowment. A constant value.
- C: the total consumption.
- p: price of consumption.
- w: wage rate.
- L: length of labor supply.

So we have the formula:

$$\begin{aligned} pC &= M + wL \\ pC - wL &= M \end{aligned} \tag{1.31}$$

Let's define several other variables:

- $\overline{L}$: maximum labor supply.
- $R = \overline{L} - L$: relax, or leisure.
- $\overline{R} = \overline{L}$: total leisure.
- $\overline{C} = \frac{M}{p}$: how many goods could be bought using endowment.

So we could change the formula to:

$$\begin{aligned} pC - wL &= M \\ pC + w(\overline{L} - L) &= M + w\overline{L} \\ pC + w(\overline{L} - L) &= p\overline{C} + w\overline{L} \\ pC + wR &= p\overline{C} + w\overline{R} \end{aligned} \tag{1.32}$$

So the sum of consumption and leisure equals the endowment of consumption and leisure. According to the Slutsky equation, we have

$$\frac{\partial R}{\partial \omega} = \frac{\partial R^s}{\partial \omega} + (\overline{R} - R)\frac{\partial R}{\partial m} \tag{1.33}$$

Assume leisure is a normal good, that is we prefer more leisure when income rises. There is no direction conclusion of (1.33). However, if $\overline{R}$ is large, it is possible that $\frac{\partial R}{\partial \omega}$ becomes positive. So the labor supply curve could be backward-bending

1.1.10 Intertemporal Choice

Assume there are two period T_1 and T_2, and the consumer will be given m_1 and m_2. The consumer could choose to consume c_1 and c_2 in two periods. Assume the interest rate is r. The fomula is:

$$c_1 + \frac{c_2}{1+r} = m_1 + \frac{m_2}{1+r} \tag{1.34}$$

The conclusion is:

1. The consumer is lender if $c_1 < m_1$, and borrower if $c_1 > m_1$.
2. Because the slope of budget line is $\frac{\mathrm{d}c_2}{\mathrm{d}c_1} = -(1+r)$, increasing r will make the budget line steeper.
3. According to the Slutsky equation, if $m_1 - c_1$ is negative, raising r will reduce c_1.

The formular above did not consider tax. If the tax rate is t and tax is deductible for interest payment, the r will become $(1-t)r$ in all cases.

1.1.11 Uncertainty

Definition 4 (expected utility function)**.** *For the utility in the state of nature, consumer will have contigent consumption plan, which is a specification of what to consume in each state of nature. Assumme the probability is π_1 and π_2, and the consumption is c_1 and c_2, the expected utility function, or von Neumann-Morgenstern utility function, is :*

$$u(c_1, c_2, \pi_1, \pi_2) = \pi_1 v(c_1) + \pi_2 v(c_2) \tag{1.35}$$

The expected utility function is unique up to affine transformation $f(x) = ax + b$.

This form is different from other utility function because we cannot consume two goods together. We have to always consume only one of them.

The consumer is risk averse if $v(x)$ is concave, and risk lover if it is convex.

Example 7 (insurance)**.** *Consumer has asset a, and when a bad event happens the asset would become b. The insurance cost is γ, so a γK insurance could protect K asset. The event b happens is π.*

The situations are:

$$\begin{aligned} c_1 &= a - \gamma K \\ c_2 &= b - \gamma K + K \end{aligned} \tag{1.36}$$

The expected utility is

$$(1-\pi)u(c_1) + \pi u(c_2) \tag{1.37}$$

For insurance company, its expected income is $\gamma K - \pi K - (1-\pi)0 = (\gamma - \pi)K$. If we force the insurance company to be neutual, $\gamma - \pi = 0 \Rightarrow \gamma = \pi$.

Let's optimize the Lagrange equation

$$f = (1-\pi)u(c_1) + \pi u(c_2) - L[(c_2 - b)\gamma + (c_1 - a)(1-\gamma)] \tag{1.38}$$

We have

$$\begin{aligned} \frac{\partial f}{\partial c_1} &= (1-\pi)\frac{\partial u(c_1)}{\partial c_1} + L(1-\gamma) = 0 \\ \frac{\partial f}{\partial c_2} &= \pi \frac{\partial u(c_2)}{\partial c_2} + L\gamma = 0 \end{aligned} \tag{1.39}$$

So $\frac{\partial u(c_1)}{\partial c_1} = \frac{\partial u(c_2)}{\partial c_2}$. If the consumer preference is risk averse, there exists solution. Because $u' < 0$, we have $c_1 = c_2$. so

$$\begin{aligned} a - \gamma K &= b - \gamma + K \\ K &= a - b \end{aligned} \tag{1.40}$$

It means if the consumer is risk averse, he will insure all the potential loss.

1.1.12 CAPM

In the theory of capital asset pricing model (CAPM), there are two assets: one is risk-free asset with return r_f, and the other with return rm and standard derivation σ_m. Assume $1-x$ percent is invested risk-free asset and x in risk asset. The portfolio return and risk is:

$$\begin{aligned} r_x &= x r_m + (1-x) r_f = (r_m - r_f)x + r_f \\ \sigma_x &= x\sigma_m \end{aligned} \tag{1.41}$$

So there is a trade-off between return and risk, which is the price of risk:

$$p = \frac{r_m - r_f}{\sigma_m} \tag{1.42}$$

The utility $u(\mu, \sigma)$ is convex (σ is the x-axis) because people would prefer high return for the same risk. In the optimal choice, we have

$$MRS = -\frac{\frac{\partial u}{\partial \sigma}}{\frac{\partial u}{\partial \mu}} = \frac{r_m - r_f}{\sigma_m} \tag{1.43}$$

The beta of risky asset relative to the market is:

$$\beta_i = \frac{\textbf{Cov}\left(r_i, r_m\right)}{\textbf{Var}\left[r_m\right]} \tag{1.44}$$

For an asset, its risk adjustment is

$$\beta_i \sigma_m p = \beta_i \sigma_m \frac{r_m - r_f}{\sigma_m} = \beta_i (r_m - r_f) \tag{1.45}$$

So the adjusted return of risky asset is

$$r_m - \text{risk adjustment} = r_m - \beta_i \sigma_m p = r_m - \beta_i (r_m - r_f) \tag{1.46}$$

If the adjusted return is higher than r_f, it is a good deal. So in equilibrium case the adjusted return is r_f, which means

$$r_m = r_f + \beta_i (r_m - r_f) \tag{1.47}$$

Chapter 2

Game Theory

2.1 Basics

A game is a description of strategic interaction that includes the constraints on the actions that the players can take, but does not specify the actions that they do take.

classification:

strategic game the player chooses his plan of action once and for all, and all players' decisions are made simultaneously.

extensive game specify the possible orders of events. Each player can consider his play at later time.

2.2 Static Games of Complete Information

2.2.1 Definition

Definition 5 (common knowledge). *An event E is common knowledge if:*

1. *Everyone knows E*
2. *Everyone knows that everyone knows E, and so on ad infinitum*

The assumptions we will have are:

1. Players are rational, which means they will maximize their payoff consistently.
2. Players are intelligent, which means they know everything about the game
3. Common knowledge, which means it is common knowledge that all players are rational and intelligent.
4. Self-enforcement, which means they will act independently.

Definition 6 (strategy). *A strategy is a plan of action intended to accomplish a specific goal. For example, "if A do something then I will respond with ...". The set of all strategies for player i is denoted as S_i.*

Definition 7 (strategy profile). *The strategy profile is the set of all strategies of all players.*

Definition 8 (pure strategy). *A pure strategy for a player is a deterministic plan of actions.*

Definition 9 (normal-form game). *A normal-form game consistes of three features:*

1. *A finite set of players. $N = \{1, 2, ..., n\}$.*
2. *A collection of sets of pure strategies, $\{S_1, S_2, ..., S_n\}$. The set of all strategy profiles is denoted by $S = \prod_{j \in N} S_j$.*

   A strategy profile is $s \in S$.
3. *A set of payoff functions $\{v_1, v_2, ..., v_n\}$, each assigning a payoff value to each combination of chosen strategies, that is, $v_i : S \to \mathbb{R}$.*

A normal-form game is a tuple $\Gamma = \left\langle N, \{S_i\}_{i \in N}, \{v_i\}_{i \in N} \right\rangle$. Sometimes we use actions A to stand for strategies S and $\succsim$ for v_i. So the tuple becomes $\Gamma = \langle N, A_i, \succsim_i \rangle$.

2.2.2 Dominance

Definition 10. *A strategy profile $s \in S$ pareto dominates strategy profile $m \in S$ if $\forall i \in N, v_i(s) \geq v_i(m)$ and $\exists i \in N, v_i(s) > v_i(m)$. In this case m is pareto dominated by s. A strategy profile is pareto optimal if it is not pareto dominated by any other strategy profiles.*

Definition 11 (belief). *Define $S_{-i} = \prod_{j \neq i} S_j$ and strategy profile $s_{-i} \in S_{-i}$. s_{-i} is a belief of player i. The payoff of player i for strategy profile s is now $v_i(s_i, s_{-i})$ where $s = (s_i, s_{-i})$.*

Definition 12. *For player i with strategy $n_i \in S_i$ and $m_i \in S_i$, m_i is weakly dominated by n_i , or $n_i \succeq m_i$ if*

$$\forall s_{-i} \in S_{-i}, v_i(n_i, s_{-i}) \geq v_i(m_i, s_{-i})$$

m_i is strictly dominated by n_i, or $n_i \succ m_i$, if

$$\forall s_{-i} \in S_{-i}, v_i(n_i, s_{-i}) > v_i(m_i, s_{-i})$$

Theorem 7. *A rational player will never play a strictly dominated strategy.*

Definition 13 (strictly dominant strategy). *$m_i \in S_i$ is a strictly dominant strategy if $\forall n_i \in S_i, n_i \neq m_i, m_i \succ n_i$.*

Definition 14 (strictly dominant strategy equilibrium). *The strategy profile $s^D \in S$ is a strictly dominant strategy equilibrium if $s_i^D \in S_i$ is a strictly dominant strategy for all $i \in N$.*

Theorem 8. *If a game has a strictly dominant strategy equilibrium s^D, s^D is unique.*

Definition 15 (IESDS). *The iterated elimination of strictly dominated strategies is defined as:*

1. *Define $\forall i, S_i^0 = S_i$, and $S^0 = \prod S_i^0$*
2. *For all players i, find its strictely dominated strategy $s_i \in S_i$ (if any).*
3. *For all players i, define $S_i^{k+1} = S_i^k \setminus \{s_i\}$, and $S^{k+1} = \prod S_i^{k+1}$*
4. *If $S^{k+1} = S^k$, define $S^{ES} = S^{k+1}$ and terminate.*

A strategy profile $s^{ES} \in S^{ES}$ is called iterated-elimination equilibrium.

Theorem 9. *If s^* is a strictly dominant strategy equilibrium for a game Γ, it is unique.*

2.2.3 Nash Equilibrium

Definition 16 (best reponse). *The strategy $m_i \in S_i$ is player i's best response to his opponents' strategy $s_{-i} \in S_{-i}$ if*

$$m_i \in \underset{s_i \in S_i}{argmax}\, v_i(s_i, s_{-i})$$

Definition 17 (best-reponse correspondence). *The best-response correspondence of player i is a set value function $B_i : S_{-i} \to 2^{S_i}$ that $B_i(s_{-i})$ is the best reponse of player i to s_{-i}.*

Theorem 10. *A rational player who believes his opponents are playing s_{-i} will always choose the best response to s_{-i}.*

Theorem 11. *A strictly dominated strategy cannot be the best response.*

Theorem 12. *If s^* is the strictly dominant strategy equilibrium, or if s^* is the unique strategy profile in S^{ES}, then s_i^* is the best response to s_{-i}^* for all $i \in N$.*

Definition 18 (Nash equilibrium). *The pure strategy profile s^* is a Nash equilibrium of s_i^* is a best response to s_{-i}^* for all i, that is*:

$$\forall i \in N, s_i^* \in B_i(s_{-i}^*)$$

The easiest way to find a Nash equilibrium is first to calculate the best response function of each player, then find a profile s^* of actions for which $s_i^* \in B_i(s_{-1}^*)$ for all i.

Nash equilibrium is a steady-state so no player wants to change its choice.

Theorem 13. *If s^* is either*:

1. *the strictly dominant strategy equilibrium,*
2. *the unique survivor of IESDS*
3. *the unique rationalizable strategy profile*

Then s^* is the unique Nash equilibrium.

Theorem 14. *Nash equilibrium does not guarantee Pareto optimality.*

2.2.4 Mixed Strategy

Definition 19 (mixed strategy). *Define $\triangle S_i$ as the simplex of S_i, which is the set of all probability distributions over S_i. A mixed strategy for player i is an element $\sigma_i \in \triangle S_i$, so that σ_i is a probability distribution over S_i. The same definition works for continuous probability case.*

Definition 20 (support). *For a mixed strategy σ_i for player i, a pure strategy s_i is in the support of σ_i if and only if $\sigma_i(s_i) > 0$. The same definition works for continuous probability case.*

Definition 21 (belief). *A belief for player i is a probability distribution $\pi_i \in \triangle S_{-i}$ over the strategies of his opponents. It is denoted by $\pi_i(s_{-i})$ for s_{-i}.*

Definition 22 (expected payoff). *The expected payoff of player i when he chooses the pure strategy $s_i \in S_i$ and his opponents play the mixed strategy $\sigma_{-i} \in \triangle S_{-i}$ is*:

$$v_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \times v_i(s_i, s_{-i})$$

When player i choose a mixed strategy $\sigma_i \in \triangle S_i$ and his opponents play the mixed strategy $\sigma_{-i} \in \triangle S_{-i}$, the expected payoff is:

$$\begin{aligned}
v_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) \times v_i(s_i, \sigma_{-i}) \\
&= \sum_{s_i \in S_i} \sigma_i(s_i) \times \left(\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) \times v_i(s_i, s_{-i}) \right) \\
&= \sum_{s \in S} v_i(s) \times \prod_{i \in N} \sigma_i(s_i)
\end{aligned}$$

2.2.5 Nash Equilibrium for Mixed Strategy

Definition 23. *The mixed strategy σ^* is a Nash equilibrium if for each player σ^* is the best response to σ^*_{-1}, that is:*

$$\forall i \in N, \sigma_i \in \triangle S_i, v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*)$$

Theorem 15. *If σ^* is a Nash equilibrium, and m_i and n_i are in the support of σ_i^*, then*

$$v_i(m_i, \sigma_{-i}^*) = v_i(n_i, \sigma_{-i}^*) = v_i(\sigma_i^*, \sigma_{-i}^*)$$

Proof. Assume on the contrary that $v_i(m_i, \sigma_{-i}^*) > v_i(n_i, \sigma_{-i}^*)$. We could assign higher probability to m_i to increase the expected payoff. The end result of adjustment is the probability of n_i would be 0 and n_i will not be in the support of σ_i. □

Theorem 16. *For a two players game, there can be no Nash equilibrium in which one plays a pure strategy and the other a mixed strategy.*

Proof. If player i plays a pure strategy. Player j will face different payoffs when he plays randomly, so he will choose the highest payoff and become a pure strategy. □

Definition 24 (strictly dominated)**.** *Let $\sigma_i \in \triangle S_i$ and $s_i \in S_i$ be possible strategies for player i. s_i is strictly dominated by σ_i if*

$$\forall s_{-1} \in S_{-1}, v_i(\sigma_i, s_{-i}) > v_i(s_i, s_{-i})$$

Theorem 17 (Brouwer's Fixed-point Theorem)**.** *If $f(x) : [0, 1] \to [0, 1]$ is a continuous function, then there exists a x^* that $f(x^*) = x^*$.*

Theorem 18 (Kakutani's Fixed-point Theorem)**.** *A function $f : X \to X$ has a fixed point x if four conditions are met:*

1. *X is a non-empty, compact and convex non-empty subset of $\mathbb{R}^n$.*
2. *$\forall x, f(x) \neq \emptyset$*
3. *$\forall x, f(x)$ is convex.*
4. *f has a closed graph.*

Definition 25 (collection of best-response correspondences)**.** *The collection of best-response correpsondences is a function $BR : \triangle S \to \triangle S$ that for every element $\sigma \in \triangle S$, $\big(BR(\sigma)\big)_i$ is the best response to σ_{-i}.*

Theorem 19. *A mixed-strategy profile $\sigma^* \in \triangle S$ is a Nash equilibrium if and only if it is a fixed point of the collection of best-response correspondences, $\sigma^* \in BR(\sigma^*)$.*

Theorem 20. *Every finite mixed strategy has a Nash equilibrium.*

Proof. (use Kakutani's fixed-point theorem) If σ_i and $\sigma_i^{'}$ are in $BR(\sigma_{-i})$. Their linear combinarion is also in $BR(\sigma_{-i})$. □

2.3 Dynamic Games of Complete Information

2.3.1 Extensive Games

Definition 26 (Nature's Choice)**.** *The Nature will choose an action according to a probability distribution. The uncertainty is called exogenous uncertainty.*

Definition 27 (extensive game)**.** *An extensive game with perfect information has the following components:*

1. *A set of N players.*
2. *A set H of histories, which is a finite or infinite sequence of actions a that satisfies the following 3 properties:*
   (a) $\emptyset \in H$.
   (b) *Sub-history exists: if $(a^k)_{k\le K} \in H$, and $L < K$, then $(a^k)_{k\le L} \in H$.*
   (c) *Infinite history exists: For an infinite sequence $(a^k)_{k<\infty}$, if $(a^k)_{k\le L} \in H$ for all L, then $(a^k)_{k<\infty} \in H$.*

   A history $h \in H$ is terminal if:

   (a) *it is infinite,*
   (b) *or there is no a that $(h, a) \in H$.*

   The set of all nonterminal history is Z.
3. *A player function $P : H\backslash Z \to N$ that assign each nonterminal history a member of N.*
4. *The next actions is function A that $A(h) = \{a : (h, a) \in H\}$.*
5. *For each player i a preference relation $\succsim_i$ on Z.*

The extensive game is denoted as $\Gamma = \langle N, H, P, \succsim_i \rangle$. It could be displayed as a tree called game tree.

Definition 28 (strategy)**.** *A strategy of player i is a function $s_i : \{h : P(h) = i\} \to A(h)$. So it assigns an action in $A(h)$ for each nonterminal history h that $P(h) = i$.*

Definition 29 (outcome)**.** *The outcome $O(s)$ is the terminal history if each player i follows his strategy s_i.*

The Nash equilibrium definition is the same for the extensive game.

Definition 30. *The strategic form $\left\langle N, S_i, \succsim_i^{'} \right\rangle$ of the extensive game with perfect information $\Gamma = \langle N, H, P, \succsim_i \rangle$ is that:*

1. *S_i is the set of strategies of player i in Γ.*
2. *$\succsim_i^{'}$ is defined as $h_a \succsim_i^{'} h_b$ if and only if $O(h_a) \succsim_i O(h_b)$.*

Definition 31 (reduced strategy)**.** *All histories will be classified into domains. A domain contains all histories that are linearly ordered by $\subset$. So a history may appear in multiple domains. A reduced strategy for player i is the strategy for a domain.*

Definition 32 (subgame)**.** *For the game $\Gamma = \langle N, H, P, \succsim_i \rangle$, the subgame that follows the history h is the game $\Gamma(h) = \langle N, H|_h, P|_h, \succsim_i |_h \rangle$ that:*

1. *$H|_h = \{h' : (h, h') \in H\}$.*
2. *$P|_h(h') = P(h, h')$ for each $h' \in H|_h$.*
3. *$h' \succsim_i |_h h''$ if and only if $(h, h') \succsim_i (h, h'')$.*

For subgame, we define strategy and outcome as:

1. *$s_i|_h(h') = s_i(h, h')$ for all $h' \in H|_h$.*
2. *O_h is the outcome function of $\Gamma(h)$.*

Definition 33 (subgame perfect equilibrium)**.** *A subgame perfect equilibrium is a strategy profile s^* that $s^*|_h$ is the Nash equilibrium for $\Gamma(h)$.*

Theorem 21 (one deviation property)**.** *A strategy profile s^* satisfies one derivation property if for all $h \in H$ and $P(h) = i$, we have*

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succsim_i|_h O_h(s_i^{'}|_h, s_{-i}^*|_h)$$

where $s_i^{'}$ is a strategy of player i that $s_i^{'}$ and $s_i^*|_h$ only differers at history h. Then s^* is a subgame perfect equilibrium if it satisfies one deviation property.

Proof. Assume the reverse. There is a history h and player k that $s_k^*|_h$ is not the best response to $s_{-k}^*|_h$ in $\Gamma(h)$. Find $\hat{s_k} \succ s_k^*$ so the size of $\{(h, h') \in H : s_k^*|_h(h') \neq \hat{s_k}(h')\}$ is minimal. Then find the longest history $\hat{h}$ that $s_k^*|_h(\hat{h}) \neq \hat{s_k}(\hat{h})$. $(\hat{s_k}, s_{-k}^*|_h)$ must pass through $\hat{h}$. Or $\hat{h}$ does not contribute to the outcome and we could let $s_k^*|_h(\hat{h}) = \hat{s_k}(\hat{h})$ and nothing is changed, which violate the assumption of $\hat{s_k}$. Now consider $\Gamma(\hat{h})$. $s_k^*|_h(\hat{h})$ and $\hat{s_k}(\hat{h})$ only differs in $\hat{h}$ and $s_k^*|_h(\hat{h}) \succ \hat{s_k}(\hat{h})$, a violation. □

Theorem 22 (Kuhn's theorem). *Every finite extensive game with perfect information has a subgame perfect equilibrium.*

Proof. Define $|\Gamma|$ to be the length of the longest history in Γ. If $\left|\Gamma(h)\right| = 0$, so h is a terminal history, define $R(h) = h$. Assume $R(h)$ is defined for $\left\{h \in H : \left|\Gamma(h)\right| \leq K\right\}$. Let h^* be a history that $\left|\Gamma(h^*)\right| = K + 1$. The next player is $P(h^*) = i$. Define $s_i(h^*)$ to choose the best action from $A(h^*)$. Now $R(h^*) = R(h^*, s_i(h^*))$. Now we have defined s and it is subgame perfect equilibrium according to Theorem 21. The process is called backwards induction. □

2.3.2 Other Definitions

Definition 34. *Player i has a collection of information h_i with the following properties:*

1. *If $|h_i| = 1$, player i who moves at x knows he is at x.*
2. *If $|h_i| \geq 2$, player i who moves at x does not know where he is.*
3. *If $|h_i| \geq 2$, let $x \in h_i$ and $y \in h_i$, then $A_i(x_i) = A_i(y_i)$. So the actions are indistinguishable, or player i would know where he is.*

If $|h_i| \geq 2$, we use ellipse to include all nodes within the same information set. We could also use dashed lines to connect them.

Definition 35. *A game in which every information set is a singleton and there are no moves of Nature is called a game of perfect information. Otherwise, it is called a game of imperfect information.*

Theorem 23. *Any simultaneous-move game is a game of imperfect information.*

Definition 36 (pure strategy). *Let $A_i(h_i)$ be the set of actions a player i can take at h_i, and let A_i be the set of all actions of player i. A pure strategy for player i is a function $s_i : H_i \to A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for all information set h_i. We denote by S_i the set of all pure strategies.*

Definition 37 (mixed strategy). *A mixed strategy for player i is a probability distribution over his pure strategy $s_i \in S_i$.*

Definition 38 (behavioral strategy). *A behavior strategy is $\sigma_i : H_i \to \triangle A_i(h_i)$ where $\sigma_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in informaiton set h_i.*

Comparison between mixed strategy and behavioral strategy:

- In mixed strategy, we choose a pure strategy for all possible h_i before the game is played, and follow this pure strategy
- In behavior strategy, the randomness is chosen as the game unfolds. So for each h_i we will randomly choose.

Definition 39 (perfect recall). *A game of perfect recall is one in which no player ever forgets information that he previously knew.*

Theorem 24. *Mixed and behavioral strategies are equivalent in a perfect recall game.*

Definition 40. *Let σ^* be a Nash equilibrium profile of behavioral strategies in an extensive-form game. An information set is on the equilibrium path if given σ^* it is reached with positive probability. Or it is called off the equilibrium path.*

2.4 Examples

Example 8 (Condorcet Paradox)**.** *There are players 1, 2 and 3 who vote on a, b and c. Their preferences are:*

1. *Player 1 : $a \succ b \succ c$*
2. *Player 2 : $c \succ b \succ a$*
3. *Player 3 : $b \succ a \succ c$*

The choice is made by majority vote:

1. *For a and c, player 1 and 3 vote for a*
2. *For b and c, player 1 and 2 vote for c*
3. *For a and b, player 2 and 3 vote for b*

So $a \succ b \succ c \succ a$, which is a paradox.

Example 9 (the prisoner's dilemma)**.** *Two suspects at the police station and questioning each in a different room. Each suspect is offered a deal and he will either confess, or flinks, or say nothing and remain mum.*

Players $N = \{1, 2\}$.
Strategy $S_i = \{M, F\}$. *F is flinks,, M is to remain mum.*
Payoff *Let $v_i(s_1, s_2)$ be the payoff to player i.*

- $v_1(M, M) = v_2(M, M) = -2$
- $v_1(F, F) = v_2(F, F) = -4$
- $v_1(M, F) = v_2(M, F) = -5$
- $v_1(F, M) = v_2(F, M) = -1$

Example 10 (cournot duopoly)**.** *Introduced by Augustin Cournot (1838). Two identical firms produce some goods. They will choose the quantity of production, which will determine the price and profit.*

matching pennies: has no Nash equilibrium

Bibliography

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