Homework 2- PCS

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10/11/2021

Problem 4

Give a method for generating a random variable having distribution function

$$F(x) = 1 - exp(-\alpha x^{\beta}), 0 < x < \infty$$

A random variable having such a distribution is said to be a Weibull random variable.

Let $x = F^-1(u)$, for u ~ unif(0,1). Then,

$$\begin{aligned} u &= F(x) = 1 - exp(-\alpha x^{\beta}) \\ u - 1 &= exp(-\alpha x^{\beta}) \\ 1 - u &= exp(-\alpha x^{\beta}) \\ \log(1-u) &= \log(\exp -\alpha x^{\beta}) \\ \log(1-u) &= -\alpha x^{\beta} \\ \frac{-\log(1-u)}{\alpha} &= x^{\beta} \\ (\frac{-\log(1-u)}{\alpha})^{\frac{1}{\beta}} &= x. \end{aligned}$$

.

So, $X = F^{-1}(U) = (\frac{-\log(1-U)}{\alpha})^{\frac{1}{\beta}}$.

#Problem 6 Let X be an exponential random variable with mean 1. Give an effiicient algortihm for simulating a random variable whose distribution is the conditional distribution of X given that $X < 0.05$. That is, its density function is

$$f(x) = \frac{e^{-x}}{1 - e^{-.05}}, 0 < x < 0.05$$

. Generate 1000 such variables and use them to estimate $E[X|X < 0.05]$. Then determine the exact value of $E[X|X < 0.05]$. We can see

$$\begin{aligned} F(x) = \int_0^x f(t)dt &= \int_0^x \frac{e^-t}{1 - e^{-0.05}} \\ &= \frac{-e^{-t}}{1 - e^{-0.05}}\Big|_0^x \\ &= \frac{1 - e^{-x}}{1 - e^{-0.05}}, 0 < x < 0.05 \end{aligned}$$

Using the inverse cdf method, let $u = F(x) = \frac{1-e^{-x}}{1-e^{-0.05}}$ where u ~ unif(0.1). Omitting some algebra, we find $X = F^-1(U) = -\log(1 - U(1 - e^{-0.05})$.

The exact solution for

$$E[X|X < 0.05] = \int_0^{0.05} x\frac{e^-x}{1 - e^{-0.05}} = \frac{-1}{1 - e^{-0.05}}[xe^{-x} + e^{-x}]_0^{0.05} = 0.02479167535$$

```
approximate_mean <- function(numTimes) {
  #Generate u for each simulation times
  u <- runif(numTimes)  #doubles
  #Generate X = F^_1(U)
  X <- -log(1-u*(1-exp(-0.05))) #R knows log() is ln()
  mean(X)
}

approximate_mean(1000)
```

```
## [1] 0.02484872
```

The difference between our exact solution and estimated solution is 2.02555347Ö10⁻4.

#Problem 10 A Casualty insurance company has 1000 policyholders, each of whom will independently present a claim in the next month with probability 0.05. Assuming that the amounts of the claims made are independent exponential random variables with mean $800, use simulation to estimate the probability that the sum of these claims exceeds $50,000.

Let N be the number of claims in a month, note that N~Binomial(1000, .05) and the claim amount be X_i where X_i is iid ~Exp(800). The sum of claim amount is thus $S = \sum_{i=1}^{N} X_i$.

```
sim <- function(nsim) {
  counter = 0
  for(i in 1:nsim){
    N <- rbinom(1,1000,0.05)
    X <- rexp(N, rate = 1/800)
    S <- sum(X)
    counter <- counter +sum(S>50000)
  }
  return(counter/nsim)
}

sim(1000)
```

```
## [1] 0.108
```

Our estimated probability that the sum of insurance claims exceeds $50,000 is 0.104.

Question 4

In example 5f we simulated a normal random variable by using the rejection technique with an exponential distribution with rate 1. Show that among all exponential density functions $g(x) = \lambda e^{-\lambda x}$ the number of iterations needed is minimized when $\lambda = 1$.

We know the "number of iterations of the alogrithm that are are needed is a geometric random variable with mean c. " (Ross 73) and for X ~Geometric(p), $P(X = k) = (1 - p)^{k-1}p, E(X) = \frac{1}{p}$. Thus the probability of acceptance is $\frac{1}{c}$.

Let X be the standard normal distribution random variable and X = |Z|. Then, $f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \geq 0$ and $g(x) = \lambda e^{-\lambda x}, x > 0$.

Moreover,

$$c = \max_x \frac{f(x)}{g(x)} = \frac{2}{\lambda\sqrt{2\pi}} e^{\lambda x - \frac{x^2}{2}}$$

Since

$$\max_x (\lambda x - \frac{x^2}{2}) = \lambda x - \frac{x^2}{2}\Big|_{x=\lambda} = \frac{\lambda^2}{2}$$

, $c = \frac{2}{\lambda\sqrt{2\pi}} e^{\frac{\lambda^2}{2}}$.

We must minimize c to to maximize iterations. Let $h(\lambda) = \frac{1}{\lambda} e^{\frac{\lambda^2}{2}}, \lambda > 0$. Then, $\frac{d}{d\lambda} h(\lambda) = e^{\frac{\lambda^2}{2}} (1 - \frac{1}{\lambda^2}) = 0$. Thus, $\lambda \Rightarrow 1$ $\frac{d}{d^2\lambda} h(\lambda) = e^{\frac{\lambda^2}{2}} (\lambda - \frac{1}{\lambda} + \frac{2}{\lambda^3}) \Rightarrow \frac{d}{d^2\lambda} h(1) > 0$.