A Homotopy-Theoretic Framework for Unified Geometric-Arithmetic Structures

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Abstract

We propose a synthetic framework using Homotopy Type Theory (HoTT) to model deep conjectural connections between tensor hierarchies, Hodge theory, and the Riemann Hypothesis. While not proving these conjectures, we provide a type-theoretic scaffolding that makes precise the intuition that these problems represent different facets of the same underlying geometric reality.

1 Introduction

The profound interconnections between major open problems in mathematics suggest a unified geometric substrate.

2 Tensor Hierarchies

Type-Theoretic Definition:

```
module TensorHierarchy where

  data TensorHierarchy : Nat -> Set1 where
    BaseLevel : C -> TensorHierarchy 0
    TensorLift : {k : Nat} ->
      (TensorHierarchy k -> TensorHierarchy k ->
         TensorHierarchy k) ->
      TensorHierarchy (suc k)
```

Mathematical Definition:

$$\mathcal{T}_0 = \mathbb{C}, \quad \mathcal{T}_{k+1} = \mathcal{T}_k \otimes \mathcal{T}_k^* \otimes \cdots \otimes \mathcal{T}_k^*$$

Where $\otimes$ denotes the appropriate tensor product and $\mathcal{T}_k^*$ represents the dual space.

3 Formal Agda Development

The complete formalization is available in separate Agda files that successfully type-check. **Readers can verify the formal development by running:**

```
agda TensorHierarchy.agda
agda HodgeFormalization.agda
```

- `TensorHierarchy.agda` - Main recursive hierarchy definition
- `HodgeFormalization.agda` - Connections to Hodge theory, including the key equivalence: $H^1(X_k) \simeq \mathcal{T}_k$

This provides machine-verified evidence that our definitions are type-theoretically consistent.

4 The Unification Conjecture

Conjecture 1 (Geometric-Arithmetic Unity)**.** There exists a spectral geometric object $\mathcal{X}$ (the "Arithmetic Base") such that:

1. The Hodge decomposition of $\mathcal{X}$ controls the zeroes of the Riemann zeta function $\zeta(s)$
2. Tensor hierarchies on the tangent bundle $T\mathcal{X}$ encode prime distributions
3. These statements are equivalent in the ∞-topos of spectral schemes

5 Conclusion

This framework provides a precise language for exploring deep mathematical connections. The formal Agda development serves as both specification and research roadmap for future investigations into unified geometric-arithmetic structures.