CHAPTER

# Interplanetary Trajectories

8

## CHAPTER OUTLINE



## 8.1 Introduction

In this chapter, we consider some basic aspects of planning interplanetary missions. We begin by considering Hohmann transfers, which are the easiest to analyze and the most energy efficient. The orbits of the planets involved must lie in the same plane and the planets must be positioned just right for a Hohmann transfer to be used. The time between such opportunities is derived. The method of patched conics is employed to divide the mission up into three parts: the hyperbolic departure trajectory relative to the home planet, the cruise ellipse relative to the sun, and the hyperbolic arrival trajectory relative to the target plane.

The use of patched conics is justified by calculating the radius of a planet's sphere of influence and showing how small it is on the scale of the solar system. Matching the velocity of the spacecraft at the

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home planet's sphere of influence to that required to initiate the outbound cruise phase and then specifying the periapse radius of the departure hyperbola determines the delta-v requirement at departure. The sensitivity of the target radius to the burnout conditions is discussed. Matching the velocities at the target planet's sphere of influence and specifying the periapse of the arrival hyperbola yields the delta-v at the target for a planetary rendezvous or the direction of the outbound hyperbola for a planetary flyby. Flyby maneuvers are discussed, including the effect of leading and trailing-side flybys, and some noteworthy examples of the use of gravity assist maneuvers are presented.

The chapter concludes with an analysis of the situation in which the planets' orbits are not coplanar and the transfer ellipse is tangent to neither orbit. This is akin to the chase maneuver in Chapter 6 and requires the solution of Lambert's problem using Algorithm 5.2.

## 8.2 Interplanetary Hohmann transfers

As can be seen from Table A.1, the orbits of most of the planets in the solar system lie very close to the earth's orbital plane (the ecliptic plane). The innermost planet, Mercury, and the outermost dwarf planet, Pluto, differ most in inclination (7° and 17°, respectively). The orbital planes of the other planets lie within 3.5° of the ecliptic. It is also evident from Table A.1 that most of the planetary orbits have small eccentricities, the exceptions once again being Mercury and Pluto. To simplify the beginning of our study of interplanetary trajectories, we will assume that all the planets' orbits are circular and coplanar. Later on, in Section 8.10, we will relax this assumption.

The most energy-efficient way for a spacecraft to transfer from one planet's orbit to another is to use a Hohmann transfer ellipse (Section 6.2). Consider Figure 8.1, which shows a Hohmann transfer from an inner planet 1 to an outer planet 2. The departure point $D$ is at periapsis (perihelion) of the transfer ellipse and the arrival point is at apoapsis (aphelion). The circular orbital speed of planet 1 relative to the sun is given by Eqn (2.63),

$$V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} \tag{8.1}$$

The specific angular momentum $h$ of the transfer ellipse relative to the sun is found from Eqn (6.2), so that the speed of the space vehicle on the transfer ellipse at the departure point $D$ is

$$V_D^{(v)} = \frac{h}{R_1} = \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1+R_2)}} \tag{8.2}$$

This is greater than the speed of the planet. Therefore the required delta-v at $D$ is

$$\Delta V_D = V_D^{(v)} - V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}} - 1\right) \tag{8.3}$$

Likewise, the delta-v at the arrival point $A$ is

$$\Delta V_A = V_2 - V_A^{(v)} = \sqrt{\frac{\mu_{\text{sun}}}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1+R_2}}\right) \tag{8.4}$$

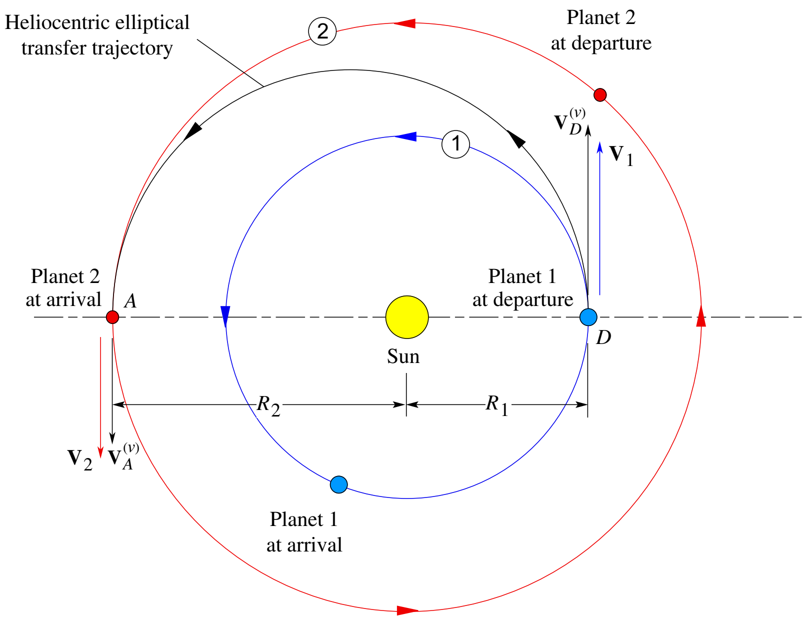


**FIGURE 8.1**

Hohmann transfer from inner planet 1 to outer planet 2.

This velocity increment, like that at point $D$, is positive since planet 2 is traveling faster than the spacecraft at point $A$.

For a mission from an outer planet to an inner planet, as illustrated in Figure 8.2, the delta-vs computed using Eqns (8.3) and (8.4) will both be negative instead of positive. This is because the departure point and the arrival point are now at aphelion and perihelion, respectively, of the transfer ellipse. The speed of the spacecraft must be reduced for it to drop into the lower energy transfer ellipse at the departure point $D$, and it must be reduced again at point $A$ in order to arrive in the lower energy circular orbit of planet 2.

## 8.3 Rendezvous opportunities

The purpose of an interplanetary mission is for the spacecraft to not only intercept a planet's orbit but also to rendezvous with the planet when it gets there. For rendezvous to occur at the end of a Hohmann transfer, the location of planet 2 in its orbit at the time of the spacecraft's departure from planet 1 must be such that planet 2 arrives at the apse line of the transfer ellipse at the same time as the spacecraft does. Phasing maneuvers (Section 6.5) are clearly not practical, especially for manned missions, due to the large periods of the heliocentric orbits.

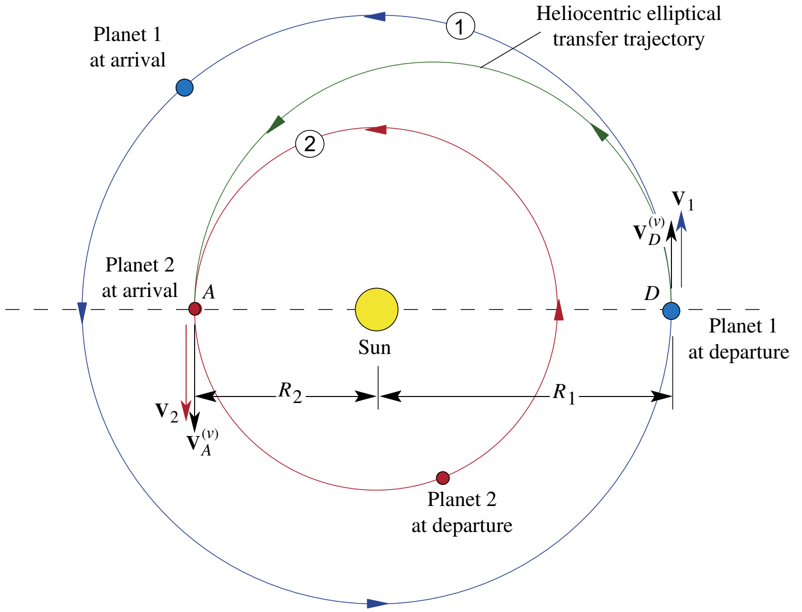


**FIGURE 8.2**

Hohmann transfer from outer planet 1 to inner planet 2.

Consider planet 1 and planet 2 in circular orbits around the sun, as shown in Figure 8.3. Since the orbits are circular, we can choose a common horizontal apse line from which to measure the true anomaly $\theta$. The true anomalies of planets 1 and 2, respectively, are

$$\theta_1 = \theta_{1_0} + n_1 t \tag{8.5}$$

$$\theta_2 = \theta_{2_0} + n_2 t \tag{8.6}$$

where $n_1$ and $n_2$ are the mean motions (angular velocities) of the planets and $\theta_{1_0}$ and $\theta_{2_0}$ are their true anomalies at time $t = 0$. The phase angle between the position vectors of the two planets is defined as

$$\phi = \theta_2 - \theta_1 \tag{8.7}$$

$\phi$ is the angular position of planet 2 relative to planet 1. Substituting Eqns (8.5) and (8.6) into Eqn (8.7) we get

$$\phi = \phi_0 + (n_2 - n_1)t \tag{8.8}$$

$\phi_0$ is the phase angle at time zero. $n_2 - n_1$ is the orbital angular velocity of planet 2 relative to planet 1. If the orbit of planet 1 lies inside that of planet 2, as in Figure 8.3(a), then $n_1 > n_2$. Therefore, the relative angular velocity $n_2 - n_1$ is negative, which means planet 2 moves clockwise relative to planet 1. On the other hand, if planet 1 is outside of planet 2 then $n_2 - n_1$ is positive, so that the relative motion is counterclockwise.

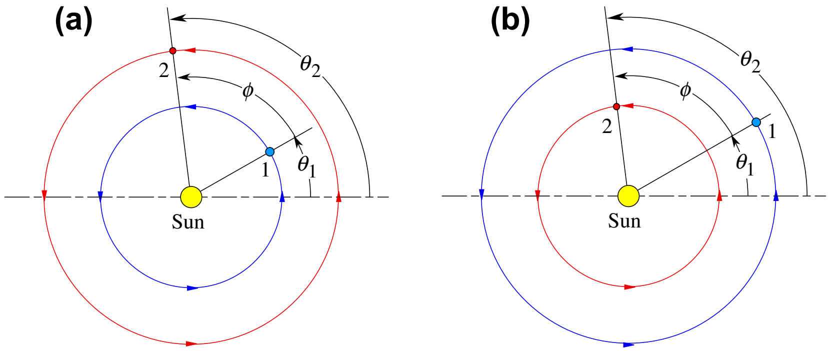


**FIGURE 8.3**

Planets in circular orbits around the sun. (a) Planet 2 outside the orbit of planet 1. (b) Planet 2 inside the orbit of planet 1.

The phase angle obviously varies linearly with time according to Eqn (8.8). If the phase angle is $\phi_0$ at $t=0$, how long will it take to become $\phi_0$ again? The answer: when the position vector of planet 2 rotates through $2\pi$ radians relative to planet 1. The time required for the phase angle to return to its initial value is called the synodic period, which is denoted $T_{syn}$. For the case shown in Figure 8.3(a) in which the relative motion is clockwise, $T_{syn}$ is the time required for $\phi$ to change from $\phi_0$ to $\phi_0 - 2\pi$. From Eqn (8.8) we have

$$\phi_0 - 2\pi = \phi_0 + (n_2 - n_1)T_{syn}$$

so that

$$T_{syn} = \frac{2\pi}{n_1 - n_2} \quad (n_1 > n_2)$$

For the situation illustrated in Figure 8.3(b) ($n_2 > n_1$), $T_{syn}$ is the time required for $\phi$ to go from $\phi_0$ to $\phi_0 + 2\pi$, in which case Eqn (8.8) yields

$$T_{syn} = \frac{2\pi}{n_2 - n_1} \quad (n_2 > n_1)$$

Both cases are covered by writing

$$T_{syn} = \frac{2\pi}{|n_1 - n_2|} \tag{8.9}$$

Recalling Eqn (3.9), we can write $n_1 = 2\pi/T_1$ and $n_2 = 2\pi/T_2$. Thus, in terms of the orbital periods of the two planets,

$$T_{syn} = \frac{T_1 T_2}{|T_1 - T_2|} \tag{8.10}$$

Observe that $T_{syn}$ is the orbital period of planet 2 relative to planet 1.

## EXAMPLE 8.1

Calculate the synodic period of Mars relative to that of the earth.

**Solution**

In Table A.1 we find the orbital periods of earth and Mars:

$$T_{\text{earth}} = 365.26 \text{ days } (1 \text{ year})$$
$$T_{\text{Mars}} = 1 \text{ year } 321.73 \text{ days} = 687.99 \text{ days}$$

Hence,

$$T_{\text{syn}} = \frac{T_{\text{earth}}T_{\text{Mars}}}{|T_{\text{earth}} - T_{\text{Mars}}|} = \frac{365.26 \times 687.99}{|365.26 - 687.99|} = \boxed{777.9 \text{ days}}$$

These are earth days (1 day = 24 h). Therefore it takes 2.13 years for a given configuration of Mars relative to the earth to occur again.

Figure 8.4 depicts a mission from planet 1 to planet 2. Following a heliocentric Hohmann transfer, the spacecraft intercepts and undergoes rendezvous with planet 2. Later it returns to planet 1 by means of another Hohmann transfer. The major axis of the heliocentric transfer ellipse is the sum of the radii of the two planets' orbits, $R_1 + R_2$. The time $t_{12}$ required for the transfer is one-half the period of the ellipse. Hence, according to Eqn (2.83),

$$t_{12} = \frac{\pi}{\sqrt{\mu_{\text{sun}}}}\left(\frac{R_1 + R_2}{2}\right)^{\frac{3}{2}} \tag{8.11}$$

During the time it takes the spacecraft to fly from orbit 1 to orbit 2, through an angle of $\pi$ radians, planet 2 must move around its circular orbit and end up at a point directly opposite of planet 1's

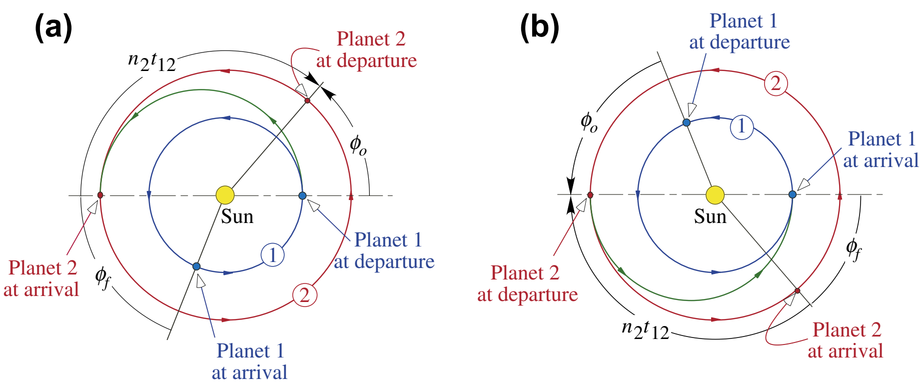


**FIGURE 8.4**

Round-trip mission, with layover, to planet 2. (a) Departure and rendezvous with planet 2. (b) Return and rendezvous with planet 1.

position when the spacecraft departed. Since planet 2's angular velocity is $n_2$, the angular distance traveled by the planet during the spacecraft's trip is $n_2 t_{12}$. Hence, as can be seen from Figure 8.4(a), the initial phase angle $\phi_0$ between the two planets is

$$\phi_0 = \pi - n_2 t_{12} \tag{8.12}$$

When the spacecraft arrives at planet 2, the phase angle will be $\phi_f$, which is found using Eqns (8.8) and (8.12).

$$\phi_f = \phi_0 + (n_2 - n_1)t_{12} = (\pi - n_2 t_{12}) + (n_2 - n_1)t_{12}$$

$$\phi_f = \pi - n_1 t_{12} \tag{8.13}$$

For the situation illustrated in Figure 8.4, planet 2 ends up being behind planet 1 by an amount equal to the magnitude of $\phi_f$.

At the start of the return trip, illustrated in Figure 8.4(b), planet 2 must be $\phi'_0$ radians ahead of planet 1. Since the spacecraft flies the same Hohmann transfer trajectory back to planet 1, the time of flight is $t_{12}$, the same as the outbound leg. Therefore, the distance traveled by planet 1 during the return trip is the same as the outbound leg, which means

$$\phi'_0 = -\phi_f \tag{8.14}$$

In any case, the phase angle at the beginning of the return trip must be the negative of the phase angle at arrival from planet 1. The time required for the phase angle to reach its proper value is called the wait time, $t_{\text{wait}}$. Setting time equal to zero at the instant we arrive at planet 2, Eqn (8.8) becomes

$$\phi = \phi_f + (n_2 - n_1)t$$

$\phi$ becomes $-\phi_f$ after the time $t_{\text{wait}}$. That is,

$$-\phi_f = \phi_f + (n_2 - n_1)t_{\text{wait}}$$

or

$$t_{\text{wait}} = \frac{-2\phi_f}{n_2 - n_1} \tag{8.15}$$

where $\phi_f$ is given by Eqn (8.13). Equation (8.15) may yield a negative result, which means the desired phase relation occurred in the past. Therefore, we must add or subtract an integral multiple of $2\pi$ to the numerator in order to get a positive value for $t_{\text{wait}}$. Specifically, if $N = 0$, 1, 2,…, then

$$t_{\text{wait}} = \frac{-2\phi_f - 2\pi N}{n_2 - n_1} \quad (n_1 > n_2) \tag{8.16}$$

$$t_{\text{wait}} = \frac{-2\phi_f + 2\pi N}{n_2 - n_1} \quad (n_1 < n_2) \tag{8.17}$$

where $N$ is chosen to make $t_{\text{wait}}$ positive. $t_{\text{wait}}$ would probably be the smallest positive number thus obtained.

## EXAMPLE 8.2

Calculate the minimum wait time for initiating a return trip from Mars to earth.

**Solution**

From Tables A.1 and A.2 we have

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$
$$R_{\text{Mars}} = 227.9 \times 10^6 \text{ km}$$
$$\mu_{\text{sun}} = 132.71 \times 10^9 \text{ km}^3/\text{s}^2$$

According to Eqn (8.11), the time of flight from earth to Mars is

$$t_{12} = \frac{\pi}{\sqrt{\mu_{\text{sun}}}}\left(\frac{R_{\text{earth}} + R_{\text{Mars}}}{2}\right)^{\frac{3}{2}}$$
$$= \frac{\pi}{\sqrt{132.71 \times 10^9}}\left(\frac{149.6 \times 10^6 + 227.9 \times 10^6}{2}\right)^{\frac{3}{2}} = 2.2362 \times 10^7 \text{ s}$$

or

$$t_{12} = 258.82 \text{ days}$$

From Eqn (3.9) and the orbital periods of earth and Mars (see Example 8.1 above) we obtain the mean motions of the earth and Mars.

$$n_{\text{earth}} = \frac{2\pi}{365.26} = 0.017202 \text{ rad/day}$$
$$n_{\text{Mars}} = \frac{2\pi}{687.99} = 0.0091327 \text{ rad/day}$$

The phase angle between earth and Mars when the spacecraft reaches Mars is given by Eqn (8.13).

$$\phi_f = \pi - n_{\text{earth}} t_{12} = \pi - 0.017202 \cdot 258.82 = -1.3107 \text{ (rad)}$$

Since $n_{\text{earth}} > n_{\text{Mars}}$, we choose Eqn (8.16) to find the wait time.

$$t_{\text{wait}} = \frac{-2\phi_f - 2\pi N}{n_{\text{Mars}} - n_{\text{earth}}} = \frac{-2(-1.3107) - 2\pi N}{0.0091327 - 0.017202} = 778.65N - 324.85 \text{ (days)}$$

$N = 0$ yields a negative value, which we cannot accept. Setting $N = 1$, we get

$$\boxed{t_{\text{wait}} = 453.8 \text{ days}}$$

This is the minimum wait time. Obviously, we could set $N = 2, 3, \ldots$ to obtain longer wait times.

In order for a spacecraft to depart on a mission to Mars by means of a Hohmann (minimum energy) transfer, the phase angle between earth and Mars must be that given by Eqn (8.12). Using the results of Example 8.2, we find it to be

$$\phi_o = \pi - n_{\text{Mars}} t_{12} = \pi - 0.0091327 \cdot 258.82 = 0.7778 \text{ rad} = 44.57^\circ$$

This opportunity occurs once every synodic period, which we found to be 2.13 years in Example 8.1. In Example 8.2, we found that the time to fly to Mars is 258.8 days, followed by a wait time of

453.8 days, followed by a return trip time of 258.8 days. Hence, the minimum total time for a manned Mars mission is

$$t_{\text{total}} = 258.8 + 453.8 + 258.8 = 971.4 \text{ days} = 2.66 \text{ years}$$

## 8.4 Sphere of influence

The sun, of course, is the dominant celestial body in the solar system. It is over 1000 times more massive than the largest planet, Jupiter, and has a mass of over 300,000 earths. The sun's gravitational pull holds all the planets in its grasp according to Newton's law of gravity, Eqn (1.40). However, near a given planet, the influence of its own gravity exceeds that of the sun. For example, at its surface the earth's gravitational force is over 1600 times greater than the sun's. The inverse-square nature of the law of gravity means that the force of gravity $F_g$ drops off rapidly with distance $r$ from the center of attraction. If $F_{g_0}$ is the gravitational force at the surface of a planet with radius $r_0$, then Figure 8.5 shows how rapidly the force diminishes with distance. At ten body radii, the force is 1% of its value at the surface. Eventually, the force of the sun's gravitational field overwhelms that of the planet.

In order to estimate the radius of a planet's gravitational sphere of influence, consider the three-body system comprising a planet $p$ of mass $m_p$, the sun $s$ of mass $m_s$ and a space vehicle $v$ of mass $m_v$ illustrated in Figure 8.6. The position vectors of the planet and spacecraft relative to an inertial frame centered at the sun are $\mathbf{R}$ and $\mathbf{R}_v$, respectively. The position vector of the space vehicle relative to the planet is $\mathbf{r}$. (Throughout this chapter we will use upper case letters to represent position, velocity, and acceleration measured relative to the sun and lower case letters when they are measured relative to a planet.) The gravitational force exerted on the vehicle by the planet is denoted $\mathbf{F}_p^{(v)}$, and that exerted

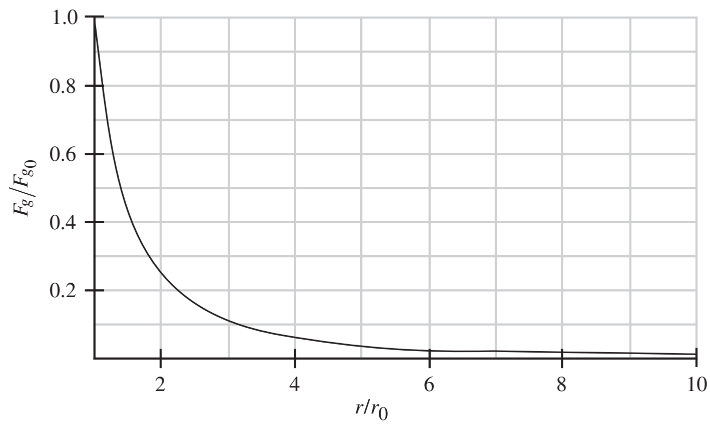


**FIGURE 8.5**

Decrease of gravitational force with distance from a planet's surface.

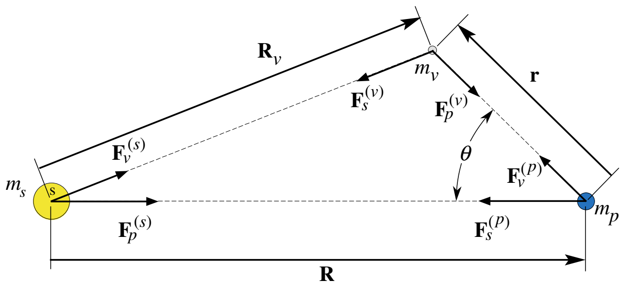


**FIGURE 8.6**

Relative position and gravitational force vectors among the three bodies.

by the sun is $\mathbf{F}_s^{(v)}$. Likewise, the forces on the planet are $\mathbf{F}_s^{(p)}$ and $\mathbf{F}_v^{(p)}$, whereas on the sun we have $\mathbf{F}_v^{(s)}$ and $\mathbf{F}_p^{(s)}$. According to Newton's law of gravitation (Eqn (2.10)), these forces are

$$\mathbf{F}_p^{(v)} = -\frac{Gm_v m_p}{r^3}\mathbf{r} \tag{8.18a}$$

$$\mathbf{F}_s^{(v)} = -\frac{Gm_v m_s}{{R_v}^3}\mathbf{R}_v \tag{8.18b}$$

$$\mathbf{F}_s^{(p)} = -\frac{Gm_p m_s}{R^3}\mathbf{R} \tag{8.18c}$$

Observe that

$$\mathbf{R}_v = \mathbf{R} + \mathbf{r} \tag{8.19}$$

From Figure 8.6 and the law of cosines we see that the magnitude of $\mathbf{R}_v$ is

$$R_v = \left(R^2 + r^2 - 2Rr\cos\theta\right)^{\frac{1}{2}} = R\left[1 - 2\frac{r}{R}\cos\theta + \left(\frac{r}{R}\right)^2\right]^{\frac{1}{2}} \tag{8.20}$$

We expect that within the planet's sphere of influence, $r/R \ll 1$. In that case, the terms involving $r/R$ in Eqn (8.20) can be neglected, so that, approximately

$$R_v = R \tag{8.21}$$

The equation of motion of the spacecraft relative to the sun-centered inertial frame is

$$m_v\ddot{\mathbf{R}}_v = \mathbf{F}_s^{(v)} + \mathbf{F}_p^{(v)}$$

Solving for $\ddot{\mathbf{R}}_v$ and substituting the gravitational forces given by Eqns (8.18a) and (8.18b), we get

$$\ddot{\mathbf{R}}_v = \frac{1}{m_v}\left(-\frac{Gm_v m_s}{R_v^3}\mathbf{R}_v\right) + \frac{1}{m_v}\left(-\frac{Gm_v m_p}{r^3}\mathbf{r}\right) = -\frac{Gm_s}{R_v^3}\mathbf{R}_v - \frac{Gm_p}{r^3}\mathbf{r} \tag{8.22}$$

Let us write this as

$$\ddot{\mathbf{R}}_v = \mathbf{A}_s + \mathbf{P}_p \tag{8.23}$$

where

$$\mathbf{A}_s = -\frac{Gm_s}{R_v^3}\mathbf{R}_v \quad \mathbf{P}_p = -\frac{Gm_p}{r^3}\mathbf{r} \tag{8.24}$$

$\mathbf{A}_s$ is the primary gravitational acceleration of the vehicle due to the sun, whereas $\mathbf{P}_p$ is the secondary or perturbing acceleration due to the planet. The magnitudes of $\mathbf{A}_s$ and $\mathbf{P}_p$ are

$$A_s = \frac{Gm_s}{R^2} \quad P_p = \frac{Gm_p}{r^2} \tag{8.25}$$

where we made use of the approximation given by Eqn (8.21). The ratio of the perturbing acceleration to the primary acceleration is, therefore,

$$\frac{P_p}{A_s} = \frac{\dfrac{Gm_p}{r^2}}{\dfrac{Gm_s}{R^2}} = \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2 \tag{8.26}$$

The equation of motion of the planet relative to the inertial frame is

$$m_p\ddot{\mathbf{R}} = \mathbf{F}_v^{(p)} + \mathbf{F}_s^{(p)}$$

Solving for $\ddot{\mathbf{R}}$, noting that $\mathbf{F}_v^{(p)} = -\mathbf{F}_p^{(v)}$, and using Eqns (8.18b) and (8.18c) yields

$$\ddot{\mathbf{R}} = \frac{1}{m_p}\left(\frac{Gm_v m_p}{r^3}\mathbf{r}\right) + \frac{1}{m_p}\left(-\frac{Gm_p m_s}{R^3}\mathbf{R}\right) = \frac{Gm_v}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{R} \tag{8.27}$$

Subtracting Eqn (8.27) from Eqn (8.22) and collecting terms, we find

$$\ddot{\mathbf{R}}_v - \ddot{\mathbf{R}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v^3}\left[\mathbf{R}_v - \left(\frac{R_v}{R}\right)^3\mathbf{R}\right]$$

Recalling Eqn (8.19), we can write this as

$$\ddot{\mathbf{r}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v^3}\left\{\mathbf{r} + \left[1 - \left(\frac{R_v}{R}\right)^3\right]\mathbf{R}\right\} \tag{8.28}$$

This is the equation of motion of the vehicle relative to the planet. By using Eqn (8.21) and the fact that $m_v \ll m_p$, we can write this in approximate form as

$$\ddot{\mathbf{r}} = \mathbf{a}_p + \mathbf{p}_s \tag{8.29}$$

where

$$\mathbf{a}_p = -\frac{Gm_p}{r^3}\mathbf{r} \quad \mathbf{p}_s = -\frac{Gm_s}{R^3}\mathbf{r} \tag{8.30}$$

In this case, $\mathbf{a}_p$ is the primary gravitational acceleration of the vehicle due to the planet, and $\mathbf{p}_s$ is the perturbation caused by the sun. The magnitudes of these vectors are

$$a_p = \frac{Gm_p}{r^2} \quad p_s = \frac{Gm_s}{R^3}r \tag{8.31}$$

The ratio of the perturbing acceleration to the primary acceleration is

$$\frac{p_s}{a_p} = \frac{Gm_s \dfrac{r}{R^3}}{\dfrac{Gm_p}{r^2}} = \frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 \tag{8.32}$$

For motion relative to the planet, the ratio $p_s/a_p$ is a measure of the deviation of the vehicle's orbit from the Keplerian orbit arising from the planet acting by itself ($p_s/a_p = 0$). Likewise, $P_p/A_s$ is a measure of the planet's influence on the orbit of the vehicle relative to the sun. If

$$\frac{p_s}{a_p} < \frac{P_p}{A_s} \tag{8.33}$$

then the perturbing effect of the sun on the vehicle's orbit around the planet is less than the perturbing effect of the planet on the vehicle's orbit around the sun. We say that the vehicle is therefore within the planet's sphere of influence. Substituting Eqns (8.26) and (8.32) into Eqn (8.33) yields

$$\frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 < \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2$$

which means

$$\left(\frac{r}{R}\right)^5 < \left(\frac{m_p}{m_s}\right)^2$$

or

$$\frac{r}{R} < \left(\frac{m_p}{m_s}\right)^{\frac{2}{5}}$$

Let $r_{\text{SOI}}$ be the radius of the sphere of influence. Within the planet's sphere of influence, defined by

$$\frac{r_{\text{SOI}}}{R} = \left(\frac{m_p}{m_s}\right)^{\frac{2}{5}} \tag{8.34}$$

the motion of the spacecraft is determined by its equations of motion relative to the planet (Eqn (8.28)). Outside the sphere of influence, the path of the spacecraft is computed relative to the sun (Eqn (8.22)).

The sphere of influence radius presented in Eqn (8.34) is not an exact quantity. It is simply a reasonable estimate of the distance beyond which the sun's gravitational attraction dominates that of a planet. The spheres of influence of all the planets and the earth's moon are listed in Table A.2.

## EXAMPLE 8.3

Calculate the radius of the earth's sphere of influence.

In Table A.1 we find

$$m_{\text{earth}} = 5.974 \times 10^{24} \text{ kg}$$
$$m_{\text{sun}} = 1.989 \times 10^{30} \text{ kg}$$
$$R_{\text{earth}} = 149.6 \times 10^{6} \text{ km}$$

Substituting this data into Eqn (8.34) yields

$$r_{\mathrm{SOI}} = 149.6 \times 10^6 \left(\frac{5.974 \times 10^{24}}{1.989 \times 10^{24}}\right)^{\frac{2}{5}} = 925 \times 10^6 \text{ km}$$

Since the radius of the earth is 6378 km,

$$r_{\mathrm{SOI}} = 145 \text{ earth radii}$$

Relative to the earth, its sphere of influence is very large. However, relative to the sun it is tiny, as illustrated in Figure 8.7.

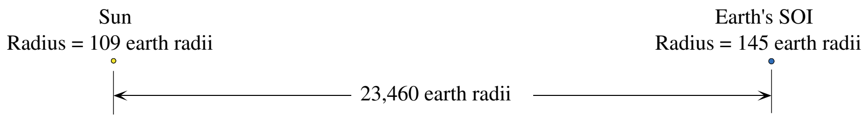


**FIGURE 8.7**

The earth's sphere of influence and the sun, drawn to scale.

## 8.5 Method of patched conics

"Conics" refers to the fact that two-body, or Keplerian, orbits are conic sections with the focus at the attracting body. To study an interplanetary trajectory, we assume that when the spacecraft is outside the sphere of influence of a planet it follows an unperturbed Keplerian orbit around the sun. Because interplanetary distances are so vast, for heliocentric orbits we may neglect the size of the spheres of influence and consider them, like the planets they surround, to be just points in space coinciding with the planetary centers. Within each planetary sphere of influence, the spacecraft travels an unperturbed Keplerian path about the planet. While the sphere of influence appears as a mere speck on the scale of the solar system, from the point of view of the planet it is very large indeed and may be considered to lie at infinity.

To analyze a mission from planet 1 to planet 2 using the method of patched conics, we first determine the heliocentric trajectory—such as the Hohmann transfer ellipse discussed in Section 8.2—that will intersect the desired positions of the two planets in their orbits. This trajectory takes the spacecraft from the sphere of influence of planet 1 to that of planet 2. At the spheres of influence, the heliocentric velocities of the transfer orbit are computed relative to the planet to establish the velocities "at infinity," which are then used to determine planetocentric departure trajectory at planet 1 and arrival trajectory at planet 2. In this way, we "patch" together the three conics, one centered at the sun and the other two centered at the planets in question.

Whereas the method of patched conics is remarkably accurate for interplanetary trajectories, such is not the case for lunar rendezvous and return trajectories. The orbit of the moon is determined primarily by the earth, whose sphere of influence extends well beyond the moon's 384,400 km orbital radius. To apply patched conics to lunar trajectories we ignore the sun and consider the motion of a spacecraft as influenced by just the earth and moon, as in the restricted three-body problem discussed in Section 2.12. The size of the moon's sphere of influence is found using Eqn (8.34), with the earth playing the role of the sun:

$$r_{\mathrm{SOI}} = R\left(\frac{m_{\mathrm{moon}}}{m_{\mathrm{earth}}}\right)^{\frac{2}{5}}$$

where $R$ is the radius of the moon's orbit. Thus, using Table A.1,

$$r_{\mathrm{SOI}} = 384{,}400\left(\frac{73.48\times10^{21}}{5974\times10^{21}}\right)^{\frac{2}{5}} = 66{,}200 \ \ \mathrm{km}$$

as recorded in Table A.2. The moon's sphere of influence extends out to over one-sixth of the distance to the earth. We can hardly consider it to be a mere speck relative to the earth. Another complication is the fact that the earth and the moon are somewhat comparable in mass, so that their center of mass lies almost three-quarters of an earth radius from the center of the earth. The motion of the moon cannot be accurately described as rotating around the center of the earth.

Complications such as these place the analysis of cislunar trajectories beyond the scope of this chapter (in Example 2.18, we did a lunar trajectory calculation not by using patched conics but by integrating the equations of motion of a spacecraft within the context of the restricted three-body problem). Extensions of the patched conic technique to lunar trajectories may be found in references such as Bate, Mueller, and White (1971), Kaplan (1976), and Battin (1999).

## 8.6 Planetary departure

In order to escape the gravitational pull of a planet, the spacecraft must travel a hyperbolic trajectory relative to the planet, arriving at its sphere of influence with a relative velocity $\mathbf{v}_\infty$ (hyperbolic excess velocity) greater than zero. On a parabolic trajectory, according to Eqn (2.91), the spacecraft will arrive at the sphere of influence ($r = \infty$) with a relative speed of zero. In that case, the spacecraft remains in the same orbit as the planet and does not embark upon a heliocentric elliptical path.

Figure 8.8 shows a spacecraft departing on a Hohmann trajectory from planet 1 toward a target planet 2, which is farther away from the sun (as in Figure 8.1). At the sphere of influence crossing, the heliocentric velocity $\mathbf{V}_D^{(v)}$ of the spacecraft is parallel to the asymptote of the departure hyperbola as well as to the planet's heliocentric velocity vector $\mathbf{V}_1$. $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_1$ must be parallel and in the same direction for a Hohmann transfer such that $\Delta V_D$ in Eqn (8.3) is positive. Clearly, $\Delta V_D$ is the hyperbolic excess speed of the departure hyperbola,

$$v_\infty = \sqrt{\frac{\mu_{\mathrm{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}} - 1\right) \tag{8.35}$$

It would be well at this point for the reader to review Section 2.9 on hyperbolic trajectories and compare Figure 8.8 and Figure 2.25. Recall that point $C$ is the center of the hyperbola.

A space vehicle is ordinarily launched into an interplanetary trajectory from a circular parking orbit. The radius of this parking orbit equals the periapse radius $r_p$ of the departure hyperbola.

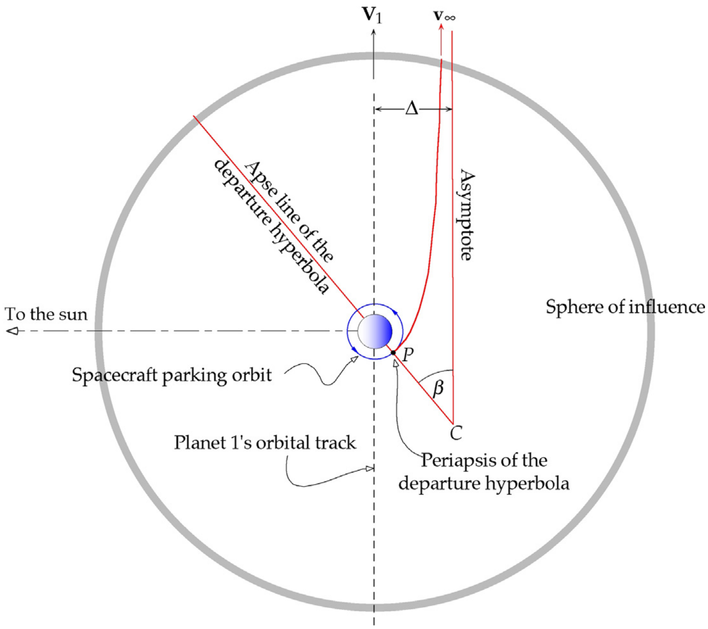


**FIGURE 8.8**

Departure of a spacecraft on a mission from an inner planet to an outer planet.

According Eqn (2.50), the periapsis radius is given by

$$r_p = \frac{h^2}{\mu_1}\frac{1}{1+e} \tag{8.36}$$

where $h$ is the angular momentum of the departure hyperbola (relative to the planet), $e$ is the eccentricity of the hyperbola, and $\mu_1$ is the planet's gravitational parameter. The hyperbolic excess speed is found in Eqn (2.115), from which we obtain

$$h = \frac{\mu_1\sqrt{e^2-1}}{v_\infty} \tag{8.37}$$

Substituting this expression for the angular momentum into Eqn (8.36) and solving for eccentricity yields

$$e = 1 + \frac{r_p v_\infty^2}{\mu_1} \tag{8.38}$$

We place this result back into Eqn (8.37) to obtain the following expression for the angular momentum:

$$h = r_p\sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}} \tag{8.39}$$

Since the hyperbolic excess speed is specified by the mission requirements (Eqn (8.35)), choosing a departure periapsis $r_p$ yields the parameters $e$ and $h$ of the departure hyperbola. From the angular momentum, we get the periapsis speed,

$$v_p = \frac{h}{r_p} = \sqrt{v_\infty^2 + \frac{2\mu_1}{r_p}} \tag{8.40}$$

which can also be found from an energy approach using Eqn (2.113). With Eqn (8.40) and the speed of the circular parking orbit (Eqn (2.63)),

$$v_c = \sqrt{\frac{\mu_1}{r_p}} \tag{8.41}$$

we can calculate the delta-v required to put the vehicle onto the hyperbolic departure trajectory,

$$\Delta v = v_p - v_c = v_c\left(\sqrt{2 + \left(\frac{v_\infty}{v_c}\right)^2} - 1\right) \tag{8.42}$$

The location of periapsis, where the delta-v maneuver must occur, is found using Eqn (2.99) and Eqn (8.38),

$$\beta = \cos^{-1}\left(\frac{1}{e}\right) = \cos^{-1}\left(\frac{1}{1 + \dfrac{r_p v_\infty^2}{\mu_1}}\right) \tag{8.43}$$

$\beta$ gives the orientation of the apse line of the hyperbola to the planet's heliocentric velocity vector.

It should be pointed out that the only requirement on the orientation of the plane of the departure hyperbola is that it must contain the center of mass of the planet as well as the relative velocity vector $\mathbf{v}_\infty$. Therefore, as shown in Figure 8.9, the hyperbola can be rotated about a line *A–A,* which passes through the planet's center of mass and is parallel to $\mathbf{v}_\infty$ (or $\mathbf{V}_1$, which of course is parallel to $\mathbf{v}_\infty$ for Hohmann transfers). Rotating the hyperbola in this way sweeps out a surface of revolution on which all possible departure hyperbolas lie. The periapsis of the hyperbola traces out a circle which, for the specified periapsis radius $r_p$, is the locus of all possible points of injection into a departure trajectory toward the target planet. This circle is the base of a cone with vertex at the center of the planet. From Figure 2.25 we can determine that its radius is $r_p \sin\beta$, where $\beta$ is given just above in Eqn (8.43).

The plane of the parking orbit, or direct ascent trajectory, must contain the line *A–A* and the launch site at the time of launch. The possible inclinations of a prograde orbit range from a minimum of $i_{\min}$, where $i_{\min}$ is the latitude of the launch site, to $i_{\max}$, which cannot exceed 90°. Launch site safety considerations may place additional limits on this range. For example, orbits originating from the Kennedy Space Center in Florida, USA (latitude 28.5°), are limited to inclinations between 28.5° and 52.5°. For the scenario illustrated in Figure 8.12, the location of the launch site limits access to just the departure trajectories having periapses lying between *a* and *b*. The figure shows that there are two times per day—when the planet rotates the launch site through positions 1 and 1′—that a spacecraft can be launched into a parking orbit. These times are closer together (the launch window is smaller) the lower the inclination of the parking orbit.

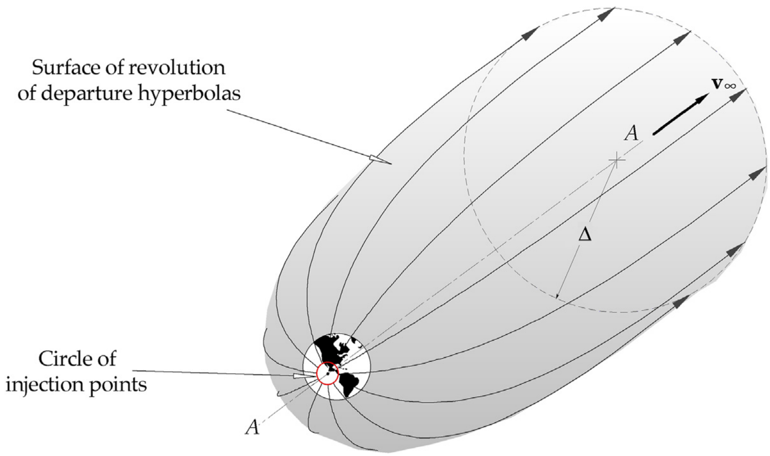


**FIGURE 8.9**

Locus of possible departure trajectories for a given $v_\infty$ and $r_p$.

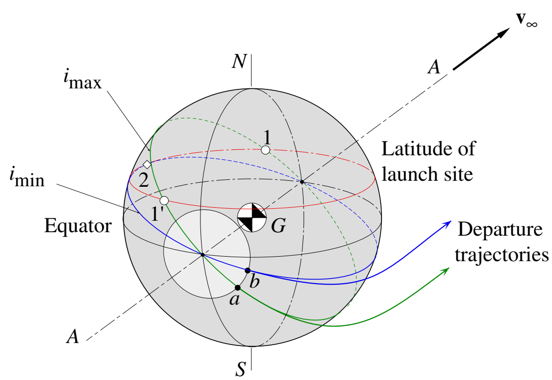


**FIGURE 8.10**

Parking orbits and departure trajectories for a launch site at a given latitude.

Once a spacecraft is established in its parking orbit, then an opportunity for launch into the departure trajectory occurs at each orbital circuit.

If the mission is to send a spacecraft from an outer planet to an inner planet, as in Figure 8.2, then the spacecraft's heliocentric speed $V_D^{(v)}$ at departure must be less than that of the planet. That means the spacecraft must emerge from the backside of the sphere of influence with its relative velocity vector $\mathbf{v}_\infty$ directed opposite to $\mathbf{V}_1$, as shown in Figure 8.11. Figures 8.9 and 8.10 apply to this situation as well.

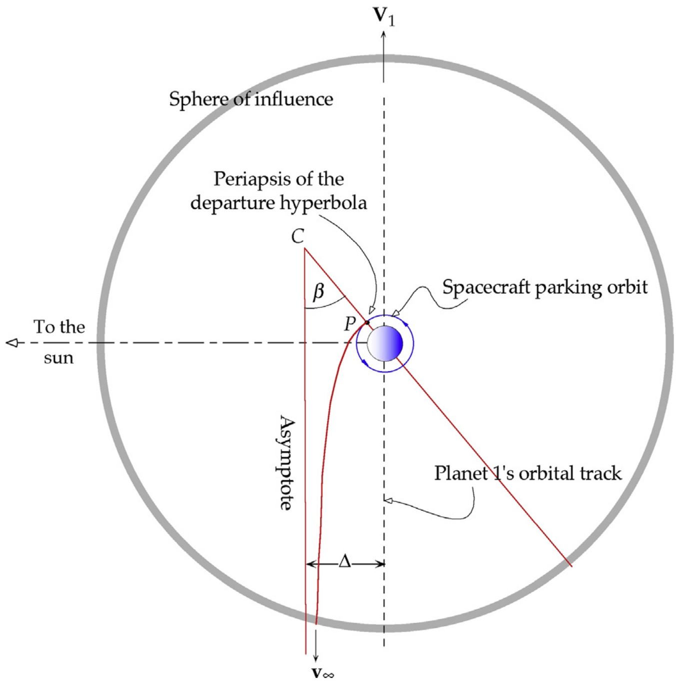


**FIGURE 8.11**

Departure of a spacecraft on a trajectory from an outer planet to an inner planet.

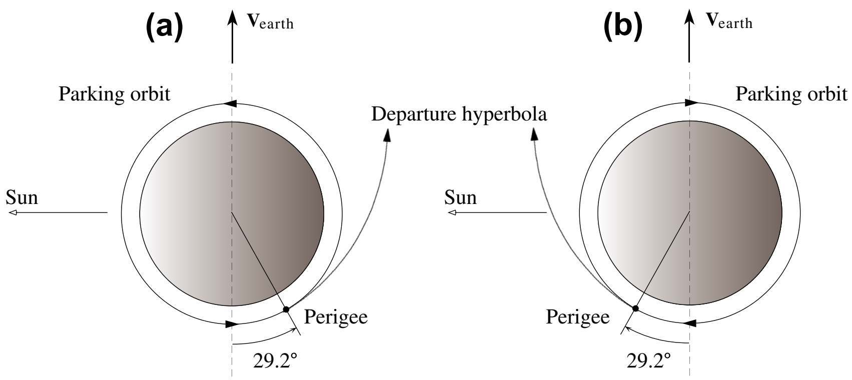


**FIGURE 8.12**

Departure trajectory to mars initiated from (a) the dark side and (b) the sunlit side of the earth.

## EXAMPLE 8.4

A spacecraft is launched on a mission to Mars starting from a 300 km circular parking orbit. Calculate (a) the delta-v required, (b) the location of perigee of the departure hyperbola, and (c) the amount of propellant required as a percentage of the spacecraft mass before the delta-v burn, assuming a specific impulse of 300 s.

### Solution

From Tables A.1 and A.2, we obtain the gravitational parameters for the sun and the earth,

$$\mu_{\text{sun}} = 1.327 \times 10^{11} \ \text{km}^3/\text{s}^2$$
$$\mu_{\text{earth}} = 398{,}600 \ \text{km}^3/\text{s}^2$$

and the orbital radii of the earth and Mars,

$$R_{\text{earth}} = 149.6 \times 10^6 \ \text{km}$$
$$R_{\text{Mars}} = 227.9 \times 10^6 \ \text{km}$$

**(a)** According to Eqn (8.35), the hyperbolic excess speed is

$$v_\infty = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}}\left(\sqrt{\frac{2R_{\text{Mars}}}{R_{\text{earth}} + R_{\text{Mars}}}} - 1\right) = \sqrt{\frac{1.327 \times 10^{11}}{149.6 \times 10^6}}\left(\sqrt{\frac{2(227.9 \times 10^6)}{149.6 \times 10^6 + 227.9 \times 10^6}} - 1\right)$$

from which

$$v_\infty = 2.943 \ \text{km/s}$$

The speed of the spacecraft in its 300 km circular parking orbit is given by Eqn (8.41),

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_{\text{earth}} + 300}} = \sqrt{\frac{398{,}600}{6678}} = 7.726 \ \text{km/s}$$

Finally, we use Eqn (8.42) to calculate the delta-v required to step up to the departure hyperbola.

$$\Delta v = v_p - v_c = v_c\left(\sqrt{2 + \left(\frac{v_\infty}{v_c}\right)^2} - 1\right) = 7.726\left(\sqrt{2 + \left(\frac{2.943}{7.726}\right)^2} - 1\right)$$

$$\boxed{\Delta v = 3.590 \ \text{km/s}}$$

**(b)** Perigee of the departure hyperbola, relative to the earth's orbital velocity vector, is found using Eqn (8.43),

$$\beta = \cos^{-1}\left(\frac{1}{1 + \dfrac{r_p v_\infty^2}{\mu_{\text{earth}}}}\right) = \cos^{-1}\left(\frac{1}{1 + \dfrac{6678 \cdot 2.943^2}{368{,}600}}\right)$$

$$\boxed{\beta = 29.16^\circ}$$

Figure 8.12 shows that the perigee can be located on either the sunlit or the dark side of the earth. It is likely that the parking orbit would be a prograde orbit (west to east), which would place the burnout point on the dark side.

**(c)** From Eqn (6.1), we have

$$\frac{\Delta m}{m} = 1 - e^{-\frac{\Delta v}{I_{sp} g_o}}$$

Substituting $\Delta v = 3.590$ km/s, $I_{sp} = 300$ s, and $g_o = 9.81 \times 10^{-3}$ km/s$^2$, this yields

$$\boxed{\frac{\Delta m}{m} = 0.705}$$

That is, prior to the delta-v maneuver, over 70% of the spacecraft mass must be propellant.

## 8.7 Sensitivity analysis

The initial maneuvers required to place a spacecraft on an interplanetary trajectory occur well within the sphere of influence of the departure planet. Since the sphere of influence is just a point on the scale of the solar system, one may ask what effects small errors in position and velocity at the maneuver point have on the trajectory. Assuming the mission is from an inner to an outer planet, let us consider the effect that small changes in the burnout velocity $v_p$ and radius $r_p$ have on the target radius $R_2$ of the heliocentric Hohmann transfer ellipse (see Figures 8.1 and 8.8).

$R_2$ is the radius of aphelion, so we use Eqn (2.70) to obtain

$$R_2 = \frac{h^2}{\mu_{\text{sun}}}\frac{1}{1-e}$$

Substituting $h = R_1 V_D^{(v)}$ and $e = (R_2 - R_1)/(R_2 + R_1)$, and solving for $R_2$, yields

$$R_2 = \frac{R_1^2\left[V_D^{(v)}\right]^2}{2\mu_{\text{sun}} - R_1\left[V_D^{(v)}\right]^2} \tag{8.44}$$

(This expression holds as well for a mission from an outer to inner planet.) The change $\delta R_2$ in $R_2$ due to a small variation $\delta V_D^{(v)}$ of $V_D^{(v)}$ is

$$\delta R_2 = \frac{\mathrm{d}R_2}{\mathrm{d}V_D^{(v)}}\delta V_D^{(v)} = \frac{4R_1^2\mu_{\text{sun}}}{\left\{2\mu_{\text{sun}} - R_1\left[V_D^{(v)}\right]^2\right\}^2} V_D^{(v)}\delta V_D^{(v)}$$

Dividing this equation by Eqn (8.44) leads to

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \dfrac{R_1\left[V_D^{(v)}\right]^2}{2\mu_{\text{sun}}}}\frac{\delta V_D^{(v)}}{V_D^{(v)}} \tag{8.45}$$

The departure speed $V_D^{(v)}$ of the space vehicle is the sum of the planet's speed $V_1$ and excess speed $v_\infty$.

$$V_D^{(v)} = V_1 + v_\infty$$

We can solve Eqn (8.40) for $v_\infty$,

$$v_\infty = \sqrt{v_p^2 - \frac{2\mu_1}{r_p}}$$

Hence

$$V_D^{(v)} = V_1 + \sqrt{v_p^2 - \frac{2\mu_1}{r_p}} \tag{8.46}$$

The change in $V_D^{(v)}$ due to variations $\delta r_p$ and $\delta v_p$ of the burnout position (periapse) $r_p$ and speed $v_p$ is given by

$$\delta V_D^{(v)} = \frac{\partial V_D^{(v)}}{\partial r_p}\delta r_p + \frac{\partial V_D^{(v)}}{\partial v_p}\delta v_p \tag{8.47}$$

From Eqn (8.46), we obtain

$$\frac{\partial V_D^{(v)}}{\partial r_p} = \frac{\mu_1}{v_\infty r_p^2} \qquad \frac{\partial V_D^{(v)}}{\partial v_p} = \frac{v_p}{v_\infty}$$

Therefore,

$$\delta V_D^{(v)} = \frac{\mu_1}{v_\infty {r_p}^2}\delta r_p + \frac{v_p}{v_\infty}\delta v_p$$

Once again making use of Eqn (8.40), this can be written as follows:

$$\frac{\delta V_D^{(v)}}{V_D^{(v)}} = \frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \dfrac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p} \tag{8.48}$$

Substituting this into Eqn (8.45) finally yields the desired result, an expression for the variation of $R_2$ due to variations in $r_p$ and $v_p$.

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \dfrac{R_1\left[V_D^{(v)}\right]^2}{2\mu_{\text{sun}}}}\left(\frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \dfrac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p}\right) \tag{8.49}$$

Consider a mission from earth to Mars, starting from a 300 km parking orbit. We have

$$\mu_{\text{sun}} = 1.327 \times 10^{11} \ \text{km}^3/\text{s}^2$$
$$\mu_1 = \mu_{\text{earth}} = 398{,}600 \ \text{km}^3/\text{s}^2$$
$$R_1 = 149.6 \times 10^6 \ \text{km}$$
$$R_2 = 227.9 \times 10^6 \ \text{km}$$
$$r_p = 6678 \ \text{km}$$

In addition, from Eqns (8.1) and (8.2),

$$V_1 = V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} = \sqrt{\frac{1.327 \times 10^{11}}{149.6 \times 10^6}} = 29.78 \ \text{km/s}$$

$$V_D^{(v)} = \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1 + R_2)}} = \sqrt{2 \cdot 1.327 \times 10^{11}}\sqrt{\frac{227.9 \times 10^6}{149.6 \times 10^6(149.6 \times 10^6 + 227.9 \times 10^6)}}$$
$$= 32.73 \ \text{km/s}$$

Therefore,

$$v_\infty = V_D^{(v)} - V_{\text{earth}} = 2.943 \text{ km/s}$$

and, from Eqn (8.40),

$$v_p = \sqrt{v_\infty^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{2.943^2 + \frac{2 \cdot 398{,}600}{6678}} = 11.32 \text{ km/s}$$

Substituting these values into Eqn (8.49) yields

$$\frac{\delta R_2}{R_2} = 3.127\frac{\delta r_p}{r_p} + 6.708\frac{\delta v_p}{v_p}$$

This expression shows that a 0.01% variation (1.1 m/s) in the burnout speed $v_p$ changes the target radius $R_2$ by 0.067% or 153,000 km! Likewise, an error of 0.01% (0.67 km) in burnout radius $r_p$ produces an error of over 70,000 km. Thus, small errors that are likely to occur in the launch phase of the mission must be corrected by midcourse maneuvers during the coasting flight along the elliptical transfer trajectory.

## 8.8 Planetary rendezvous

A spacecraft arrives at the sphere of influence of the target planet with a hyperbolic excess velocity $\mathbf{v}_\infty$ relative to the planet. In the case illustrated in Figure 8.1, a mission from an inner planet 1 to an outer planet 2 (e.g., earth to Mars), the spacecraft's heliocentric approach velocity $\mathbf{V}_A^{(v)}$ is smaller in magnitude than that of the planet, $\mathbf{V}_2$. Therefore, it crosses the forward portion of the sphere of influence, as shown in Figure 8.13. For a Hohmann transfer, $\mathbf{V}_A^{(v)}$ and $\mathbf{V}_2$ are parallel, so the magnitude of the hyperbolic excess velocity is, simply,

$$v_\infty = V_2 - V_A^{(v)} \tag{8.50}$$

If the mission is as illustrated in Figure 8.2, from an outer planet to an inner one (e.g., earth to Venus), then $V_A^{(v)}$ is greater than $V_2$, and the spacecraft must cross the rear portion of the sphere of influence, as shown in Figure 8.14. In that case

$$v_\infty = V_A^{(v)} - V_2 \tag{8.51}$$

What happens after crossing the sphere of influence depends on the nature of the mission. If the goal is to impact the planet (or its atmosphere), the aiming radius $\Delta$ of the approach hyperbola must be such that hyperbola's periapsis radius $r_p$ equals essentially the radius of the planet. If the intent is to go into orbit around the planet, then $\Delta$ must be chosen so that the delta-v burn at periapsis will occur at the correct altitude above the planet. If there is no impact with the planet and no drop into a capture orbit around the planet, then the spacecraft will simply continue past periapsis on a flyby trajectory, exiting the sphere of influence with the same relative speed $v_\infty$ it entered but with the velocity vector rotated through the turn angle $\delta$, given by Eqn (2.100),

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) \tag{8.52}$$

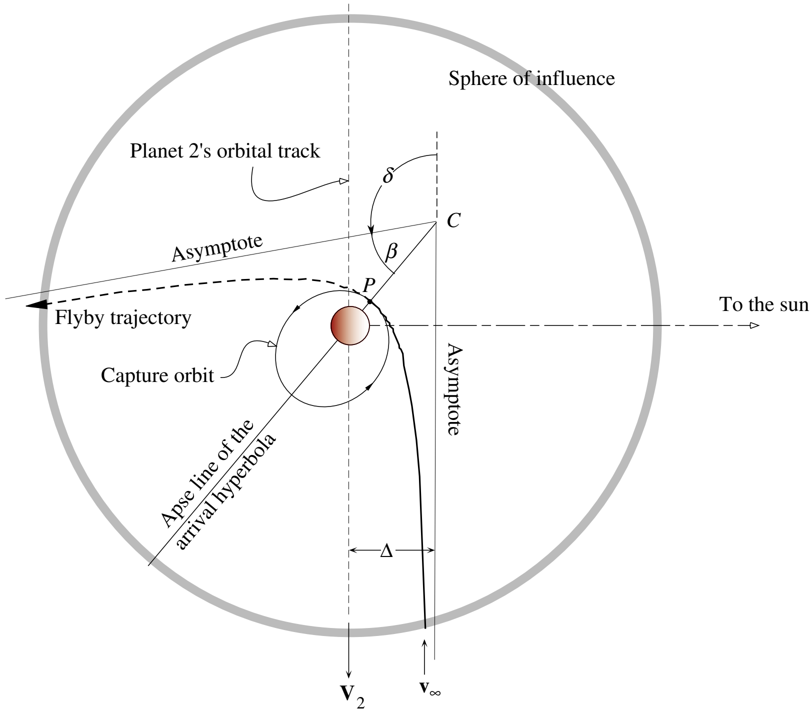


**FIGURE 8.13**

Spacecraft approach trajectory for a hohmann transfer to an outer planet from an inner one. *P* is the periapse of the approach hyperbola.

With the hyperbolic excess speed $v_\infty$ and the periapse radius $r_p$ specified, the eccentricity of the approach hyperbola is found from Eqn (8.38),

$$e = 1 + \frac{r_p v_\infty^2}{\mu_2} \tag{8.53}$$

where $\mu_2$ is the gravitational parameter of planet 2. Hence, the turn angle is

$$\delta = 2\sin^{-1}\left(\frac{1}{1 + \dfrac{r_p v_\infty^2}{\mu_2}}\right) \tag{8.54}$$

We can combine Eqns (2.103) and (2.107) to obtain the following expression for the aiming radius:

$$\Delta = \frac{h^2}{\mu_2}\frac{1}{\sqrt{e^2 - 1}} \tag{8.55}$$

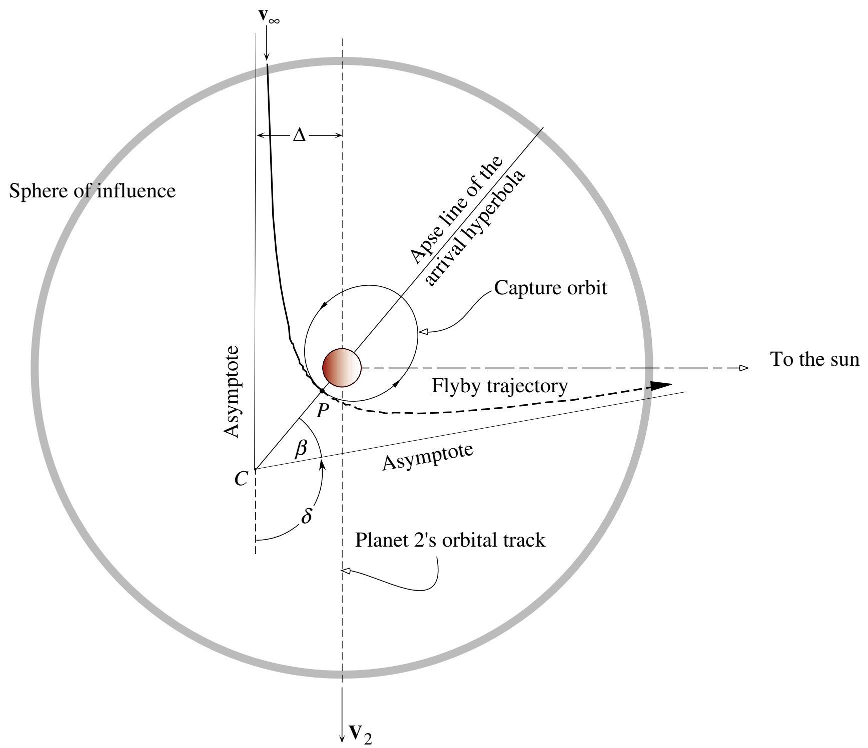


**FIGURE 8.14**

Spacecraft approach trajectory for a hohmann transfer to an inner planet from an outer one. *P* is the periapse of the approach hyperbola.

The angular momentum of the approach hyperbola relative to the planet is found using Eqn (8.39),

$$h = r_p\sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}} \tag{8.56}$$

Substituting Eqns (8.53) and (8.56) into Eqn (8.55) yields the aiming radius in terms of the periapse radius and the hyperbolic excess speed,

$$\Delta = r_p\sqrt{1 + \frac{2\mu_2}{r_p v_\infty^2}} \tag{8.57}$$

Just as we observed when discussing departure trajectories, the approach hyperbola does not lie in a unique plane. We can rotate the hyperbolas illustrated in Figures 8.11 and 8.12 about a line *A–A* parallel to $\mathbf{v}_\infty$ and passing through the target planet's center of mass, as shown in Figure 8.15. The

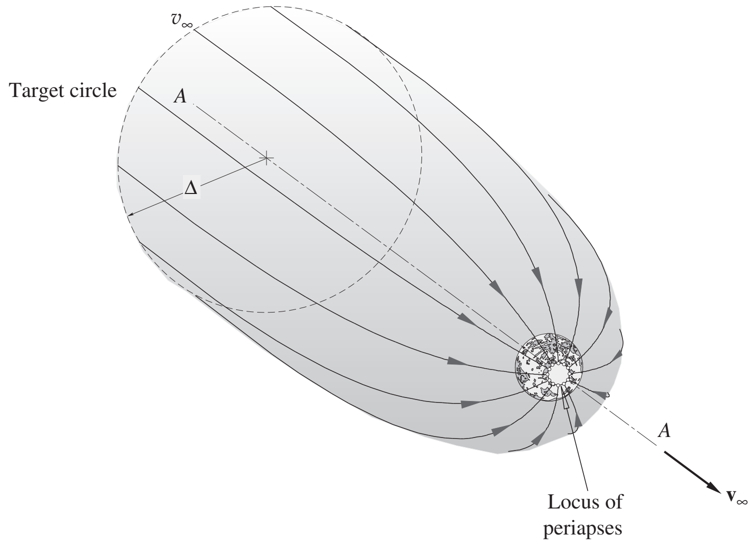


**FIGURE 8.15**

Locus of approach hyperbolas to the target planet.

approach hyperbolas in that figure terminate at the circle of periapses. Figure 8.16 is a plane through the solid of revolution revealing the shape of hyperbolas having a common $v_\infty$ but varying $\Delta$.

Let us suppose that the purpose of the mission is to enter an elliptical orbit of eccentricity $e$ around the planet. This will require a delta-v maneuver at periapsis $P$ (Figures 8.13 and 8.14), which is also periapsis of the ellipse. The speed in the hyperbolic trajectory at periapsis is given by Eqn (8.40),

$$v_p\big)_{\text{hyp}} = \sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}} \tag{8.58}$$

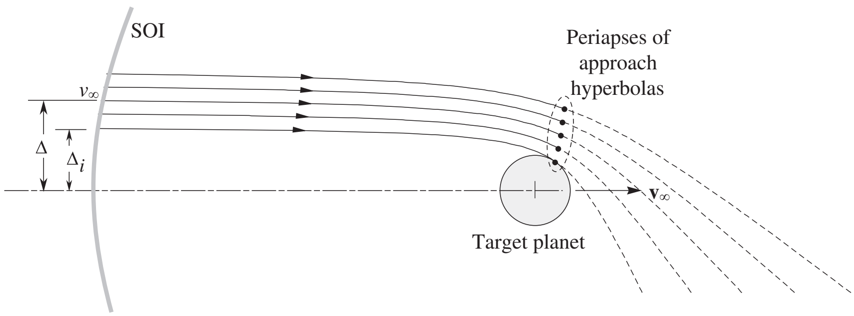


**FIGURE 8.16**

Family of approach hyperbolas having the same $v_\infty$ but different $\Delta$.

The velocity at periapsis of the capture orbit is found by setting $h = r_p v_p$ in Eqn (2.50) and solving for $v_p$.

$$v_p\Big)_{\text{capture}} = \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.59}$$

Hence, the required delta-v is

$$\Delta v = v_p\Big)_{\text{hyp}} - v_p\Big)_{\text{capture}} = \sqrt{v_\infty^2 + \frac{2\mu_2}{r_p}} - \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.60}$$

For a given $v_\infty$, $\Delta v$ clearly depends on the choice of periapse radius $r_p$ and capture orbit eccentricity $e$. Requiring the maneuver point to remain the periapsis of the capture orbit means that $\Delta v$ is maximum for a circular capture orbit and decreases with increasing eccentricity until $\Delta v = 0$, which, of course, means no capture (flyby).

In order to determine optimal capture radius, let us write Eqn (8.60) in nondimensional form as

$$\frac{\Delta v}{v_\infty} = \sqrt{1 + \frac{2}{\xi}} - \sqrt{\frac{1+e}{\xi}} \tag{8.61}$$

where

$$\xi = \frac{r_p v_\infty^2}{\mu_2} \tag{8.62}$$

The first and second derivatives of $\Delta v / v_\infty$ with respect to $\xi$ are

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\Delta v}{v_\infty} = \left(-\frac{1}{\sqrt{\xi+2}} + \frac{\sqrt{1+e}}{2}\right)\frac{1}{\xi^{\frac{3}{2}}} \tag{8.63}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{\Delta v}{v_\infty} = \left[\frac{2\xi+3}{(\xi+2)^{\frac{3}{2}}} - \frac{3}{4}\sqrt{1+e}\right]\frac{1}{\xi^{\frac{5}{2}}} \tag{8.64}$$

Setting the first derivative equal to zero and solving for $\xi$ yields

$$\xi = 2\frac{1-e}{1+e} \tag{8.65}$$

Substituting this value of $\xi$ into Eqn (8.64), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{\Delta v}{v_\infty} = \frac{\sqrt{2}}{64}\frac{(1+e)^3}{(1-e)^{\frac{3}{2}}} \tag{8.66}$$

This expression is positive for elliptical orbits ($0 \le e < 1$), which means that when $\xi$ is given by Eqn (8.65), $\Delta v$ is a minimum. Therefore, from Eqn (8.62), the optimal periapse radius as far as fuel expenditure is concerned is

$$r_p = \frac{2\mu_2}{v_\infty^2}\frac{1-e}{1+e} \tag{8.67}$$

We can combine Eqns (2.50) and (2.70) to get

$$\frac{1-e}{1+e} = \frac{r_p}{r_a} \tag{8.68}$$

where $r_a$ is the apoapsis radius. Thus, Eqn (8.67) implies

$$r_a = \frac{2\mu_2}{v_\infty^2} \tag{8.69}$$

That is, the apoapsis of this capture ellipse is independent of the eccentricity and equals the radius of the optimal circular orbit.

Substituting Eqn (8.65) back into Eqn (8.61) yields the minimum $\Delta v$,

$$\Delta v = v_\infty \sqrt{\frac{1-e}{2}} \tag{8.70}$$

Finally, placing the optimal $r_p$ into Eqn (8.57) leads to an expression for the aiming radius required for minimum $\Delta v$,

$$\Delta = 2\sqrt{2}\,\frac{\sqrt{1-e}}{1+e}\frac{\mu_2}{v_\infty^2} = \sqrt{\frac{2}{1-e}}\,r_p \tag{8.71}$$

Clearly, the optimal $\Delta v$ (and periapsis height) are reduced for highly eccentric elliptical capture orbits ($e \to 1$). However, it should be pointed out that the use of optimal $\Delta v$ may have to be sacrificed in favor of a variety of other mission requirements.

## EXAMPLE 8.5

After a Hohmann transfer from earth to Mars, calculate

**(a)** the minimum delta-v required to place a spacecraft in orbit with a period of 7 h.
**(b)** the periapsis radius.
**(c)** the aiming radius.
**(d)** the angle between periapsis and Mars' velocity vector.

### Solution

The following data are required from Tables A.1 and A.2:

$$\mu_{\text{sun}} = 1.327 \times 10^{11}\ \text{km}^3/\text{s}^2$$
$$\mu_{\text{Mars}} = 42{,}830\ \text{km}^3/\text{s}^2$$
$$R_{\text{earth}} = 149.6 \times 10^6\ \text{km}$$
$$R_{\text{Mars}} = 227.9 \times 10^6\ \text{km}$$
$$r_{\text{Mars}} = 3396\ \text{km}$$

**(a)** The hyperbolic excess speed is found using Eqn (8.4),

$$v_\infty = \Delta V_A = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Mars}}}}\left(1 - \sqrt{\frac{2R_{\text{earth}}}{R_{\text{earth}} + R_{\text{Mars}}}}\right) = \sqrt{\frac{1.327 \times 10^{11}}{227.9 \times 10^6}}\left(1 - \sqrt{\frac{2\cdot 149.6 \times 10^6}{149.6 \times 10^6 + 227.9 \times 10^6}}\right)$$

$$v_\infty = 2.648\ \text{km/s}$$

We can use Eqn (2.83) to express the semimajor axis $a$ of the capture orbit in terms of its period $T$,

$$a = \left(\frac{T\sqrt{\mu_{\text{Mars}}}}{2\pi}\right)^{\frac{2}{3}}$$

Substituting $T = 7 \cdot 3600$ s yields

$$a = \left(\frac{25{,}200\sqrt{42{,}830}}{2\pi}\right)^{\frac{2}{3}} = 8832 \text{ km}$$

From Eqn (2.73) we obtain

$$a = \frac{r_p}{1-e}$$

On substituting the optimal periapsis radius, Eqn (8.67), this becomes

$$a = \frac{2\mu_{\text{Mars}}}{v_\infty^2}\frac{1}{1+e}$$

from which

$$e = \frac{2\mu_{\text{Mars}}}{av_\infty^2} - 1 = \frac{2 \cdot 42{,}830}{8832 \cdot 2.648^2} - 1 = 0.3833$$

Thus, using Eqn (8.70), we find

$$\Delta v = v_\infty\sqrt{\frac{1-e}{2}} = 2.648\sqrt{\frac{1-0.3833}{2}} = \boxed{1.470 \text{ km/s}}$$

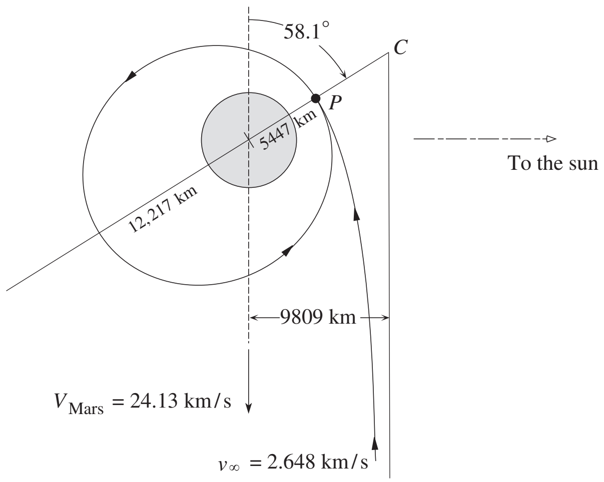


**FIGURE 8.17**

An optimal approach to a mars capture orbit with a seven hour period. $r_{\text{Mars}} = 3396$ km.

**(b)** From Eqn (8.66), we obtain the periapse radius

$$r_p = \frac{2\mu_{Mars}}{v_\infty^2}\frac{1-e}{1+e} = \frac{2 \cdot 42{,}830}{2.648^2}\frac{1-0.3833}{1+0.3833} = \boxed{5447 \text{ km}}$$

**(c)** The aiming radius is given by Eqn (8.71),

$$\Delta = r_p\sqrt{\frac{2}{1-e}} = 5447\sqrt{\frac{2}{1-0.3833}} = \boxed{9809 \text{ km}}$$

**(d)** Using Eqn (8.43), we get the angle to periapsis

$$\beta = \cos^{-1}\left(\frac{1}{1+\dfrac{r_p v_\infty^2}{\mu_{Mars}}}\right) = \cos^{-1}\left(\frac{1}{1+\dfrac{5447 \cdot 2.648^2}{42{,}830}}\right) = \boxed{58.09^\circ}$$

Mars, the approach hyperbola, and the capture orbit are shown to scale in Figure 8.17. The approach could also be made from the dark side of the planet instead of the sunlit side. The approach hyperbola and capture ellipse would be the mirror image of that shown, as is the case in Figure 8.12.

## 8.9 Planetary flyby

A spacecraft that enters a planet's sphere of influence and does not impact the planet or go into orbit around it will continue in its hyperbolic trajectory through periapsis $P$ and exit the sphere of influence. Figure 8.18 shows a hyperbolic flyby trajectory along with the asymptotes and apse line of the hyperbola. It is a leading-side flyby because the periapsis is on the side of the planet facing into the direction of motion. Likewise, Figure 8.19 illustrates a trailing-side flyby. At the inbound crossing point, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the spacecraft equals the planet's heliocentric velocity $\mathbf{V}$ plus the hyperbolic excess velocity $\mathbf{v}_{\infty_1}$ of the spacecraft (relative to the planet),

$$\mathbf{V}_1^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_1} \tag{8.72}$$

Similarly, at the outbound crossing, we have

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} \tag{8.73}$$

The change $\Delta\mathbf{V}^{(v)}$ in the spacecraft's heliocentric velocity is

$$\Delta\mathbf{V}^{(v)} = \mathbf{V}_2^{(v)} - \mathbf{V}_1^{(v)} = (\mathbf{V} + \mathbf{v}_{\infty_2}) - (\mathbf{V} + \mathbf{v}_{\infty_1})$$

which means

$$\Delta\mathbf{V}^{(v)} = \mathbf{v}_{\infty_2} - \mathbf{v}_{\infty_1} = \Delta\mathbf{v}_\infty \tag{8.74}$$

The excess velocities $\mathbf{v}_{\infty_1}$ and $\mathbf{v}_{\infty_2}$ lie along the asymptotes of the hyperbola and are therefore inclined at the same angle $\beta$ to the apse line (see Figure 2.25), with $\mathbf{v}_{\infty_1}$ pointing toward and $\mathbf{v}_{\infty_2}$ pointing away from the center $C$. They both have the same magnitude $v_\infty$, with $\mathbf{v}_{\infty_2}$ having rotated relative to $\mathbf{v}_{\infty_1}$ by the turn angle $\delta$. Hence, $\Delta\mathbf{v}_\infty$—and therefore $\Delta\mathbf{V}^{(v)}$—is a vector that lies along the

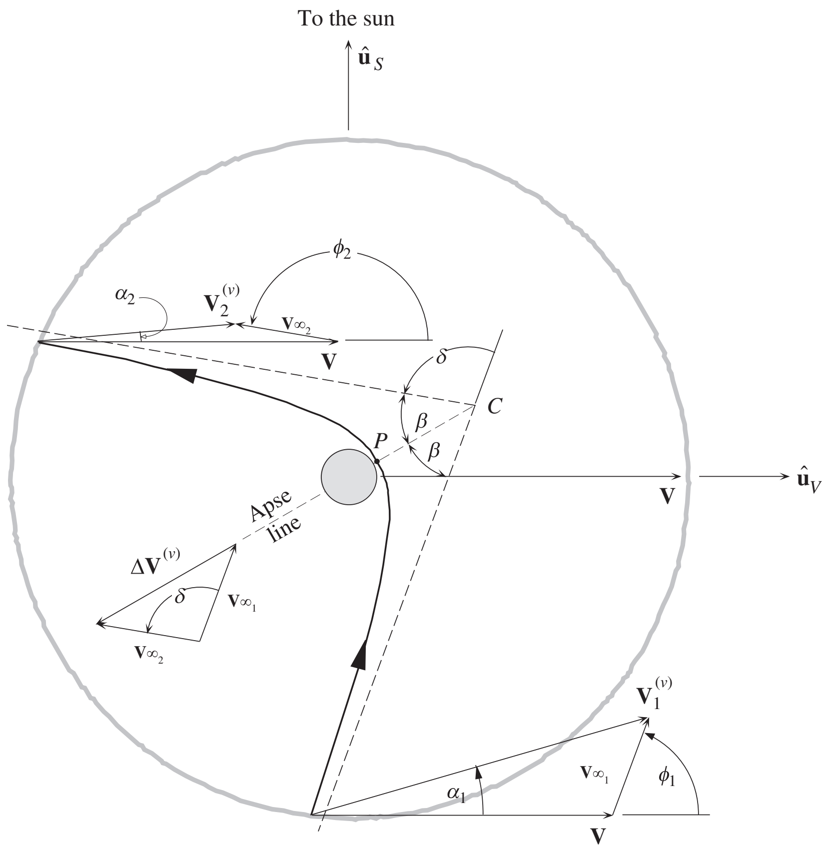


**FIGURE 8.18**

Leading-side planetary flyby.

apse line and always points away from periapsis, as illustrated in Figures 8.18 and 8.19. From these figures it can be seen that, in a leading-side flyby, the component of $\Delta\mathbf{V}^{(v)}$ in the direction of the planet's velocity is negative, whereas for the trailing-side flyby, it is positive. This means that a leading-side flyby results in a decrease in the spacecraft's heliocentric speed. On the other hand, a trailing-side flyby increases that speed.

In order to analyze a flyby problem, we proceed as follows. First, let $\hat{\mathbf{u}}_V$ be the unit vector in the direction of the planet's heliocentric velocity $\mathbf{V}$ and let $\hat{\mathbf{u}}_S$ be the unit vector pointing from the planet to the sun. At the inbound crossing of the sphere of influence, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the spacecraft is

$$\mathbf{V}_1^{(v)} = \left[V_1^{(v)}\right]_V \hat{\mathbf{u}}_V + \left[V_1^{(v)}\right]_S \hat{\mathbf{u}}_S \tag{8.75}$$

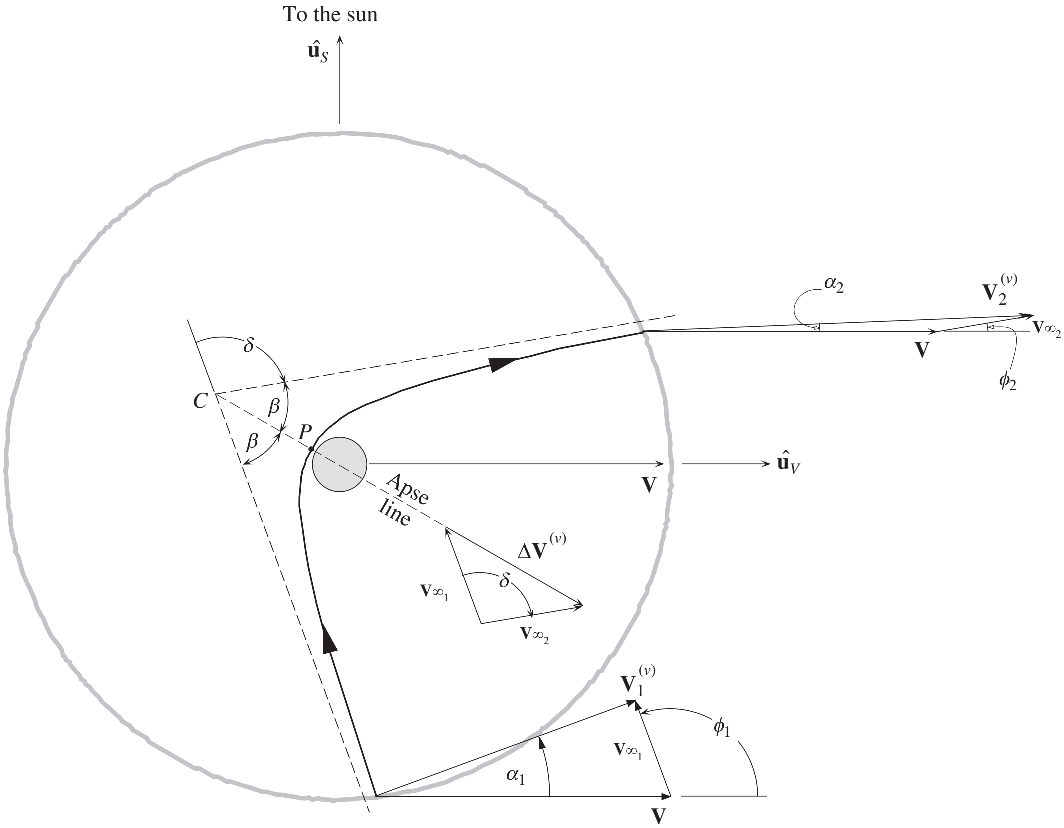


**FIGURE 8.19**

Trailing-side planetary flyby.

where the scalar components of $\mathbf{V}_1^{(v)}$ are

$$\left[V_1^{(v)}\right]_V = V_1^{(v)} \cos \alpha_1 \quad \left[V_1^{(v)}\right]_S = V_1^{(v)} \sin \alpha_1 \tag{8.76}$$

$\alpha_1$ is the angle between $\mathbf{V}_1^{(v)}$ and $\mathbf{V}$. All angles are measured positive counterclockwise. Referring to Figure 2.12, we see that the magnitude of $\alpha_1$ is the flight path angle $\gamma$ of the spacecraft's heliocentric trajectory when it encounters the planet's sphere of influence (a mere speck) at the planet's distance $R$ from the sun. Furthermore,

$$\left[V_1^{(v)}\right]_V = V_{\perp_1} \quad \left[V_1^{(v)}\right]_S = -V_{r_1} \tag{8.77}$$

$V_{\perp_1}$ and $V_{r_1}$ are furnished by Eqns (2.48) and (2.49),

$$V_{\perp_1} = \frac{\mu_{\text{sun}}}{h_1}(1 + e_1 \cos \theta_1) \quad V_{r_1} = \frac{\mu_{\text{sun}}}{h_1} e_1 \sin \theta_1 \tag{8.78}$$

in which $e_1$, $h_1$, and $\theta_1$ are the eccentricity, angular momentum, and true anomaly of the heliocentric approach trajectory, respectively.

The velocity of the planet relative to the sun is

$$\mathbf{V} = V\hat{\mathbf{u}}_V \tag{8.79}$$

where $V = \sqrt{\mu_{sun}/R}$. At the inbound crossing of the planet's sphere of influence, the hyperbolic excess velocity of the spacecraft is obtained from Eqn (8.72),

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V}$$

Using this we find

$$\mathbf{v}_{\infty_1} = (v_{\infty_1})_V\hat{\mathbf{u}}_V + (v_{\infty_1})_S\hat{\mathbf{u}}_S \tag{8.80}$$

where the scalar components of $\mathbf{v}_{\infty_1}$ are

$$(v_{\infty_1})_V = V_1^{(v)}\cos\alpha_1 - V \quad (v_{\infty_1})_S = V_1^{(v)}\sin\alpha_1 \tag{8.81}$$

$v_\infty$ is the magnitude of $\mathbf{v}_{\infty_1}$,

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1}\cdot\mathbf{v}_{\infty_1}} = \sqrt{\left[V_1^{(v)}\right]^2 + V^2 - 2V_1^{(v)}V\cos\alpha_1} \tag{8.82}$$

At this point, $v_\infty$ is known, so that on specifying the periapsis radius $r_p$ we can compute the angular momentum and eccentricity of the flyby hyperbola (relative to the planet), using Eqns (8.38) and (8.39).

$$h = r_p\sqrt{v_\infty^2 + \frac{2\mu}{r_p}} \qquad e = 1 + \frac{r_p v_\infty^2}{\mu} \tag{8.83}$$

where $\mu$ is the gravitational parameter of the planet.

The angle between $\mathbf{v}_{\infty_1}$ and planet's heliocentric velocity is $\phi_1$. It is found using the components of $\mathbf{v}_{\infty_1}$ in Eqn (8.81),

$$\phi_1 = \tan^{-1}\frac{(v_{\infty_1})_S}{(v_{\infty_1})_V} = \tan^{-1}\frac{V_1^{(v)}\sin\alpha_1}{V_1^{(v)}\cos\alpha_1 - V} \tag{8.84}$$

At the outbound crossing, the angle between $\mathbf{v}_{\infty_2}$ and $\mathbf{V}$ is $\phi_2$, where

$$\phi_2 = \phi_1 + \delta \tag{8.85}$$

For the leading-side flyby in Figure 8.18, the turn angle is $\delta$ positive (counterclockwise), whereas in Figure 8.19 it is negative. Since the magnitude of $\mathbf{v}_{\infty_2}$ is $v_\infty$, we can express $\mathbf{v}_{\infty_2}$ in components as

$$\mathbf{v}_{\infty_2} = v_\infty\cos\phi_2\hat{\mathbf{u}}_V + v_\infty\sin\phi_2\hat{\mathbf{u}}_S \tag{8.86}$$

Therefore, the heliocentric velocity of the spacecraft at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = \left[V_2^{(v)}\right]_V\hat{\mathbf{u}}_V + \left[V_2^{(v)}\right]_S\hat{\mathbf{u}}_S \tag{8.87}$$

where the components of $\mathbf{V}_2^{(v)}$ are

$$\left[V_2^{(v)}\right]_V = V + v_\infty \cos\phi_2 \quad \left[V_2^{(v)}\right]_S = v_\infty \sin\phi_2 \tag{8.88}$$

From this we obtain the radial and transverse heliocentric velocity components,

$$V_{\perp_2} = \left[V_2^{(v)}\right]_V \quad V_{r_2} = -\left[V_2^{(v)}\right]_S \tag{8.89}$$

Finally, we obtain the three elements $e_2$, $h_2$, and $\theta_2$ of the new heliocentric departure trajectory by means of Eqn (2.21),

$$h_2 = RV_{\perp_2} \tag{8.90}$$

Eqn (2.45),

$$R = \frac{h_2^2}{\mu_{\text{sun}}}\frac{1}{1 + e_2\cos\theta_2} \tag{8.91}$$

and Eqn (2.49),

$$V_{r_2} = \frac{\mu_{\text{sun}}}{h_2} e_2 \sin\theta_2 \tag{8.92}$$

Notice that the flyby is considered to be an impulsive maneuver during which the heliocentric radius of the spacecraft, which is confined within the planet's sphere of influence, remains fixed at $R$. The heliocentric velocity analysis is similar to that described in Section 6.7.

## EXAMPLE 8.6

A spacecraft departs earth with a velocity perpendicular to the sun line on a flyby mission to Venus. Encounter occurs at a true anomaly in the approach trajectory of $-30°$. Periapsis altitude is to be 300 km.

**(a)** For an approach from the dark side of the planet, show that the postflyby orbit is as illustrated in Figure 8.20.
**(b)** For an approach from the sunlit side of the planet, show that the postflyby orbit is as illustrated in Figure 8.21.

**Solution**

The following data is found in Tables A.1 and A.2:

$$\mu_{\text{sun}} = 1.3271 \times 10^{11} \text{ km}^3/\text{s}^2$$

$$\mu_{\text{Venus}} = 324{,}900 \text{ km}^3/\text{s}^2$$

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$

$$R_{\text{Venus}} = 108.2 \times 10^6 \text{ km}$$

$$r_{\text{Venus}} = 6052 \text{ km}$$

*Preflyby ellipse (orbit 1)*

Evaluating the orbit formula, Eqn (2.45), at aphelion of orbit 1 yields

$$R_{\text{earth}} = \frac{h_1^2}{\mu_{\text{sun}}}\frac{1}{1 - e_1}$$

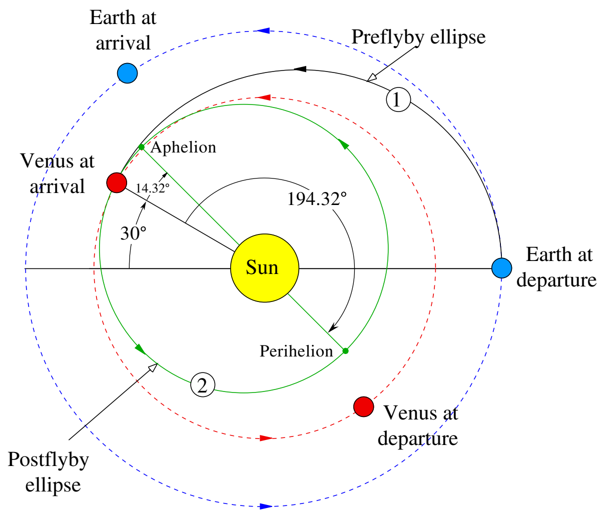


**FIGURE 8.20**

Spacecraft orbits before and after a flyby of venus, approaching from the dark side.

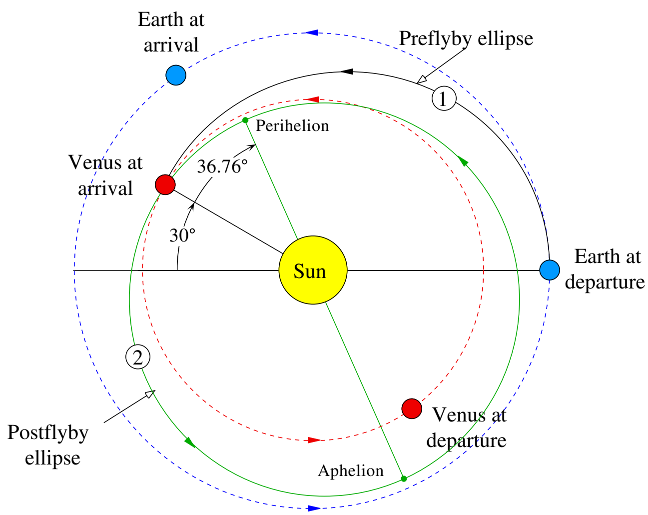


**FIGURE 8.21**

Spacecraft orbits before and after a flyby of venus, approaching from the sunlit side.

Thus,

$$h_1^2 = \mu_{sun} R_{earth}(1 - e_1) \tag{a}$$

At intercept,

$$R_{Venus} = \frac{h_1^2}{\mu_{sun}} \frac{1}{1 + e_1 \cos(\theta_1)}$$

Substituting Eqn (a) and $\theta_1 = -30°$ and solving the resulting expression for $e_1$ leads to

$$e_1 = \frac{R_{earth} - R_{Venus}}{R_{earth} + R_{Venus}\cos(\theta_1)} = \frac{149.6 \times 10^6 - 108.2 \times 10^6}{149.6 \times 10^6 + 108.2 \times 10^6 \cos(-30°)} = 0.1702$$

With this result, Eqn (a) yields

$$h_1 = \sqrt{1.327 \times 10^{11} \cdot 149.6 \times 10^6 (1 - 0.1702)} = 4.059 \times 10^9 \ \text{km}^2/\text{s}$$

Now we can use Eqns (2.31) and (2.49) to calculate the radial and transverse components of the spacecraft's heliocentric velocity at the inbound crossing of Venus' sphere of influence.

$$V_{\perp_1} = \frac{h_1}{R_{Venus}} = \frac{4.059 \times 10^9}{108.2 \times 10^6} = 37.51 \ \text{km/s}$$

$$V_{r_1} = \frac{\mu_{sun}}{h_1} e_1 \sin(\theta_1) = \frac{1.327 \times 10^{11}}{4.059 \times 10^9} \cdot 0.1702 \cdot \sin(-30°) = -2.782 \ \text{km/s}$$

The flight path angle, from Eqn (2.51), is

$$\gamma_1 = \tan^{-1} \frac{V_{r_1}}{V_{\perp_1}} = \tan^{-1}\left(\frac{-2.782}{37.51}\right) = -4.241°$$

The negative sign is consistent with the fact that the spacecraft is flying toward perihelion of the preflyby elliptical trajectory (orbit 1).

The speed of the space vehicle at the inbound crossing is

$$V_1^{(v)} = \sqrt{v_{r_1}^2 + v_{\perp_1}^2} = \sqrt{(-2.782)^2 + 37.51^2} = 37.62 \ \text{km/s} \tag{b}$$

*Flyby hyperbola*

From Eqns (8.75) and (8.77), we obtain

$$\mathbf{V}_1^{(v)} = 37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S \ \ (\text{km/s})$$

The velocity of Venus in its presumed circular orbit around the sun is

$$\mathbf{V} = \sqrt{\frac{\mu_{sun}}{R_{Venus}}}\hat{\mathbf{u}}_V = \sqrt{\frac{1.327 \times 10^{11}}{108.2 \times 10^6}}\hat{\mathbf{u}}_V = 35.02\hat{\mathbf{u}}_V \ \ (\text{km/s}) \tag{c}$$

Hence

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V} = (37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S) - 35.02\hat{\mathbf{u}}_V = 2.490\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S \ \ (\text{km/s}) \tag{d}$$

It follows that

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1} \cdot \mathbf{v}_{\infty_1}} = 3.733 \ \text{km/s}$$

The periapsis radius is

$$r_p = r_{\text{Venus}} + 300 = 6352 \text{ km}$$

Equations (8.38) and (8.39) are used to compute the angular momentum and eccentricity of the planetocentric hyperbola.

$$h = 6352\sqrt{v_\infty^2 + \frac{2\mu_{\text{Venus}}}{6352}} = 6352\sqrt{3.733^2 + \frac{2\cdot 324{,}900}{6352}} = 68{,}480 \text{ km}^2/\text{s}$$

$$e = 1 + \frac{r_p v_\infty^2}{\mu_{\text{Venus}}} = 1 + \frac{6352\cdot 3.733^2}{324{,}900} = 1.272$$

The turn angle and true anomaly of the asymptote are

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.272}\right) = 103.6°$$

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) = \cos^{-1}\left(-\frac{1}{1.272}\right) = 141.8°$$

From Eqns (2.50), (2.103), and (2.107), the aiming radius is

$$\Delta = r_p\sqrt{\frac{e+1}{e-1}} = 6352\sqrt{\frac{1.272+1}{1.272-1}} = 18{,}340 \text{ km} \tag{e}$$

Finally, from Eqn (d) we obtain the angle between $\mathbf{v}_{\infty_1}$ and $\mathbf{V}$,

$$\phi_1 = \tan^{-1}\frac{2.782}{2.490} = 48.17° \tag{f}$$

There are two flyby approaches, as shown in Figure 8.22. In the dark side approach, the turn angle is counterclockwise (+102.9°), whereas for the sunlit side approach, it is clockwise (−102.9°).

*Dark side approach*

According to Eqn (8.85), the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\text{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 + \delta = 48.17° + 103.6° = 151.8°$$

Hence, by Eqn (8.86),

$$\mathbf{v}_{\infty_2} = 3.733(\cos 151.8°\hat{\mathbf{u}}_V + \sin 151.8°\hat{\mathbf{u}}_S) = -3.289\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S \text{ (km/s)}$$

Using this and Eqn (c) above, we compute the spacecraft's heliocentric velocity at the outbound crossing.

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = 31.73\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S \text{ (km/s)}$$

It follows from Eqn (8.89) that

$$V_{\perp_2} = 31.73 \text{ km/s} \quad V_{r_2} = -1.766 \text{ km/s} \tag{g}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{v_{r_2}^2 + v_{\perp_2}^2} = \sqrt{(-1.766)^2 + 31.73^2} = 31.78 \text{ km/s}$$

This is 5.83 km/s less than the inbound speed.

*Postflyby ellipse (orbit 2) for the dark side approach*

For the heliocentric postflyby trajectory, labeled orbit 2 in Figure 8.20, the angular momentum is found using Eqn (8.90)

$$h_2 = R_{\text{Venus}}V_{\perp_2} = \left(108.2\times 10^6\right)\cdot 31.73 = 3.434\times 10^9 \left(\text{km}^2/\text{s}\right) \tag{h}$$

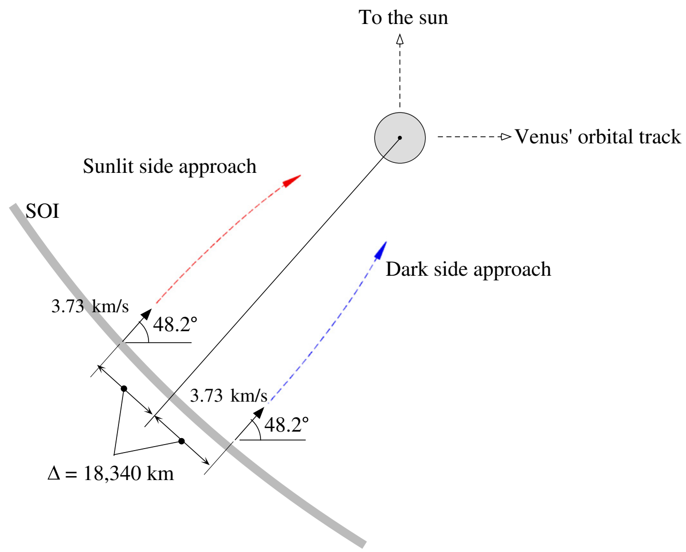


**FIGURE 8.22**

Initiation of a sunlit side approach and dark side approach at the inbound crossing.

From Eqn (8.91),

$$e\cos\theta_2 = \frac{h_2^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{\left(3.434\times10^6\right)^2}{1.327\times10^{11}\cdot108.2\times10^6} - 1 = -0.1790 \qquad \text{(i)}$$

and from Eqn (8.92)

$$e\sin\theta_2 = \frac{V_{r_2} h_2}{\mu_{\text{sun}}} = \frac{-1.766\cdot3.434\times10^9}{1.327\times10^{11}} = -0.04569 \qquad \text{(j)}$$

Thus

$$\tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{-0.045{,}69}{-0.1790} = 0.2553 \qquad \text{(k)}$$

which means

$$\theta_2 = 14.32^\circ \quad \text{or} \quad 194.32^\circ \qquad \text{(l)}$$

But $\theta_2$ must lie in the third quadrant since, according to Eqns (i) and (j), both the sine and cosine are negative. Hence,

$$\theta_2 = 194.32^\circ \qquad \text{(m)}$$

With this value of $\theta_2$, we can use either Eqn (i) or Eqn (j) to calculate the eccentricity,

$$e_2 = 0.1847 \qquad \text{(n)}$$

Perihelion of the departure orbit lies 194.32° clockwise from the encounter point (so that aphelion is 14.32° therefrom), as illustrated in Figure 8.20. The perihelion radius is given by Eqn (2.50),

$$R_{\text{perihelion}} = \frac{h_2^2}{\mu_{\text{sun}}}\frac{1}{1+e_2} = \frac{\left(3.434\times10^9\right)^2}{1.327\times10^{11}}\frac{1}{1+0.1847} = 74.98\times10^6 \ \text{km}$$

which is well within the orbit of Venus.

*Sunlit side approach*

In this case, the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\text{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 - \delta = 48.17° - 103.6° = -55.44°$$

Therefore,

$$\mathbf{v}_{\infty_2} = 3.733[\cos(-55.44°)\hat{\mathbf{u}}_V + \sin(-55.44°)\hat{\mathbf{u}}_S] = 2.118\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S \ \text{(km/s)}$$

The spacecraft's heliocentric velocity at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V}_{\text{Venus}} + \mathbf{v}_{\infty_2} = 37.14\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S \ \text{(km/s)}$$

which means

$$V_{\perp_2} = 37.14 \ \text{km/s} \quad V_{r_2} = 3.074 \ \text{km/s}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{3.074^2 + v_{\perp_2}^2} = \sqrt{3.050^2 + 37.14^2} = 37.27 \ \text{km/s}$$

This speed is just 0.348 km/s less than inbound crossing speed. The relatively small speed change is due to the fact that the apse line of this hyperbola is nearly perpendicular to Venus' orbital track, as shown in Figure 8.23. Nevertheless, the periapses of both hyperbolas are on the leading side of the planet.

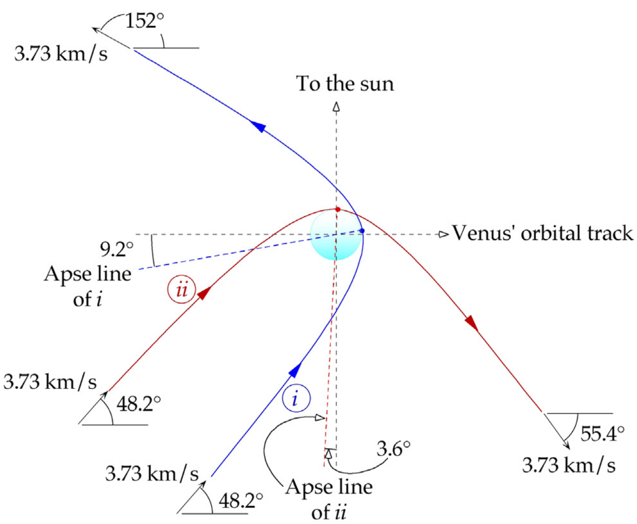


**FIGURE 8.23**

Hyperbolic flyby trajectories for (i) the dark side approach and (ii) the sunlit side approach.

*Postflyby ellipse (orbit 2) for the sunlit side approach*

To determine the heliocentric postflyby trajectory, labeled orbit 2 in Figure 8.21, we repeat Steps (h) through (n) above.

$$h_2 = R_{\text{Venus}} V_{\perp_2} = \left(108.2 \times 10^6\right) \cdot 37.14 = 4.019 \times 10^9 \ \ (\text{km}^2/\text{s})$$

$$e \cos \theta_2 = \frac{h_2^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{\left(4.019 \times 10^9\right)^2}{1.327 \times 10^{11} \cdot 108.2 \times 10^6} - 1 = 0.1246 \tag{o}$$

$$e \sin \theta_2 = \frac{V_{r_2} h_2}{\mu_{\text{sun}}} = \frac{3.074 \cdot 4.019 \times 10^9}{1.327 \times 10^{11}} = 0.09309 \tag{p}$$

$$\tan \theta_2 = \frac{e \sin \theta_2}{e \cos \theta_2} = \frac{0.09309}{0.1246} = 0.7469$$

$$\theta_2 = 36.08^\circ \quad \text{or} \quad 216.08^\circ$$

$\theta_2$ must lie in the first quadrant since both the sine and cosine are positive. Hence,

$$\theta_2 = 36.76^\circ \tag{q}$$

With this value of $\theta_2$, we can use either Eqn (o) or Eqn (p) to calculate the eccentricity,

$$e_2 = 0.1556$$

Perihelion of the departure orbit lies 36.76° clockwise from the encounter point as illustrated in Figure 8.21. The perihelion radius is

$$R_{\text{perihelion}} = \frac{h_2{}^2}{\mu_{\text{sun}}} \frac{1}{1 + e_2} = \frac{\left(4.019 \times 10^9\right)^2}{1.327 \times 10^{11}} \frac{1}{1 + 0.1556} = 105.3 \times 10^6 \ \text{km}$$

which is just within the orbit of Venus. Aphelion lies between the orbits of earth and Venus.

---

Gravity assist maneuvers are used to add momentum to a spacecraft over and above that available from a spacecraft's onboard propulsion system. A sequence of flybys of planets can impart the delta-v needed to reach regions of the solar system that would be inaccessible using only existing propulsion technology. The technique can also reduce the flight time. Interplanetary missions using gravity assist flybys must be carefully designed in order to take advantage of the relative positions of planets.

The 260-kg spacecraft Pioneer 11, launched in April 1973, used a December 1974 flyby of Jupiter to gain the momentum required to carry it to the first ever flyby encounter with Saturn on September 1, 1979.

Following its September 1977 launch, Voyager 1 likewise used a flyby of Jupiter (March 1979) to reach Saturn in November 1980. In August 1977, Voyager 2 was launched on its "grand tour" of the outer planets and beyond. This involved gravity assist flybys of Jupiter (July 1979), Saturn (August 1981), Uranus (January 1986), and Neptune (August 1989), after which the spacecraft departed the solar system at an angle of 30° to the ecliptic.

With a mass nine times that of Pioneer 11, the dual-spin Galileo spacecraft departed on October 18, 1989, for an extensive international exploration of Jupiter and its satellites lasting until September 2003. Galileo used gravity assist flybys of Venus (February 1990), earth (December 1990), and earth again (December 1992) before arriving at Jupiter in December 1995.

The international Cassini mission to Saturn also made extensive use of gravity assist flyby maneuvers. The Cassini spacecraft was launched on October 15, 1997, from Cape Canaveral, Florida, and

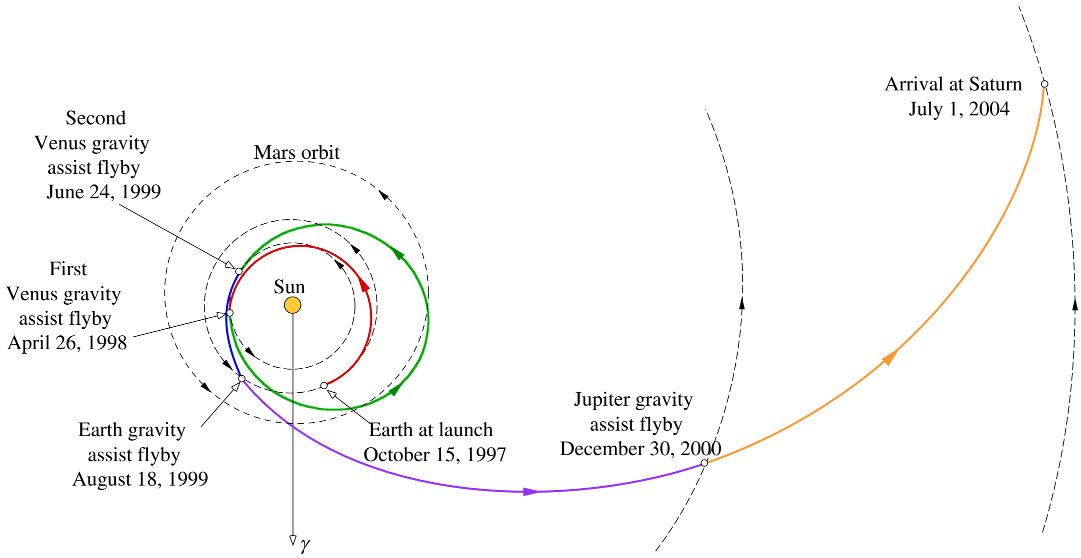


**FIGURE 8.24**

Cassini's 7-year mission to saturn.

arrived at Saturn nearly 7 years later, on July 1, 2004. The mission involved four flybys, as illustrated in Figure 8.24. A little over 8 months after launch, on April 26, 1998, Cassini flew by Venus at a periapsis altitude of 284 km and received a speed boost of about 7 km/s. This placed the spacecraft in an orbit which sent it just outside the orbit of Mars (but well away from the planet) and returned it to Venus on June 24, 1999, for a second flyby, this time at an altitude of 600 km. The result was a trajectory that vectored Cassini toward the earth for an August 18, 1999, flyby at an altitude of 1171 km. The 5.5 km/s speed boost at earth sent the spacecraft toward Jupiter for its next flyby maneuver. This occurred on December 30, 2000, at a distance of 9.7 million km from Jupiter, boosting Cassini's speed by about 2 km/s and adjusting its trajectory so as to rendezvous with Saturn about three and a half years later.

## 8.10 Planetary ephemeris

The state vector **R**, **V** of a planet is defined relative to the heliocentric ecliptic frame of reference illustrated in Figure 8.25. This is very similar to the geocentric equatorial frame of Figure 4.7. The sun replaces the earth as the center of attraction, and the plane of the ecliptic replaces the earth's equatorial plane. The vernal equinox continues to define the inertial $X$-axis.

In order to design realistic interplanetary missions, we must be able to determine the state vector of a planet at any given time. Table 8.1 provides the orbital elements of the planets and their rates of change per century (Cy) with respect to the J2000 epoch (January 1, 2000, 12 h UT). The table, covering the years 1800–2050, is sufficiently accurate for our needs. From the orbital elements, we can infer the state vector using Algorithm 4.5.

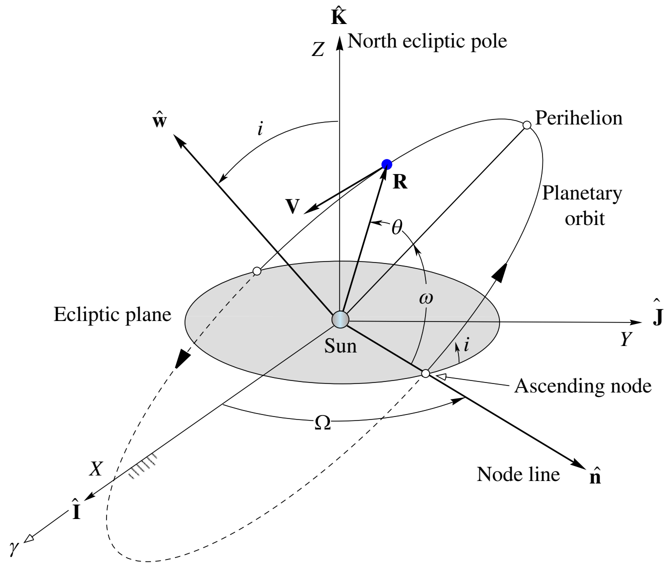


**FIGURE 8.25**

Planetary orbit in the heliocentric ecliptic frame.

In order to interpret Table 8.1, observe the following:

1 astronomical unit (1 AU) is $1.49597871 \times 10^8$ km, the average distance between the earth and the sun.
1 arcsecond (1″) is 1/3600 of a degree.
$a$ is the semimajor axis.
$e$ is the eccentricity.
$i$ is the inclination to the ecliptic plane.
Ω is the right ascension of the ascending node (relative to the J2000 vernal equinox).
$\varpi$, the longitude of perihelion, is defined as $\varpi = \omega + \Omega$, where $\omega$ is the argument of perihelion.
$L$, the mean longitude, is defined as $L = \varpi + M$, where $M$ is the mean anomaly.
$\dot{a}, \dot{e}, \dot{\Omega}$, etc., are the rates of change of the above orbital elements per Julian Cy. One Cy equals 36,525 days.

## ALGORITHM 8.1

Determine the state vector of a planet at a given date and time. All angular calculations must be adjusted so that they lie in the range 0°−360°. Recall that the gravitational parameter of the sun is $\mu = 1.327 \times 10^{11}$ km$^3$/s$^2$. This procedure is implemented in MATLAB as the function *planet_elements_and_sv.m* in Appendix D.35.

1. Use Eqns (5.47) and (5.48) to calculate the Julian day number $JD$.
2. Calculate $T_0$, the number of Julian centuries between J2000 and the date in question (Eqn (5.49))

$$T_0 = \frac{JD - 2{,}451{,}545.0}{36{,}525} \tag{8.93a}$$

3. If $Q$ is any one of the six planetary orbital elements listed in Table 8.1, then calculate its value at $JD$ by means of the formula

$$Q = Q_0 + \dot{Q}T_0 \tag{8.93b}$$

where $Q_0$ is the value listed for J2000 and $\dot{Q}$ is the tabulated rate. All angular quantities must be adjusted to lie in the range 0°−360°.
4. Use the semimajor axis $a$ and the eccentricity $e$ to calculate the angular momentum $h$ at $JD$ from Eqn (2.71)

$$h = \sqrt{\mu a(1 - e^2)}$$

5. Obtain the argument of perihelion $\omega$ and mean anomaly $M$ at $JD$ from the results of Step 3 by means of the definitions

$$\omega = \varpi - \Omega$$
$$M = L - \varpi$$

6. Substitute the eccentricity $e$ and the mean anomaly $M$ at $JD$ into Kepler's equation (Eqn (3.14)) and calculate the eccentric anomaly $E$.
7. Calculate the true anomaly $\theta$ using Eqn (3.13).
8. Use $h$, $e$, $\Omega$, $i$, $\omega$, and $\theta$ to obtain the heliocentric position vector $\mathbf{R}$ and velocity $\mathbf{V}$ by means of Algorithm 4.5, with the heliocentric ecliptic frame replacing the geocentric equatorial frame.

## EXAMPLE 8.7

Find the distance between earth and Mars at 12 h UT on August 27, 2003. Use Algorithm 8.1.

Step 1:

According to Eqn (5.48), the Julian day number $J_0$ for midnight (0 h UT) of this date is

$$\begin{aligned} J_0 &= 367 \cdot 2003 - \mathrm{INT}\left\{\frac{7\left[2003 + \mathrm{INT}\left(\frac{8+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275 \cdot 8}{9}\right) + 27 + 1{,}721{,}013.5 \\ &= 735{,}101 - 3507 + 244 + 27 + 1{,}721{,}013.5 \\ &= 2{,}452{,}878.5 \end{aligned}$$

**Table 8.1** Planetary Orbital Elements and Their Centennial Rates

| | $a$(AU)<br>$\dot{a}$(AU/Cy) | $e$<br>$\dot{e}$(1/Cy) | $i$(°)<br>$\dot{i}$(°/Cy) | $\Omega$(°)<br>$\dot{\Omega}$(°/Cy) | $\varpi$(°)<br>$\dot{\varpi}$(°/Cy) | $L$(°)<br>$\dot{L}$(°/Cy) |
|---|---|---|---|---|---|---|
| Mercury | 0.38709927 | 0.20563593 | 7.00497902 | 48.33076593 | 77.45779628 | 252.25032350 |
| | 0.00000037 | 0.00001906 | −0.00594749 | −0.12534081 | 0.16047689 | 149,472.67411175 |
| Venus | 0.72333566 | 0.00677672 | 3.39467605 | 76.67984255 | 131.60246718 | 181.97909950 |
| | 0.00000390 | −0.00004107 | −0.00078890 | −0.27769418 | 0.00268329 | 58,517.81538729 |
| Earth | 1.00000261 | 0.01671123 | −0.00001531 | 0.0 | 102.93768193 | 100.46457166 |
| | 0.00000562 | −0.00004392 | −0.01294668 | 0.0 | 0.32327364 | 35,999.37244981 |
| Mars | 1.52371034 | 0.09339410 | 1.84969142 | 49.55953891 | −23.94362959 | −4.55343205 |
| | 0.0001847 | 0.00007882 | −0.00813131 | −0.29257343 | 0.44441088 | 19,140.30268499 |
| Jupiter | 5.20288700 | 0.04838624 | 1.30439695 | 100.47390909 | 14.72847983 | 34.39644501 |
| | −0.00011607 | −0.00013253 | −0.00183714 | 0.20469106 | 0.21252668 | 3034.74612775 |
| Saturn | 9.53667594 | 0.05386179 | 2.48599187 | 113.66242448 | 92.59887831 | 49.95424423 |
| | −0.00125060 | −0.00050991 | 0.00193609 | −0.28867794 | −0.41897216 | 1222.49362201 |
| Uranus | 19.18916464 | 0.04725744 | 0.77263783 | 74.01692503 | 170.95427630 | 313.23810451 |
| | −0.00196176 | −0.00004397 | −0.00242939 | 0.04240589 | 0.40805281 | 428.48202785 |
| Neptune | 30.06992276 | 0.00859048 | 1.77004347 | 131.78422574 | 44.96476227 | −55.12002969 |
| | 0.00026291 | 0.00005105 | 0.00035372 | −0.00508664 | −0.32241464 | 218.45945325 |
| (Pluto) | 39.48211675 | 0.24882730 | 17.14001206 | 110.30393684 | 224.06891629 | 238.92903833 |
| | −0.00031596 | 0.00005170 | 0.00004818 | −0.01183482 | −0.04062942 | 145.20780515 |

*Source: From Standish et al. (2013). Used with permission.*

At $UT = 12$, the Julian day number is

$$JD = 2{,}452{,}878.5 + \frac{12}{24} = 2{,}452{,}879.0$$

Step 2:

The number of Julian centuries between J2000 and this date is

$$T_0 = \frac{JD - 2{,}451{,}545}{36{,}525} = \frac{2{,}452{,}879 - 2{,}451{,}545}{36{,}525} = 0.036523 \text{ Cy}$$

Step 3:

Table 8.1 and Eqn (8.93b) yield the orbital elements of earth and Mars at 12 h UT on August 27, 2003.

| | $a$ (km) | $e$ | $i$ (°) | $\Omega$ (°) | $\varpi$ (°) | $L$ (°) |
|---|---|---|---|---|---|---|
| Earth | $1.4960 \times 10^8$ | 0.016710 | −0.00048816 | 0.0 | 102.95 | 335.27 |
| Mars | $2.2794 \times 10^8$ | 0.093397 | 1.8494 | 49.549 | 336.07 | 334.51 |

Step 4:

$$h_{\text{earth}} = 4.4451 \times 10^9 \ \text{km}^2/\text{s}$$
$$h_{\text{Mars}} = 5.4760 \times 10^9 \ \text{km}^2/\text{s}$$

Step 5:

$$\omega_{\text{earth}} = (\varpi - \Omega)_{\text{earth}} = 102.95 - 0 = 102.95°$$
$$\omega_{\text{Mars}} = (\varpi - \Omega)_{\text{Mars}} = 336.07 - 49.549 = 286.52°$$
$$M_{\text{earth}} = (L - \varpi)_{\text{earth}} = 335.27 - 102.95 = 232.32°$$
$$M_{\text{Mars}} = (L - \varpi)_{\text{Mars}} = 334.51 - 336.07 = -1.56°(358.43°)$$

Step 6:

$$E_{\text{earth}} - 0.016710 \sin E_{\text{earth}} = 232.32°(\pi/180) \Rightarrow E_{\text{earth}} = 231.57°$$
$$E_{\text{Mars}} - 0.093397 \sin E_{\text{Mars}} = 358.43°(\pi/180) \Rightarrow E_{\text{Mars}} = 358.27°$$

Step 7:

$$\theta_{\text{earth}} = 2\tan^{-1}\left(\sqrt{\frac{1+0.016710}{1-0.016710}}\tan\frac{231.57°}{2}\right) = -129.18° \Rightarrow \theta_{\text{earth}} = 230.82°$$
$$\theta_{\text{Mars}} = 2\tan^{-1}\left(\sqrt{\frac{1+0.093397}{1-0.093397}}\tan\frac{358.27°}{2}\right) = -1.8998° \Rightarrow \theta_{\text{Mars}} = 358.10°$$

Step 8:

From Algorithm 4.5,

$$\mathbf{R}_{\text{earth}} = \left(135.59\hat{\mathbf{I}} - 66.803\hat{\mathbf{J}} - 0.00056916\hat{\mathbf{K}}\right) \times 10^6 \ (\text{km})$$
$$\mathbf{V}_{\text{earth}} = 12.680\hat{\mathbf{I}} + 26.610\hat{\mathbf{J}} - 0.00022672\hat{\mathbf{K}} \ (\text{km/s})$$
$$\mathbf{R}_{\text{Mars}} = \left(185.95\hat{\mathbf{I}} - 89.959\hat{\mathbf{J}} - 6.4534\hat{\mathbf{K}}\right) \times 10^6 \ (\text{km})$$
$$\mathbf{V}_{\text{Mars}} = 11.478\hat{\mathbf{I}} + 23.881\hat{\mathbf{J}} + 0.21828\hat{\mathbf{K}} \ (\text{km/s})$$

The distance $d$ between the two planets is therefore

$$d = \|\mathbf{R}_{\text{Mars}} - \mathbf{R}_{\text{earth}}\|$$
$$= \sqrt{(185.95 - 135.59)^2 + [-89.959 - (-66.803)]^2 + (-6.4534 - 0.00056916)^2} \times 10^6$$

or

$$\boxed{d = 55.80 \times 10^6 \ \text{km}}$$

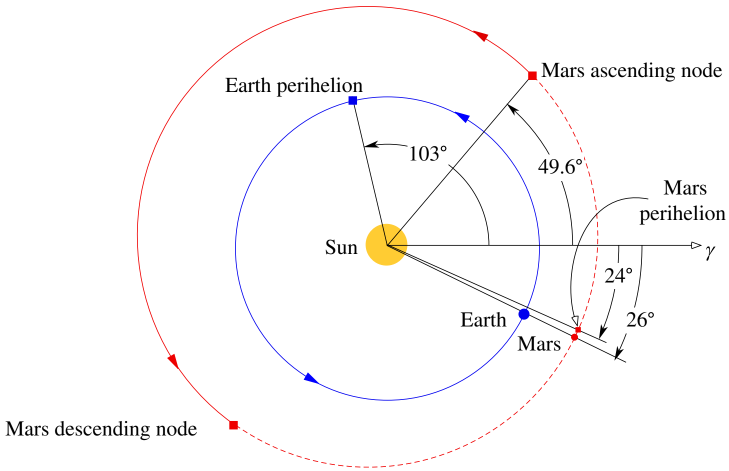


**FIGURE 8.26**

Earth and mars on august 27, 2003. Angles shown are heliocentric latitude, measured in the plane of the ecliptic counterclockwise from the vernal equinox of J2000.

The positions of earth and Mars are illustrated in Figure 8.26. It is a rare event for Mars to be in opposition (lined up with earth on the same side of the sun) when Mars is at or near perihelion. The two planets had not been this close in recorded history.

## 8.11 Non-Hohmann interplanetary trajectories

To implement a systematic patched conic procedure for three-dimensional trajectories, we will use vector notation and the procedures described in Sections 4.4 and 4.6 (Algorithms 4.2 and 4.5), together with the solution of Lambert's problem presented in Section 5.3 (Algorithm 5.2). The mission is to send a spacecraft from planet 1 to planet 2 in a specified time $t_{12}$. As previously discussed in this chapter, we break the mission down into three parts: the departure phase, the cruise phase, and the arrival phase. We start with the cruise phase.

The frame of reference that we use is the heliocentric ecliptic frame shown in Figure 8.27. The first step is to obtain the state vector of planet 1 at departure (time $t$) and the state vector of planet 2 at arrival (time $t + t_{12}$). This is accomplished by means of Algorithm 8.1.

The next step is to determine the spacecraft's transfer trajectory from planet 1 to planet 2. We first observe that, according to the patched conic procedure, the heliocentric position vector of the spacecraft at time $t$ is that of planet 1 ($\mathbf{R}_1$) and at time $t + t_{12}$ its position vector is that of planet 2 ($\mathbf{R}_2$). With $\mathbf{R}_1$, $\mathbf{R}_2$ and the time of flight $t_{12}$ we can use Algorithm 5.2 (Lambert's problem) to obtain the

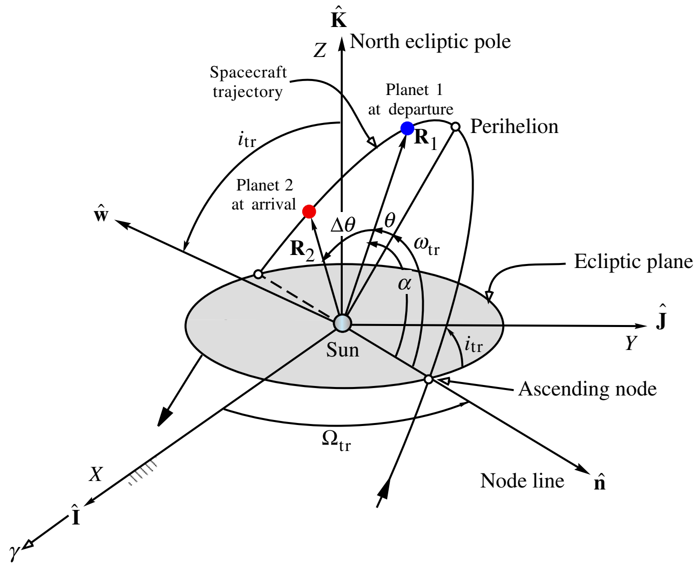


**FIGURE 8.27**

Heliocentric orbital elements of a three-dimensional transfer trajectory from planet 1 to planet 2.

spacecraft's departure and arrival velocities $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_A^{(v)}$ relative to the sun. Either of the state vectors $\mathbf{R}_1,\mathbf{V}_D^{(v)}$ or $\mathbf{R}_2,\mathbf{V}_A^{(v)}$ can be used to obtain the transfer trajectory's six orbital elements by means of Algorithm 4.2.

The spacecraft's hyperbolic excess velocity on exiting the sphere of influence of planet 1 is

$$\mathbf{v}_\infty)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_1 \tag{8.94a}$$

and its excess speed is

$$v_\infty)_{\text{Departure}} = \|\mathbf{V}_D^{(v)} - \mathbf{V}_1\| \tag{8.94b}$$

Likewise, at the sphere of influence crossing at planet 2,

$$\mathbf{v}_\infty)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_2 \tag{8.95a}$$

$$v_\infty)_{\text{Arrival}} = \|\mathbf{V}_A^{(v)} - \mathbf{V}_2\| \tag{8.95b}$$

## ALGORITHM 8.2

Given the departure and arrival dates (and, therefore, the time of flight), determine the trajectory for a mission from planet 1 to planet 2. This procedure is implemented as the MATLAB function *interplanetary.m* in Appendix D.36.

1. Use Algorithm 8.1 to determine the state vector $\mathbf{R}_1$, $\mathbf{V}_1$ of planet 1 at departure and the state vector $\mathbf{R}_2$, $\mathbf{V}_2$ of planet 2 at arrival.
2. Use $\mathbf{R}_1$, $\mathbf{R}_2$ and the time of flight in Algorithm 5.2 to find the spacecraft velocity $\mathbf{V}_D^{(v)}$ at departure from planet 1's sphere of influence and its velocity $\mathbf{V}_A^{(v)}$ upon arrival at planet 2's sphere of influence.
3. Calculate the hyperbolic excess velocities at departure and arrival using Eqns (8.94) and (8.95).

## EXAMPLE 8.8

A spacecraft departs earth's sphere of influence on November 7, 1996 (0 h UT), on a prograde coasting flight to Mars, arriving at Mars' sphere of influence on September, 12, 1997 (0 h UT). Use Algorithm 8.2 to determine the trajectory and then compute the hyperbolic excess velocities at departure and arrival.

### Solution

Step 1:

Algorithm 8.1 yields the state vectors for earth and Mars.

$$\mathbf{R}_{\text{earth}} = 1.0499 \times 10^8\hat{\mathbf{I}} + 1.0465 \times 10^8\hat{\mathbf{J}} + 716.93\hat{\mathbf{K}} \ \text{(km)} \quad \left(R_{\text{earth}} = 1.4824 \times 10^8 \ \text{km}\right)$$

$$\mathbf{V}_{\text{earth}} = -21.515\hat{\mathbf{I}} + 20.986\hat{\mathbf{J}} + 0.00014376\hat{\mathbf{K}} \ \text{(km/s)} \quad \left(V_{\text{earth}} = 30.055 \ \text{km/s}\right)$$

$$\mathbf{R}_{\text{Mars}} = -2.0858 \times 10^7\hat{\mathbf{I}} - 2.1842 \times 10^8\hat{\mathbf{J}} - 4.06244 \times 10^6\hat{\mathbf{K}} \ \text{(km)} \quad \left(R_{\text{Mars}} = 2.1945 \times 10^8 \ \text{km}\right)$$

$$\mathbf{V}_{\text{Mars}} = 25.037\hat{\mathbf{I}} - 0.22311\hat{\mathbf{J}} - 0.62018\hat{\mathbf{K}} \ \text{(km/s)} \quad \left(V_{\text{Mars}} = 25.046 \ \text{km/s}\right)$$

Step 2:

The position vector $\mathbf{R}_1$ of the spacecraft at crossing the earth's sphere of influence is just that of the earth,

$$\mathbf{R}_1 = \mathbf{R}_{\text{earth}} = 1.0499 \times 10^8\hat{\mathbf{I}} + 1.0465 \times 10^8\hat{\mathbf{J}} + 716.93\hat{\mathbf{K}} \ \text{(km)}$$

On arrival at Mars' sphere of influence, the spacecraft's position vector is

$$\mathbf{R}_2 = \mathbf{R}_{\text{Mars}} = -2.0858 \times 10^7\hat{\mathbf{I}} - 2.1842 \times 10^8\hat{\mathbf{J}} - 4.06244 \times 10^6\hat{\mathbf{K}} \ \text{(km)}$$

According to Eqns (5.47) and (5.48)

$$JD_{\text{Departure}} = 2{,}450{,}394.5$$

$$JD_{\text{Arrival}} = 2{,}450{,}703.5$$

Hence, the time of flight is

$$t_{12} = 2{,}450{,}703.5 - 2{,}450{,}394.5 = 309 \text{ days}$$

Entering $\mathbf{R}_1$, $\mathbf{R}_2$, and $t_{12}$ into Algorithm 5.2 yields

$$\mathbf{V}_D^{(v)} = -24.429\hat{\mathbf{I}} + 21.782\hat{\mathbf{J}} + 0.94810\hat{\mathbf{K}} \text{ (km/s)} \quad \left[V_D^{(v)} = 32.743 \text{ km/s}\right]$$

$$\mathbf{V}_A^{(v)} = 22.157\hat{\mathbf{I}} - 0.19959\hat{\mathbf{J}} - 0.45793\hat{\mathbf{K}} \text{ (km/s)} \quad \left[V_A^{(v)} = 22.162 \text{ km/s}\right]$$

Using the state vector $\mathbf{R}_1, \mathbf{V}_D^{(v)}$, we employ Algorithm 4.2 to find the orbital elements of the transfer trajectory.

$$\boxed{\begin{aligned} &h = 4.8456 \times 10^6 \text{ km}^2/\text{s} \\ &e = 0.20581 \\ &\Omega = 44.898^\circ \\ &i = 1.6622^\circ \\ &\omega = 19.973^\circ \\ &\theta_1 = 340.04^\circ \\ &a = 1.8475 \times 10^8 \text{ km} \end{aligned}}$$

Step 3:

At departure, the hyperbolic excess velocity is

$$\mathbf{v}_\infty)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_{\text{earth}} = -2.9138\hat{\mathbf{I}} + 0.79525\hat{\mathbf{J}} + 0.94796\hat{\mathbf{K}} \text{ (km/s)}$$

Therefore, the hyperbolic excess speed is

$$v_\infty)_{\text{Departure}} = \left\|\mathbf{v}_\infty)_{\text{Departure}}\right\| = \boxed{3.1656 \text{ km/s}} \qquad \text{(a)}$$

Likewise, at arrival

$$\mathbf{v}_\infty)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_{\text{Mars}} = -2.8805\hat{\mathbf{I}} + 0.023514\hat{\mathbf{J}} + 0.16254\hat{\mathbf{K}} \text{ (km/s)}$$

so that

$$v_\infty)_{\text{Arrival}} = \left\|\mathbf{v}_\infty)_{\text{Arrival}}\right\| = \boxed{2.8852 \text{ km/s}} \qquad \text{(b)}$$

For the previous example, Figure 8.28 shows the orbits of earth, Mars, and the spacecraft from directly above the ecliptic plane. Dotted lines indicate the portions of an orbit that are below the plane. $\lambda$ is the heliocentric longitude measured counterclockwise from the vernal equinox of J2000. Also shown are the position of Mars at departure and the position of the earth at arrival.

The transfer orbit resembles that of the Mars Global Surveyor, which departed earth on November 7, 1996, and arrived at Mars 309 days later, on September 12, 1997.

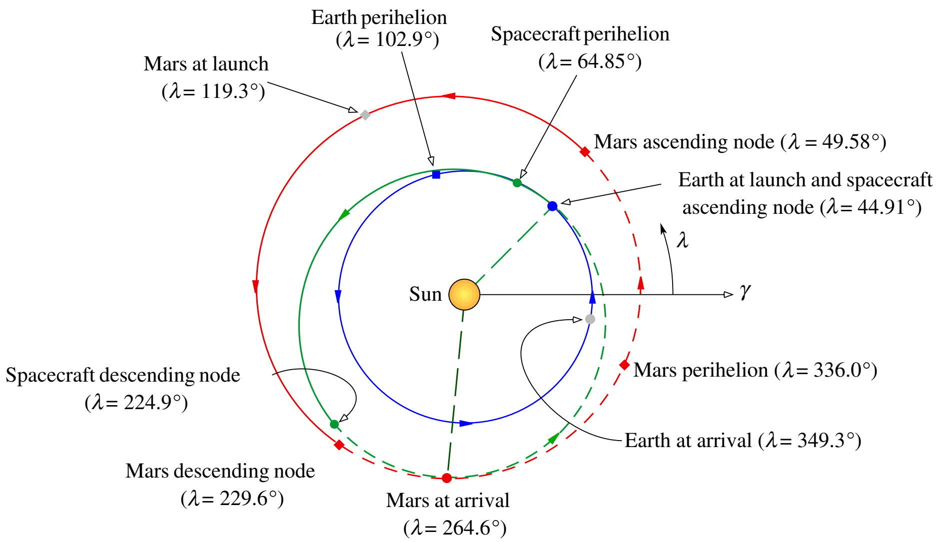


**FIGURE 8.28**

The transfer trajectory, together with the orbits of earth and mars, as viewed from directly above the plane of the ecliptic.

## EXAMPLE 8.9

In Example 8.8, calculate the delta-v required to launch the spacecraft onto its cruise trajectory from a 180 km circular parking orbit. Sketch the departure trajectory.

### Solution

Recall that

$$r_{earth} = 6378 \text{ km}$$
$$\mu_{earth} = 398{,}600 \text{ km}^3/\text{s}^2$$

The radius to periapsis of the departure hyperbola is the radius of the earth plus the altitude of the parking orbit,

$$r_p = 6378 + 180 = 6558 \text{ km}$$

Substituting this and Eqn (a) from Example 8.8 into Eqn (8.40) we get the speed of the spacecraft at periapsis of the departure hyperbola,

$$v_p = \sqrt{\left[(v_\infty)_{\text{Departure}}\right]^2 + \frac{2\mu_{earth}}{r_p}} = \sqrt{3.1651^2 + \frac{2\cdot 398{,}600}{6558}} = 11.47 \text{ km/s}$$

The speed of the spacecraft in its circular parking orbit is

$$v_c = \sqrt{\frac{\mu_{earth}}{r_p}} = \sqrt{\frac{398{,}600}{6558}} = 7.796 \text{ km/s}$$

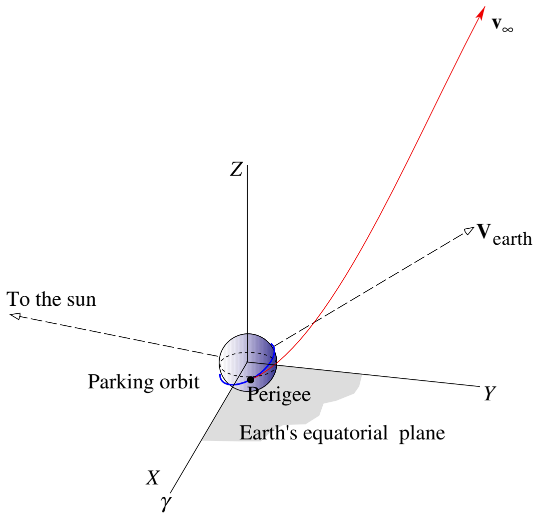


**FIGURE 8.29**

The departure hyperbola, assumed to be at 28° inclination to earth's equator.

Hence, the delta-v requirement is

$$\Delta v = v_p - v_c = \boxed{3.674 \text{ km/s}}$$

The eccentricity of the hyperbola is given by Eqn (8.38),

$$e = 1 + \frac{r_p v_\infty{}^2}{\mu_{\text{earth}}} = 1 + \frac{6558 \cdot 3.1656^2}{398{,}600} = 1.165$$

If we assume that the spacecraft is launched from a parking orbit of 28° inclination, then the departure appears as shown in the three-dimensional sketch in Figure 8.29.

## EXAMPLE 8.10

In Example 8.8, calculate the delta-v required to place the spacecraft in an elliptical capture orbit around Mars with a periapsis altitude of 300 km and a period of 48 h. Sketch the approach hyperbola.

### Solution

From Tables A.1 and A.2, we know that

$$r_{\text{Mars}} = 3380 \text{ km}$$
$$\mu_{\text{Mars}} = 42{,}830 \text{ km}^3/\text{s}^2$$

The radius to periapsis of the arrival hyperbola is the radius of Mars plus the periapsis of the elliptical capture orbit,

$$r_p = 3380 + 300 = 3680 \text{ km}$$

According to Eqn (8.40) and Eqn (b) of Example 8.8, the speed of the spacecraft at periapsis of the arrival hyperbola is

$$\left.v_p\right)_{\text{hyp}} = \sqrt{\left[(v_\infty)_{\text{Arrival}}\right]^2 + \frac{2\mu_{\text{Mars}}}{r_p}} = \sqrt{2.8852^2 + \frac{2 \cdot 42{,}830}{3680}} = 5.621 \text{ km/s}$$

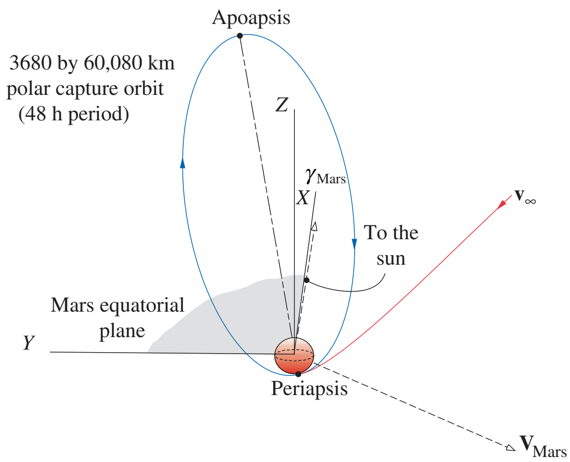


**FIGURE 8.30**

The approach hyperbola and capture ellipse.

To find the speed $v_p)_{ell}$ at periapsis of the capture ellipse, we use the required period (48 h) to determine the ellipse's semimajor axis, using Eqn (2.83),

$$a_{ell} = \left(\frac{T\sqrt{\mu_{Mars}}}{2\pi}\right)^{\frac{3}{2}} = \left(\frac{48 \cdot 3600 \cdot \sqrt{42{,}830}}{2\pi}\right)^{\frac{3}{2}} = 31{,}880 \text{ km}$$

From Eqn (2.73), we obtain

$$e_{ell} = 1 - \frac{r_p}{a_{ell}} = 1 - \frac{3680}{31{,}880} = 0.8846$$

Then Eqn (8.59) yields

$$v_p)_{ell} = \sqrt{\frac{\mu_{Mars}}{r_p}(1 + e_{ell})} = \sqrt{\frac{42{,}830}{3680}(1 + 0.8846)} = 4.683 \text{ km/s}$$

Hence, the delta-v requirement is

$$\Delta v = v_p)_{hyp} - v_p)_{ell} = \boxed{0.9382 \text{ km/s}}$$

The eccentricity of the approach hyperbola is given by Eqn (8.38),

$$e = 1 + \frac{r_p v_\infty^2}{\mu_{Mars}} = 1 + \frac{3680 \cdot 2.8851^2}{42{,}830} = 1.715$$

Assuming that the capture ellipse is a polar orbit of Mars, then the approach hyperbola is as illustrated in Figure 8.30. Note that Mars' equatorial plane is inclined 25° to the plane of its orbit around the sun. Furthermore, the vernal equinox of Mars lies at an angle of 85° from that of the earth.

## PROBLEMS

### Section 8.2

**8.1** Find the total delta-v required for a Hohmann transfer from earth's orbit to Saturn's orbit.
{Ans.: }

**8.2** Find the total delta-v required for a Hohmann transfer from Mars' orbit to Jupiter's orbit.
{Ans.: }

### Section 8.3

**8.3** Calculate the synodic period of Venus relative to the earth.
{Ans.: }

**8.4** Calculate the synodic period of Jupiter relative to Mars.
{Ans.: }

### Section 8.4

**8.5** Calculate the radius of the spheres of influence of Mercury, Venus, Mars, and Jupiter.
{Ans.: See Table A.2}

**8.6** Calculate the radius of the spheres of influence of Saturn, Uranus, and Neptune.
{Ans.: See Table A.2}

### Section 8.6

**8.7** On a date when the earth was $147.4 \times 10^6$ km from the sun, a spacecraft parked in a 200 km altitude circular earth orbit was launched directly into an elliptical orbit around the sun with perihelion of $120 \times 10^6$ km and aphelion equal to the earth's distance from the sun on the launch date. Calculate the delta-v required and $v_\infty$ of the departure hyperbola.
{Ans.: $v_\infty = 1.579$ km/s, $\Delta v = 3.34$ km/s}

**8.8** Calculate the propellant mass required to launch a 2000-kg spacecraft from a 180 km circular earth orbit on a Hohmann transfer trajectory to the orbit of Saturn. Calculate the time required for the mission and compare it to that of Cassini. Assume the propulsion system has a specific impulse of 300 s.
{Ans.: 6.03 year; 21,810 kg}

### Section 8.7

**8.9** An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee radius due to a change of
**(a)** 1 km in the perigee radius.

**(b)** 1 m/s in the perigee speed.
{Ans.: (a) 13.27 km (b) 10.99 km}

**8.10** An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee speed due to a change of
**(a)** 1 km in the perigee radius.
**(b)** 1 m/s in the perigee speed.
{Ans.: (a) −1.81 m/s (b) −0.406 m/s}

## Section 8.8

**8.11** Estimate the total delta-v requirement for a Hohmann transfer from earth to Mercury, assuming a 150 km circular parking orbit at earth and a 150 km circular capture orbit at Mercury. Furthermore, assume that the planets have coplanar circular orbits with radii equal to the semimajor axes listed in Table A.1.
{Ans.: 13.08 km/s}

## Section 8.9

**8.12** Suppose a spacecraft approaches Jupiter on a Hohmann transfer ellipse from earth. If the spacecraft flies by Jupiter at an altitude of 200,000 km on the sunlit side of the planet, determine the orbital elements of the postflyby trajectory and the delta-v imparted to the spacecraft by Jupiter's gravity. Assume that all the orbits lie in the same (ecliptic) plane.
{Ans.: $\Delta V = 10.6$ km/s, $a = 4.79 \times 10^6$ km, $e = 0.8453$}

## Section 8.10

**8.13** Use Table 8.1 to verify the orbital elements for earth and Mars presented in Example 8.7.

**8.14** Use Table 8.1 to determine the day of the year 2005 when the earth was farthest from the sun.
{Ans.: 4 July}

## Section 8.11

**8.15** On December 1, 2005, a spacecraft left a 180 km altitude circular orbit around the earth on a mission to Venus. It arrived at Venus 121 days later on April 1, 2006, entering a 300 km by 9000 km capture ellipse around the planet. Calculate the total delta-v requirement for this mission.
{Ans.: 6.75 km/s}

**8.16** On August 15, 2005, a spacecraft in a 190 km, 52° inclination circular parking orbit around the earth departed on a mission to Mars, arriving at the red planet on March 15, 2006, whereupon retrorockets place it into a highly elliptic orbit with a periapsis of 300 km and a period of 35 h. Determine the total delta-v required for this mission.
{Ans.: 4.86 km/s}