IV.3. Let x be any positive real number. Then show that for all natural numbers n, the following inequality holds:

$$(1+x)^n \geq 1 + xn$$

Hint: Use induction on n

Base Case: (n=1)

$$(1+x)^1 \geq 1 + 1x$$

$$1 + x = 1 + x$$

Induction hypothesis:

Assume for some arbitrary positive integers $k \geq 1$

that:

$$(1+x)^k \geq 1 + kx \qquad where\ x > 0$$

We assume that statement is true for $n = k$ to prove it for $n = k + 1$

We need to show:

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

LHS = $(1+x)^{k+1} = (1+x)^k(1+x)$

x is a positive real number, so we can multiply both side of the Inductive hypothesis by $(1+x)$

And doing so will not change the direction of the inequality:

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$(1+x)^k(1+x) \geq 1 + x + kx + kx^2$$

$$(1+x)^k(1+x) \geq 1 + (k+1)x + kx^2$$

k being a positive integer and x being a positive real number making kx^2 a strictly positive $(kx^2 > 0)$

By knowing $(kx^2 > 0)$, we can state that:

$$1 + (k+1)x + kx^2 \geq 1 + (k+1)x$$

Combining the steps:

$$(1+x)^{k+1} \geq (1+kx)(1+x) \quad = \quad 1 + (k+1)x + kx^2 \geq 1 + (k+1)x$$

Thus, we have successfully show that the statement $\left((1+x)^{k+1} \geq 1 + (k+1)x\right)$ is true. ■