VIII.2. Let $A = \{1, 2, 3, 4\}$ and let $R$, $S$, $T$ and $U$ be the following relations:

$R = \{(1, 3), (3, 2), (2, 1), (4, 4)\}$,

$S = \{(2, 1), (3, 3), (4, 2)\}$,

$T = \{(4, 1), (4, 2), (3, 1), (3, 2), (1, 2)\}$,

$U = \{(x, y) \mid x > y\}$.

(a) For each of $R$, $S$, $T$ and $U$ determine whether they are functional, reflexive, symmetric, anti-symmetric or transitive.
Explain your answer in each case, showing why your answer is correct.

(b) What is the transitive closure of $R$?

(c) Explain why $R^*$, the transitive closure of $R$, is an equivalence relation. Describe the equivalence classes $E_x$ into which the relation partitions the set $A$.

# Solution:

## a.

| | Functional | Reflexive | Symmetric | Anti-symmetric | Transitive |
|---|---|---|---|---|---|
| **R** | ✓ | ✗ | ✗ | ✓ | ✗ |
| **S** | ✓ | ✗ | ✗ | ✓ | ✗ |
| **T** | ✗ | ✗ | ✗ | ✓ | ✓ |
| **U** | ✗ | ✗ | ✗ | ✓ | ✓ |

## b.

$$R^* = \{(1,3), (3,2), (2,1), (4,4), (3,1), (1,1), (3,3), (2,3), (1,2), (2,2)\}$$

## c.

$R^*$ is equivalence because it is reflexive, symmetric and transitive.

Equivalence classes:

1,2,3 are all mutually reachable, so they are equivalent.

$$E_1 = E_2 = E_3 = \{1, 2, 3\}$$

4 is isolated except for (4,4)

$$E_4 = \{4\}$$

Finally, partition of $A$ under $R^*$:

$$\{\{1, 2, 3\}, \{4\}\}$$