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Mathematics as a Constructive Activity: Exploiting Dimensions of Possible Variation

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Introduction

Mathematics is often seen by learners as a collection of concepts and techniques for solving problems assigned as homework. Learners, especially in cognate disciplines such as engineering, computer science, geography, management, economics, and the social sciences, see mathematics as a toolbox on which they are forced to draw at times in order to pursue their own discipline. They want familiarity and fluency with necessary techniques as tools to get the answers they seek. For them, learning mathematics is seen as a matter of training behaviour sufficiently to be able to perform fluently and competently on tests, and to use mathematics as a tool when necessary.

Unfortunately this pragmatic and tool-based perspective may cut people off from the creative and constructive aspects of mathematics, making it more difficult for them to know when to use mathematics, or to be flexible in their use of it. On its own, this perspective can reinforce a cycle of de-motivation and disinclination. The result is a descending spiral of inattention, minimal investment of energy and time, and absence of appreciation and understanding, leaving learners disempowered from pursuing their discipline through the use of mathematics.

By contrast, mathematicians see mathematics as a domain of creativity and discovery in its articulation, proof, and application. Full appreciation of a mathematical topic includes the exposure of underlying structure as well as the distillation and abstraction of techniques that solve classes of problems, together with component concepts. Mathematicians construct objects as examples of concepts, as illustrative worked examples of the use of techniques, and as possible examples of or counter-examples to conjectures. As Paul Halmos put it,

> A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one. … Counter-examples are examples too, of course, but they have a bad reputation: they accentuate the negative, they deny not affirm. … the difference … is more a matter of emotion. (Halmos, 1983, p. 63)

> If I had to describe my conclusion [as to a method of studying] in one word, I'd say *examples*. They are to me of paramount importance. Every time I learn a new concept I look for examples … and non-examples. … The examples should include wherever possible the typical ones and the extreme degenerate ones. (Halmos, 1985, p. 62)

Halmos is one of many authors who express a similar sentiment. Feynman expressed it this way: "I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go" (Feynman and Leighton, 1985, p. 244).

We take the view that mathematics can be presented and experienced as a constructive activity in which creativity and making choices are valued, not just for their own sake, but in order to stimulate learners to use their own powers to make sense of phenomena mathematically, whether in the domain of mathematics itself, or in cognate disciplines that make use of mathematical tools and mathematical thinking. By powers here we mean things like imagining and expressing, specialising and generalising, conjecturing and convincing, and so on (Mason, Burton, & Stacey, 1982; Mason, 2002; Mason & Johnston-Wilder, 2004a, 2004b).

For example, presenting learners with the function $x \rightarrow |x|$ as an example (often the only one) of a function which is differentiable everywhere on the reals except at one point, invites them to see it as a monster to be barred (Lakatos, 1976), as a pathological object to be ignored (MacHale, 1980). Yet there is an opportunity to invite learners to use it to construct whole classes of functions which have the same property at (0, 0), such as $x \rightarrow \lambda|x| + xg(x)$ for any differentiable function g. They can also construct for themselves a function that is differentiable everywhere except at some other specified point, and then extend this to non-differentiability at several points (e.g., $x \rightarrow \lambda|x-a| + \mu|x-b|$ where $a \neq b$ and $\lambda\mu \neq 0$). Not only does this invite learners to make use of their control over functions through translation and scaling, but it demonstrates that any example can be expanded to whole classes of functions.

We suggest that seeing mathematics as a constructive activity requires a small but important shift in thinking that can have significant impact on the interest and commitment of learners, the way they use their acquired concepts and tools, and hence the way they use mathematics in their own discipline. After many years of teaching in high school and university contexts, often working with disaffected students, and countless experiences working with others on mathematics, we take it as axiomatic that people who are encouraged to use their powers not only experience pleasure, but sharpen and extend those powers. They can become more motivated to enquire further. Learners who are encouraged to be creative and to exercise choice respond by becoming more committed to understanding rather than merely automating behavioural practices.

Finally, we suggest that mathematics seen and presented as a creative and constructive activity can engage and motivate those who might pursue mathematics further. It can also yield insight into how learners are thinking in ways that complement what teachers discover from learners' responses to routine tests on standard exercises.

This chapter begins by showing that many, or most, tasks and exercises presented to students for pedagogic purposes can be seen as constructions, and that adopting this view can assist learners in appreciating what the problem is asking, as well as finding solutions and also motivating them. The second section shows how the notion of dimensions-of-possible-variation can be used to inform the structuring of tasks, so that learners can be induced to move from exercising in order to train their behaviour, to exploring in order to sharpen their thinking and deepen their awareness of underlying mathematical structure. The third section moves beyond exercises as tasks, and considers problems in the two forms identified by Pólya. The fourth section shows how the same principle of dimensions-of-possible-variation can inform the way learners encounter mathematical concepts as they develop rich example-spaces which will, it is hoped, come to mind when they encounter future problems. The fifth section briefly comments on the need to prompt and provoke learners to shift from the metaphor of learning mathematics as exercise, to learning mathematics as exploration.

The chapter is theoretical in nature, presenting possibilities for practice based on mathematical structures. Thus, the warrants for this approach are to be found in mathematics itself rather than in empirical research[1]. Nevertheless, the practices arise from our extensive, evaluated, self-critical experience as mathematicians, teachers and educators, and they have been used with learners ranging from adolescents to adults, in school and university. Our method of enquiry is to observe the behaviour of people as they explore and learn mathematics: our own behaviours, as well as those of our students and colleagues. We intentionally construct, and put ourselves and colleagues into, mathematical situations that appear to be analogous to what learners encounter. From these we gain insight into strategies based on mathematical thinking of which learners may be unaware. We test, in many contexts, both the strategies and ways of drawing them to the attention of learners who do not yet use them spontaneously. For example, this chapter arises from some eight years of work during which we focused on the potential contribution the construction of mathematical objects can make to an enhanced experience of mathematics. We always seek feedback from workshop and teaching sessions. A particularly powerful form of feedback comes from teachers who are energised to use our ideas in their

[1] Our method (Mason 2002a) relates to both action research and design research paradigms. It takes place in naturalistic settings; we evaluate and develop actions in a deliberate and cyclic manner. Ideas are not only put back into practice, but also to critical audiences of peers and learners; the path of our enquiry branches frequently; the opportunities to work on ideas arise during our normal working lives. To those who want detail about numbers of cycles, numbers of students, numbers of occasions we can only answer 'hundreds'.

next teaching sessions and report back to us of their experiences. We do not carry out quantitative controlled studies because these are not appropriate in a domain that is highly sensitive to the world views of teacher and learners, the relationship between them, and their propensity to reflect on their experience.

Pedagogic Phenomena Being Addressed

The particular pedagogic phenomena we are addressing can be summarised as the tendency for many learners to do as little as necessary in order to complete what they are asked to do, and the widespread assumption that 'doing' the tasks they are set will mean that they will learn. This is the basis for the implicit *didactic contract* between teacher and learners described and elaborated by Brousseau (1984, 1997). Too often 'doing' means 'getting answers' rather than using the task to appreciate general concepts and methods as exemplified by the particular. There is a tendency for learners to be satisfied with the particular rather than trying to see through the particular to the general (Whitehead, 1919), or, on the other hand, leaping to vast generalisations without checking particular cases. Learners are often satisfied by assenting to what they are told, by trying to work out what the teacher thinks rather than (re-) constructing for themselves, by using worked examples as templates in doing assignments without probing the underlying reasoning, and by internalising as little as possible. This response to the *didactic contract* is rarely sufficient to reach what Skemp (1976) described as *instrumental understanding* (knowing enough to succeed only at routine tasks), much less the more desirable *relational understanding* (appreciating connections and knowing enough to be able to respond flexibly in novel situations). Christiansen and Walther (1986) observed that "Even when students work on assigned tasks supported by carefully established educational contexts and by corresponding teacher-actions, learning as intended does not follow automatically from their activity on tasks" (p. 262). As human beings, learners respond well to being asked to make choices, including creating mathematical objects for themselves. It gives them a sense of involvement and it enriches the space of examples to which they have access when someone else is talking in generalities. Indeed, understanding of concepts and techniques can usefully be thought of in terms of the richness of the *example space* that comes to mind (Watson & Mason, 2002, 2005), and the complexity of transformations learners can use to modify those examples. We have found that with prolonged exposure to an atmosphere in which learners are expected to construct examples, their example spaces become both more extensive and more richly interconnected.

We have also found that learners respond well to being called upon and expected to use their own powers to specialise and to generalise (Pólya, 1945, 1962), to imagine and to express, to conjecture and to convince, to organise and to characterise. This is born out by numerous studies, such as Boaler (2002), Senk & Thompson (2003), and Watson, De Geest & Prestage (2003): when learners' powers are engaged, they display behaviour beyond what is normally expected. Thus the challenge is to promote a movement from merely *assenting* to what they are told or asked to do, to taking the initiative and *asserting* (in the form of making, testing and validating conjectures, constructing examples which illustrate conditions, and generalising particular tasks to a class of 'types' of tasks) through using and developing their natural powers (Mason & Johnston-Wilder, 2004a).

An atmosphere of competitive seeking of single correct answers does little to foster mathematical thinking. Calling upon learners to make choices, to act creatively and to use their powers is best supported in a *conjecturing atmosphere*, in which what is said by anyone is treated as a conjecture uttered with the intention of possibly modifying it according to critique and counter-examples. There are close analogies with what Legrand (1993) calls *scientific debate*. Among other things this implies that learners are constantly challenging, constantly seeking examples and counter-examples. It means that learners use examples in order to re-construct generalisations and to appreciate mathematical reasoning for themselves.

Our theoretical position can be summarised as follows:

- promoting mathematical thinking improves motivation and confidence, and hence both competence and effective use of mathematics as a tool, even among those only taking mathematics as a service subject.
- trusting learners to make choices and so to exercise creativity and to explore available freedoms enriches their mathematical understanding and appreciation of concepts, techniques, and heuristics, as well as fostering involvement in and getting pleasure from learning mathematics.

Exercises as Construction Opportunities

A set of exercises may be seen by a learner as an obstacle, a necessary hurdle to be overcome as quickly as possible, but it can also be seen as an opportunity to develop fluency and facility by reducing the amount of attention required to get solutions using a standard technique, to the point of automating that technique. A set of exercises can also be taken as an opportunity to seek generality which encompasses all of the particulars. This turns it into a construction task: to re-construct the *question space*[2] from which the particular questions have been drawn. This view of exercises as an opportunity to appreciate generality has a long history, since the earliest written records contain worked examples of mathematical calculations. Babylonian cuneiform tablets and Egyptian papyri often include statements such as 'do thou like this' and 'thus it is done' (Gillings, 1972) suggesting that the learner is intended to do more than simply 'follow' the template. The point of doing exercises is to appreciate the generality of a method, and to internalise and automate its functioning. Girolamo Cardano (1501-1576) writing in Latin in his famous work *Artis Magnae Sive de Regulis Algebraicis* includes phrases such as

> In accordance with these demonstrations, we will formulate three rules and we attach a jingle in order to help remember them; We have used a variety of examples so that you may understand that the same can be done in other cases and will be able to try them out for the two rules that follow, even though we will there be content with only two examples; It must always be observed as a general rule … ; So let this be an example to you; by this is shown the *modus operandi* …. (quoted in Cardano, 1545/1969, p. 36–41)

Thus the point of a set of exercises can be seen as a behavioural aspect of the *didactic contract*, or it can be seen as an opportunity to construct not only solutions to the particular tasks, but the general class of which these are particulars.

Take for example, the factoring of the difference of two squares. The basic idea is straightforward, but it can be masked by varying different features. Asking learners to factor the following expressions

$$x^2 - 1, \quad a^2 - b^2, \quad 4x^2 - 9y^2, \quad 64a^4 - 81b^4, \quad 4x^2a^6 - 25y^4b^{10}, \quad (2x + 3a)^2 - (3x - 2a)^2$$

could be intended to expose learners to the variety of possibilities for factoring the difference of two squares. Most texts would have several examples of each type; some would mix them up, and others would arrange similar ones together. The former are probably intending learners to detect similarities from amongst apparent differences, whereas in the latter case the intention is probably to get learners to detect what is the same about several so as to appreciate a range of possibilities. In both cases there is an implicit aim towards speed and accuracy, but to gain speed one has to recognise and exploit similarity, while to gain accuracy one has to focus on specific details. But unless learners are prompted to reflect on *what* is changing and *how*, they are unlikely to appreciate the various aspects which can change, nor what changes are permissible to maintain the underlying structure. Here, constants and multiple letters can change, but each must be raised to an even power. Perhaps it is no wonder that learners often fail to recognise the difference of two squares outside of the section of text devoted to it.

Seeing sets of exercises as construction opportunities opens the way to further meta-tasks intended to promote reflection and construal by learners. Karp (2004) provides an excellent illustration in the context of quadratic equations that have been obscured by the use of reciprocals, linear expressions, and square roots. In almost every topic, learners can be asked to construct:

- another example like the ones they have done;
- one which obscures the use of the technique as much as possible;
- one which shows that they know how to do 'questions of this type' (and they can be asked to describe how they would recognise a 'question of this type');
- one which would be a good test for learners next year;
- one which they think might challenge the teacher (or some other relative expert);
- the most general question of this type;

and so on. Focus and emphasis switch from doing particular questions to appreciating the technique as a general

[2] A term introduced to us by Chris Sangwin (personal communication) arising from his project to use software to generate random questions for learners from a space of possible questions but paying attention to internal structure and constraints within that space (e.g., quadratic equations with real or complex roots).

method and appreciating the whole exercise as representative of a space of possibilities. This is where Marton's notion of *dimensions of variation* is so useful.

Dimensions of Possible Variation, and Related Ranges of Permissible Change

Marton proposed that learning can be seen as extending awareness of what constitutes an example (Marton & Booth, 1997; Marton & Trigwell, 2000; Marton & Tsui, 2004, see also Runesson, 2005). He observed differences in learning according to the nature and range of variation to which learners were exposed. To capture this, he introduced the notion of *dimensions of variation* to refer to the different aspects of an object which can be varied and still that object remains an example of a specified concept. For example, in meeting limits, learners need to be aware that a limit can be approached from one or the other side only, or from both sides, depending on the context. This constitutes a dimension of variation that is vital to appreciating limits. Other more elementary dimensions include the fact that the variable expressing the limit can be something other than x or h; that the point being approached can be specific (a number) or general (a letter); that the function whose limit is sought may have other variables and parameters present; and that in any case, there may or may not be a limit. In group theory, learners need to be aware that not all groups can be displayed as the symmetries of objects in two or three dimensional space; that the same group can be generated by different actions on different objects; that a finite cyclic group can be of any finite order; and that the generators can be denoted by any symbol.

Since teachers and learners are usually aware of different dimensions, and since the same person may be aware of different dimensions of variation at different times, in Watson & Mason (2002, 2005) we extended the notion of dimensions to *dimensions of possible variation* of which someone may be aware. Thus a teacher may be aware of many more possibilities than learners, but may not be aware of this difference. By being explicitly aware of important dimensions of possible variation, a teacher can choose to direct learner attention to relevant dimensions as they develop their sense of a concept. Furthermore, within each dimension of possible variation, learners may be aware of different *ranges of permissible change*. For example, learners might have encountered $|x - a|$ for various integer values of a but it may never have occurred to them that a could be any real number. Indeed, their sense of 'any real number' may itself have a limited range of permissible variation. If learners only see systems of equations with integer coefficients, they may never think of the possibility that they could be other numbers.

The notion of dimension of possible variation is, in some sense, the dual to an important and ubiquitous theme in mathematics, that of *invariance in the midst of change*. Any theorem in mathematics can be seen as a statement of what remains (relatively) invariant when other things are permitted to change. Usually it is some relationship or property which is being preserved. However, it is not always clear that learners are aware of what it is possible to change.

Seeing generality through particular instances is basically detecting some features to keep invariant while others are permitted to change. A useful pedagogic strategy that calls upon this theme is to ask learners to look for what is the same, and what is different between two or more objects (Watson & Mason, 1998; Coles & Brown, 1999; Brown & Coles, 2000) as well as with the problem solving strategy of asking yourself 'what if ... were to change' (Brown & Walter, 1983).

Applied to a set of exercises, looking for dimensions of possible variation and the associated range of permissible change within each dimension calls upon the learner to go beyond merely solving each individual exercise. It draws attention to the exercises as representative of a potential question space and invites learner re-construction of such a space. Next time learners encounter a similar question, they are much more likely to recognise the type and so have access to a solution approach or technique than learners who have contented themselves with obtaining solutions solely to the particular questions in the exercise set.

For learners, aspects of exercises that could vary include context, explicit numbers, implicit structural numbers, and choices of which elements of a task are given as data and which are to be found. This last opens the mathematical theme of *doing & undoing*, in which learners attempt to characterise all the similar questions that have the same answer, and all the possible answers to similar types of questions. A learner who has explored dimensions of possible variation is much more likely to recognise the structure of a novel task rather than being misled by superficial similarities with previously solved tasks.

For teachers, in addition to these dimensions, the order of presentation can take into account a supposed hierarchy of complexity, a supposed order of conceptual development, or significant aspects, which is essential for learners to appreciate in order to understand the topic, concept or technique (as illustrated in Karp, 2004).

Problems as Construction Opportunities

Users of mathematics are more in need of experience of using mathematics to resolve problems in their own domain of interest than they are of rehearsing to mastery a collection of techniques. Techniques can be quickly automated if there is need for repeated and efficient use. But the same is true of the mathematics student: "The mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics *really* consists of is problems and solutions." (Halmos, 1980, p. 519)

What constitutes a *problem* is a non-trivial matter. What is a problem to learners is not usually problematic for the author or setter. For example, in a recent study Lara Alcock & Keith Weber asked learners in a university course to prove that if f is an increasing function on R to R, then there is no real number c that is a global maximum for f (Alcock & Weber, 2005). An experienced mathematician has an instant intuition that can be translated into a formal argument with only a minimum of refinement. A learner may not find the technical terms 'speak to them', may lack intuition or insight and so may have to explore, and is very likely to be unfamiliar with writing formal reasoning. They may experience some or all of this as 'not knowing what is expected of them'.

The 'problem' for most learners is simply 'how to get the answer as quickly and painlessly as possible', whereas the author or setter sees it as an opportunity for learners to re-construct and appreciate techniques and concepts, and to experience mathematical thinking. Learners need to be supported in moving from a 'don't – can't – won't try' frame to one of 'can take on as a challenge to use my powers of thinking' (Dweck, 1999). If the word *problem* is only applied to tasks set by a teacher, we lose the importance of the learner's perception of worthwhile challenge. It is the learner who problematises tasks, for until the learner experiences the task as a mathematical problem to understand and make sense of (rather than a problem of getting an answer), tasks are simply tasks that 'have to be done'. Tasks which bring learners into contact with significant themes and concepts without requiring teacher intervention are known as *adidactical situations* (Brousseau, 1997).

When you encounter an unfamiliar problem, or when a problem involves unfamiliar technical terms, it can be very helpful to construct a specific example. To be useful, such an example must be confidence inspiring if not familiar. The purpose of this *specialising* as Pólya called it (Pólya 1945, 1962) is to develop confidence and to try to see what is going on so that you can re-generalise for yourself (Floyd, Burton, James & Mason, 1981, Mason & Johnston-Wilder, 2004b). You don't just 'do calculations', you watch what you do and try to see what is particular and what is general. By seeing through the particular to the general (Whitehead 1919) learners come to appreciate the generality, and to solve not just the particular problem, but other problems like it. Clearly experience of and disposition towards constructing examples both help in tackling new and unfamiliar problems. George Pólya (1945) famously divided problems into two kinds: *problems to find* (something), and *problems to prove*. In the following subsections we argue that both kinds can usefully be seen as construction opportunities.

Problems to Find

The language of *find* has the potential to summon up an image of someone looking around in dusty corners for something that someone, probably someone else, has mislaid. It is of course intended to signal the creative side of mathematics, but most often such problems come across to learners as a requirement to 'find the right technique and to apply it correctly'. An alternative perspective is to see 'problems to find' as construction tasks. Asked to solve some simultaneous equations in n variables, the challenge is to construct an n-tuple which satisfies all those constraints; to solve a differential equation the challenge is to construct functions which meet the constraint; to find an extremal value of a differentiable function on an interval the challenge is to construct a point that lies on the function and is a local extremum for that function; to find the definite or indefinite integral of some function the challenge is to construct a number, but more than that, a way of looking at the function which makes it amenable to one or more integration techniques.

Seeing a 'problem to find' as a construction task, which may be facilitated by the use of some familiar, and perhaps some not so familiar tools, is completely different psychologically from trying to work out the answers at the back of the book and from worked examples, what sequences of symbols of varying degrees of meaningfulness to write down and in what order, so as to satisfy the marker. Consider for example the following apparently simple 'problem': find $\int_0^2 (x-1)dx$. This might appear at the beginning of a collection of integrals of polynomials, with the intention that learners rehearse integration and evaluation of definite integrals. It is hardly 'a problem', except to learners who perhaps have no mental images and so are unaware that geometrically the answer is clearly zero, or to learners who are not overly confident in integrating and substituting correctly. Seen as one of many hurdles to overcome in sequence, it requires getting an answer. Alternatively, the task can be seen as a challenge to construct something. Seen this way, questions begin to arise about what sort of an object is sought (a number) and what this number might mean (apart from being 'the answer'). The number being sought is not just any number, but one which meets the constraint that it measures the area calculated by the integral.

Thinking in terms of dimensions of possible variation leads to questions such as how the number depends on the various parts of the question that could be altered. What if one or both of the limits were changed? What about changing the constant in the integrand? A learner approaching the task in this vein might be encouraged to find a general class of integrals, all of which give the same answer, such as $\int_a^b \left(x-\frac{a+b}{2}\right)dx$, which highlights the role of the arithmetic mean of the limits of integration and stimulates questions such as: 'what about changing the integrand?'; 'what possibilities are there for a general quadratic with the same answer?' One could go further and try to use geometric insight to predict the shape of multiple integrals and line integrals that also give zero as the answer, and to ask oneself about the significance of the zero. It is well known for example, that learners are often perplexed by 'negative areas' (Mason 2002). Suddenly what looks like a routine question turns into a possibility for exploration, for encountering important mathematical ideas, and for clarifying the mathematical meaning of 'area'. In the process of investigation, learners are likely to do several integrations (presumably the original intention) while at the same time making choices as to what functions to integrate, and using their own powers to specialise and generalise, conjecture and convince.

Problems 'to find' may sometimes have only a single answer, but often there are several or even infinitely many possibilities, all of which constitute a potential example space. Where learners are asked to construct not just a single example, but several examples from that space, one after another, we have found that they begin to appreciate the extent of that space, so that their thinking changes from being satisfied with the first (usually simple) example that comes to mind, to looking for the scope and breadth of generality possible. As this appreciation grows, as they consider more and more extreme or 'peculiar' examples (Bills, 1996a; Bills, 1996b) they can give rein to their creativity. For example, learners exposed to $|x|$ as a continuous function differentiable everywhere except at one point often do not appreciate the class of functions in the teacher's example space. By prompting them to use that idea to construct others, such as $|x| + |x-1| + |x-2|$, inserting coefficients, and then creating other cusps for themselves from other functions, they are likely to appreciate both cognitively and affectively the plethora of examples over which they have control (as distinct from a foreboding sense of functions lurking in the shadows of which they have no idea!). Appreciating the significance of a general solution to a differential equation as producing a class of functions which (in the case of first order at least) do not intersect and which 'cover' the space, makes the selection of a particular example, according to initial conditions, much more meaningful. Comparing what is the same and what different about different members of a general class can help learners comprehend the differential equation as a property being satisfied by a class of functions that differ in other ways.

Problems 'To Prove'

Pólya's distinction between 'find' and 'prove' is not rigid, because of course you can argue that a problem 'to prove' is a problem 'to find a proof', but this playfulness fails to appreciate the significant psychological difference between trying to find something you don't know, and trying to find a chain of reasoning that justifies something you do know (or conjecture). Perhaps the clearest example of the distinction is found in induction problems. There is a huge difference between

(a) Find $\displaystyle\sum_{k=2}^{n} \frac{1}{k(k-1)}$.

(b) Prove that $\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}$.

The former requires an element of creativity, with access to 'partial fractions' coming to mind, or else the construction of some particular cases and the formulation of a conjecture on the basis of these. Once there is a conjecture, the problem becomes one of proving. The problem 'to prove' expects learners to have 'induction' come to mind, or else to deconstruct $1-\frac{1}{n}$ as a telescoping sum:

$$1-\frac{1}{n}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\left(\frac{1}{n-1}-\frac{1}{n}\right).$$

Note that during the induction step it is often difficult to gain insight into 'why' the proof works, into the structural underpinnings that make the statement true. See Harel and Brown (this volume) for a further discussion of students' conceptions of inductive proofs.

Learners who are satisfied with getting the answers to a suite of exercises are likely to learn much less than learners who develop the confidence that they could 'do another question of this type' in the future. The former can dwell in the particular, but the latter are engaging with the general, with "what does it mean to be a problem of 'this type'?". In our view, this is what constitutes mathematical thinking, as opposed to clerical proficiency. This perspective is supported by Vadim Krutetskii's research (Krutetskii, 1976). He found that learners who were quick at mathematics tended to remember numbers and structure, whereas learners who were not so quick tended to remember contexts and other surface features of the tasks they encountered.

Constructive Approach to Solving

When asked to solve a particular problem, usually through the use of a technique recently encountered, a typical strategy for learners is to find a worked example which looks similar. As Chi & Bassock (1989), Sweller & Cooper (1985), and Anthony (1994) have pointed out, the problem with worked examples is that choices are made as to what action to take at each step, and the basis for that choice may not always be clear to learners. Consequently, using the worked example as a template may achieve a solution, perhaps involving some tinkering due to slightly different setting, but it may not shed any light on 'how the technique is used', much less on when and why it 'works'. When problem solving is seen as construction, metaphors such as *bricolage* are available, where an attempt at a solution is assembled from available bits and pieces, which might at least shed light on where the real difficulty lies, or even on how to then set about constructing solutions systematically.

Alcock (2004) found through interviewing mathematicians that learners do not use examples in the same ways as mathematicians. Whereas mathematicians use examples to illuminate through instantiation, to see through the particular to the general, and to consider possible counter-examples, many learners see them at best as templates and at worst as 'more stuff to learn'. They often select examples for reasons which do not appear to be mathematically robust, relevant or informative, and to use them as demonstrations as if this constituted a proof. She goes on to suggest ways in which learners could be supported to make more effective use of the examples they do encounter, and in particular to get learners to construct their own examples (see also Dahlberg & Housman 1997, Watson & Mason 2002, 2005).

Freedom and Constraint

A pervasive theme throughout mathematics concerns the imposition of constraints on some otherwise general object, leading to characterising the collection of objects (if any) that satisfy those constraints. For example, starting with the freedom of an arbitrary pair of numbers, you can follow Diophantos (1964) and impose constraints such as that the sum is given, or perhaps that the sum of their squares is itself a square. An additional constraint can be included, such as that the difference is given, or that the numbers are also to be in a given ratio. Instead of searching around for a template to deal with the specific task, learners can become accustomed to starting from an unconstrained generality, and then imposing the constraints sequentially, seeking as general a solution as possible at each stage. For example,

- to find a number that leaves a remainder of 1 on dividing by 2, 3, 4, and 5, it can be useful to start with n as any integer, then to impose the constraint the remainder be 1 on dividing by 2, and to construct the most general number possible, and so on.

- to construct a solution to three simultaneous linear equations in three unknowns, it can be useful to think first in terms of a general triple of numbers with no constraint, as a point roaming through three dimensions. Imposing one constraint limits the movement to a plane, and expressing one variable in terms of the others effectively 'solves' one equation. A second constraint then confines the point to the intersection of two planes (if they are not parallel), and the first solution can be substituted in the second to yield a general expression for all those points.

The effect may be the same as using a prepared technique, but the psychological experience can be quite different as the learners' powers are made use of to develop a resolution. Sometimes the heuristic of trying particular cases can be useful as a means to locate underlying structure; sometimes a single particular case can be used as a generic example, as Hilbert is reported to have done:

He [Hilbert] was a most concrete, intuitive mathematician who invented, and very consciously used, a principle; namely, if you want to solve a problem first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as can be made, without losing any of its punch, and then you solve it. The generalization is a triviality which you don't have to pay much attention to. This principle of Hilbert's proved extremely useful for him and also for others who learned it from him. Unfortunately, it has been forgotten. (Courant, 1981, p. 161)

By attending to the actions taken in a particular case, it is sometimes possible to create your own template that can be generalised.

Seeing a problem in terms of constructing an object that satisfies a number of constraints not only evokes a spirit of construction, but opens the way to identifying and dealing with those constraints. Starting from the most general unconstrained object can be followed by expressing the most general object satisfying one constraint, then two constraints, and so on until a solution is found or shown not to exist. This is certainly powerful in algebra, where it can help learners to locate and express constraints symbolically in order to produce a symbolic statement of the problem. Isaac Newton was one of those who worked at a time when the mathematical focus of solving problems shifted from expressing constraints in symbols, to developing techniques for solving the resulting equations and inequalities. Many learners may not realise that the techniques they encounter in textbooks and lectures are the fruits of this kind of labour. But techniques are only useful once the problem has been expressed symbolically.

Geometry is a domain in which it often helps to see construction tasks such as: 'given three distinct concurrent lines, construct all the triangles for which these are the medians, or the altitudes, or the angle bisectors'. Even the problem that stimulated much of Schoenfeld's (1985) work, 'find a circle tangent to a given pair of lines and passing through a specified point on one of those lines', can fruitfully be seen not just as a construction task, but as a task with constraints imposed on initial freedom. The problem is to construct a circle. That is easy enough, but what matters is a sense of the freedom available: choice of centre and radius. Then it must pass through a specified point. Then it must be tangent to a given line through that point. Then it must also be tangent to another given line as well. By becoming aware of the possibilities at each stage, the solver not only gets a sense of the impact of the constraints, but also, by looking for the most general class of solutions at each stage, may find access to the consequences of each constraint in turn. Choosing the constraints in a different order is sometimes more helpful, but learners need to be aware that they are dealing with a sequence of constraints before they can change the order.

Some geometry problems succumb to the removal of one constraint and the construction of a locus that captures the freedom available without that constraint. Where the locus is recognisable (a straight line, a circle, ...) it may suggest a conjecture that can then be used to complete the construction (Love, 1996). The algebraic version of this is to let go of one or more constraints and express the general class of all objects before imposing further constraints.

Concept Development as Construction

Exercises are the most visible, but by no means the most significant, aspect of learners' pedagogic experience (see the chapter by Weber, Porter, and Housman in this volume for a further description of the pedagogical advantages of using examples). Every concept — indeed every idea — has behind it a culturally rich collection of images and connections that Tall and Vinner (1981) call the *concept-image* (see also the chapter by Edwards and Ward in this volume for a further description of concept images). Indeed, it seems a reasonable conjecture that every technical term in mathematics signals an important shift in the way of perceiving and thinking that someone made in the past, and has to be re-experienced by each learner. A significant component of a person's concept image is the collection of examples and non-examples to which they have access, what Watson and Mason (2005) call their *example space*. Awareness of dimensions of possible variation can inform both the teacher in preparing encounters for learners, and learners as ways to probe their understanding of concepts.

The notion of dimensions of possible variation is particularly powerful when learning is seen as appreciating variation. Only if you know what can be varied can you appreciate the delicate relationship between particular and general, between an example and that which is exemplified.

Trying to understand a mathematical concept or theorem is much like trying to make sense of a mathematical problem. You have to ground yourself in something familiar, and this usually takes the form of an example that is sufficiently familiar to enable you to proceed with confidence. By following the theoretical or the general through the particularities of an example, it is possible to get a sense of what the theorem is saying or what the concept is encapsulating.

However, when a teacher offers an example and works it through, it is the teacher's example. Learners mostly assent to what is asserted. In a textbook, the words and examples become 'yet more to be learned', or ignored: yet more to which to assent, *en route* to the tasks. When learners construct their own examples, they take a completely different stance towards the concept or theorem. They 'assert'; they actively seek to make sense of underlying relationships, properties and structure which form the substance of the theorem or concept.

Developing appreciation of a mathematical concept involves finding yourself using the concept to express what you are thinking. As has been pointed out by many people (e.g., Lakoff, 1987), concepts are inextricably entwined with examples of those concepts. The richer the range of examples and the more extensive the sense of how to construct examples, the richer the appreciation of the concept. Halmos raises the important question of sources:

> Where can we find examples, non-examples, and counter-examples? Answer: the same place where we find the definitions, theorems, proofs and all other aspects of mathematics — in the works of those who came before us, and in our own thoughts. ... we find them first, foremost, and above all, in ourselves, by creative thinking. (Halmos, 1985, p. 64)

Furthermore, awareness of, or being able to construct examples which lie 'just over the boundary' of the concept, or providing counter-examples to weakened conditions of a theorem, alerts the learner to difficulties that may arise when the theorem is applied. A task or mathematical situation initiates a *space of examples* that may be given in the text (Michener, 1978), and may be enriched through learners being stimulated to construct their own examples (Watson & Mason, 2002, 2005). The richer and more complex that space, the richer the connections and sense of appreciation of the concept.

If learners are in the habit of constructing their own examples, and if they are supported in seeking generalities that encompass a range of examples, then they are likely to feel more secure in the use of the concept, as well as having access to a range of possible examples on which to test conjectures and through which to seek structure. For example, as was pointed out in the introduction, many learners treat $f(x) = |x|$ as an isolated, even perverse, example of a function, designed just to 'give them trouble' because they want continuous functions to be differentiable. Whereas it is easy and even natural to *monster-bar* a single example, it is much harder to do this to a huge class of examples. Whenever a counter-example is offered, what matters is whether learners are aware of how the single counter-example is just one illustration of a class or space of such examples, and how those other examples could be constructed. Klymchuk (2005) provides not only a wealth of counter-examples to erroneous but common learner conjectures and assumptions in the calculus, but also points to ways of making the most of the examples.

From Exercising to Exploring

One of the features of developing facility in the use of a technique is to reduce the amount of attention required to carry out the technique. The novice requires full attention to each step, with the result that they may not have enough free attention to watch out for slips, and they may lose their sense of direction overall. By contrast, having automated a technique, the expert has free attention both to catch slips and unusual features arising, and also to maintain a sense of the overall plan. As Jerome Bruner (1986) points out, a skilled teacher can act as *consciousness for two* by retaining a sense of the overall plan while the learner engages with particular details. Developing fluency in a technique involves becoming familiar with the overall flow and procedure. Gaining facility requires practice.

Traditionally, facility is developed through the doing of a large number of repetitive exercises, literally 'exercising' the fluent use of a technique or way of thinking about a problem. The trouble is that it is possible to do repetitive tasks without actually gaining facility, indeed without even thinking very much about anything! When doing a set of exercises, learners are rarely interested in the answers, but only whether they conform with those provided at the back of the book. Learners are not often supported and provoked into treating exercises as the raw material for mathematical development.

A much more effective approach advocated by Caleb Gattegno (1987) and developed by, among others Dave Hewitt (1994), is to engage learners in an exploration which, *en passant*, involves them in constructing examples for themselves in order to try to see what is going on. For example, finding the points in the plane through which a specified number of tangents can be drawn to a given curve such as $y = x(x - 1)$ or $y = x(x - 1)(x + 1)$ invites learners to choose points, to construct lines, to make them tangent to the curves, and to try to locate some overall pattern to the results which they might then be able to justify. In the process they are likely to gain considerable practice in differentiating and constructing straight lines. There is an opportunity to appreciate the growth of functions as x gets large (positively and negatively) in relation to potential tangents. Similarly, finding the distance between pairs of straight lines in three dimensions can be tedious as a set of exercises, but finding the minimum distance between two ruled surfaces amounts to the same thing, except that the learner is the one who constructs the pairs of straight lines, and is furthermore induced to work as generally as possible rather than being content with particular distances between particular pairs of lines.

If the particular examples that learners construct for themselves require the use of a technique with which they have some fluency, then it is in their interest to 'get the answers' because it will contribute to working out what is going on in the exploration. Thus learners can be induced to rehearse a technique on examples constructed by them rather than imposed from outside, and whose solutions matter to them.

Summary

In order to appreciate a concept, it is vital to be aware of the variation it entails, and the invariance that is preserved. In order to make sense of a technique for solving a class of problems, it is vital to be aware of the range and scope of that class. Learners who focus their attention on getting through tasks as quickly and painlessly as possible do themselves an enormous disservice, for they hasten the moment when they decide that mathematics is too much for them, and they cut themselves off from the pleasure of creativity which mathematics affords, and which is necessary in order to use mathematical concepts and techniques flexibly.

In order to promote learners' encounters with mathematical creativity, teachers can look for opportunities for learners to make choices, to construct objects as examples, and to articulate generalisations of particulars. When exercises, problems, examples of concepts, and counter-examples to obvious conjectures always come from the teacher or the text, learners are cut off from access to the creative and constructive aspects of mathematics, which are the sources of pleasure and the desire to find out more. Learners are dis-empowered and it becomes even more difficult for learners to shift from mere assenting to full engagement. We have argued, above all, that by seeing and posing tasks as construction tasks, using whatever familiar mathematical objects and techniques are at hand, learners' whole attitude toward the learning of mathematics can be altered for the good.

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