Construction of q-ary Constant Weight Sequences using a Knuth-like Approach

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About this work

- Extension on “Encoding and decoding of balanced q-ary sequences using a Gray code prefix,” ISIT 2016.
- ”A Construction for Balancing Non-Binary Sequences Based on Gray Code Prefixes”, arXiv:1706.00852v1.
- Received the chancellor’s medal award for best master dissertation.
- Thanks to my advisor and external examiners.

Overview

Definitions

- Consider a q-ary information sequence $\boldsymbol{x} = x_0 x_1 x_2 \ldots x_{k-1}$, $x_i \in \{0, 1, \ldots, q-1\}$, of length k.
- Let the prefix that will be appended to $\boldsymbol{x}$ be of length r; and let the information and the prefix together be denoted by $\boldsymbol{c} = c_0 c_1 c_2 \ldots c_{k-1}$, $c_i \in \{0, 1, \ldots, q-1\}$, of length $n = k + r$.
- The weight of $\boldsymbol{c}$, $w(\boldsymbol{c})$ is defined as

$$w(\boldsymbol{c}) = \sum_{i=0}^{k-1} c_i.$$

- $\boldsymbol{c}$ is called constant weight (CW) sequence with weight, $w(\boldsymbol{c})$ and it is said to be balanced if $w(\boldsymbol{c}) = \beta_{n,q} = \frac{n(q-1)}{2}$.

Balancing of q-ary sequences

- It has been proven [1], that $\boldsymbol{x}$, can always be balanced by adding modulo q one sequence from a set of balancing sequences $\boldsymbol{b}(s,p) = b_1 b_2 \dots b_k$ generated as follows:

$$b_i = \begin{cases} s, & i-1 \geq p, \\ s+1 \pmod{q}, & i-1 < p, \end{cases} \quad \text{where} \quad \begin{cases} 0 \leq s \leq q-1, \\ 0 \leq p \leq k-1. \end{cases}$$

- Let z be the iterator through these balancing sequences, with $z = sk + p$, $0 \leq z \leq kq - 1$. $\boldsymbol{b}(s,p)$ and $\boldsymbol{b}(z)$ refers to the same.
- Let $\boldsymbol{y}$ denote the sequence after a balancing sequence is added, $\boldsymbol{y} = \boldsymbol{x} \oplus_q \boldsymbol{b}(z)$. At least one $\boldsymbol{b}(z)$ will lead to a balanced output $\boldsymbol{y}$.

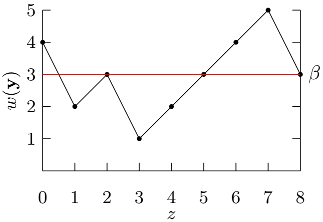
[1] T. G. Swart and J. H. Weber, "Efficient balancing of q-ary sequences with parallel decoding," in *Proc. IEEE Int. Symp. Inform. Theory*, Seoul, Korea, 2009.

Balancing of q-ary sequences (Cont'd)

Example 1

For $q = 3$, $k = 3$, consider the sequence $\boldsymbol{x} = 202$. The balancing value is $\beta_{k,q} = 3$.

z	$\boldsymbol{b}(z)$	$\boldsymbol{x} \oplus_q \boldsymbol{b}(z) = \boldsymbol{y}$	$w(\boldsymbol{y})$
0	000	$202 \oplus_3 000 = 202$	4
1	100	$202 \oplus_3 100 = 002$	2
2	110	$202 \oplus_3 110 = 012$	**3**
3	111	$202 \oplus_3 111 = 010$	1
4	211	$202 \oplus_3 211 = 110$	2
5	221	$202 \oplus_3 221 = 120$	**3**
6	222	$202 \oplus_3 222 = 121$	4
7	022	$202 \oplus_3 022 = 221$	5
8	002	$202 \oplus_3 002 = 201$	**3**



q-ary Gray Codes

- Invented by Frank Gray [2]; originally used to solve problems in pulse code communication; and extended to several other fields.
- $\boldsymbol{d} = d_1 d_2 \ldots d_{r'}$ denotes a sequence amongst the set of q-ary sequences of length r' listed in lexicographic order. They are mapped to Gray code sequences, $\boldsymbol{g} = g_1 g_2 \ldots g_{r'}$. Any two adjacent sequences differ in only one symbol position, with weight difference of either -1 or $+1$.
- *4-ary Gray code of length 2*

z	$\boldsymbol{d}$	$\boldsymbol{g}$	z	$\boldsymbol{d}$	$\boldsymbol{g}$	z	$\boldsymbol{d}$	$\boldsymbol{g}$	z	$\boldsymbol{d}$	$\boldsymbol{g}$
0	00	00	4	10	13	8	20	20	12	30	33
1	01	01	5	11	12	9	21	21	13	31	32
2	02	02	6	12	11	10	22	22	14	32	31
3	03	03	7	13	10	11	23	23	15	33	30

[2] F. Gray, "Pulse code communication," *U. S. Patent 2632058*, 1953.

Encoding and Decoding of q-ary Gray codes [3]

Gray code encoding algorithm The parity of the sum S_i of the first $i-1$ digits of $\boldsymbol{g}$ determines the Gray code symbols, where $2 \leq i \leq r'$ and $g_1 = d_1$, then

$$S_i = \sum_{j=1}^{i-1} g_j, \quad \text{and} \quad g_i = \begin{cases} d_i, & \text{if } S_i \text{ is even,} \\ q-1-d_i, & \text{if } S_i \text{ is odd.} \end{cases}$$

Gray code decoding algorithm

$$S_i = \sum_{j=1}^{i-1} g_j, \quad \text{and} \quad d_i = \begin{cases} g_i, & \text{if } S_i \text{ is even,} \\ q-1-g_i, & \text{if } S_i \text{ is odd.} \end{cases}$$

[3] D.-J. Guan, "Generalized Gray codes with applications," in *Proc. National Science Council, Republic of China, Part A*, 1998.

Applications of CW sequences

- They play an important role in communication system where high security and confidentiality are needed, because of various properties such as correlations, balanced value distributions and strong linear complexity.
- Frequency hopping in GSM networks.
- Detection of unidirectional errors and threshold setting in barcode implementations.
- DNA sequences (Biology field).
- In VLC (visible light communication), to eliminate flickering in CSK and performing dimming in FSK, OOK.

Research goal

- Construction of CW sequences through an efficient encoding and decoding scheme.

Encoding

Generating q-ary CW Sequences

- We want to construct an (n, k, W, q) CW sequence of length n, weight W with k information symbols.
- The length of Gray code prefix is, $r' = \log_q(kq) = \log_q(k) + 1$; such that cardinalities of the set of Gray code prefix and that of weighting sequences are equal.
- Lemma 1. For any q-ary information sequence $\boldsymbol{x}$ of length k, where parameters k and q are not coprime, we can find a $\boldsymbol{b}(z)$ such that the weight of $\boldsymbol{y} = \boldsymbol{x} \oplus_q \boldsymbol{b}(z)$ is $\omega_1 \leq w(\boldsymbol{y}) \leq \omega_2$, where $\omega_1 = \beta_{k,q} - (q-1)$ and $\omega_2 = \beta_{k,q} + (q-1)$.
- Theorem 1. An (n, k, W, q) CW sequence can be constructed from any q-ary information sequence $\boldsymbol{x}$ of length k where

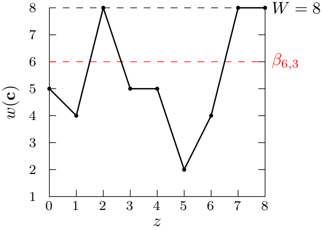
$$\frac{(k-2)(q-1)}{2} \leq W \leq \frac{(k+2r'+4)(q-1)}{2}. \quad (1)$$

Encoding (Cont'd)

Example 2

Encoding the ternary sequence $\boldsymbol{x} = 212$ into a CW sequence of weight $W = 8$. The condition $k = q^t$, is imposed and the Gray code prefix length is $r' = log_3 3 + 1 = 2$.

z	$\boldsymbol{x} \oplus_q \boldsymbol{b}(z) = \boldsymbol{y}$	$\boldsymbol{c} = [u \vert \boldsymbol{g} \vert \boldsymbol{y}]$	$w(\boldsymbol{c})$
0	$212 \oplus_3 000 = 212$	$\underline{\mathbf{0}00}212$	5
1	$212 \oplus_3 100 = 012$	$\underline{\mathbf{0}01}012$	4
2	$212 \oplus_3 110 = 022$	$\underline{\mathbf{2}02}022$	**8**
3	$212 \oplus_3 111 = 020$	$\underline{\mathbf{0}12}020$	5
4	$212 \oplus_3 211 = 120$	$\underline{\mathbf{0}11}120$	5
5	$212 \oplus_3 221 = 100$	$\underline{\mathbf{0}10}100$	2
6	$212 \oplus_3 222 = 101$	$\underline{\mathbf{0}20}101$	4
7	$212 \oplus_3 022 = 201$	$\underline{\mathbf{2}21}201$	**8**
8	$212 \oplus_3 002 = 211$	$\underline{\mathbf{0}22}211$	**8**



Encoding (Cont'd)

Generating q-ary CW Sequences with extended weight range

- Appending a redundant vector $\boldsymbol{u}$ of length e to $\boldsymbol{c}' = [\boldsymbol{g}|\boldsymbol{y}]$, then the output sequence becomes $\boldsymbol{c} = [\boldsymbol{u}|\boldsymbol{g}|\boldsymbol{y}]$. This leads to (n, k, W, q) CW sequences where $n = k + r' + e$.
- This will lead to an increase of weight range as $w(\boldsymbol{u}) \in [0, e(q-1)]$.
- Theorem 2. Any q-ary information sequence of length k can generate an (n, k, W, q) CW sequence where

$$\frac{(k-2)(q-1)}{2} < W < \frac{(k+2r'+2e+1)(q-1)}{2}. \tag{2}$$

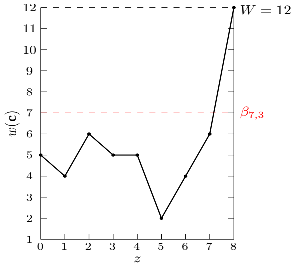
- The redundant vector $\boldsymbol{u} = u_1 u_2 \ldots u_e$ is such that $u_i \in \{0, 1, \ldots, q-1\}$ and $w(\boldsymbol{u}) = W - w(\boldsymbol{c}')$ if and only if $W \geq w(\boldsymbol{c}')$, otherwise $\boldsymbol{u} = \boldsymbol{0}$.

Encoding (Cont'd)

Example 3

Consider the same ternary information sequence $\boldsymbol{x} = 212$ of length 3 as in Example 2. We would like to generate a $(7, 3, 12, 3)$ CW sequence of weight $W = 12$ and $n = 7$.

z	$\boldsymbol{x} \oplus_q \boldsymbol{b}(z) = \boldsymbol{y}$	$\boldsymbol{c} = [\boldsymbol{u} \vert \boldsymbol{g} \vert \boldsymbol{y}]$	$w(\boldsymbol{c})$
0	$212 \oplus_3 000 = 212$	0000212	5
1	$212 \oplus_3 100 = 012$	0001012	4
2	$212 \oplus_3 110 = 022$	0002022	6
3	$212 \oplus_3 111 = 020$	0012020	5
4	$212 \oplus_3 211 = 120$	0011120	5
5	$212 \oplus_3 221 = 100$	0010100	2
6	$212 \oplus_3 222 = 101$	0020101	4
7	$212 \oplus_3 022 = 201$	0021201	6
8	$212 \oplus_3 002 = 211$	2222211	**12**



Range has been extended from [2, 10] to [2, 12].

Encoding (Cont'd)

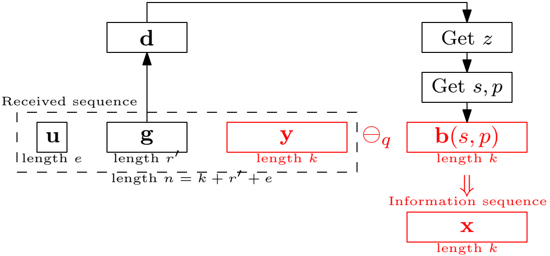
Parameters evaluation

- W was calculated according to equation (2).

	t	$k = q^t$	W	n	r'	e
$q = 2$	2	4	$[2, 8]$	10	3	3
	3	8	$[4, 12]$	16	4	4
	4	16	$[8, 16]$	24	5	3
$q = 3$	1	3	$[2, 11]$	7	2	2
	2	9	$[8, 21]$	15	3	3
	3	27	$[26, 41]$	34	4	3
$q = 4$	1	4	$[4, 18]$	8	2	2
	2	16	$[22, 42]$	22	3	3
	3	64	$[94, 120]$	72	4	4

Decoding

- The redundant vector $\boldsymbol{u}$ is dropped, then the r' symbols are extracted as the Gray code prefix and converted to corresponding iterator z.
- z is used to determine the parameters s and p, then $\boldsymbol{b}(s, p)$ can be derived.
- Finally, the original sequence is recovered through $\boldsymbol{x} = \boldsymbol{y} \ominus_q \boldsymbol{b}(s, p)$.



Decoding (Cont'd)

Decoding of $(2, 4)$-Gray code

Gray code ($\boldsymbol{g}$)	Sequence ($\boldsymbol{d}$)	z	s, p	$\boldsymbol{b}(s, p)$
00	00	0	$0, 0$	0000
01	01	1	$0, 1$	1000
02	02	2	$0, 2$	1100
03	03	3	$0, 3$	1110
13	10	4	$1, 0$	1111
12	11	5	$1, 1$	2111
11	12	6	$1, 2$	2211
10	13	7	$1, 3$	2221
20	20	8	$2, 0$	2222
21	21	9	$2, 1$	3222
22	22	10	$2, 2$	3322
23	23	11	$2, 3$	3332
33	30	12	$3, 0$	3333
32	31	13	$3, 1$	0333
31	**32**	**14**	$\mathbf{3, 2}$	**0033**
30	33	15	$3, 3$	0003

Decoding (Cont'd)

Example 4

Consider the decoding of the $(7, 4, 14, 4)$ CW sequence, $\underline{231}3113$.

- The redundant symbol $u = 2$ is dropped. Then the Gray code sequence of length 2, is extracted as 31.
- The Gray code $\boldsymbol{g} = 31$ corresponds to $\boldsymbol{d} = 32$, and index $z = 14$. This implies that $s = 3$ and $p = 2$, therefore $\boldsymbol{b}(3, 2) = 0033$ (presented in the previous table).
- Finally, the information sequence is recovered as

$$\boldsymbol{x} = \boldsymbol{y} \ominus_q \boldsymbol{b}(s, p) = 3113 \ominus_3 0033 = 3120.$$

Cardinality study

- $\mathcal{N}_1$ is the cardinality of q-ary CW sequences for specific W of length n and $\mathcal{N}_2$, the cardinality of q-ary information sequences of length k.
- To construct an (n, k, W, q) CW sequence, one clearly requires enough parity bits r such that $\mathcal{N}_1 \geq \mathcal{N}_2 = q^k$, where $n = k + r$.

W		q	n	k	$\mathcal{N}_1$	$\mathcal{N}_2$
$\beta_{n,q} - q + 1$	3	2	7	4	35	16
	5	2	12	8	792	256
	10	2	21	16	352716	65536
	3	3	5	3	30	27
	10	3	12	9	58278	19683
	6	4	6	4	336	256
$\beta_{n,q}$	4	2	7	4	35	16
	6	2	12	8	924	256
	11	2	21	16	352716	65536
	5	3	5	3	51	27
	12	3	12	9	737789	19683
	9	4	4	6	580	256
$\beta_{n,q} + q$	6	2	8	4	28	16
	9	2	13	8	715	256
	13	2	22	16	497420	65536
	9	3	6	3	50	27
	16	3	13	9	129844	19683
	15	4	7	4	728	256

Redundancy and complexity

- The redundancy is $r = \log_q k + e + 1 \Rightarrow k = q^{r-1-e}$.
- For $e = 1$, the redundancy becomes $r = \log_q k + 2$, which is similar as that presented in [4].
- The addition of the vector $\boldsymbol{u}$ does not change the complexity of this construction compared to the one in [4].
- This method requires $\mathcal{O}(qk \log_q k)$ digit operations for the encoding and $\mathcal{O}(k)$ digit operations for the decoding process.
- Comparison of our scheme with other constructions based on redundancy and complexity can be found in [4].

[4] E. N. Mambou and T. G. Swart, "Encoding and decoding of balanced q-ary sequences using a Gray code prefix," in *Proc. IEEE Int. Symp. Inform. Theory*, Barcelona, Spain, 2016.

Conclusion

- An efficient algorithm was proposed for encoding and decoding (n, k, W, q) CW sequences based on Gray code prefixes with a method to extend the achievable CW range.
- The construction does not make use of memory-consuming lookup tables, and only simple operations such as addition and subtraction are needed.
- The decoding process can be performed mostly in parallel.
- As the proposed method is only applicable to information sequences of length k where $k = q^t$, the improvement would be to extend this algorithm to the case where $k \neq q^t$.

Thanks for your attention!

"We cannot solve our problems with the same thinking we used when we created them." Albert Einstein

QUESTIONS AND COMMENTS!!

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NRF National Research Foundation