Math, Problem Set #2, Inner Product Spaces

Elysa Strunin

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3.1

i

$0.25(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2)$
$= 0.25(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle)$
$= 0.25(\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle - \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{y}\rangle)$
$= 0.25(4\langle \mathbf{x}, \mathbf{y}\rangle)$
$\langle \mathbf{x}, \mathbf{y}\rangle$

ii

$0.5(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2)$
$= 0.5(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle)$
$= 0.5(\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle)$
$= 0.5(2\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{y}, \mathbf{y}\rangle)$
$= \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$
$= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

3.2

$= 0.25(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}-i\mathbf{y}\|^2 - i\|\mathbf{x}+i\mathbf{y}\|^2)$
$= 0.25(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle + i\langle \mathbf{x}-i\mathbf{y}, \mathbf{x}-i\mathbf{y}\rangle - i\langle \mathbf{x}+i\mathbf{y}, \mathbf{x}+i\mathbf{y}\rangle)$
$= 0.25(\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle - \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{y}\rangle + i(\langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{x}, i\mathbf{y}\rangle + \langle i\mathbf{y}, i\mathbf{y}\rangle) - i(\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, i\mathbf{y}\rangle$
$+\langle i\mathbf{y}, i\mathbf{y}\rangle))$
$= 0.25(4\langle \mathbf{x}, \mathbf{y}\rangle + i(\langle \mathbf{x}, -i\mathbf{y}\rangle + \langle -i\mathbf{y}, \mathbf{x}\rangle) - i(\langle \mathbf{x}, i\mathbf{y}\rangle + \langle i\mathbf{y}, \mathbf{x}\rangle))$
$= 0.25(4\langle \mathbf{x}, \mathbf{y}\rangle + 0 + 0)$ because $\langle -i\mathbf{y}, \mathbf{x}\rangle) = \langle \mathbf{x}, i\mathbf{y}\rangle)$ etc
$\Rightarrow = \langle \mathbf{x}, \mathbf{y}\rangle$

3.3

i

$cos\theta = \frac{\langle x, x^5\rangle}{\|x\|\|x^5\|} = \frac{\langle x, x^5\rangle}{\sqrt{\langle x, x\rangle\langle x^5, x^5\rangle}} = \sqrt{33}/7$ by calculating the specified inner product
$\theta = arccos\sqrt{33}/7 = 0.608$

ii

$cos\theta = \frac{\langle x^2,x^4\rangle}{\|x^2\|\|x^4\|} = \frac{\langle x^2,x^4\rangle}{\sqrt{\langle x^2,x^4\rangle\langle x^2,x^4\rangle}} = \sqrt{45}/7$ by calculating the specified inner product
$\theta = arccos\sqrt{45}/7 = 0.290$

3.8

i

Inner product of each basis function and itself is 1:
$\langle cost, cost\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2tdt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+cos2t)dt = 1$ by double-angle formula
$\langle sint, sint\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2tdt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-cos2t)dt = 1$
$\langle cos2t, cos2t\rangle = 1$ by double-angle formula and change of variable
Similarly, $\langle sin2t, sin2t\rangle = 1$ by double-angle formula and change of variable
Inner product of each basis function and every other basis function is 0:
$\langle cost, sint\rangle = 0$ by change of variable
$\langle cos2t, sin2t\rangle = 0$ by change of variable
$\langle sint, sin2t\rangle = 0$ by double-angle formula and change of variable
$\langle cost, sin2t\rangle = 0$ by double-angle formula and change of variable $\langle cost, cos2t\rangle = 0$ by double-angle formula, change of variable, and periodic nature of cos^3t (s.t. area under the curve from $-\pi$ to π is zero)
$\langle sint, cos2t\rangle = 0$ by double-angle formula, change of variable, and periodic nature of sin^3t (s.t. area under the curve from $-\pi$ to π is zero)

ii

$\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2dt}$
$= \sqrt{2\pi^2/3} = \frac{\pi\sqrt{6}}{3}$

iii

$proj_X cos3t = \sum_{i=1}^{4}\langle x_i, cos3t\rangle x_i$
$= \langle cost, cos3t\rangle cost + \langle sint, cos3t\rangle sint +$ $\langle cos2t, cos3t\rangle cos2t + \langle sin2t, cos3t\rangle sin2t = 0$ because:
$\langle cost, cos3t\rangle cost$
$= \frac{cost}{\pi}\int_{-\pi}^{\pi} costcos3tdt = \frac{cost}{\pi}\int_{-\pi}^{\pi}(4cos^4t - 3cos^2t)dt$ by triple-angle formula
$= \frac{cost}{\pi}\int_{-\pi}^{\pi} cos^2t(4cos^2t-3)dt = \frac{cost}{\pi}\int_{-\pi}^{\pi}(0.5cos4t + 0.5cos2t)dt$ by double-angle formula
$= 0$ by change of variable

$\langle sint, cos3t\rangle sint$
$= \frac{sint}{\pi}\int_{-\pi}^{\pi}(4sintcos^3t - 3sintcost)dt = 0$ by change of variable
(reduces to area under a single point)

$\langle cos2t, cos3t\rangle cos2t$
$= \frac{cos2t}{\pi}\int_{-\pi}^{\pi}(8cos^5t - 7cos^3t + 3sin^2tcost)dt$ by double- and triple-angle formulas
$= \frac{cos2t}{\pi}\int_{-\pi}^{\pi}(3sin^2tcost)dt$ periodic nature of cosˆ3t and cosˆ5t
$= 0$ by change of variable (reduces to a single point)

$\langle sin2t, cos3t\rangle sin2t = \frac{sin2t}{\pi}\int_{-\pi}^{\pi}(8sintcos^4t - 6sintcos^2t)dt$ by double- and triple-angle formulas
$= 0$ by change of variable (reduces to a single point)

iv

$proj_X cost = \sum_{i=1}^{4} \langle x_i, t \rangle x_i$
$= \langle cost, t\rangle cost + \langle sint, t\rangle sint +$ $\langle cos2t, t\rangle cos2t + \langle sin2t, t\rangle sin2t = 2sint + sin2t$ because:

$\langle cost, t\rangle cost$
$= \frac{cost}{\pi} \int_{-\pi}^{\pi} (tcost)dt$
$= \frac{cost}{\pi} ([tsint]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} sintdt)$
$= 0$

$\langle sint, t\rangle sint$
$= \frac{sint}{\pi} \int_{-\pi}^{\pi} (tsint)dt = \frac{sint}{\pi}([tcost]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} costdt) = \frac{2sint\pi}{\pi} + 0 = 2sint$

$\langle cos2t, t\rangle cos2t$
$= \frac{cos2t}{\pi} \int_{-\pi}^{\pi} (tcos2t)dt = \frac{cos2t}{\pi}([0.5tsin2t]_{-\pi}^{\pi} - 0.5 \int_{-\pi}^{\pi} sin2tdt) = 0$

$\langle sin2t, t\rangle sin2t$
$= \frac{sin2t}{\pi} \int_{-\pi}^{\pi} (tsin2t)dt = \frac{sin2t}{\pi}([0.5tsin2t]_{-\pi}^{\pi} - 0.5 \int_{-\pi}^{\pi} cos2tdt) = \frac{\pi sin2t}{\pi} = sin2t$

3.9

Orthonormal transformation if for every $\mathbf{x}, \mathbf{y} \in R^2$ have $\langle \mathbf{x}, \mathbf{y}\rangle_{R^2} = \langle L(\mathbf{x}), L(\mathbf{y})\rangle_{R^2}$
$\langle \mathbf{x}, \mathbf{y}\rangle_{R^2} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2$
Using the standard rotation matrix in R^2: $L(\mathbf{x}) =$

$$\begin{pmatrix} x_1 cos\theta - x_2 sin\theta \\ x_1 sin\theta + x_2 cos\theta \end{pmatrix}$$

$\langle L(\mathbf{x}), L(\mathbf{y})\rangle = L(\mathbf{x})^T L(\mathbf{y})$
$= (x_1 cos\theta - x_2 sin\theta)(y_1 cos\theta - y_2 sin\theta)$ $+(x_1 sin\theta + x_2 cos\theta)(y_1 sin\theta + y_2 cos\theta)$
$= x_1 y_1 (cos^2\theta + sin^2\theta) + x_2 y_2 (cos^2\theta + sin^2\theta)$
$= x_1 y_1 + x_2 y_2$

3.10

i

$Q^H Q = QQ^H = I \Rightarrow$ Q is an orthonormal matrix:
Take any $\mathbf{x}, \mathbf{y} \in F^n$:
$\langle Q\mathbf{x}, Q\mathbf{y}\rangle = (Q\mathbf{x})^H (Q\mathbf{y}) = \mathbf{x}^H (Q^H Q) y = \mathbf{x}^H \mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle$
Q is an orthonormal matrix $\Rightarrow Q^H Q = QQ^H = I$:
Orthonormal matrix implies that:
$\langle Q\mathbf{x}, Q\mathbf{y}\rangle = \mathbf{x}^H (Q^H Q) y = \mathbf{x}^H \mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle$
Therefore $(Q^H Q) = I$ and $\Rightarrow (QQ^H)Q = QI = Q \Rightarrow QQ^H = I$

ii

Q is orthonormal matrix: $\|Q\mathbf{x}\| = \sqrt{\langle Q\mathbf{x}, Q\mathbf{x}} = \sqrt{\langle \mathbf{x}, \mathbf{x}}$ because orthonormal $= \|\mathbf{x}\|$

iii

Have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle Q\mathbf{x}, Q\mathbf{y} \rangle$
$\langle Q^{-1}\mathbf{x}, Q^{-1}\mathbf{y} \rangle = \langle Q^{-1}Q\mathbf{x}, Q^{-1}Q\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \Rightarrow Q^{-1}$ is orthonormal

iv

From (i), have $Q^H Q = I$
Consider the columns of Q and the rows of Q^H (i.e., the columns of Q again)
Clear from I that the columns of Q are an orthonormal set and fulfill the δ_{ij} definition of orthonormal

v

For orthonormal matrix Q:
$1 = det(I_n) = det(Q^H Q) = det(Q^H)det(Q) = (detQ)^2$ since $det(Q) = det(Q^H)$
$\Rightarrow |det(Q)| = 1$
No, the converse is not true. Counterexample: consider the following matrix A, $det(A) = 1$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Clearly A is not orthonormal.

vi

Have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle Q_2\mathbf{x}, Q_2\mathbf{y} \rangle$
$\langle Q_1\mathbf{x}, Q_1\mathbf{y} \rangle = \langle Q_1Q_2\mathbf{x}, Q_1Q_2\mathbf{y} \rangle$
$= \langle \mathbf{x}, \mathbf{y} \rangle$ because Q_1 is orthonormal
$\Rightarrow Q_1Q_2$ is orthonormal also

3.11

Take linearly dependent vectors $\mathbf{x_1}, \mathbf{x_2}$
Apply Gram-Schmidt:
Normalize $\mathbf{x_1} \Rightarrow \mathbf{q_1}$
Projecting $\mathbf{x_2}$ onto $\mathbf{q_1}$ yields $\mathbf{x_2}$ because the two vectors are linearly dependent
$\mathbf{q_2} = \frac{\mathbf{x_2} - \mathbf{x_2}}{\|\mathbf{x_2} - \mathbf{x_2}\|} = 0/0$, which is problematic

3.16

i

From the QR decomposition, $A = QR$
Consider $A = (QD)(D^{-1}R) = QR$
Must show (1) and (2):
(1) That QD is orthonormal and distinct from Q:
Clear that for each column $\mathbf{q_n}$ of Q, and each diagonal element d_n of D,
each column of QD can be written as $d_n\mathbf{q_n}$; i.e., column $\mathbf{q_n}$ scaled by the corresponding diagonal element

Know that $\langle \mathbf{q_i}, \mathbf{q_j} \rangle = 0$ for $i \neq j$; clearly this holds regardless of scalars d_i, d_j
Know that $\langle \mathbf{q_i}, \mathbf{q_j} \rangle = 1$ for $i = j$
Want $\langle d_i \mathbf{q_i}, d_j \mathbf{q_j} \rangle = 1$ for $i = j$
$\Rightarrow$ Can have $|d_i| = 1$
So with the right selection of $d_i = 1, -1 (D \neq I$ is required), can construct a QD that is orthonormal and distinct from Q
(2) That $D^{-1}R$ is upper triangular:
Clear that it is upper triangular

ii

With respect to columns $\mathbf{q_n}$ of Q and rows $\mathbf{r_n}$ of R:
Can scale $\mathbf{q_k}$ by α and $\mathbf{r_k}$ by $1/\alpha$ without changing QR.
$\Rightarrow$ Can continue obtaining different QR factorizations by multiplying column k of Q by -1 (since it doesn't change the magnitude of the column vector) and row k of Q by -1.
But the positive diagonal of R eliminates this opportunity and ensures uniqueness.
A must be invertible because a noninvertible A would imply ≥ 1 zero diagonal element of R, which would be unaffected by the scalar multiplication.

3.17

Solving $\mathbf{x} = (A^H A)^{-1} A^H \mathbf{b}$ is equivalent to solving $(R^H R)\mathbf{x} = (R^H R)(A^H A)^{-1} A^H \mathbf{b}$
Likewise, solving $R\mathbf{x} = Q^H \mathbf{b}$ is equivalent to solving $R^H R\mathbf{x} = R^H Q^H \mathbf{b}$
Show: $(R^H R)(A^H A)^{-1} A^H = R^H Q^H$ by using $A = QR$
$(R^H R)(A^H A)^{-1} A^H$
$= (R^H R)(R^H Q^H Q R)^{-1} R^H Q^H$
$= (R^H R)(R^H R)^{-1} R^H Q^H$
$= R^H Q^H$

3.23

For any $\mathbf{x}, \mathbf{y} \in V$:
Consider a triangle with the following side lengths: $\|\mathbf{x}\|, \|\mathbf{y}\|, \|\mathbf{x} - \mathbf{y}\|$
Apply triangle inequality:
$\|\mathbf{x}\| \leq \|\mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\|$
$\Rightarrow \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$
Apply triangle inequality:
$\|\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\|$
$\Rightarrow \|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|$
So $\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \ \forall \mathbf{x}, \mathbf{y} \in V$

3.24

i

Positivity: $\int_a^b |f(t)| dt \geq 0$ because absolute value
Scale Preservation: $\|\alpha f\|_{L^1} = \int_a^b |\alpha f(t)| dt = |\alpha| \int_a^b |f(t)| dt$

Triangle Inequality: $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$ because:
$\int_a^b |f(t)+g(t)|dt \leq \int_a^b |f(t)|dt + \int_a^b |g(t)|dt$ because absolute value

ii

Positivity: $(\int_a^b |f(t)|^2 dt)^{1/2} \geq 0$ because absolute value squared
Scale Preservation: $\|\alpha f\|_{L^2} = (\int_a^b |\alpha f(t)|^2 dt)^{1/2} = (|\alpha| \int_a^b |f(t)|^2 dt)^{1/2}$
Triangle Inequality: $\|f+g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$ because:
$(\int_a^b |f(t)+g(t)|^2 dt)^{1/2} \leq (\int_a^b |f(t)|^2 dt)^{1/2} + (\int_a^b |, g(t)|^2 dt)^{1/2}$ Because triangle inequality applies to absolute value, and the other transformations following absolute value are monotonic

iii

Positivity: $sup_{x\in[a,b]}|f(x)| \geq 0$ because absolute value
Scale Preservation: $\|\alpha f\|_{L^{\inf}} = sup_{x\in[a,b]}|\alpha f(x)| = |\alpha| sup_{x\in[a,b]}|f(x)|$ because α is not a function of x
Triangle Inequality: $sup_{x\in[a,b]}|f(x)+g(x)| \leq sup_{x\in[a,b]}|f(x)| + sup_{x\in[a,b]}|g(x)|$ because it is clear that LHS $<$ RHS, unless the same choice of x maximizes both functions, AND both functions are ≥ 0.

3.26

$\|\cdot\|_a, \|\cdot\|_b$ are T.E. if $\exists\ 0 < m \leq M$ s.t. $m\|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq M\|\mathbf{x}\|_a, \forall \mathbf{x} \in X$
Topological equivalence is an equivalence relation:
Reflexive:
$\|\cdot\|_a, \|\cdot\|_a$ are T.E.:
Clear that $\forall\ 0 < m \leq 1 \leq M$,
$m\|\mathbf{x}\|_a \leq \|\mathbf{x}\|_a \leq M\|\mathbf{x}\|_a, \forall \mathbf{x} \in X$
Symmetric:
$\|\cdot\|_a, \|\cdot\|_b$ are T.E. $\Rightarrow \|\cdot\|_b, \|\cdot\|_a$ are T.E:
$\|\cdot\|_a, \|\cdot\|_b$ are T.E. $\Rightarrow \frac{1}{M}\|\mathbf{x}\|_b \leq \|\mathbf{x}\|_a \leq \frac{1}{m}\|\mathbf{x}\|_b, \forall \mathbf{x} \in X$ Clear that $0 < 1/M \leq 1/m$
Transitive:
$\|\cdot\|_a, \|\cdot\|_b$ are T.E. (with m_1, M_1) and $\|\cdot\|_b, \|\cdot\|_c$ are T.E. (with m_2, M_2) $\Rightarrow \|\cdot\|_a, \|\cdot\|_c$ are T.E.:
Clear that $\frac{1}{M_1 M_2}\|\mathbf{x}\|_c \leq \|\mathbf{x}\|_a \leq \frac{1}{m_1 m_2}\|\mathbf{x}\|_c, \forall \mathbf{x} \in X$
$\Rightarrow \|\cdot\|_c, \|\cdot\|_a$ are T.E. $\Rightarrow \|\cdot\|_a, \|\cdot\|_c$ are T.E. by symmetric property

The stated p-norms are T.E. via the following and the transitive property:

i

ii