Reasoning in Propositional Logic

Contents

Natural Deduction

- **Consider the following argument stated in natural language:**
  - Today is Tuesday or Wednesday
  - The doctor office is open today
  - If it is Wednesday, then the doctor office is closed.
  - Therefore, Today is Tuesday

Formalizing this logical argument into PL:

$$\frac{\begin{array}{c} t \vee w \\ d \\ w \rightarrow \neg d \end{array}}{t}$$

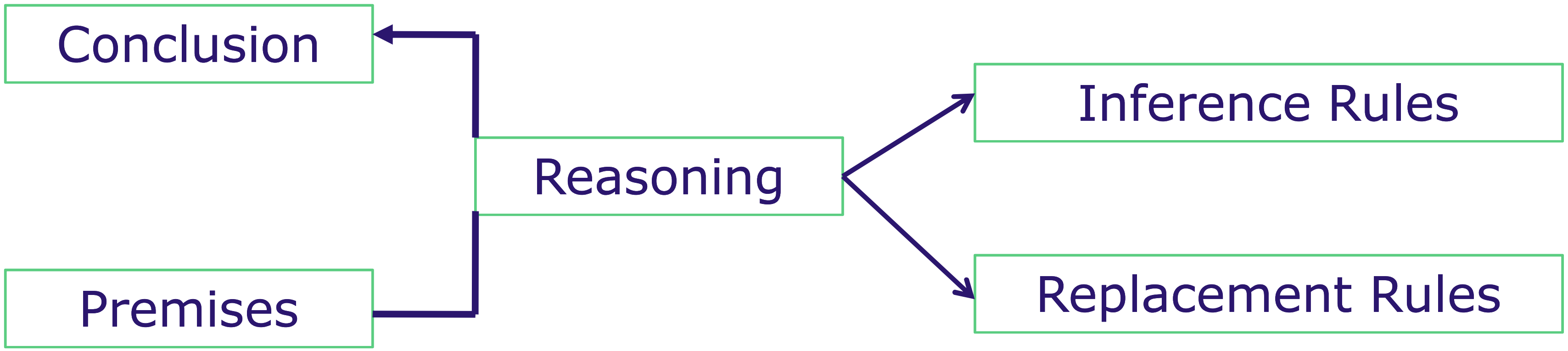
Natural Deduction

❖ **Chain of reasoning from premises to conclusion:**

1. Today is Tuesday or Wednesday (Premise)
2. The doctor office is open today (Premise)
3. If it is Wednesday, then the doctor office is closed (Premise)
4. Today is not Wednesday (from 2,3)
5. Today is Tuesday (from 1,4)

We need reasoning procedure in PL to simulate this human reasoning

Reasoning Procedures in PL



Inference Rules:

1. Modus ponens (MP)
2. Modus tollens (MT)
3. Disjunctive syllogism (DS)
4. Addition (Add.)
5. Simplification (Simp.)
6. Conjunction (Conj.)
7. Hypothetical syllogism (HS)
8. Constructive dilemma (CD)
9. Absorption (Abs.)

Inference Rules

❖ Modus ponens (MP)

$$\frac{\begin{array}{c}\alpha \rightarrow \beta \\ \alpha\end{array}}{\beta}$$

It is also called the rule of "→-elimination".

❖ Modus tollens (MT)

$$\frac{\begin{array}{c}\alpha \rightarrow \beta \\ \neg\beta\end{array}}{\neg\alpha}$$

It is also called the rule of "→-elimination".

❖ Disjunctive syllogism (DS)

$$\frac{\begin{array}{c}\alpha \vee \beta \\ \neg\alpha\end{array}}{\beta}$$

It is also called the rule of :"v-elimination".

Inference Rules

❖ Addition (Add.)

$$\frac{\alpha}{\alpha \vee \beta}$$

It is also called the rule:"v-Introduction".

❖ Simplification (Simp.)

$$\frac{\alpha \ \& \ \beta}{\alpha} \qquad\qquad \frac{\alpha \ \& \ \beta}{\beta}$$

It is also called the rule:"&-elimination".

Inference Rules

❖ Conjunction (Conj.)

$$\frac{\begin{array}{c}\alpha \\ \beta\end{array}}{\alpha \ \& \ \beta}$$

It is also called the rule:"&- introduction".

❖ Hypothetical syllogism (HS)

$$\frac{\alpha \rightarrow \beta \qquad \beta \rightarrow \gamma}{\alpha \rightarrow \gamma}$$

It is also called the rule:"chain deduction".

❖ Constructive dilemma (CD)

$$\frac{(\alpha \rightarrow \gamma) \,\&\, (\beta \rightarrow \delta) \qquad \alpha \vee \beta}{\gamma \vee \delta}$$

❖ Absorption (Abs.)

$$\frac{\alpha \rightarrow \beta}{\alpha \rightarrow (\alpha \,\&\, \beta)}$$

Rules of replacement

1. Double negation (DN)
2. Commutativity (Com.)
3. Associativity (Assoc.)
4. Tautology (Taut.)
5. DeMorgan's Laws (DM)
6. Transposition (Trans.)
7. Material Implication (Impl.)
8. Distribution (Dist.)
9. Material Equivalence (Equiv.)

Rules of replacement

Double negation (DN):

$\neg\neg\alpha$ is interreplaceable with α

Commutativity (Com.):

$\alpha \,\&\, \beta$ is interreplaceable with $\beta \,\&\, \alpha$

$\alpha \vee \beta$ is interreplaceable with $\beta \vee \alpha$

Associativity (Assoc.):

$(\alpha \,\&\, \beta) \,\&\, \gamma$ is interreplaceable with $\alpha \,\&\, (\beta \,\&\, \gamma)$

$(\alpha \vee \beta) \vee \gamma$ is interreplaceable with $\alpha \vee (\beta \vee \gamma)$

Rules of replacement

Tautology (Taut.):

α is interreplaceable with $\alpha \,\&\, \alpha$

α is interreplaceable with $\alpha \vee \alpha$

DeMorgan's Laws (DM):

$\neg(\alpha \,\&\, \beta)$ is interreplaceable with $\neg \alpha \vee \neg\beta$

$\neg(\alpha \vee \beta)$ is interreplaceable with $\neg \alpha \,\&\, \neg\beta$

Transposition (Trans.):

$\alpha \rightarrow \beta$ is interreplaceable with $\neg\beta \rightarrow \neg \alpha$

Rules of replacement

Material Implication (Impl.):

$\alpha \rightarrow \beta$ is interreplaceable with $\neg \alpha \vee \beta$

Material Equivalence (Equiv.):

$\alpha \leftrightarrow \beta$ is interreplaceable with $(\alpha \rightarrow \beta) \,\&\, (\beta \rightarrow \alpha)$

Distribution (Dist.):

$\alpha \,\&\, (\beta \vee \gamma)$ is interreplaceable with $(\alpha \,\&\, \beta) \vee (\alpha \,\&\, \gamma)$

$\alpha \vee (\beta \,\&\, \gamma)$ is interreplaceable with $(\alpha \vee \beta) \,\&\, (\alpha \vee \gamma)$

(1) Direct Deductions (DD)

A direct deduction of a conclusion from a set of premises consists of an ordered sequence of wffs such that each one is either

(1) a premise,

(2) derived from previous members of the sequence by one of the inference rules,

(3) derived from a previous member of the sequence by the replacement rules,

(4) The conclusion is the final step of the sequence

(1) Direct Deductions

- Applying direct deduction on this argument:

$$\begin{array}{c} t \vee w \\ d \\ w \rightarrow \neg d \\ \hline t \end{array}$$

1) $t \vee w$	Premise
2) d	Premise
3) $w \rightarrow \neg d$	Premise
4) $\neg w$	2,3 MT
5) t	1,4 DS #

(1) Direct Deductions

Example_2

$$\begin{array}{c} c \vee d \\ c \rightarrow o \\ d \rightarrow m \\ \neg o \\ \hline m \end{array}$$

1) $c \vee d$	Premise
2) $c \rightarrow o$	Premise
3) $d \rightarrow m$	Premise
4) $\neg o$	Premise
5) $\neg c$	2,4 MT
6) d	1,5 DS
7) m	2,6 MP #

(1) Direct Deductions

- Example_2 :
- There is no unique derivation for a given conclusion from a given set of premises.

1) $c \vee d$	Premise
2) $c \rightarrow o$	Premise
3) $d \rightarrow m$	Premise
4) $\neg o$	Premise
5) $(c \rightarrow o)$ & $(d \rightarrow m)$	2,3 Conj.
6) $o \vee m$	1,5 CD
7) m	4,6 DS #

(2) Conditional proof (CP)

- A conditional proof is a derivation technique used to establish a conditional wff, i.e., a wff whose main operator is the sign '→'.

- This is done by constructing a sub-derivation within a derivation in which the antecedent of the conditional is assumed as a hypothesis.

- If, by using the inference rules and rules of replacement, it is possible to arrive at the consequent, it is permissible to end the sub-derivation and conclude the truth of the conditional statement within the main derivation.

(2) Conditional proof

$$p \rightarrow (q \lor r)$$
$$p \rightarrow \neg s$$
$$\underline{s \leftrightarrow q}$$
$$p \rightarrow r$$

Establish the validity of this argument using CP

(2) Conditional proof

1) $p \rightarrow (q \vee r)$	Premise
2) $p \rightarrow \neg s$	Premise
3) $s \leftrightarrow q$	Premise
4) p	Assumption
5) $q \vee r$	1,4 MP
6) $\neg s$	2,4 MP
7) $(s \rightarrow q)$ & $(q \rightarrow s)$	3 Equiv.
8) $q \rightarrow s$	7 Simp.
9) $\neg q$	6,8 MT
10) r	5,9 DS
11) $p \rightarrow r$	4-10 CP #

(3) Indirect proof (IP)

- In an indirect proof ('IP' for short), our goal is to demonstrate that a certain wff is false on the basis of the premises.

- Again, we make use of a sub-derivation; here, we begin by assuming the opposite of that which we're trying to prove.
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- If on the basis of this assumption, we can demonstrate an obvious contradiction, i.e., a statement of the form $\alpha \,\&\, \neg\, \alpha$, we can conclude that the assumed statement must be false, because anything that leads to a contradiction must be false.

(3) Indirect proof

For example, consider the following argument:

$$\begin{array}{c} p \rightarrow q \\ \underline{p \rightarrow (q \rightarrow \neg p)} \\ \neg p \end{array}$$

(3) Indirect proof

1) $p \rightarrow q$	Premise
2) $p \rightarrow (q \rightarrow \neg p)$	Premise
3) p	Assumption
4) q	1,3 MP
5) $q \rightarrow \neg p$	2,3 MP
6) $\neg p$	4,5 MP
7) $p \,\&\, \neg p$	3,6 Conj.
8) $\neg p$	3-7 IP #

Assignment_2

Check the validity for the following Inference Rule:

Modus ponens (MP)

$$\frac{\alpha \rightarrow \beta \qquad \alpha}{\beta}$$

Using

a) Truth table Method

b) Direct proof Method


Thank You !