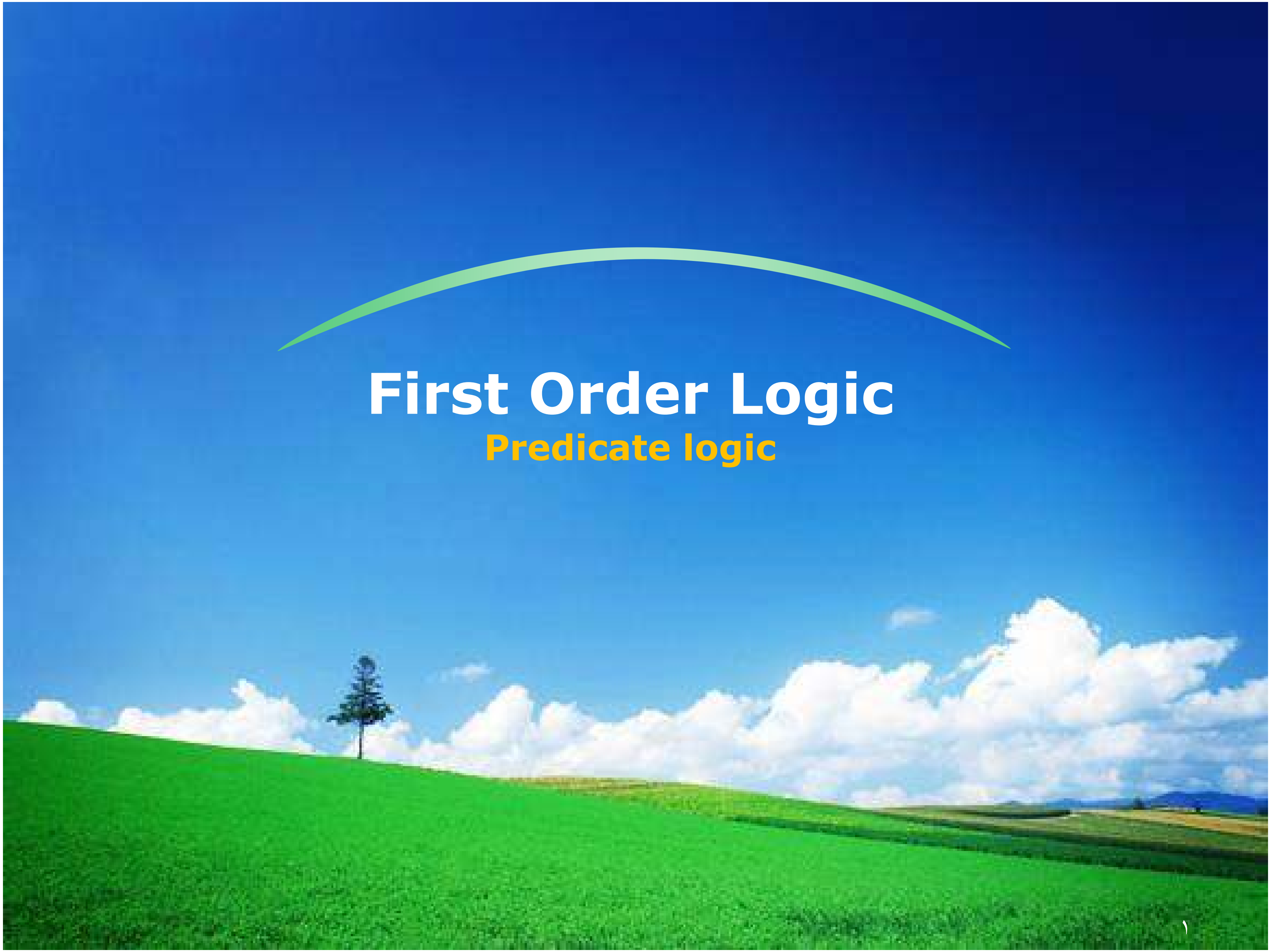
First Order Logic
Predicate logic

Contents

PL drawbacks

- **PL can not deal with variables**

  The assertion "x is greater than 1", where x is a variable, is not a proposition because you can not tell whether it is true or false unless you know the value of x. Thus the propositional logic can not deal with such sentences.

- **PL can not represent all logical patterns**

  The pattern involved in the following logical equivalences can not be captured by the propositional logic, EX:

  "Not all birds fly" is equivalent to "Some birds don't fly“

PL drawbacks

- **To solve PL drawbacks we introduce two new features into PL:**

  - ✓ **Predicates**
  - ✓ **Quantifiers**

Predicate

- **A predicate:** is a verb phrase template that describes a property of objects, or a relationship among objects represented by variables , EX:
  - The sky is blue
  - The cover of book is blue

- The phrase "is blue" is a predicate and it describes the property of being blue.
  - blue(sky)
  - b(book's cover)

Predicate

- **A predicate:** is a verb phrase template that describes a property of objects, or a relationship among objects represented by variables , EX:
  - Ahmed is the parent of Ali
  - Ahmed give book to Ali
- The phrase “parent" and the phrase “give ” are predicates describe a relationship between objects
  - parent(ahmed, ali)
  - give(ahmed, book, ali)

Predicate

- A predicate with variables called an atomic formula. P(X)

- Every atomic formula has a degree (arity).

Predicates with Connectives

- If John is a freshman, then tom is a freshman

  $$Fresh(john) \rightarrow Fresh(tom)$$

- john is not a Freshman

  $$\neg Fresh(john)$$

- neither john nor tom is a Freshman

  $$\neg Fresh(john) \& \neg Fresh(tom)$$

Quantifiers

- **The universe of discourse is the set of objects of interest**
- **The universe is the domain of the (individual) variables**
- **It can be the set of real numbers, the set of integers, the set of all cars on a parking lot, the set of all students in a classroom, etc..**

Universal Quantifier

- The expression: $\forall x\ P(X)$, denotes the universal quantification of the atomic formula P(X)

- $\forall$ is called the universal quantifier,

- $\forall x$ means all the objects x in the universe.

- It mean that P(X) is true for **every** object x in the universe.

Universal Quantifier Vs. & (and)

- **If all the elements in the universe of discourse can be listed then the universal quantification $\forall x\, P(X)$ is equivalent to the conjunction:**

$$P(x_1)\ \&\ P(x_2)\ \&\ P(x_3)\ \&\ \ldots\ P(x_n)$$

Existential Quantifier

- The expression: $\exists x P(X)$, denotes the existential quantification of the atomic formula P(X).

- $\exists$ is called the existential quantifier,

- $\exists x$ means at least one object x in the universe.

- It mean that P(X) is true for **at least one** object x of the universe.

Existential Quantifier Vs. v (OR)

- **If all the elements in the universe of discourse can be listed, then the existential quantification $\exists xP(X)$ is equivalent to the disjunction:**

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots P(x_n)$$

How to read quantified formulas

P(X) denote: "x is Perfect"

- **$\forall x\ P(X)$ translated to:**
  - "For every object ***x*** : ***x*** is perfect
  - "Every thing is perfect"
- **$\exists x\ P(X)$ translated to:**
  - "For some object **x**: **x** is perfect
  - " Some thing is perfect"
- **$\forall\ x \neg P(X)$ translated to:**

  " Every things is not perfect"

How to read quantified formulas

P(X) denote: "x is Perfect"

- **$\neg\forall x\, P(X)$ translated to:**

  "Not every thing is perfect"

- **P(book) translated to:**

  "this book is perfect"

Quantifiers

How to read quantified formulas

L(X,Y) denote: "X likes Y"

- **$\forall x\ \forall y$ L(X, Y) translated to:**

  " Every one Like every one"

- **$\forall x$ L(X, ali) translated to:**

  " Every one Like ali"

- **$\forall x\ \exists y$ L(X, Y) translated to:**

  " Every one Like some one"

Quantifiers

How to read quantified formulas

- " **Ahmed like every one who Ali like "**

  $\forall x\ [L(ali,X) \rightarrow L(ahmed,X)]$

- " **Ahmed like every one who like him"**

  $\forall x\ [L(X,ahmed) \rightarrow L(ahmed,X)]$

- " **Ahmed like some one who like him"**

  $\exists\ x\ [L(ahmed,X)\ \&\ L(X,ahmed)]$

❖ **FOL syntax rules:**

1. **Every atomic formula is wff**
2. If α is a well-formed formula, then so is [α]
3. If α is a well-formed formula, then so is ¬α
4. If α and β are well-formed formulas, then:

   $\alpha \,\&\, \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$:

   are wffs.
5. **If x is a variable and α is well-formed formula, then:**

   $\forall x\, \alpha$, $\exists x\, \alpha$ **are wffs**
6. Nothing else is a wff

Evaluation

- Suppose we have two predicates $p(x)$, $q(x,y)$, and the universe of discourse is the set $\{a,b\}$
- Suppose the following truth assignment:

$p(a)$: T	$p(b)$: F
$q(a,a)$: T	$q(b,b)$: F
$q(a,b)$: F	$q(b,a)$: T

- The truth assignment for the wff: $\forall \mathbf{x}\ [\mathbf{p(X)} \rightarrow \mathbf{q(X,X)}]$ is:

$$[p(a) \rightarrow q(a,a)\ \&\ p(b) \rightarrow q(b,b)]$$
$$[(T \rightarrow T)\ \&\ (F \rightarrow F)]$$
$$[\ T\ \&\ (F \rightarrow F)]$$
$$[\ T\ \&\ T]$$
$$T$$

First order logic (FOL)
Semantic

Evaluation

- Suppose we have two predicates p(x), q(x,y), and the universe of discourse is the set {a,b}
- Suppose the following truth assignment:

p(a): T	p(b): F
q(a,a): T	q(b,b): F
q(a,b): F	q(b,a): T

- The truth assignment for the wff: **$\forall$x $\exists$y q(X,Y)** is:

$$[\exists y\ q(a,Y)\ \&\ \exists y\ q(b,Y)\ ]$$
$$[\ [q(a,a) \vee q(a,b)]\ \&\ [q(b,a) \vee q(b,b)]\ ]$$
$$[\ [T \vee F]\ \&\ [T \vee F]\ ]$$
$$[\ T\ \&\ [T \vee F]\ ]$$
$$[\ T\ \&\ T\ ]$$
$$T$$

First order logic (FOL)
Semantic

Satisfaction

- Suppose we have two predicates p(x), q(x), and the universe of discourse is the set {a,b}.
- The truth table for the wffs:

p(a) ∨ p(b)

∃x q(X)

∀x [p(X) → q(X)]

$p(a)$	$p(b)$	$q(a)$	$q(b)$	$p(a) \vee p(b)$	$\forall x.(p(x) \Rightarrow q(x))$	$\exists x.q(x)$
1	1	1	1	1	1	1
1	1	1	0	1	0	1
1	1	0	1	1	0	1
1	1	0	0	1	0	0
1	0	1	1	1	1	1
1	0	1	0	1	1	1
1	0	0	1	1	0	1
1	0	0	0	1	0	0
0	1	1	1	1	1	1
0	1	1	0	1	0	1
0	1	0	1	1	1	1
0	1	0	0	1	0	0
0	0	1	1	0	1	1
0	0	1	0	0	1	1
0	0	0	1	0	1	1
0	0	0	0	0	1	0

Truth table size:

$$2^{m*n^k}$$

For a language with *n* constants and *m* relation of arity *k*

First order logic (FOL)

- A wff is said to be satisfiable if there exists at least one assignment makes it true.

- A wff is valid if it is true for all possible assignments.

- A wff is unsatisfiable if it is false for all possible assignments.

Reasoning in FOL

❖ Axioms of predicate logic

- Universal Instantiation (UI)
- Universal Generalization (UG)
- Existential Instantiation (EI)
- Existential Generalization (EG)
- Negation of quantified statement

❖ Axioms of predicate logic

1. **Universal Instantiation** (UI):

$\forall x\, P(x)$

$P(c)$

where c is some arbitrary element of the universe.

Axioms of predicate logic

2. **Universal Generalization** (UG):

$$\frac{P(c)}{\forall x\, P(x)}$$

where *P(c)* holds for every element c of the universe of discourse.

❖ Axioms of predicate logic

3. **Existential Instantiation (EI):**

$\exists x\ P(x)$

$P(c)$

where *c* is some element of the universe of discourse. It is not arbitrary but must be one for which P(c) is true.

❖ Axioms of predicate logic

4. **Existential Generalization:** (EG):

$$\frac{P(c)}{\exists x\, P(x)}$$

where *c* is an element of the universe.

❖ Axioms of predicate logic

Negation of quantified statement

- $\neg \forall x P(x) \leftrightarrow \exists x \neg P(x)$
- $\neg \exists x P(x) \leftrightarrow \forall x \neg P(x)$

Reasoning in FOL

Prove the validity for the following argument:

$$\frac{\forall x[P(X)\&G(X)]}{\forall x P(X)\ \&\ \forall x\ G(X)}$$

- **Predicates :**
  - **P(x): x is Positive**
  - **G(x): x is greater than zero**
  - **The universe is the set {1,2,3}**

Reasoning in FOL

$$\frac{\forall x[P(X)\&G(X)]}{\forall x P(X) \& \forall x\, G(X)}$$

1) $\forall x[P(X)\&G(X)]$	Premise
2) $P(1)\&G(1)$	3 UI
3) $P(1)$	2 Simpl.
4) $G(1)$	2 Simpl.
5) $\forall x P(X)$	3 UG
6) $\forall x\, G(X)$	4 UG
7) $\forall x P(X) \& \forall x\, G(X)$	5,6 Conj #

Reasoning in FOL

consider the following argument:

$$C(\text{this\_check})$$
$$\neg T(\text{this\_check})$$
$$\forall x\ [\ [C(X) \& \neg T(X)] \rightarrow V(X)]$$
$$\forall x\ [\ [C(X) \& V(X)] \rightarrow \neg S(X)]$$

$$\exists x\ [\ C(X) \& \neg S(X)]$$

- **Predicates :**
  - **C(x): x is a check.**
  - **T(x): x has been cashed within 30 days.**
  - **V(x): x is void.**
  - **S(x): x can be cashed.**

Reasoning in FOL

$C(this\_check)$

$\neg T(this\_check)$

$\forall x\ [\ [C(X)\&\ \neg T(X)] \rightarrow V(X)]$

$\forall x\ [\ [C(X)\&V(X)] \rightarrow \neg S(X)]$

$\exists x\ [\ C(X)\&\ \neg S(X)]$

1)	$C(this\_check)$	Premise
2)	$\neg T(this\_check)$	Premise
3)	$\forall x\ [\ [C(X)\&\ \neg T(X)] \rightarrow V(X)]$	Premise
4)	$\forall x\ [\ [C(X)\&V(X)] \rightarrow \neg S(X)]$	Premise
5)	$C(this\_check)\ \&\neg T(this\_check)$	1,2 Conj.
6)	$[C(this\_check)\&\ \neg T(this\_check)] \rightarrow V(this\_check)$	3 UI
7)	$V(this\_check)$	5,6 MP
8)	$C(this\_check)\ \&\ V(this\_check)$	1,7 Conj.
9)	$C(this\_check)\&\ V(this\_check) \rightarrow \neg S(this\_check)$	4 UI
10)	$\neg S(this\_check)$	9,8 MP
11)	$C(this\_check)\ \&\ \neg S(this\_check)$	1,10 Conj.
12)	$\exists x\ [\ C(X)\&\ \neg S(X)]$	11 EG #

Assignment_3

Consider this initial configuration for Sukoshi game:

	4		1
2			
			3
		4	

Suppose we have predicates:

- Cell(X_coordinate, Y_coordinate ,Value) : to define the value in a given cell
- Same(X, Y) : to state that X is the same as Y

Use predicate logic syntax to formalize the following:

a) **The Initial configuration for the game.**
b) **Every cell must contain at least one value.**
c) **No numeral is repeated in any row or column.**


Thank You !