Propositional Logic

Contents

Quiz

Formalize the following sentences into PL:

- If I go to Harry's or to the country, I will not go shopping

$$(h \vee c) \rightarrow \neg s$$

- The automated reply cannot be sent when the file system is full

$$f \rightarrow \neg s$$

- You can access the internet from campus if only if you are a computer science major or you are not a freshman

$$a \leftrightarrow (c \vee \neg f)$$

PL Semantic

- ***Conjunction:***
  - The conjunction of two statements α and β, written in PL as (α & β), is true if both α and β are true, and is false otherwise.

α	β	(α & β)
T	T	T
T	F	F
F	T	F
F	F	F

❖ *Disjunction :*

- The disjunction of two statements α and β, written in PL as (α v β), is false if *both* α and β are false and is true otherwise.

α	β	(α v β)
T	T	T
T	F	T
F	T	T
F	F	F

PL Semantic

❖ *Implication :*

- This truth-function is represented in language PL with the sign '→'. A statement of the form (α → β), is false if α is true and β is false, and is true otherwise.

α	β	(α→β)
T	T	T
T	F	F
F	T	T
F	F	T

PL Semantic

❖ *Equivalence :*

- This truth-function is represented in language PL with the sign '$\leftrightarrow$'. A statement of the form ($\alpha \leftrightarrow \beta$) is regarded as true if α and β are either both true or both false, and is regarded as false if they have different truth-values.

α	β	$(\alpha \leftrightarrow \beta)$
T	T	T
T	F	F
F	T	F
F	F	T

PL Semantic

❖ *Negation :*

- The negation of statement α, simply written ¬α in language PL, is regarded as true if α is false, and false if α is true.

α	¬α
T	F
F	T

PL Semantic

Apply PL precedence's rules for the following sentences

Sentence	Precedence
$p \vee q \wedge r$	$((p \vee q) \wedge r)$
$\neg p \wedge q$	$((\neg p) \wedge q)$
$p \rightarrow q \rightarrow \mathrm{r}$	$((\mathrm{p} \rightarrow \mathrm{q}) \rightarrow \mathrm{r})$
$p \rightarrow q \Leftrightarrow \mathrm{r}$	$((\mathrm{p} \rightarrow \mathrm{q}) \Leftrightarrow \mathrm{r})$

PL Semantic

Evaluation : the process of determining the truth values of a compound sentence given a truth assignment for the truth values of it's proposition constants.

Compute the truth-value for the sentence:
(p ∨ **q)** ∧ **(¬q** ∨ **r)** where **p**,**q** and **r** are **T**, **F**, and **T** respectively.

$$(p \lor q) \land (\neg\, q \lor r)$$
$$(T \lor F) \land (\neg\, F \lor T)$$
$$T \land (\neg\, F \lor T)$$
$$T \land (T \lor T)$$
$$T \land T$$
$$T$$

PL Semantic

Satisfaction : Satisfaction is the opposite of evaluation, we begin with one or more compound sentences and try to figure out which truth assignments satisfy those sentences.

Compute the truth-value for the sentence:

$$(p \,\&\, q) \rightarrow \neg r$$

PL Semantic

Compute the truth-value for the sentence:
$(p \,\&\, q) \rightarrow \neg r$

- ***Truth assignment*** *is* the possible assignment of truth-values **T** or **F** to different statement letters making up a wff or series of wffs.

- If a wff has ***n*** distinct statement letters the number of possible assignments is $\mathbf{2^{n}}$.

- $(p \,\&\, q) \rightarrow \neg r$ has **3** statement letters, ‘p', ‘q' and ‘r', and so there are **8** truth-value assignments.

P	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

PL Semantic

$(p \,\&\, q) \rightarrow \neg r$

p	q	r	p&q	¬ r	(p & q) → ¬r
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	F	F	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	F	T	T

PL Semantic

❖ How to represent Exclusive or in PL ?

- ❖ **PL has no simple sign for 'or' in *exclusive sense*.**
- ❖ **($\alpha \leftrightarrow \beta$) could be used to represent *exclusive or logical relation by :***

α	β		$\alpha \leftrightarrow \beta$	$\neg(\alpha \leftrightarrow \beta)$
T	T		T	F
T	F		F	T
F	T		F	T
F	F		T	F

Example

- Formalize the following argument using PL, and then apply satisfaction process for the sentence?

  "If I go to the mall or go to the movies, then I will not go to the gym."

Let

p: I go to the mall

q : I go to the movies

r : I will go to the gym

$$(p \lor q) \rightarrow \neg r$$

p	q	r	$(p \lor q)$	$\neg r$	$(p \lor q) \longrightarrow \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

1) Tautology (Valid)

a wff is a tautology if and only if it is true for all possible assignments to the statement letters making it up.

EX: $p \vee \neg p$

p		$\neg p$	$p \vee \neg p$
T		F	T
F		T	T

2)Self-contradiction (unsatisfiable)

a wff is a self-contradiction if and only if it is false for all possible truth-value assignments to the statement letters making it up.

EX: $p\&\neg p$

p	¬p	p&¬p
T	F	F
F	T	F

Logical Properties and Relationships

3) Contingent

- a wff is that is neither self-contradictory nor tautological is called a *contingent* WFF.
- A contingent statement is true for some assignments to its statement letters and false for others.

EX_1 : $p \,\&\, q$

EX_2 : $(p \,\&\, q) \rightarrow \neg r$

Logical Properties and Relationships

4) Consistent

Two wffs are consistent if and only if there is at least one possible assignment to the statement letters making them up that makes both wffs true.

EX: $p \vee q$ and $\neg(p \leftrightarrow \neg q)$

p	**q**	**p∨q**	**¬q**	**p↔¬q**	**¬(p ↔ ¬q)**
T	T	T	F	F	T
T	F	T	T	T	F
F	T	T	F	T	F
F	F	F	T	F	T

Logical Properties and Relationships

5) Inconsistent

two wffs are inconsistent if and only if there is no assignment to the statement letters making them up that makes them both true.

<u>EX:</u> $(p \rightarrow q) \& p$ and $\neg(q \vee \neg p)$

p	q		p→q	(p→q)&p		¬p	q∨¬p	¬(q ∨¬p)
T	T		T	T		F	T	F
T	F		F	F		F	F	T
F	T		T	F		T	T	F
F	F		T	F		T	T	F

Logical Properties and Relationships

6) Logically equivalent

two wffs are logically equivalent if and only if all possible assignments to the statement letters making them up make them identical

EX: $p \rightarrow q$ **and** $\neg p \vee q$

p	q	$p \rightarrow q$	$\neg p$	$(\neg p \vee q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Use truth table method to proof the following argument:

pvq

¬p

---

q

p	q	[redacted]	pvq	¬p	(pvq)&¬p	((pvq)&¬p)→q
T	T	[redacted]	T	F	F	T
T	F	[redacted]	T	F	F	T
F	T	[redacted]	T	T	T	T
F	F	[redacted]	F	T	F	T

Truth table method drawback

- Computational complexity
- Over the years, researchers have proposed ways to improve the performance of truth table checking. However, the best approach to dealing with large vocabularies is to use symbolic manipulation (i.e. logical reasoning and proofs) in place of truth table checking.

PL’

It is possible to define all operators of PL using the signs '¬' and '→'

PL'

It is possible to define all operators of PL using the signs '¬' and '→'

❖ OR

- The form $\neg\alpha \rightarrow \beta$ always has the same truth-value as the corresponding statement of the form $(\alpha \vee \beta)$

$$(\alpha \vee \beta) \equiv \neg\alpha \rightarrow \beta$$

α	β	$\alpha \vee \beta$	$\neg\alpha$	$\neg\alpha \rightarrow \beta$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

PL′

❖ AND

- The form ¬(α → ¬β) always has the same truth-value as the corresponding statement of the form (α & β)

$(\alpha \,\&\, \beta) \equiv \neg(\alpha \rightarrow \neg\beta)$

α	β		α & β	¬β	α→¬β	¬(α → ¬β)
T	T		T	F	F	T
T	F		F	T	T	F
F	T		F	F	T	F
F	F		F	T	T	F

PL'

❖ EQUIVALENT

- The form $\neg((\alpha \rightarrow \beta) \rightarrow \neg(\beta \rightarrow \alpha))$ always has the same truth-value as the corresponding statement of the form $(\alpha\leftrightarrow\beta)$

$$(\alpha\leftrightarrow\beta) \equiv \neg((\alpha \rightarrow \beta) \rightarrow \neg(\beta \rightarrow \alpha))$$

α	β		$\alpha\leftrightarrow\beta$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\neg(\beta\rightarrow\alpha)$	$(\alpha\rightarrow\beta)\rightarrow\neg(\beta\rightarrow\alpha)$	$\neg((\alpha \rightarrow \beta) \rightarrow \neg(\beta \rightarrow \alpha))$
T	T		T	T	T	F	F	T
T	F		F	F	T	F	T	F
F	T		F	T	F	T	T	F
F	F		T	T	T	F	F	T

Other forms of PL'

- **It is possible to define all operators of PL using the signs '¬' and 'v'**
  - (α & β) would be defined as $\neg(\neg\alpha \vee \neg\beta)$,
  - ($\alpha \rightarrow \beta$) would be defined as $\neg\alpha \vee \beta$,
  - ($\alpha \leftrightarrow \beta$) would be defined as $\neg(\neg(\neg\alpha \vee \beta) \vee \neg(\neg\beta \vee \alpha))$.

- **Similarly, we could use signs '¬' and '&'**
  - ($\alpha \vee \beta$) would be defined as $\neg(\neg\alpha \mathbin{\&} \neg\beta)$,
  - ($\alpha \rightarrow \beta$) would be defined as $\neg(\alpha \mathbin{\&} \neg\beta)$,
  - ($\alpha \leftrightarrow \beta$) would be defined as $\neg(\alpha \mathbin{\&} \neg\beta) \mathbin{\&} \neg(\beta \mathbin{\&} \neg\alpha)$.

PL’’

- Reducing all logical operators down to a *single* primitive operator.
- The sign '|' is called the “Sheffer stroke”, and is equivalent to **NAND**.

α	β	$(\alpha \mid \beta)$
T	T	F
T	F	T
F	T	T
F	F	T

PL''

- It is possible to define all operators of PL using the signs '|'

  - $(\neg\alpha)$ would be defined as $(\alpha \mid \alpha)$
  - $(\alpha \,\&\, \beta)$ would be defined as $(\alpha \mid \beta) \mid (\alpha \mid \beta)$
  - $(\alpha \vee \beta)$ would be defined as $(\alpha \mid \alpha) \mid (\beta \mid \beta)$
  - $(\alpha \rightarrow \beta)$ would be defined as $\alpha \mid (\beta \mid \beta)$
  - $(\alpha \leftrightarrow \beta)$ would be defined as $((\alpha \mid \alpha) \mid (\beta \mid \beta)) \mid (\alpha \mid \beta)$

Other forms of PL"

❖ The sign '↓' is called the "Sheffer dagger", and is equivalent to **NOR**.

α	β	$(\alpha \downarrow \beta)$
T	T	F
T	F	F
F	T	F
F	F	T

PL''

- **It is possible to define all operators of PL using the signs ' ↓ `**

  - $(\neg\alpha)$ would be defined as $(\alpha \downarrow \alpha)$
  - $(\alpha \,\&\, \beta)$ would be defined as $(\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta)$
  - $(\alpha \vee \beta)$ would be defined as $(\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta)$
  - $(\alpha \rightarrow \beta)$ would be defined as $((\alpha \downarrow \alpha) \downarrow \beta) \downarrow ((\alpha \downarrow \alpha) \downarrow \beta)$

PL, PL', PL''

PL

'¬', '&','v', '→', and '↔'.

PL'

'¬', and '→'

'¬' and 'v'

'¬' and '&'

PL''

'|'

'↓'

Assignment_1

Use Truth table method to prove the validity for logical sentence in pages (30, 32).

[you can choose only **3** sentences…]


Thank You !