Chp. 4: Classification

Classification

Predict a qualitative response for an observation based on X.

The 'Default' dataset:

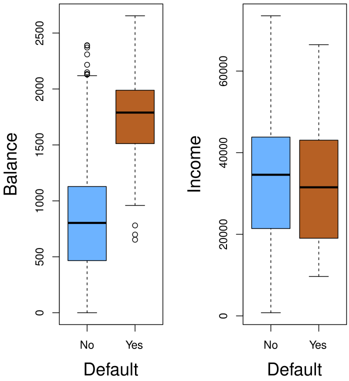


Figure 1: Fig 4.1

Fig. 4.1

Classification

Predict a qualitative response for an observation based on X.

The 'Default' dataset:

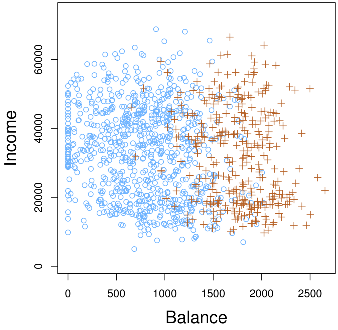
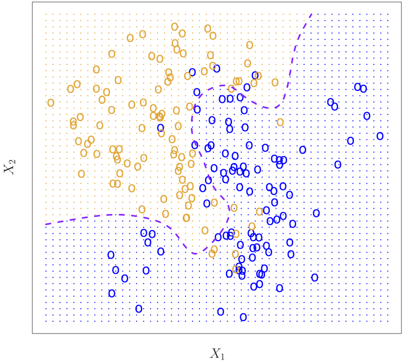


Figure 2: Fig 4.1

Fig. 4.1

Classification

- The test error rate is minimized on average by a simple classifier (the Bayes classifier) that *assigns each observation to the most likely class, given its predictor values.*
- With 2 classes, Bayes decision boundary corresponds to predicting class one if $Pr(Y = 1|X = x_0) > 0.5$ and class two otherwise.



Classification

In reality, we never know the conditional probability of Y given X! The Bayes Classifier boundary is therefore an unattainable 'gold standard'. But, we can estimate it using various methods (logistic regression, KNN classification, LDA, QDA).

Linear regression?

What if we choose

$$Y = \begin{cases} 0 & \text{if Default} = \text{No} \\ 1 & \text{if Default} = \text{Yes} \end{cases}$$

fit a linear regression model, and predict 'Yes' if $\hat{Y} > 0.5$ and 'No' otherwise.

Why not linear regression?

- No natural ordering of response variables with >2 classes.
- Estimates fall outside [0,1].

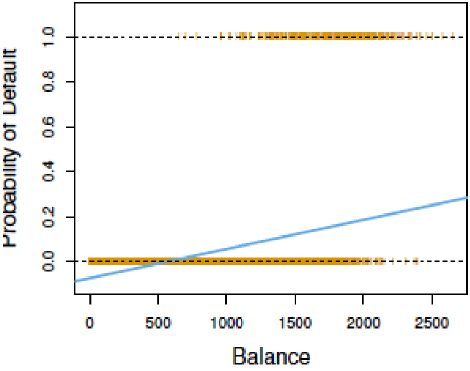


Figure 4: Fig 4.2a

Logistic regression

- Instead of trying to predict Y, let's try to predict $Pr(Y = 1|X)$, or $p(X)$ for short.
- Then, predict default = Yes if $p(X) > 0.5$.
- This threshold can be changed if we want to be more conservative

Logistic regression

- With logistic regression, we consider the *log-odds* or *logit* transformation of the response, and model the log odds as a linear combination of the predictor variables:

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

Logistic regression

- Consider the odds, written as: $\frac{p(X)}{1-p(X)}$
- e.g. if 1 in 5 people default, the probability of defaulting is 0.2.
- The odds of defaulting will be 1:4, since $p(X) = 0.2$ and $\frac{0.2}{1-0.2} = 1/4$

Logistic regression

By using the logit transformation, our function $p(X)$ takes on an S-shape:

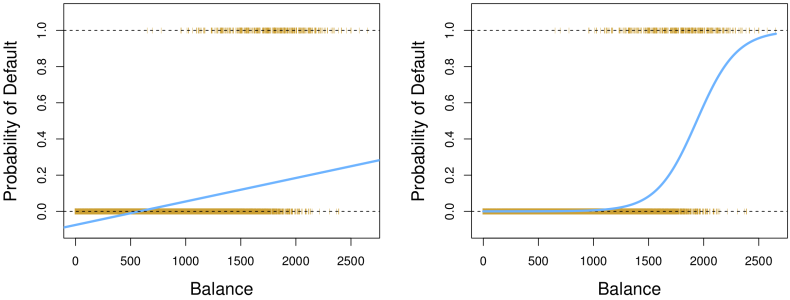


Figure 5: Fig 4.2

Fig. 4.2

Why this results in an S-shaped curve?

Start with the model that we fit for logistic regression:

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

Exponentiate and take multiplicative inverse of both sides:

$$\frac{1-p(X)}{p(X)} = \frac{1}{e^{\beta_0+\beta_1 X}}$$

Partial out fraction and add 1 to each side:

$$\frac{1}{p(X)} = 1 + \frac{1}{e^{\beta_0+\beta_1 X}}$$

Why this results in an S-shaped curve?

$$\frac{1}{p(X)} = 1 + \frac{1}{e^{\beta_0+\beta_1 X}}$$

Change 1 to a common denominator:

$$\frac{1}{p(X)} = \frac{1 + e^{\beta_0+\beta_1 X}}{e^{\beta_0+\beta_1 X}}$$

Take multiplicative inverse of both sides:

$$p(X) = \frac{e^{\beta_0+\beta_1 X}}{1 + e^{\beta_0+\beta_1 X}}$$

Why this results in an S-shaped curve?

This is the form of the logistic function:

$$S(x) = \frac{e^x}{1 + e^x}$$

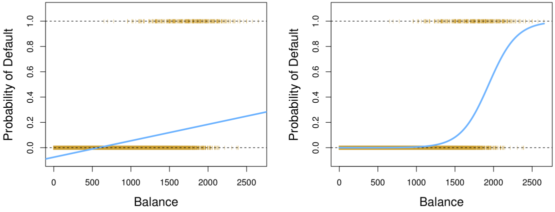


Figure 6: Fig 4.2

Estimating coefficients

- With logistic regression, we consider the *log-odds* or *logit* transformation of the response, and model the log odds as a linear combination of the predictor variables:

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

- How did we estimate β_0 and β_1 for linear regression (Chp. 3)?

Estimating coefficients

- For logistic regression, we use *maximum likelihood* to choose β_0 and β_1 such that the predicted probability $\hat{p}(x_i)$ of the response for each individual is as close as possible to the observed response.
- β_0 and β_1 are chosen to *maximize* the likelihood function (ISL p. 133).

Fitting a logistic regression model

```
library(ISLR)
data(Default)
summary(glm(default ~ balance, data=Default, family = binon
```

```
##
## Call:
## glm(formula = default ~ balance, family = binomial, data
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.2697  -0.1465  -0.0589  -0.0221   3.7589
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.065e+01  3.612e-01  -29.49   <2e-16 ***
## balance      5.499e-03  2.204e-04   24.95   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

Interpreting the coefficients:

β_1:

- For every one unit increase in 'balance', the expected change in the log odds is 0.005.
- For every one unit increase in 'balance', we expect $e^{0.0055} = 1.0055$ or ~0.55% increase in odds of defaulting on a credit card payment.

Great discussion of interpreting coefficients on UCLA Data Analysis Examples website! See here for website or here for logistic regression specifically

Making predictions

Once we have estimated the regression coefficients, we can compute the probability of default for a person with a credit card balance of \$2,000:

$$\widehat{p}(X) = \frac{e^{-10.651+0.0055\times 2{,}000}}{1 + e^{-10.651+0.0055\times 2{,}000}} = 0.586$$

Fitting a logistic regression model

- Just as for linear regression, we can generalize logistic regression to the case of multiple predictors.

$$p(X) = \frac{e^{\beta_0+\beta_1 X+\ldots+\beta_p X_p}}{1+e^{\beta_0+\beta_1 X+\ldots+\beta_p X_p}}$$

NOTE: $p(X)$, the probability of Y given X, is not the same as p, the number of predictors!!!

Fitting a logistic regression model

```
##
## Call:
## glm(formula = default ~ balance + student, family = binc
##     data = Default)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.4578  -0.1422  -0.0559  -0.0203   3.7435
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.075e+01  3.692e-01 -29.116  < 2e-16 ***
## balance      5.738e-03  2.318e-04  24.750  < 2e-16 ***
## studentYes  -7.149e-01  1.475e-01  -4.846 1.26e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
##
## (Dispersion parameter for binomial family taken to be 1)
```

Fitting a logistic regression model

$$\widehat{Pr}(\text{default=Yes}|\text{student=Yes}) = \frac{e^{-10.75+0.0057\times 2{,}000+-0.71\times 1}}{1+e^{-10.75+0.0057\times 2{,}000+-0.71\times 1}}$$

$$= 0.48$$

$$\widehat{Pr}(\text{default=Yes}|\text{student=No}) = \frac{e^{-10.75+0.0057\times 2{,}000+-0.71\times 0}}{1+e^{-10.75+0.0057\times 2{,}000+-0.71\times 0}}$$

$$= 0.66$$

Given the same credit card balance, a student is less likely to default on the payment than a non-student.

A machine learning view on logistic regression

https://towardsdatascience.com/breaking-it-down-logistic-regression-e5c3f1450bd#6a7a

Alternatives to logistic regression

- When classes are well-separated, parameter estimates for logistic regression can be unstable.
- If n is small and distribution of the predictors is approximately normal in each of the classes, **linear discriminant** model is more stable than logistic regression.
- Extensions to >2 classes exist (e.g. multinomial logistic regression) but less popular