AV: Autonomous Vehicles
Kalman filter

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Introduction

The goal of the practical work was the simulation of the Kalman filter (KF) on a DC motor driven by a square wave input voltage using Matlab. Angular position and velocity of the motor is modelled with position feedback from an incremental encoder.

Kalman filter and stationary Kalman filter (SKF) are simulated and compared for a correctly and incorrectly modelled system.

Chapter 1

Kalman Filter

Input voltage

The input voltage $u(t)$ is a zero-mean square wave with period $\Delta = 100ms$ and peak-to-peak amplitude $A = 0.1V$. This signal is sampled with sample time $T_s = 1ms$. Matlab function *square* is used to generate input square voltage. A sample input voltage waveform is shown in figure (1.1).

System modelling and simulation

State vector consists of angular position $\theta(t)$ and angular velocity $\Theta(t)$ of the motor and is defined as follows:

$$x(t) = \begin{bmatrix} \theta(t) \\ \Omega(t) \end{bmatrix} \tag{1.1}$$

The state space representation in continuous time is defined below.

$$\dot{x} = Ax + Bu$$

$$\theta = Cx + Du$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix}; \mathrm{B} = \begin{bmatrix} 0 \\ \frac{G}{T} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \mathrm{D} = [0]$$

The system model is transformed to discrete system using Matlab function $c2dm$.

$$x_{n+1} = \tilde{A}x_n + \tilde{B}u_n$$

$$y_n = \tilde{C}x_n + \tilde{D}u_n$$

The measurement y_n is quantization of angular position θ_n that is measured by incremental encoder. The measurement is modelled using the state vector. In order to discretize, θ is used to find encoder counts using the relation $\frac{\theta L}{2\pi}$and is then rounded off. The rounded off value is converted back to angular position.

$$y_n = round\left(\frac{\theta L}{2\pi}\right) \cdot 2\frac{\pi}{L} \tag{1.2}$$

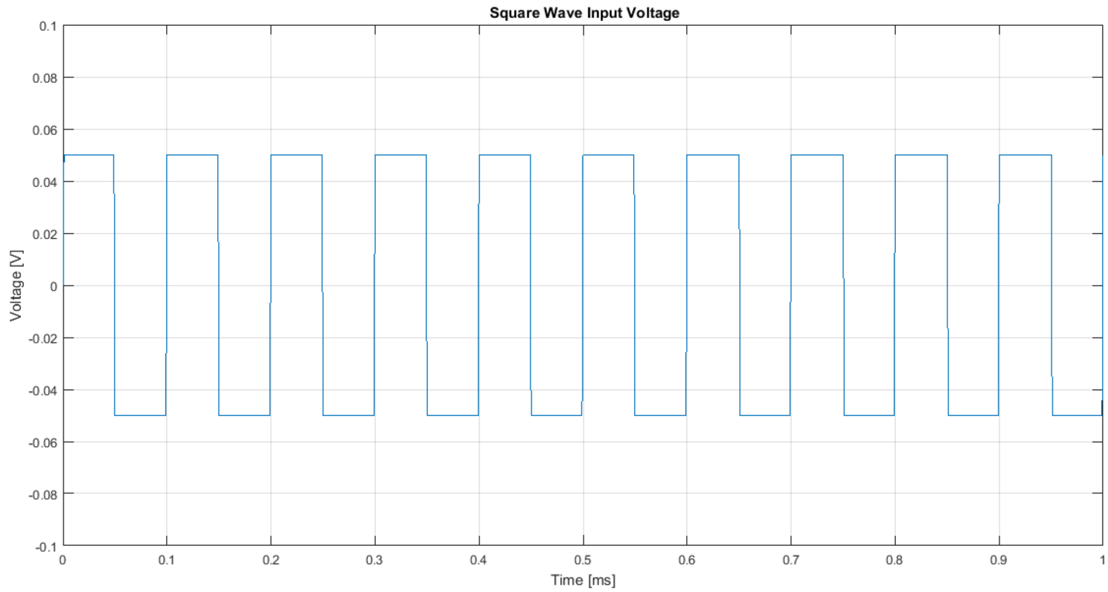


Figure 1.1: Input voltage for a duration of $1s$.

Evolution of state estimates and measurement is shown for the input voltage in figure (1.1) and $G = 50\frac{rad}{sV}$ and $T = 20ms$ in figure (1.2).

Measurement y can be seen as discrete steps. The angular velocity Ω initially rises to about $2.25rads^{-1}$, however in subsequent oscillations, maximum value of Ω stays constant and at about $2.1rads^{-1}$. This difference is caused by a different starting point, where initially Ω rises from 0 and in subsequent oscillations from $-2.1rads^{-1}$.

Kalman filter

Kalman filter is an algorithm that uses a series of measurements observed over time, containing statistical noise and other inaccuracies, and produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone[1]. Based of previously described system model, input signal and measurement noise is modelled, where the input noise is defined as white noise with zero mean and variance q.

$$v \sim W(0, q) \tag{1.3}$$

Measurement noise is modelled as quantization noise with zero mean and variance r.

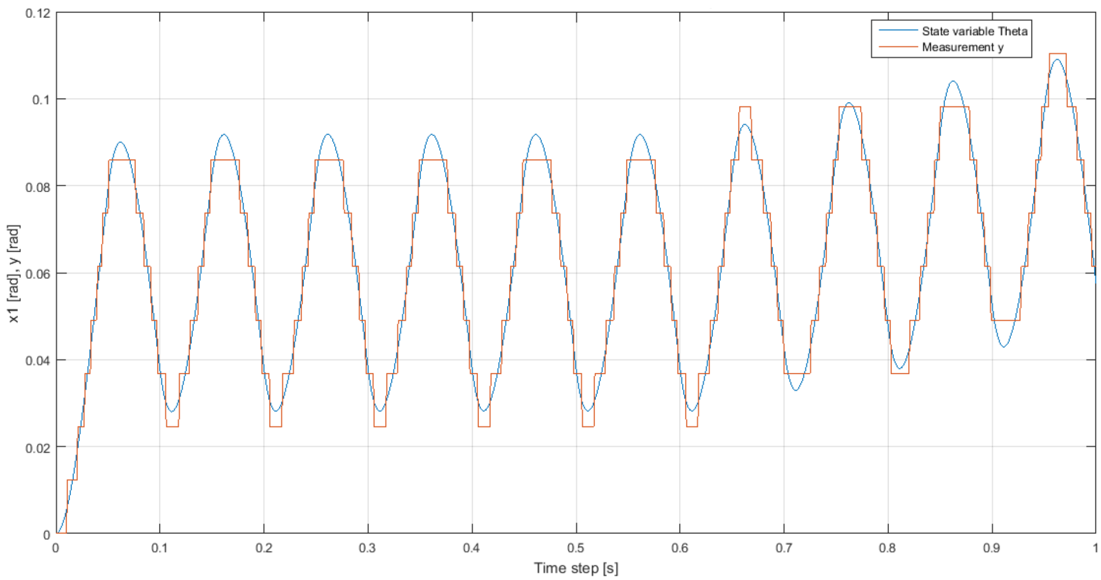
$$w \sim \Delta(0, r) \tag{1.4}$$

Markov model

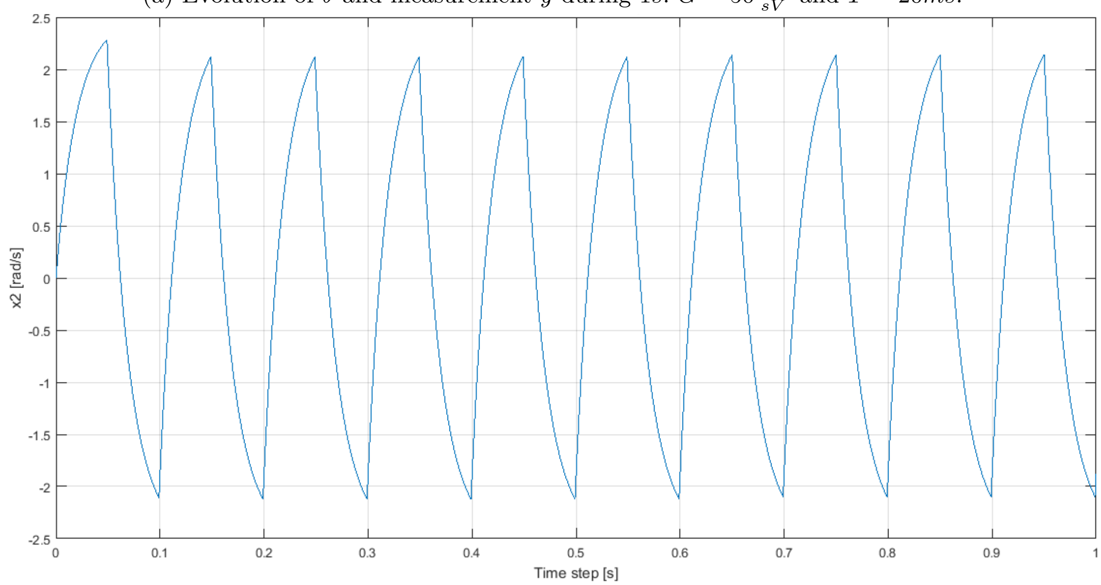
Markov equations for the presented system model are defined as follows

$$x[n+1] = \tilde{A}x[n] + \tilde{B}(u[n] + v[n])$$

[1] e-Study Guide for: Introduction to Probability Theory and Stochastic Processes



(a) Evolution of θ and measurement y during $1s$. $G = 50\frac{rad}{sV}$ and $T = 20ms$.



(b) Evolution of Ω during $1s$. $G = 50\frac{rad}{sV}$ and $T = 20ms$.

Figure 1.2: Evolution of state vector and measurement y during $1s$. $G = 50\frac{rad}{sV}$ and $T = 20ms$.

$$y[n] = \tilde{C}x[n] + w[n]$$

Following components need to be initialised for implementation of a Kalman filter:

1. measurement noise variance r.
2. initial state estimate $\hat{X}_{1/0}$,
3. initial state covariance matrix $P_{1/0}$,
4. process noise covariance matrix Q_α and input noise covariance matrix Q_β.

Measurement noise variance r

This is a qualitative measure of how uncertain the measurement y is. The incremental encoder has a precision of $L = 512$ *angles/lap*. The angle spacing between two counts noted by encoder is $\frac{2\pi}{L}$. Since the encoder is unable to measure or deduce any information for this angle range, the noise can be considered uniformly distributed. Thus noise variance r can be defined in equation (1.5).

$$r = \frac{(2\pi/L)^2}{12} \tag{1.5}$$

Initial state estimate $\hat{X}_{1/0}$

The initial estimate of the state vector required for the first iteration of the Kalman filter is set at $\Omega_{1/0} = 0$, since the motor is initially stopped. The initial angular position $\hat{\theta}_{1/0}$ however, is completely unknown. The system is modelled with initial angular position equal to zero. For Kalman filter, we take the initial estimate to be 0.05 for $\hat{\theta}_{1/0}$.

$$\hat{X}_{1/0} = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix} \tag{1.6}$$

Initial state covariance matrix $P_{1/0}$

Initial state covariance matrix is based on the initialization error of the state. Since the initial angular velocity is accurately known, its variance should is taken zero. For initial angular position, it can be assumed that there is a uniform distribution for the range of angles 2π. According to variance of uniform distribution, variance for initial angular position can be assumed equal to $\frac{(2\pi)^2}{12}$. Thus, $P_{1/0}$ is defined as

$$P_{1/0} = \begin{bmatrix} \frac{(2\pi)^2}{12} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.7}$$

Process noise covariance matrix Q_α and input noise covariance matrix Q_β

These covariance matrices give estimates of uncertainty in the process and input model. Since information about noise in process is unavailable, it is assumed that the process model is accurate and therefore the process noise covariance matrix is set as $Q_\alpha = 0$. It is assumed that the noise is only due to the input. As discussed before, input noise has variance q. Noise covariance matrix Q_β is defined as

$$Q_\beta = \tilde{B}q\tilde{B}' \tag{1.8}$$

Value of variance q is tuned in the Kalman filter model.

Implementation

Two versions of Kalman filter were implemented. One with time varying Kalman gain and another with constant Kalman gain, referred to as stationary Kalman filter. Based on the initial estimates and noise variances, it can be stated that the noise $v[n]$ and $w[n]$ and initial state covariance $P_{[1|0]}$ are not modelled with the Gaussian assumption. Therefore, the Kalman filter used is an implementation of the linear minimum mean square error (LMMSE) estimator.

The stationary Kalman filter asymptotically behaves like the conventional KF. This means that initial transition of state estimate is slow.

From the intial estimates and noise variances, it can be said that the the noise $v[n]$ and $w[n]$ and initial state covariance $P_{[1|0]}$ are not modelled with the Gaussian assumption. Thus, the Kalman filter (KF) implemented here is an implementation of the LMMSE estimator. Time varying Kalman gain is computed as follows:

Performance of both KF and SKF was studied for different values of q for perfect and rough models of system.

Simulation

Simulations were conducted using the Matlab software.

Accurate model

The model of the system is ideal:

- $G_{actual} = G_{filter} = 50\frac{rad}{sV}$
- $T_{actual} = T_{filter} = 20ms$

Since the model is assumed to be ideal, the model noise is 0. Thus, $q = 0$ yields best result and no error. However to study the influence of q, 4 different values of q are evaluated: $q = 0.1$, $q = 0.001$, $q = 0.00001$, $q = 0.0000001$. Evolution of Kalman filter and stationary Kalman filter were plotted for both state estimates, θ and Ω (1.3a), (1.4), (1.5a), (1.6a). Also logarithms for their respective Kalman gains were plotted (1.3b), (1.4b), (1.5b), (1.6b).

First, the state estimate of angular position θ for Kalman and stationary Kalman filter (1.3a), (1.5a) is analyzed. The smaller the value of q, the smaller the error between the state estimate and the actual state. The largest error is thus obtained when $q = 0.1$, represented by orange line, that is more biased towards the measurement reading than the actual state. The state estimates using the KF soon start to converge (initial state estimate error was assumed to be 0.05 deg). On the contrary the stationary Kalman filter (1.5a) takes comparably long to converge. The error is most profound for $q = 0.0000001$ shown using the blue line. This behaviour clearly illustrates the drawback of using constant Kalman gain.

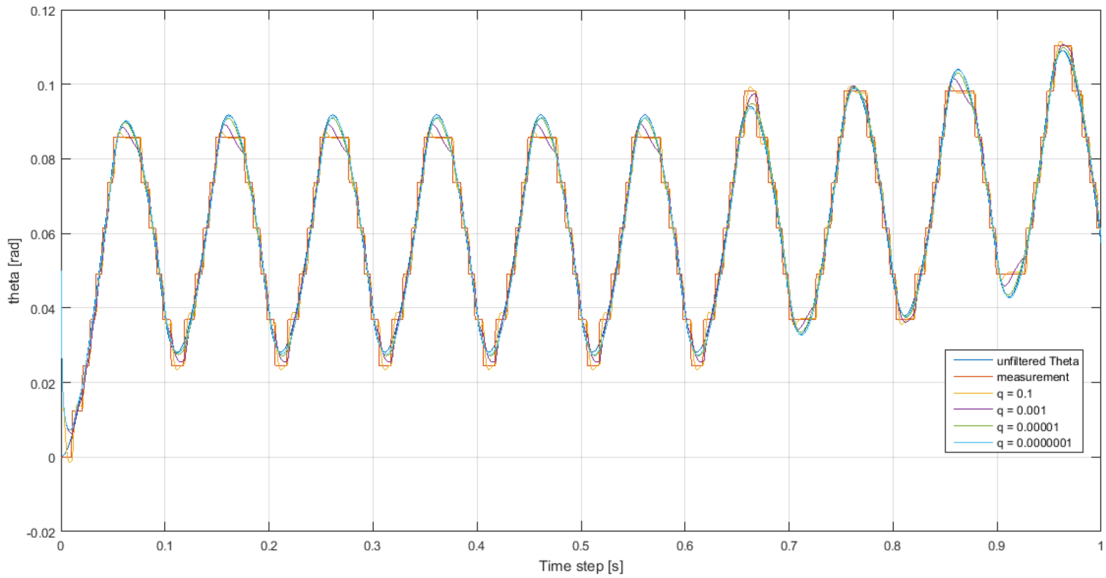
Second, the state estimate of angular velocity Ω for both Kalman filter variants (1.4a), (1.6a) is analyzed. The smaller the value of q, the smaller the error between the state estimate and the actual state. The large oscillatory errors are obtained when $q = 0.1$, represented by red line. This indicates the value of q is too high for the system. Initial state estimate error was assumbed to be 0 for Ω. Thus Kalman filter estimate soon converges. However, stationary Kalman filter (1.6a) takes $0 \div 0.05s$ to converge depending on value of q. Also, $q = 0.1$ generates very large error in the first few iteration of stationary Kalman filter.

Third, logarithms for Kalman filter gain are studied. It can be seen that the Kalman filter gain for θ (1.3b) takes $0.02 \div 0.6s$ to converge depending on the value of q. The smaller the value of q, the longer it takes for the filter to converge. Similarly, Kalman filter gain for $Omega$ (1.4b) takes $0.02 \div 0.4s$ to converge for the four values of q. Meanwhile, the stationary Kalman filter gain (1.5b), (1.6b) are constant and are already equal to the converged value.

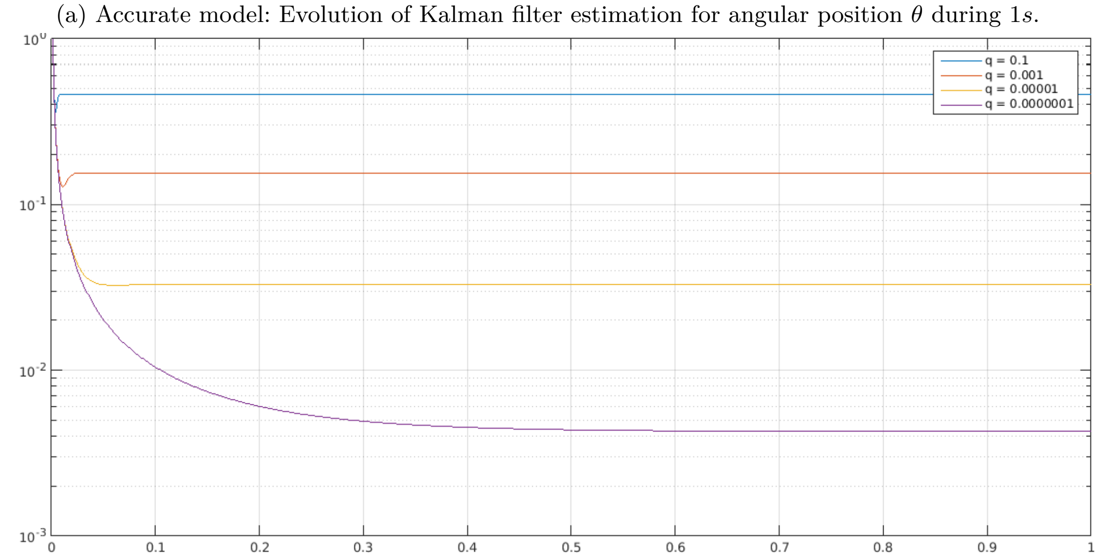
One way to qualify the quality of an estimator is to compute its mean square error (MSE). MSE is the mean value of the square from the Euclidean norm of the estimation error. It is calculated as follows:

$$MSE(X) = E(\|\hat{x}(Y) - X\|_2{}^2 \mid X) \tag{1.9}$$

Figure (1.7) shows MSE computed for both Kalman filters and for different values of q. For $q = 0.1$ MSE is largest for both KF and SKF in the figure. This is because, with a large value of q, the filter tends to rely more on the measurement than the system model. For $q = 0.001$ the MSE reduces largely, however there is still significant difference of 0.06 between Kalman filter and stationary Kalman filter. For $q = 0.00001$ the MSE is very small, however a difference of 0.001 is present between the two filters. For $q = 0.0000001$ the MSE is about 0.01 for both KF and SKF. Thus it can be concluded that when q is very small in the correct model, KF and SKF give similar performance. In the accurate model it is assumed that the system model is ideal. Thus error is easily removed by taking $q = 0$ and increasing the value of q results in errors as the filter tends to rely on measurements.

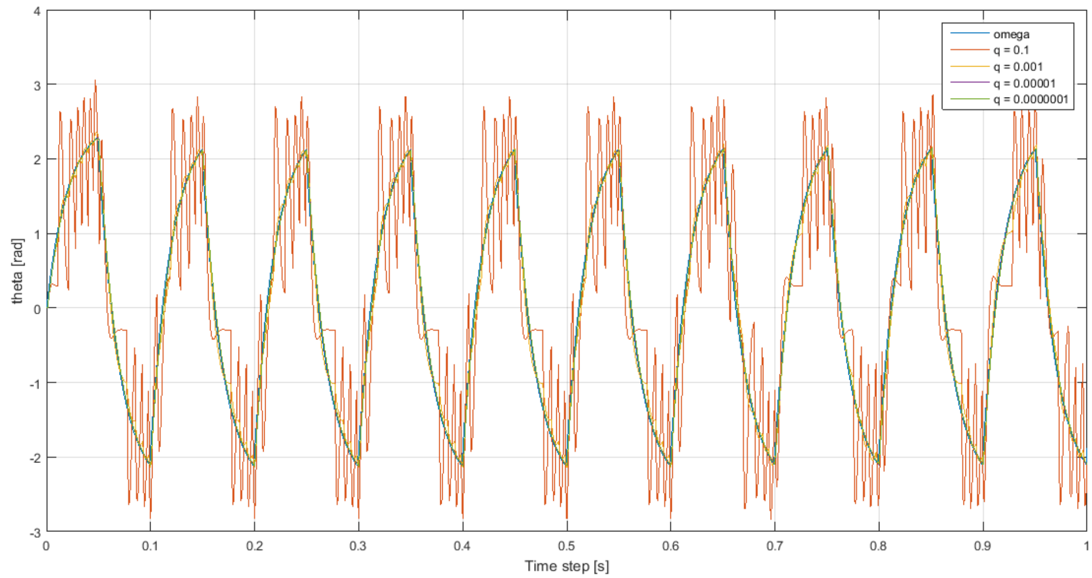


(a) Accurate model: Evolution of Kalman filter estimation for angular position θ during $1s$.

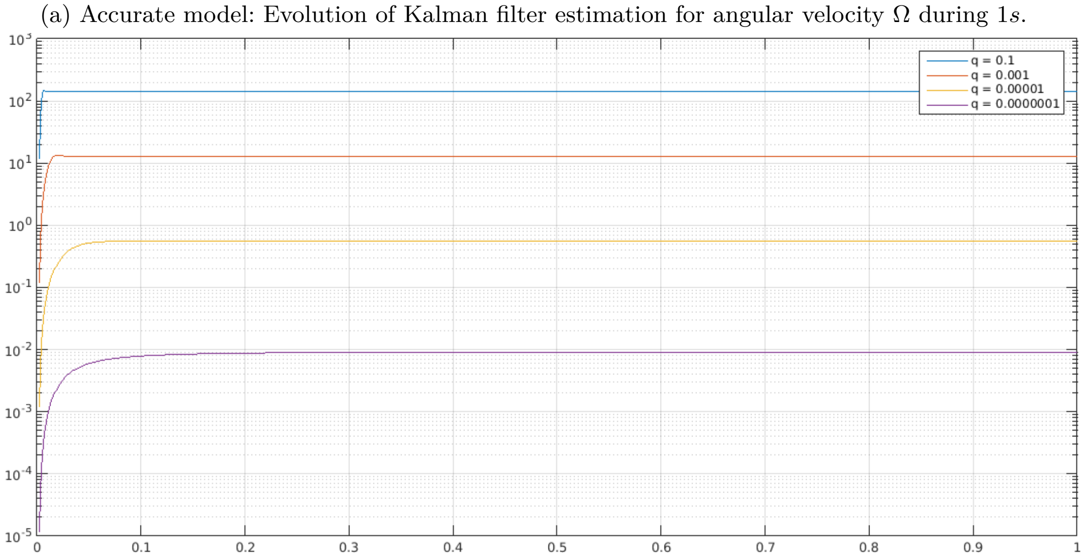


(b) Accurate model: Evolution of Kalman filter gain for angular position θ during $1s$.

Figure 1.3: Accurate model: Evolution of Kalman filter estimation and Kalman filter gain for angular position θ during $1s$. $G = 50\frac{rad}{sV}$ and $T = 20ms$.

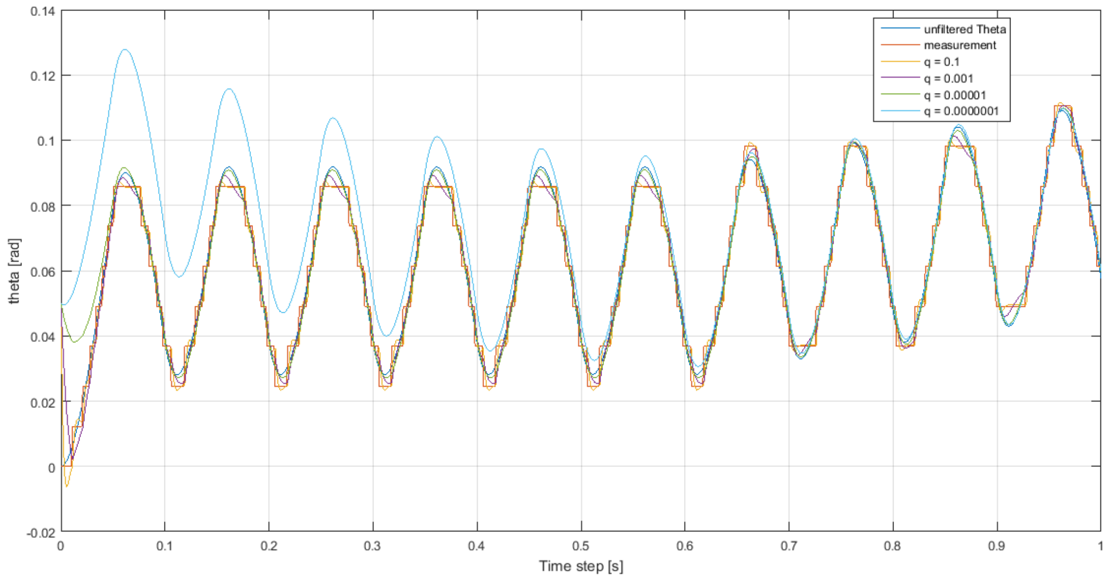


(a) Accurate model: Evolution of Kalman filter estimation for angular velocity Ω during $1s$.

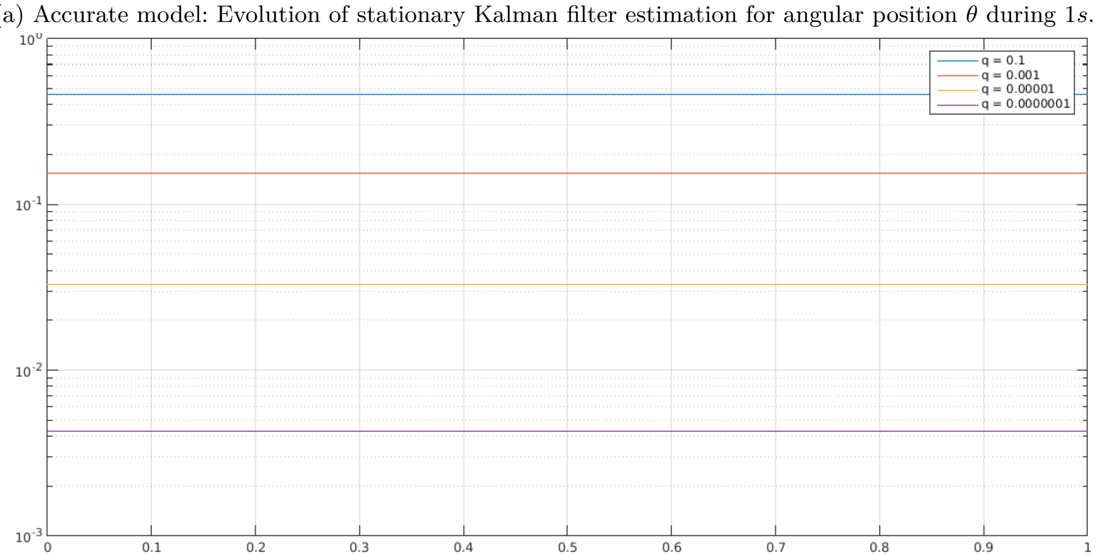


(b) Accurate model: Evolution of Kalman filter gain for angular velocity Ω during $1s$.

Figure 1.4: Accurate model: Evolution of Kalman filter estimation and Kalman filter gain for angular velocity Ω during $1s$.

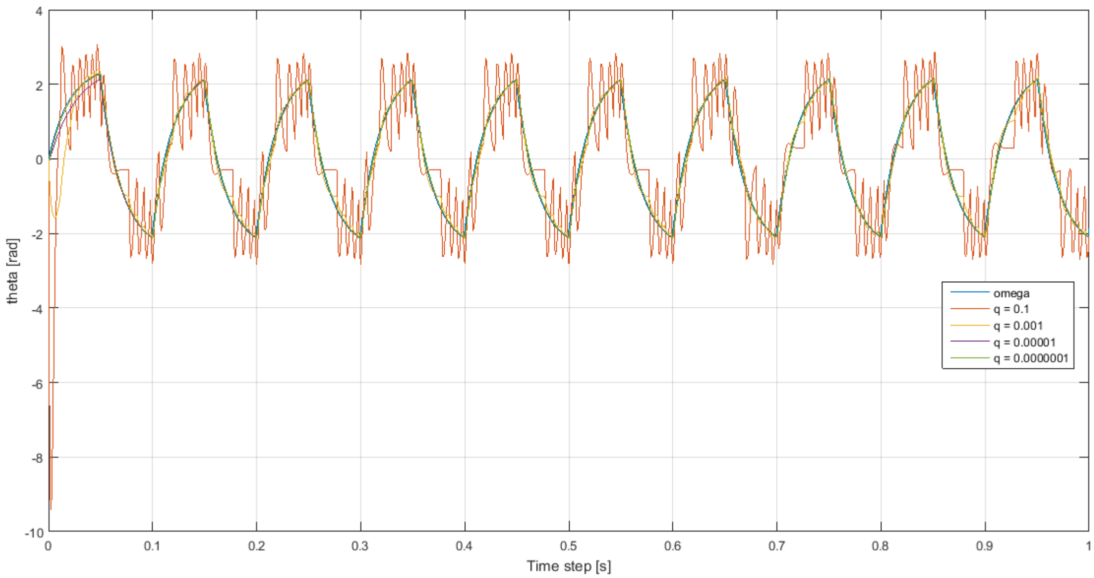


(a) Accurate model: Evolution of stationary Kalman filter estimation for angular position θ during $1s$.

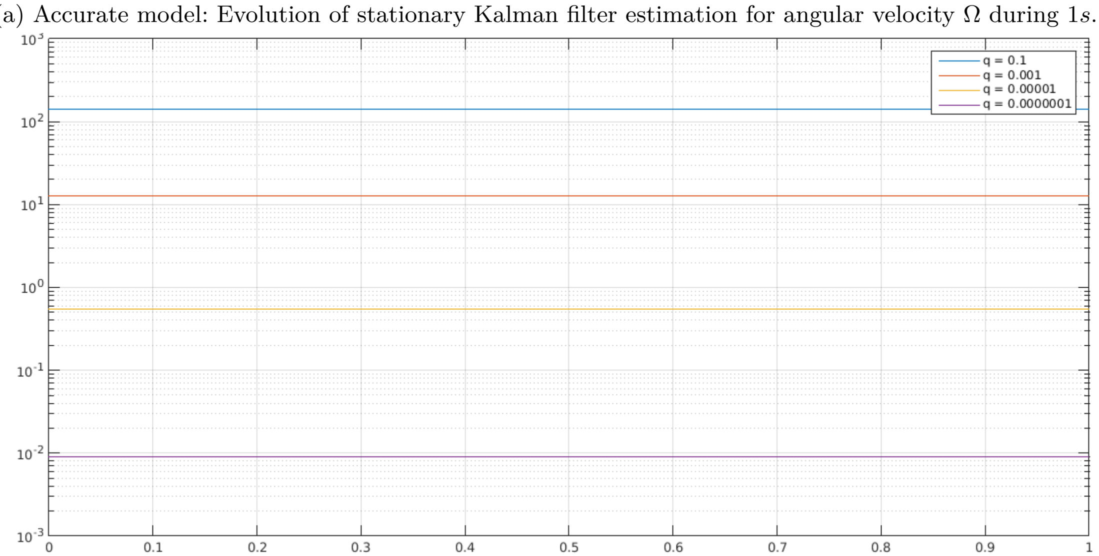


(b) Accurate model: Evolution of stationary Kalman filter gain for angular position θ during $1s$.

Figure 1.5: Accurate model: Evolution of stationary Kalman filter estimation and stationary Kalman filter gain for angular position θ during $1s$.



(a) Accurate model: Evolution of stationary Kalman filter estimation for angular velocity Ω during $1s$.



(b) Accurate model: Evolution of stationary Kalman filter gain for angular velocity Ω during $1s$.

Figure 1.6: Accurate model: Evolution of stationary Kalman filter estimation and stationary Kalman filter gain for angular velocity Ω during $1s$.

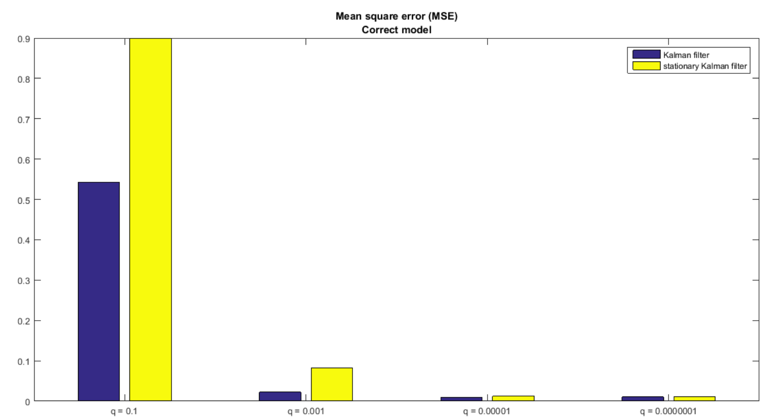


Figure 1.7: Accurate model: Mean square error (MSE) of Kalman filter and stationary Kalman filter estimation for different values of q.

Inaccurate model

The previous accurate model was changed by modifying the time to a new value $T = 25ms$.
The model of the system is rough and imprecise:

- $G_{actual} = G_{filter} = 50\frac{rad}{sV}$
- $T_{actual} = 20ms$
- $T_{filter} = 25ms$

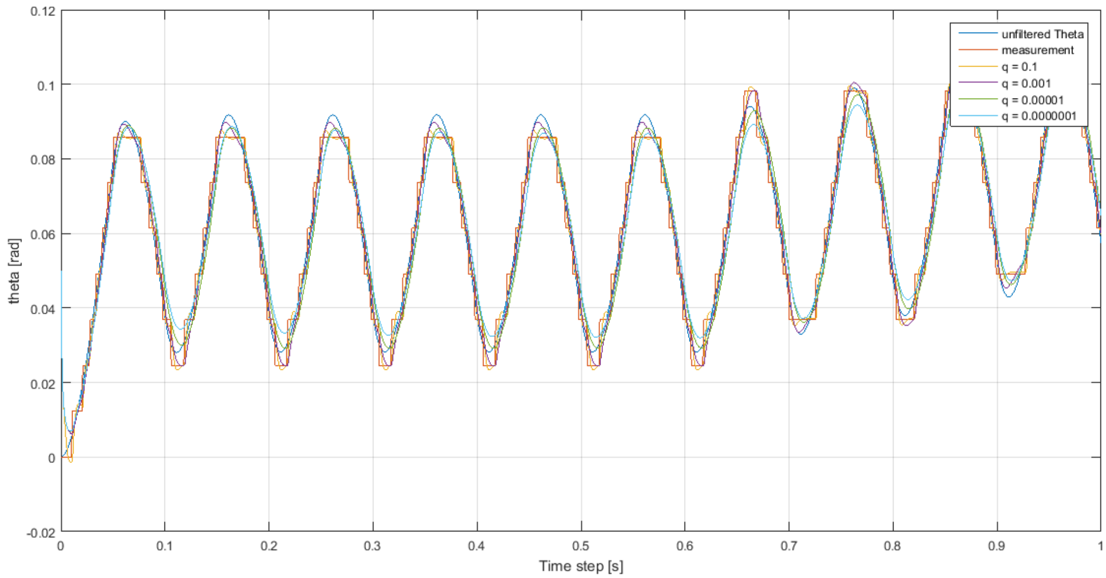
Since the model is assumed rough, it can be understood intuitively, that q must be greater than 0 and therefore it must be tuned. Similar to the accurate model, 4 different values of q are evaluated: $q = 0.1$, $q = 0.001$, $q = 0.00001$, $q = 0.0000001$. Charts for Kalman filter and stationary Kalman filter were plotted for both state estimates, θ and Ω (1.8a), (1.9), (1.10a), (1.11a). Also logarithms for their respective Kalman gains were plotted in figures (1.8b), (1.9b), (1.10b), (1.11b).

First, the state estimate θ for Kalman and stationary Kalman filter (1.8a), (1.10a) are studied. For the same values of q the Kalman filter does not converge well. The largest error is obtained when $q = 0.1$, represented by orange line. The orange line is more biased towards the measurement reading than the state estimate. After $q = 0.1$, the value $q = 0.0000001$ gives the most error. The best tuned value for q can be found between $q = 0.001$ and $q = 0.00001$. For stationary Kalman filter (1.10a) the error is most profound for $q = 0.0000001$, as shown using the blue line. This is of course because of the constant Kalman gain. Here again, once the estimate converges, best value for q can be found between $q = 0.001$ and $q = 0.00001$.

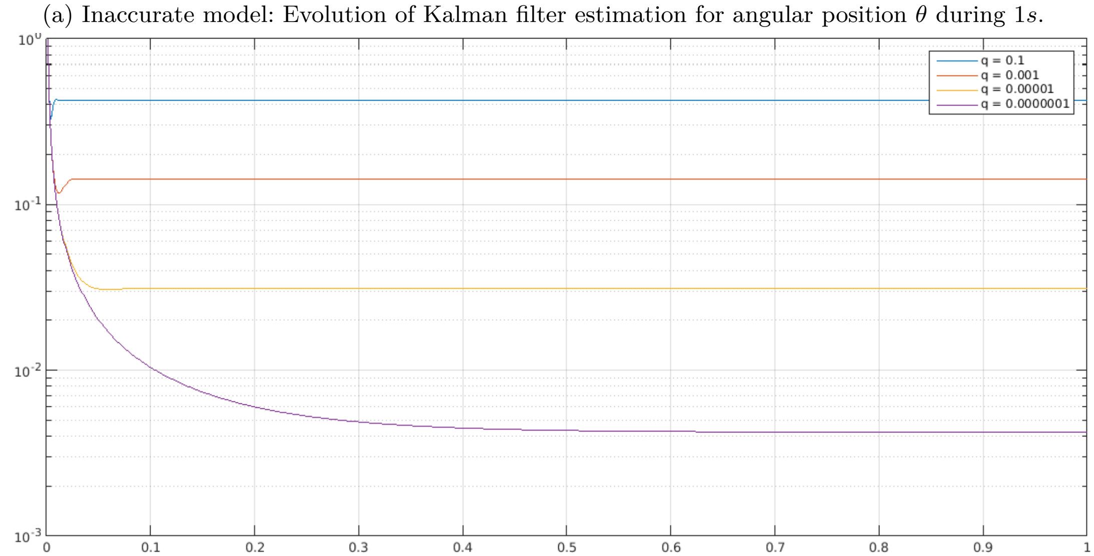
Second, the state estimate Ω for Kalman and stationary Kalman filter (1.9a), (1.11a) are studied. The smallest error in Ω can be seen when $q = 0.001$. The large oscillatory errors are obtained when $q = 0.1$, represented by red line. This indicates the value of q is too large for the system. Initial state estimate error was set as 0 for Ω. Thus Kalman filter estimate soon converges. However, the stationary Kalman filter (1.11a) takes $0 \div 0.05s$ to converge, depending on value of q. Also, $q = 0.1$ generates very large error in the first few iterations of stationary Kalman filter.

Third, logarithms for Kalman filter gain are studied. It can be seen that the Kalman filter gain for θ (1.8b) takes $0.02 \div 0.7s$ to converge depending on the value of q. The smaller the value for q, the longer it takes for the filter to converge. Similarly, the Kalman filter gain for $Omega$ (1.9b) takes $0.02 \div 0.4s$ to converge for the four values of q. The stationary Kalman filter gain (1.10b), (1.11b) is constant and already equal to the converged value.

Similar to the accurate model, the MSE is computed 1.9. Figure (1.12) shows MSE calculated for both Kalman filters and for different values of q using the inaccurate model. For $q = 0.1$ the MSE is largest for both, conventional and stationary Kalman filter. However, compared to MSE recorded in the accurate model, the error is actually smaller. This is because with larger value of q, the inaccurate model tends to rely more on the measurement value than on the system model. Since the defined model is rough, the measurement values help in reducing the error. As the value of q is reduced from $q = 0.1$ to 0.0000001, the MSE is reduces to 0.09 for both KF and SKF. The reason for this behaviour is that with a very small value for q, the estimator tends to rely more on the system model, which is inaccurate. For value $q = 0.0000001$ both, the KF and SKF, give a similar performance.

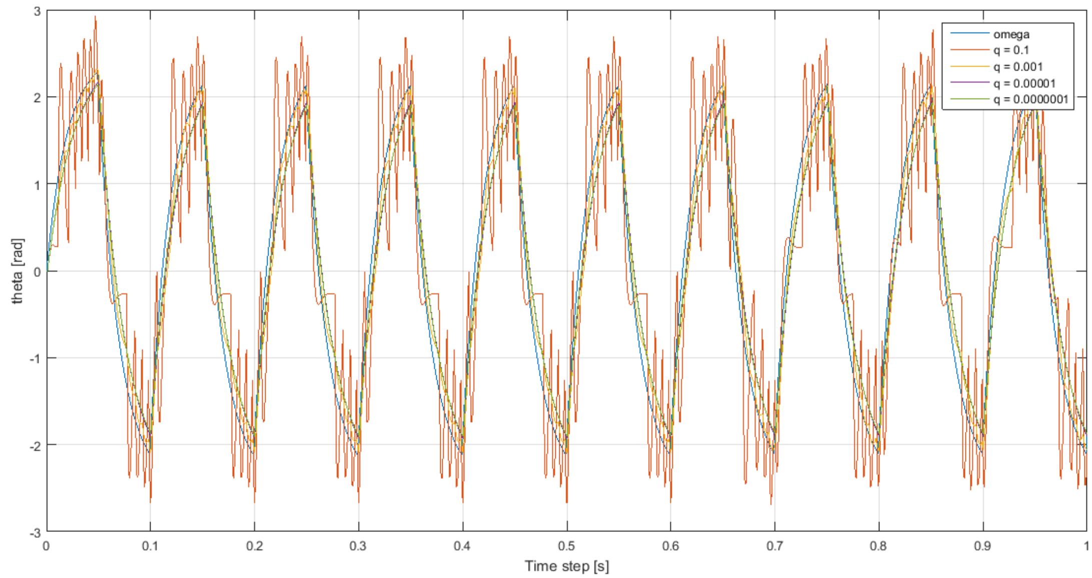


(a) Inaccurate model: Evolution of Kalman filter estimation for angular position θ during $1s$.

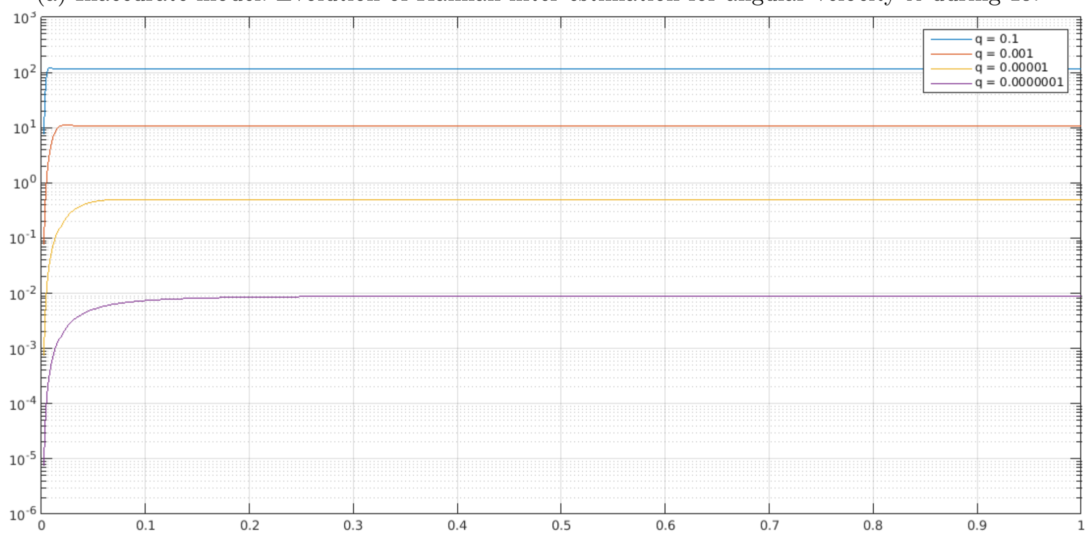


(b) Inaccurate model: Evolution of Kalman filter gain for angular position θ during $1s$.

Figure 1.8: Inaccurate model: Evolution of Kalman filter estimation and Kalman filter gain for angular position θ during $1s$.

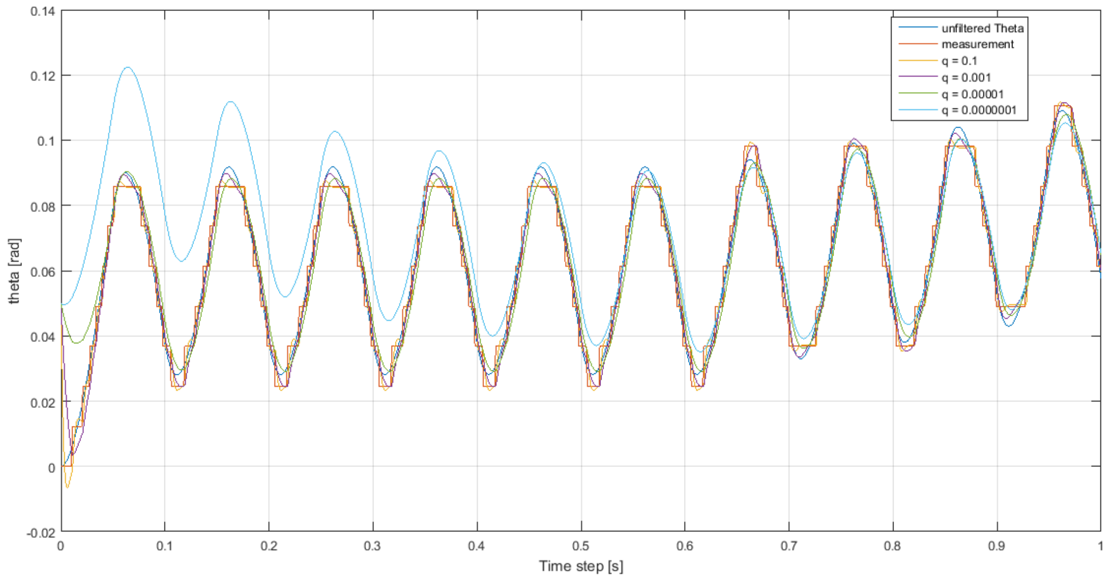


(a) Inaccurate model: Evolution of Kalman filter estimation for angular velocity Ω during $1s$.

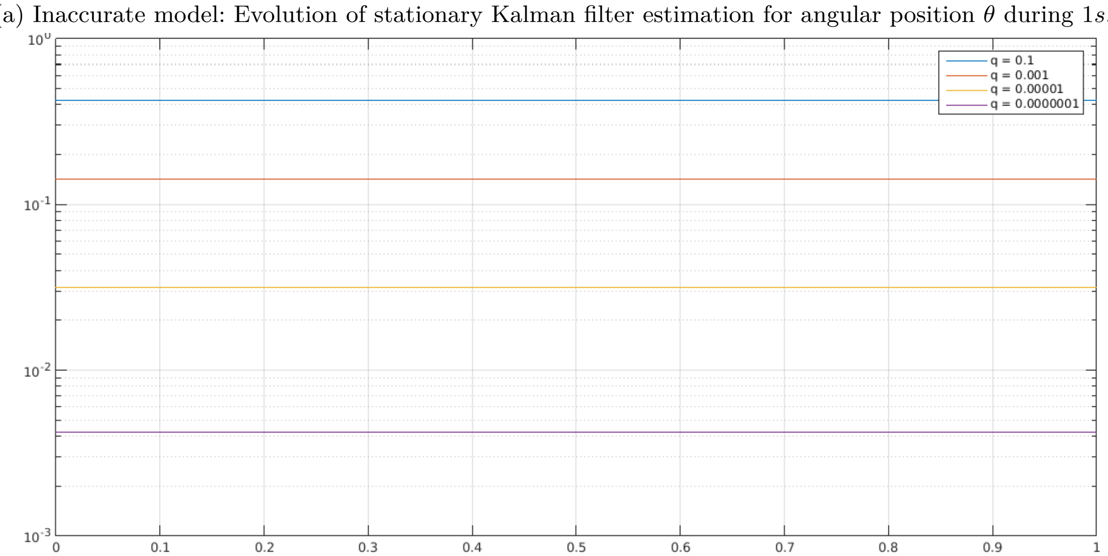


(b) Inaccurate model: Evolution of Kalman filter gain for angular velocity Ω during $1s$.

Figure 1.9: Inaccurate model: Evolution of Kalman filter estimation and Kalman filter gain for angular velocity Ω during $1s$.

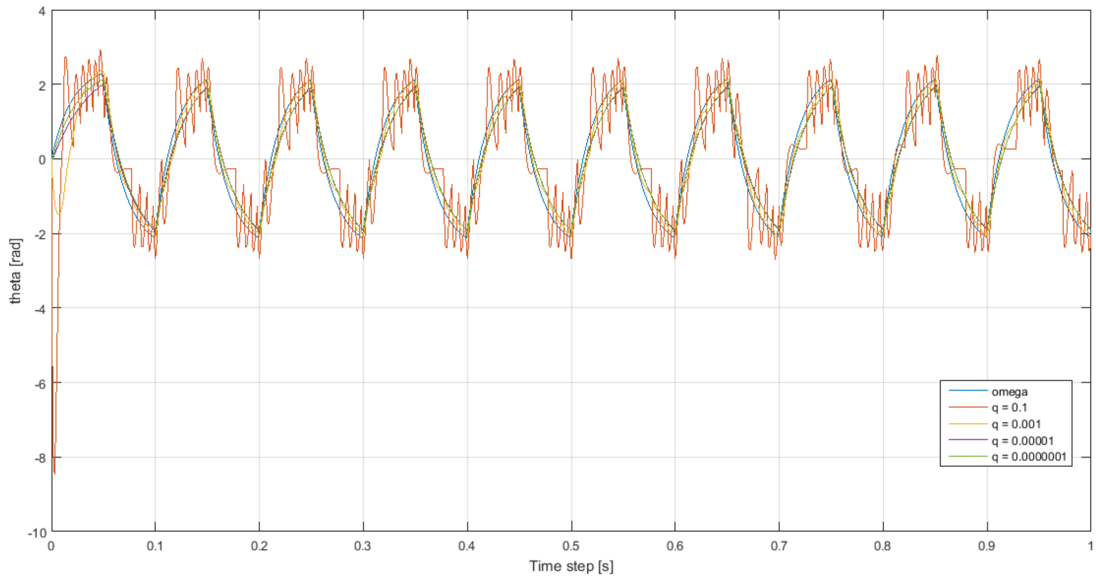


(a) Inaccurate model: Evolution of stationary Kalman filter estimation for angular position θ during $1s$.

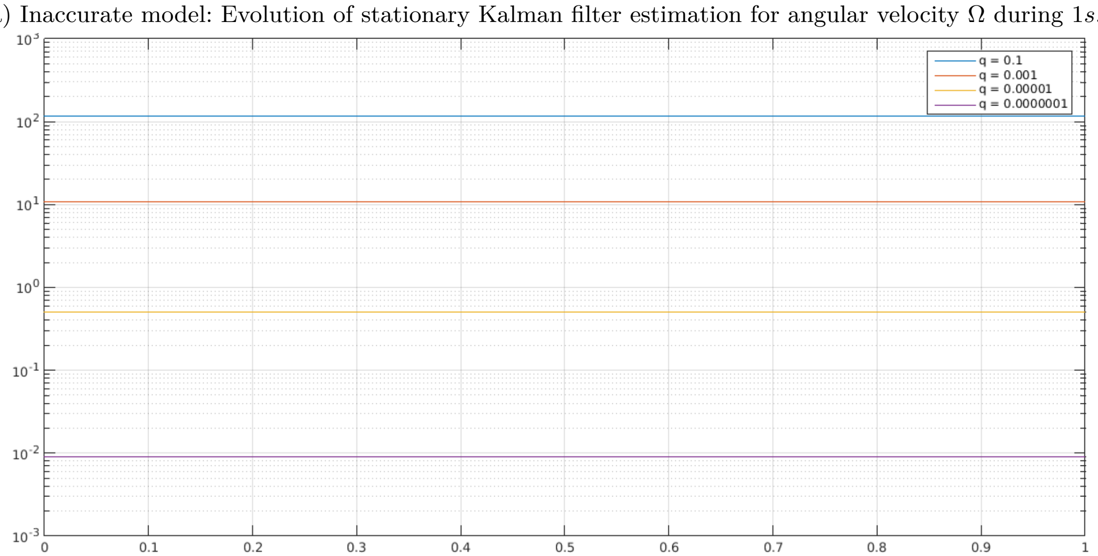


(b) Inaccurate model: Evolution of stationary Kalman filter gain for angular position θ during $1s$.

Figure 1.10: Inaccurate model: Evolution of stationary Kalman filter estimation and stationary Kalman filter gain for angular position θ during $1s$.



(a) Inaccurate model: Evolution of stationary Kalman filter estimation for angular velocity Ω during 1s.



(b) Inaccurate model: Evolution of stationary Kalman filter gain for angular velocity Ω during 1s.

Figure 1.11: Inaccurate model: Evolution of stationary Kalman filter estimation and stationary Kalman filter gain for angular velocity Ω during 1s.

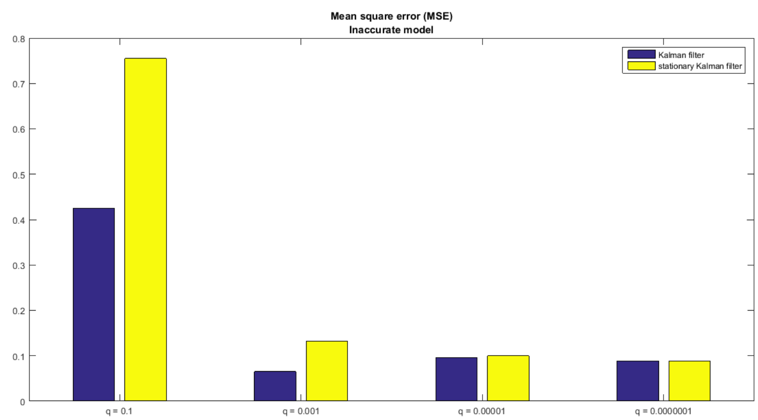


Figure 1.12: Inaccurate model: Mean square error (MSE) of Kalman filter and stationary Kalman filter estimation for different values of q.

Conclusion

Kalman filter and stationary Kalman filter for both accurate and inaccurate models of the problem statement were studied. Initialization of Kalman filter was discussed in detail, including determination of state and measurement noise, as well as initial state estimate and initial state covariance matrices. The accurate model illustrates the role of tuning the filter for reduction of process noise and input noise in a given system, in this case by varying values for variance q. However, in practice a model is never ideal and therefore, relevant parameters must be tuned to find the best estimate. Stationary Kalman filter can be used in scenarios when the Kalman filter is required to be computationally efficient or irregular behaviour of the system is very sporadic. Since Kalman gain is constant, it does not need to be computed in every iteration. For a fast convergence of the filter, time varying gain should be computed and utilized. Based on obtained results the conventional Kalman filter is also expected to perform better when the system has irregular external disturbances.