## Diffie-Hellman Protocol. (DH)

- Fix a finite cyclic group $G$ of order $n$ (e.g. $G = (\mathbb{Z}_p)^*$)
- Fix a generator $g$ in $G$ (e.g. $G = \{1, g, g^2, g^3, \ldots, g^{n-1}\}$)

| Alice | | Bob |
|---|---|---|
| choose a random $a$ in $\{1, \ldots, n\}$ | | choose a random $b$ in $\{1, \ldots, n\}$ |
| | $A = g^a \bmod p$ → | |
| | ← $B = g^b \bmod p$ | |

$$B^a_{\bmod p} = (g^b)^a = \boxed{k_{AB} = g^{ab}_{\bmod p}} = (g^a)^b = A^b_{\bmod p}$$

## Computational Diffie-Hellman (CDH)

- $G$: finite cyclic group of order $n$.
- CDH assumption holds if: $g, g^a, g^b \not\Rightarrow g^{ab}$

For all efficient algorithms $A$:

$$\Pr[A(g, g^a, g^b) = g^{ab}] < \text{negligible}$$

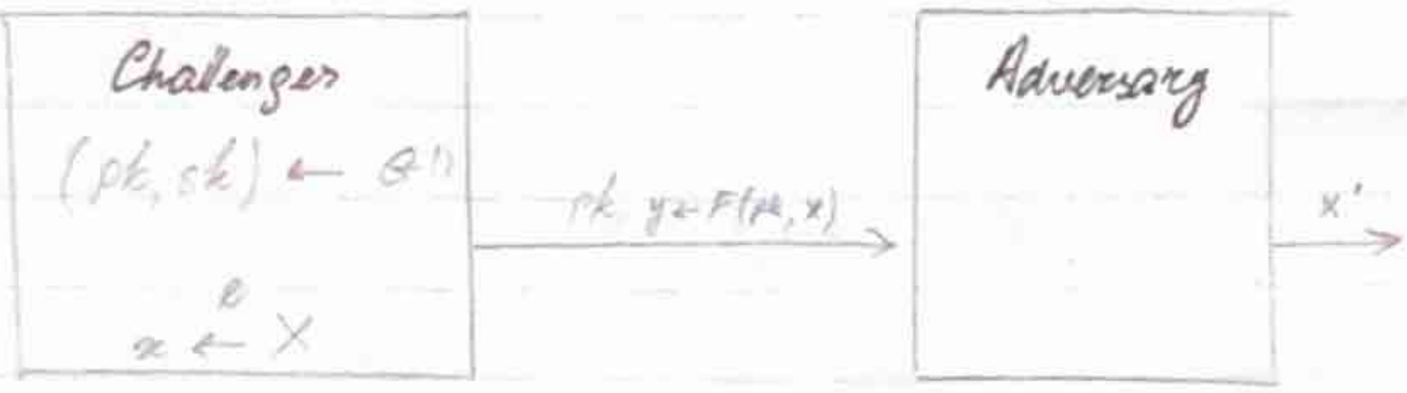
where $g \leftarrow$ a generator of $G$

$a, b \leftarrow \mathbb{Z}_n$

- Some Trapdoor Permutations (TDP's) are constructed directly from CDH

# Secure Trapdoor Functions (TDFs)

- $(G, F, F^{-1})$ is a secure TDF which can be evaluated but cannot be inverted without sk




**Def:** $(G, F, F^{-1})$ is a secure TDF if for all eff $A$:

$$Adv_{ow}[A, F] = P_r[x = x'] < negl.$$

- $(G, F, F^{-1})$ – secure TDF $X \to Y$.
- $(E_s, D_s)$ – sym. auth enc scheme over $(K, M, C)$
- $h : X \to K$ – a hash function

| $E(pk, m)$: | $D(sk, (y, c))$: |
|---|---|
| $x \leftarrow X$, $y \leftarrow F(pk, x)$ | $x \leftarrow F^{-1}(sk, y)$ |
| $k \leftarrow h(x)$, $c \leftarrow E_s(k, m)$ | $k \leftarrow h(x)$, $m \leftarrow D_s(k, c)$ |
| output $(y, c)$ | output $m$ |

- Can't apply $F$ directly to plaintext (deterministic)
  - ↳ Incorrect. $E(pk, m)$: output $c \leftarrow F(pk, m)$

## Arithmetic mod

- Let $N = pq$ where $p, q$ are prime
- $Z_N = \{0, 1, 2, \ldots, N-1\}$ $\quad (Z_N)^* = \{$ invertible elements in $Z_N\}$
- Fact $x \in Z_N$ is invertible if $\gcd(x, N) = 1$.
- Num of elements in $(Z_N)^*$ is $\varphi(N) = (p-1)(q-1) = N - p - q + 1$

- Euler's theorem: $\forall x \in (Z_N)^* : x^{\varphi(N)} = 1$

## RSA TDP

- $G()$ : choose random primes $p, q$ $\approx$ 1024 bits.
  set $N = pq$
  key generation algorithm: choose integers $e, d$ such that $e \cdot d = 1 \pmod{\varphi(N)}$
  $\Rightarrow$ output $pk = (N, e)$
  $sk = (N, d)$

- $F(pk, x) : Z_N^* \to Z_N^*$ ; $RSA(x) = x^e \quad (\in Z_N)$
  compute forward permutation

- $F^{-1}(sk, y)$ ; $y^d = RSA(x)^d = x^{ed} = x^{k\varphi(N)+1}$
  inverting the permutation $= (x^{\varphi(N)})^k \cdot x = 1 \cdot x = x$
  $x^{\varphi(N)} = 1$

**Def.** (RSA assumption) RSA is a one way permutation.

For all eff. algorithms $A$:

$$Pr[A(N, e, y) = y^{1/e}] < \text{negl}.$$

where $p, q \xleftarrow{R}$ n bit primes,
$N \leftarrow pq$
$y \leftarrow Z_N^*$ uniform