- ## Diffie-Hellman Protocol. (DH)

- Fix a finite cyclic group G of order n (e.g. $G = (\mathbb{Z}_p)^*$)
- Fix a generator g in G (e.g., $G = \{1, g, g^2, g^3, \ldots, g^{n-1}\}$)

Alice | **Bob**

choose a random a in $\{1, .., n\}$ | Choose a random b in $\{1, .., n\}$

$$\xrightarrow{A = g^a \bmod p}$$

$$\xleftarrow{B = g^b \bmod p}$$

$$B^a_{\bmod p} = (g^b)^a = \boxed{k_{AB} = g^{ab}_{\bmod p}} = (g^a)^b = A^b_{\bmod p}$$

- ## Computational Diffie-Hellman (CDH)

- G: finite cyclic group of order n.
- CDH assumption holds if: $g, g^a, g^b \not\Rightarrow g^{ab}$

For all efficient algorithms A:

$$\Pr[A(g, g^a, g^b) = g^{ab}] < \text{negligible}$$

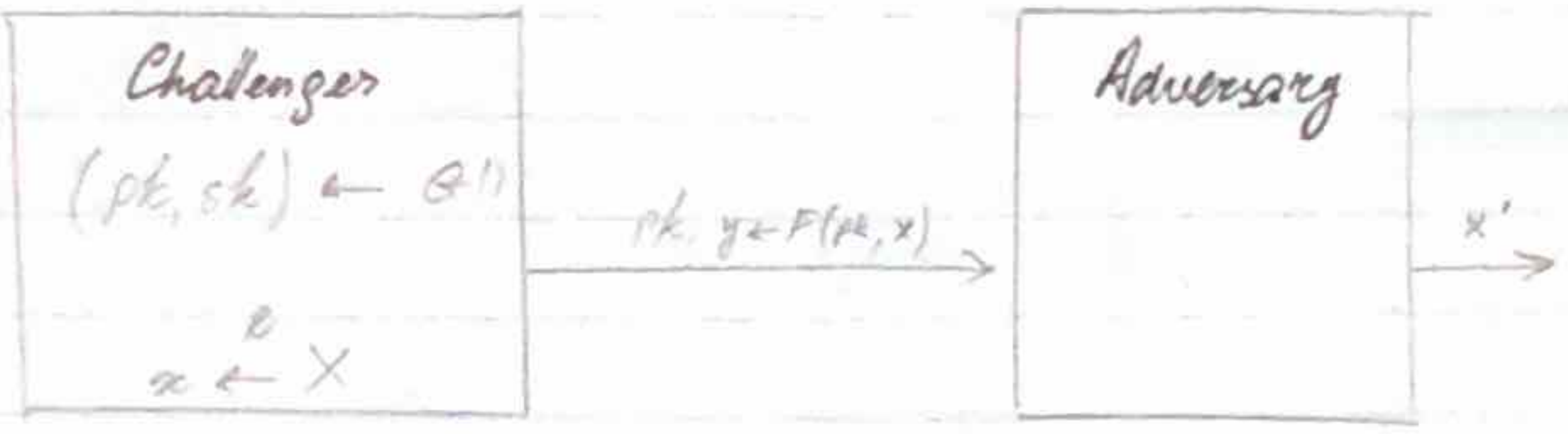
where $g \leftarrow \{\text{generators of } G\}$

$a, b \leftarrow \mathbb{Z}_n$

- Some Trapdoor Permutations (TDPs) are constructed directly from CDH

Secure Trapdoor Functions (TDFs)

- (G, F, F^{-1}) is a secure TDF which can be evaluated but cannot be inverted without sk



Def: (G, F, F^{-1}) is a secure TDF if for all eff A:

$$Adv_{ow}[A, F] = Pr[x = x'] < negl.$$

- (G, F, F^{-1}) - secure TDF $X \to Y$.
- (E_s, D_s) - sym. auth enc scheme over (K, M, C)
- $H : X \to K$ - a hash function

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E(pk, m):
    x ← X,  y ← F(pk, x)
    k ← H(x),  c ← E_s(k, m)
    output (y, c)
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D(sk, (y, c)):
    x ← F^{-1}(sk, y)
    k ← H(x),  m ← D_s(k, c)
    output m
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- Cannot apply F directly to plaintext (deterministic)

↳ Incorrect. $E(pk, m)$: output $c \leftarrow F(pk, m)$

Arithmetic mod

- Let $N = p\,q$ where p, q are prime
- $Z_N = \{0, 1, 2, \ldots, N-1\}$, $(Z_N)^* = \{$invertable elements in $Z_N\}$
- Fact $x \in Z_N$ is invertable iff $\gcd(x, N) = 1$.
- Num of elements in $(Z_N)^*$ is $\varphi(N) = (p-1)(q-1) = N - p - q + 1$
- Euler's theorem: $\forall x \in (Z_N)^* : x^{\varphi(N)} = 1$

RSA TDP

- $G()$: choose random primes p, q ~ 1024 bits.
  generation algorithm
  set $N = pq$
  choose integers e, d such that $e \cdot d = 1 \pmod{\varphi(N)}$
  => output $pk = (N, e)$
  $sk = (N, d)$
- $F(pk, x) : Z_N^* \rightarrow Z_N^*$; $RSA(x) = x^e$ (in Z_N)
  compute trapdoor permutation
- $F^{-1}(sk, y)$; $y^d = RSA(x)^d = x^{ed} = x^{k\varphi(N)+1}$
  inverting the permutation
  $= \underbrace{(x^{\varphi(N)})^k}_{1^k = 1} \cdot x = 1 \cdot x = x$

Def. (RSA assumption) RSA is a one way permutation.

For all eff. algorithms A:

$$\Pr[A(N, e, y) = y^{1/e}] < \text{negl.}$$

where $p, q \xleftarrow{R}$ n bit primes,
$N \leftarrow pq$
$y \leftarrow Z_N^*$