Control Systems - ENGR 33041

Lecture 2: Laplace Transform

Instructor:

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Slides prepared based on

Control Systems Technology, C. Johnson and H. Malki

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Recap of Lecture 1

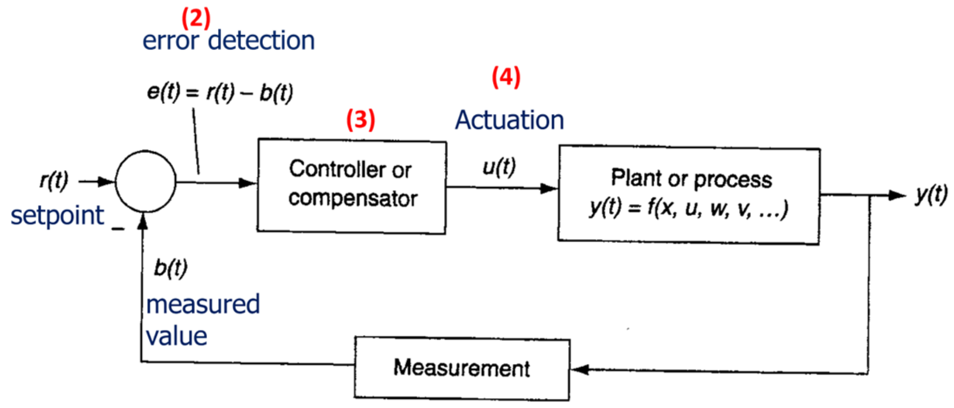


FIGURE 1.4
Basic control strategy demonstrated by a block diagram.

$$\text{Transfer Function} = \frac{\text{Block Output}}{\text{Block Input}}$$

Linear Operations

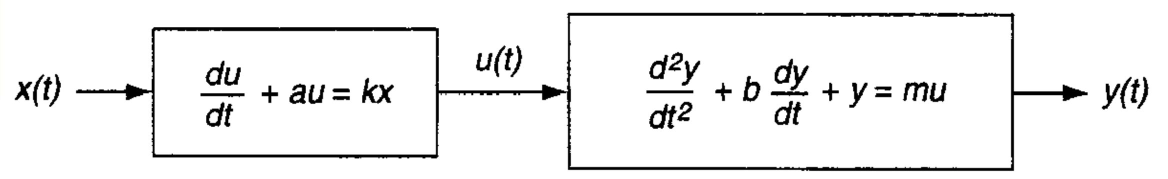
1. **Additivity** $f(x_1 + x_2) = f(x_1) + f(x_2)$

2. **Homogeneity** $f(\alpha x) = \alpha f(x)$

addition, subtraction, integration, differentiation, multiplication by constant, and division by content (nonzero) are linear functions

Purpose of Laplace Transform

- Typical transfer functions are linear differential equations with constant coefficients.



$$\frac{1}{m}\frac{d^3y}{dt^3} + \frac{a+b}{m}\frac{d^2y}{dt^2} + \frac{1+ab}{m}\frac{dy}{dt} + \frac{a}{m}y = kx$$

- Classical methods of differential equations are not able to find solutions for third order and beyond and in a practical control system there will be many more blocks (higher order systems).

- Solution:
  - Applying Laplace Transforms allows us to solve differential equations using algebra.

Transformation Concept

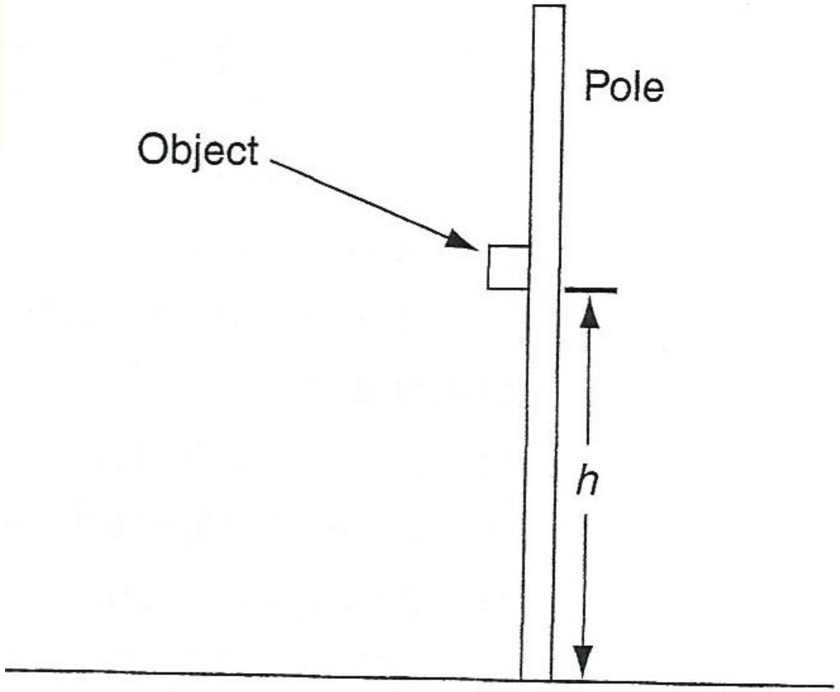


FIGURE 3.1
Measuring the height, *h*, of an object on a pole.

$$h = L\tan(\theta)$$

$$\theta = \tan^{-1}(h/L)$$

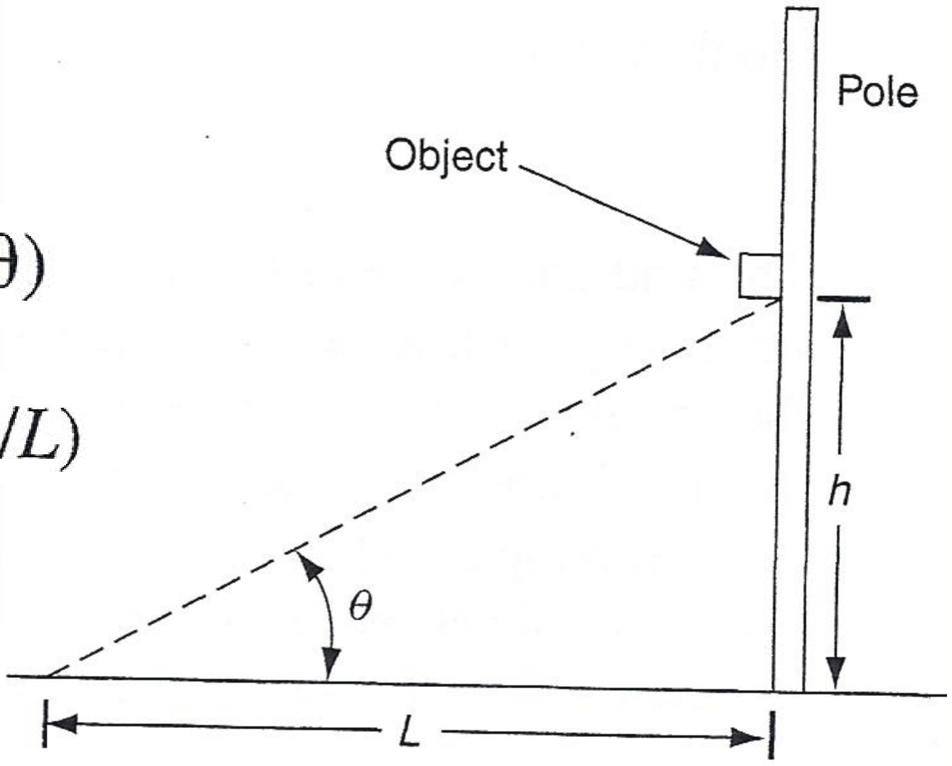


FIGURE 3.2
Transforming to a new variable, θ, to find the height.

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3.2 Introduction

- **The Laplace transform is a mathematical transformation of all functions and operations in time *t*, into functions and operations involving a complex variable *s*, which is related to frequency.**

- **Most complicated functions of time, such as exponential and trigonometric functions become simple algebraic functions of *s*.**

- **Most operations in time, including differentiation and integration, become simple algebraic operations in *s*.**

- **As a result, solving complicated time equations becomes solving algebraic equations in *s*.**

Advantages of Laplace Transforms

1. It allows us to predict many aspects of the performance of a control system without needing to completely solve the system equations.

2. Both the homogeneous and particular solutions of the differential equations can be obtained simultaneously.

Laplace Variable

$$s = \sigma + j\omega$$

The real part, σ, will be related to

exponential growth or decay in time

The imaginary part, ω, will be related to an oscillation angular frequency.

$$\omega = 2\pi f.$$

(rad/s) hertz (Hz)

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3.3 Definition of Laplace Transform

$$F(s) = \mathscr{L}[f(t)] \equiv \int_0^{\infty} f(t)e^{-st}dt$$

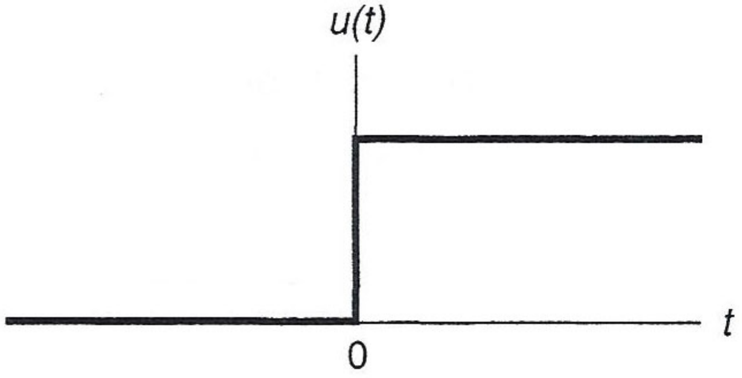
This is one-sided Laplace transform as time starts from 0 to ∞.

It assumes that the response of a system equals zero before *t* = 0.

This is valid for most of the control system applications.

Ex. 3.1 Find the Laplace transform of a unit-step function as shown in figure 3.3 and defined as follows:

$$u(t) = 0, \quad t < 0$$
$$u(t) = 1, \quad t \geq 0$$



Solution:

Similarly, Laplace transform of a constant number K:

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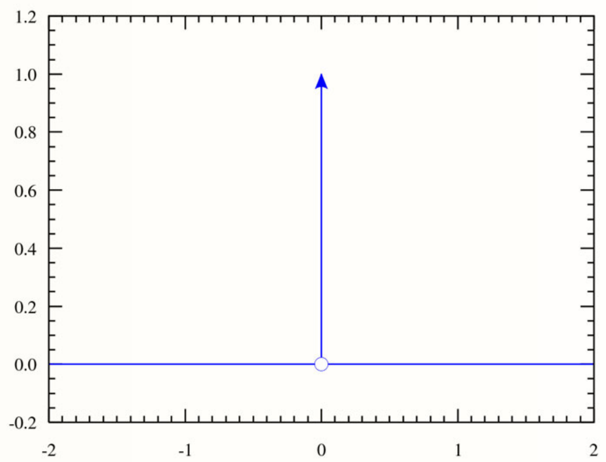
Ex. 3.2 Find the Laplace transform of an impulse function (Dirac delta function).

Impulse function or Dirac delta function is defined by the following properties:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



Solution:

$$\mathscr{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt$$

$$\mathscr{L}[\delta(t)] = 1$$

Ex. 3.3

Find the Laplace transform of exponential function.

$$f(t) = e^{-at} \text{ for } t \geq 0$$

$$f(t) = 0 \text{ otherwise}$$

Solution:

Laplace transform table

TABLE 3.1
Laplace Transforms of Functions

Time Function	Laplace Transform
Impulse, $\delta(t)$	1
Step, $u(t) = 1$	$\frac{1}{s}$
k, k a constant number	$\frac{k}{s}$
Ramp, t	$\frac{1}{s^2}$
Polynomial, t^n	$\frac{n!}{s^{n+1}}$
Exponential, e^{-at}	$\frac{1}{s+a}$
Ramp exponential, te^{-at}	$\frac{1}{(s+a)^2}$
Polynomial exponential, $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
Sine, $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine, $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped sine, $e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine, $e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Ex. 3.5

Use table 3.1 to find the Laplace transforms of the following time functions.

a. $f(t) = 14te^{-2t}$

b. $g(t) = -6e^{-4t}\sin(10t)$

TABLE 3.1
Laplace Transforms of Functions

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Step, $u(t) = 1$	$\frac{1}{s}$
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Polynomial, t^n	$\frac{n!}{s^{n+1}}$
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Solution:

Ex. 3.6 Find the Laplace transform of the following time function using MATLAB

$$f(t) = 4te^{-2t}\cos(10t + 20)$$

Solution

- **Select MATLAB Icon and double click on it**
- **In the MATLAB command window write:**

```
syms t s % variables t and s are symbolic
f = 4*t*exp(-2*t)*cos(10*t+20); % our expression
F=laplace (f)  % command to find Laplace transform
```

- **Now, if you want to simplify the Laplace transform answer and make it more readable and nicer, you can use the following two commands in MATLAB:**

```
simplify (F) % simplify our answer
pretty (ans) % make our answer readable and nicer
```

Helpful materials to learn MATLAB and Laplace transform in MATLAB

- **Introduction to MATLAB (46.5 min)**

https://www.youtube.com/watch?v=7bnVx34yQf4

- **MATLAB tutorial - Laplace transform demonstration (6 min)**

https://www.youtube.com/watch?v=rlesPBN6Whw

PROPERTIES OF LAPLACE TRANSFORMS

1. Linearity theorem

$$\mathscr{L}[a_1 f(t) \pm a_2 g(t)] = a_1 \mathscr{L}[f(t)] \pm a_2 \mathscr{L}[g(t)] \tag{3.5}$$

Example:

$$f(t) = 1 - 5e^{-3t} + 10\sin(3t)$$

Taking Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \mathcal{L}[1 - 5e^{-3t} + 10\sin(3t)] = \mathcal{L}[1] - 5\mathcal{L}[e^{-3t}] + 10\mathcal{L}[\sin(3t)]$$

$$\longrightarrow \boxed{F(s) = \frac{1}{s} - \frac{5}{s+3} + \frac{30}{s^2+9}}$$

Ex. 3.7

- **Find the Laplace transform of the following functions: (use Table 3.1 with linearity theorem)**

- $f(\mathrm{t}) = 3\mathrm{e}^{-5\mathrm{t}} + 6\mathrm{e}^{-3\mathrm{t}}$

- $f(t) = \cos 3t + 3e^{-2t} \sin 5t$

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It is important to realize that while the linearity theorem specifies that the Laplace transform of a sum/difference of time functions is the sum/difference of the individual Laplace transforms, the same is not true for products and divisions.

The Laplace transform of a product of time functions is **NOT** equal to the product of individual Laplace transforms.

Important Note

$$\mathscr{L}[f(t)g(t)] \neq \mathscr{L}[f(t)]\mathscr{L}[g(t)]$$

2. Derivative theorem

If all initial conditions are set to zero, the derivative theorem states that the n^{th} derivative operation on a time function appears as the s^n times the Laplace transform of the function:

$$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) \quad \text{(all initial conditions set to zero)} \qquad \textbf{(3.6)}$$

Ex. 3.8

Determine the Laplace transform of the following differential equation. Assume all of the initial conditions are zero.

$$\mathscr{L}\left[\frac{d^2}{dt^2}f(t) + 5\frac{d}{dt}f(t) + 3f(t) = 10\right]$$

Solution

Using the linearity and derivative properties, we have

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3. Integral theorem

$$\mathscr{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

Ex. 3.9

Find the Laplace transform of the following equation:

$$\frac{df}{dt} + 2f(t) + \int f(t)dt = 1$$

Solution

4. Final-value theorem (FVT)

$$\lim_{t \to \infty}[f(t)] = \lim_{s \to 0}[sF(s)]$$

Both conditions must hold to use FVT:

1) all non-zero roots of denominator of F(s) must have negative real parts.

2) At most one root of denominator of F(s) is at zero (origin).

If any of the above conditions is not satisfied, FVT does not hold!

Ex. 3.10

Determine the steady-state value of the following Laplace transform by using final-value theorem.

$$F(s) = \frac{5}{s(s^2 + 3s + 6)}$$

Solution

First, we need to check the FVT conditions:

- ✓ all non-zero roots of denominator of F(s) must have negative real parts
- ✓ At most one root of denominator of F(s) is at zero

Both conditions hold, so we can apply the FVT theorem:

Ex. 3.11

Determine if FVT holds for the following Laplace transforms, and if, find the final value of the time function using FVT theorem:

i. $F(s) = \frac{9}{s^2+9}$

ii. $F(s) = \frac{12}{s^2(s+5)(s^2+3s+2)}$

5. Initial-value theorem (IVT)

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]$$

Ex. 3.12

Use IVT to find the initial value of the time function represented by the Laplace transform $F(s) = \frac{3(s+2)}{s(s+7)}$

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Summary of Laplace Transform Theorems

TABLE 3.2
Laplace Transform Theorems

Time Operation	Laplace Transform Operation
Linearity, $K_1 f_1(t) + K_2 f_2(t)$ (K_1 and K_2 are constants)	$K_1 F_1(s) + K_2 F_2(s)$
nth derivative, $\dfrac{d^n f}{dt^n}$	$s^n F(s)$, if all initial conditions are zero
Integral, $\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$
Initial value theorem	$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]$
Final value theorem	$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$

INVERSE LAPLACE TRANSFORM

Ex. 3.13

Find the time functions of the following Laplace transforms using table 3.1.

a. $F(s) = \dfrac{5}{s} + \dfrac{3}{s+2}$

b. $G(s) = \dfrac{45}{s^2 + 9}$

c. $R(s) = 4 + \dfrac{2}{s} - \dfrac{6}{s^2}$

d. $Q(s) = \dfrac{3s + 6}{(s+2)^2 + 25}$

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3.5.1 Partial-Fraction Expansion

$$F(s) = \frac{9s + 39}{s^2 + 8s + 15}$$

$$F(s) = \frac{3}{s+5} + \frac{6}{s+3}$$

$$f(t) = 3e^{-5t} + 6e^{-3t}$$

Next week, we continue the Laplace transform and talk more about the partial-fraction expansion.

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HW 2, due Sep. 5, 11 AM

HW2 Instructions:

1. Introduce yourself, including your major and background.
2. Summarize your teaching, research, co-op, internship, and work experience. You may also include additional information about yourself such as any projects or academic awards you have received. The total length should be between **150** and **250** words.

- Please upload your write-up as a single Word or PDF file on Canvas.
- You may attach your CV/resume to the write-up, but it is not mandatory.
- If you bring your CV/resume to my next office hours on September 3, 1-3 PM and introduce yourself in person, you will receive an additional bonus point for this assignment.