Control Systems - ENGR 33041

Lecture 4: Control System Models

Instructor:

Hossein Mirinejad, Ph.D.

Slides prepared based on Textbook

Control Systems Technology, C. Johnson and H. Malki

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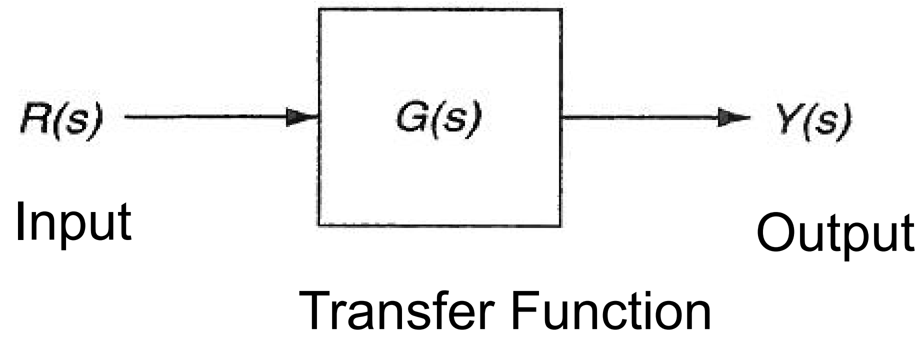
Transfer Functions

- The transfer function of a block is defined by the operation the block performs on its input to get the output.
- It is common that the input, output, and the system components (shown as blocks) are all represented by their Laplace transforms.
- Each block is represented by its Transfer Function (TF).
- Transfer function *G(s)* is the ratio of the block output to the block input in the Laplace transform domain ("s" domain).



Block Transfer Functions

Suppose a block has an input x(t) and an output y(t), where the input and output are related by the following differential equations:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 6y = 4\frac{dx}{dt} + 16x$$

Taking the Laplace Transform

(Assuming all initial conditions are zero)

$$s^2Y(s) + 8sY(s) + 6Y(s) = 4sX(s) + 16X(s)$$

$$(s^2 + 8s + 6)Y(s) = (4s + 16)X(s)$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{4(s+4)}{s^2 + 8s + 6} \qquad \textbf{(4.2)}$$

G(s): The transfer function is often called the *natural response* of the block. The transfer function is also called the *impulse response* because if the input $x(t)$ is an impulse, $x(t) = \delta(t)$, then its transform is $X(s) = 1$ and equation 4.2 shows that the output will just be the response of the system to this impulse input (impulse response):

$$Y(s)\Big|_{\text{impulse}} = G(s)X(s)\Big|_{\text{impulse}} = G(s)\cdot 1 = \frac{4(s+4)}{s^2 + 8s + 6} \qquad \textbf{(4.3)}$$

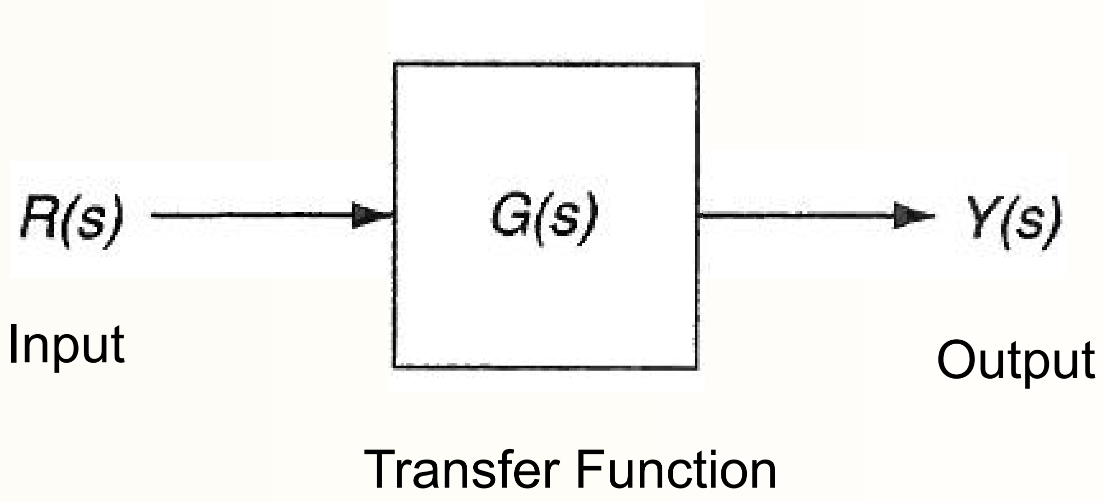


FIGURE 4.1
Representation of a block by a transfer function.

Ex. 4.1

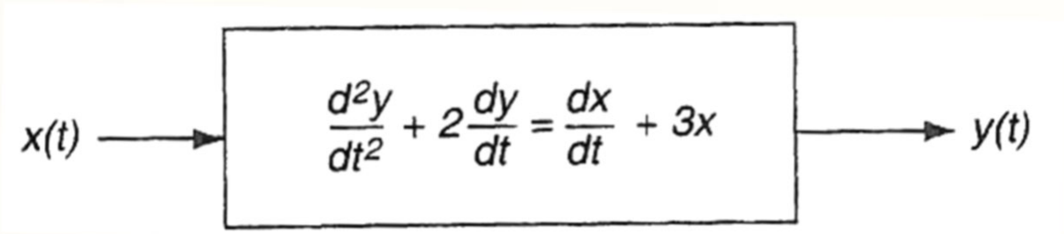


FIGURE 4.2
Differential equation block for example 4.1.

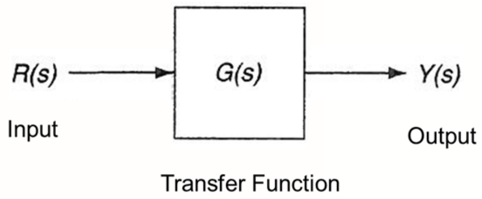
a) What is the transfer function of the block shown in Figure 4.2?

Solution:

b) Find the impulse response in time domain?

c) What is the steady-state output?

In general, the differential equation inside a block is derived from electrical, mechanical, and/or chemical properties of the physical device the block represents. Let's take a look at this example to see how the differential equation and then transfer function can be found for a physical system:

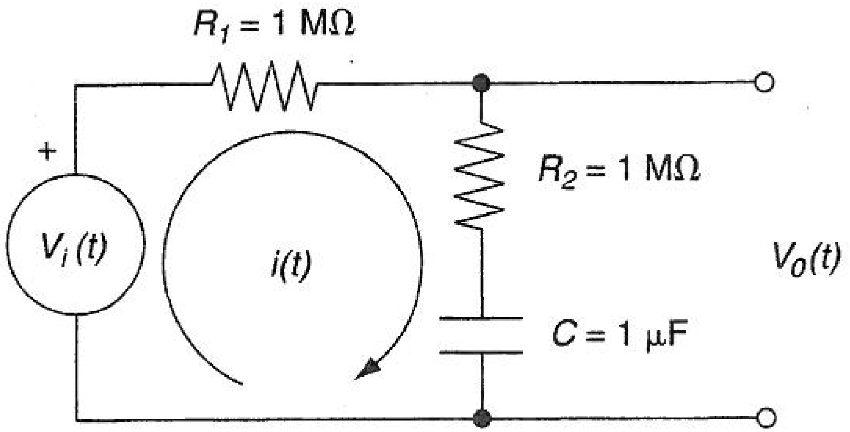


Electrical System

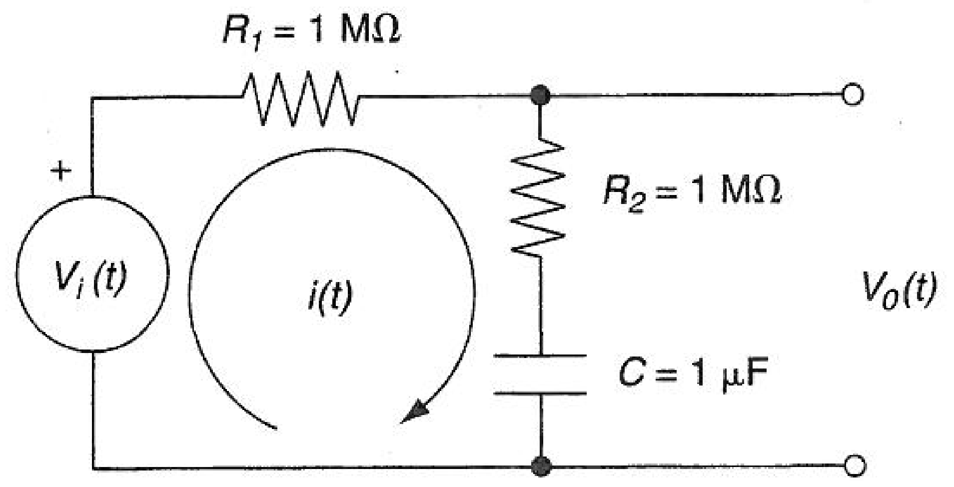
Ex. 4.2

Determine the transfer function of the RC filter circuit shown in figure 4.3.

FIGURE 4.3
Circuit for finding a transfer function in example 4.2.



Solution:



First, a clockwise loop current , *i(t)* is assigned.

Kirchhoff's Voltage Law (KVL) is used to write equations for $v_i(t)$ and $v_o(t)$

$$R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)dt - v_i(t) = 0$$

$$v_i(t) = (R_1 + R_2)i(t) + \frac{1}{C}\int i(t)dt \qquad \text{(i)}$$

Similarly, for $v_o(t)$:

$$R_2 i(t) + \frac{1}{C}\int i(t)dt - v_o(t) = 0$$

$$v_o(t) = R_2 i(t) + \frac{1}{C}\int i(t)dt \qquad \text{(ii)}$$

Taking Laplace Transform of (i) and (ii) →

$$V_i(s) = (R_1 + R_2)I(s) + \frac{1}{sC}I(s) = \left(R_1 + R_2 + \frac{1}{sC}\right)I(s)$$

$$V_o(s) = R_2 I(s) + \frac{1}{sC}I(s) = \left(R_2 + \frac{1}{sC}\right)I(s)$$

$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{R_2 + \dfrac{1}{sC}}{R_1 + R_2 + \dfrac{1}{sC}}$$

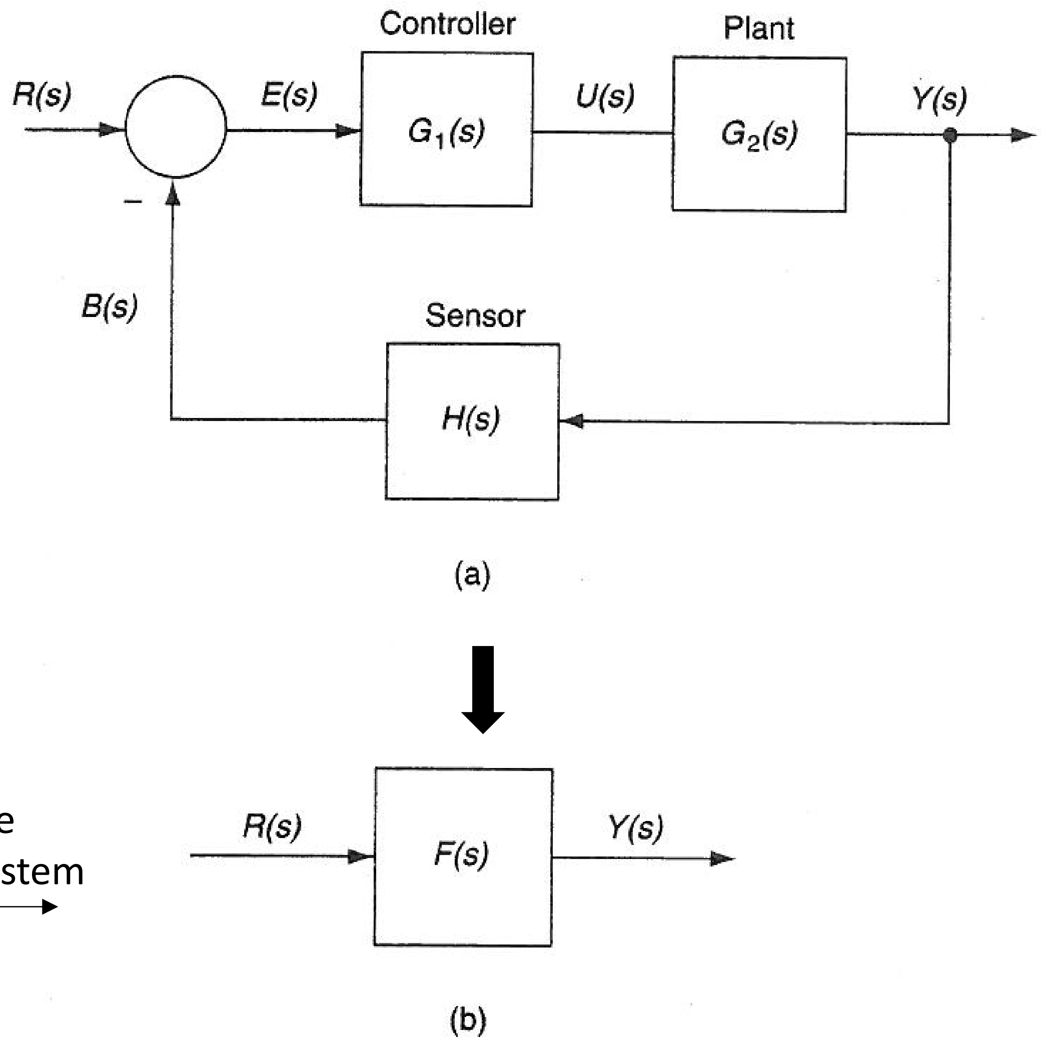
$$G(s) = \frac{sCR_2 + 1}{sCR_1 + sCR_2 + 1}$$

Substituting $R_1 = R_2 = 1\ \text{M}\Omega$ and $C = 1\ \mu F$, we have

$$G(s) = \frac{s + 1}{2s + 1} = \frac{0.5(s + 1)}{s + 0.5}$$

Transfer Function Properties

FIGURE 4.6
Generic block diagram of a control system with transfer functions.



System Transfer Function
Or
Closed-Loop Transfer Function

Single block
Representing the
entire control system

Property 1. The transfer function of any block in the control system can be obtained from the differential equation of that block by assuming all initial conditions are zero.

Ex. 4.5

The following differential equation represents a physical model of a control system block. Determine the transfer function of the system.

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 5x$$

Property 2. A differential equation can be obtained from the transfer function by applying the Laplace transform derivative theorem.

TABLE 3.2
Laplace Transform Theorems

Time Operation	Laplace Transform Operation
Linearity, $K_1 f_1(t) + K_2 f_2(t)$ (K_1 and K_2 are constants)	$K_1 F_1(s) + K_2 F_2(s)$
nth derivative, $\dfrac{d^n f}{dt^n}$	$s^n F(s)$, if all initial conditions are zero
Integral, $\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$
Initial value theorem	$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]$
Final value theorem	$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$

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Ex. 4.6

Obtain the differential equation of the following control system element transfer function,

$$\frac{Y(s)}{X(s)} = \frac{2s + 2}{s^2 + 3s + 6}$$

Property 3. Values of s for which the numerator of the system transfer function is equal to zero are called *zeros* of the system. Values of s for which the denominator of the system transfer function is equal to zero are called the *poles* of the system. In general, the denominator and numerator can be written as polynomials in s. Finding the zeros means setting the numerator polynomial equal to zero and finding the roots. Likewise, the poles are found by setting the denominator polynomial equal to zero and finding the roots.

Ex. 4.7

Find the poles and zeros of the following system transfer function.

$$F(s) = \frac{Y(s)}{R(s)} = \frac{5}{s^2 + 5s + 6}$$

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Note:

When the denominator of the system transfer function (closed-loop transfer function) is set to zero, the resulting expression is called the **Characteristic Equation** of the system.

The **Characteristic equation** is used to find the poles of the closed-loop transfer function, known as closed-loop poles. These poles determine the stability and dynamic responses of a control system, as will be shown later in Lecture 8.

What is the characteristic equations of the system in Ex 4.7?

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Ex. 4.8

Find the poles and zeros of the following system transfer function.

$$F(s) = \frac{2s + 2}{(s^2 + 3s + 6)(s + 5)}$$

Solution:

Setting the numerator equal to zero gives $2s + 2 = 2(s + 1) = 0$, so that there is a zero located at $z = -1$. Setting the denominator equal to zero provides,

$$(s^2 + 3s + 6)(s + 5) = 0$$

So, clearly, one possibility is for $s = -5$, therefore there is a pole at $p = -5$. The quadratic will provide two other values of s to get zero. These will be determined using the quadratic equation,

$$s^2 + 3s + 6 = 0 \longrightarrow p = \frac{-3 \pm \sqrt{3^2 - 4(1)(6)}}{2(1)} = -1.5 \pm \frac{\sqrt{15}}{2} j$$

Therefore, the poles of this transfer function are:

$$p = -5 \qquad p = -1.5 + \frac{\sqrt{15}}{2} j \qquad p = -1.5 - \frac{\sqrt{15}}{2} j$$

Property 4. The inverse Laplace transform of the system transfer function times the corresponding input will yield the time response of the system. Mathematically, this can be written as,

$$y(t) = \mathscr{L}^{-1}[F(s)R(s)]$$

Proof:

Ex. 4.9

Find the time response of the following transfer function assuming the input is a unit-step function [$R(s) = 1/s$].

$$F(s) = \frac{Y(s)}{R(s)} = \frac{5}{s^2 + 5s + 6}$$

Solution:

First we find the poles (roots) of the denominator, $s^2 + 5s + 6 = 0$,

$$p = \frac{-5 \pm \sqrt{25 - 4(6)}}{2} = -2.5 \pm 0.5$$

So,

$$s^2 + 5s + 6 = (s + 2)(s + 3).$$

$$\longrightarrow \quad F(s) = \frac{Y(s)}{R(s)} = \frac{5}{(s + 2)(s + 3)}$$

Applying Property 4,

$$y(t) = \mathscr{L}^{-1}[F(s)R(s)] = \mathscr{L}^{-1}\left[\frac{5}{s(s+2)(s+3)}\right]$$

Applying the partial-fraction expansion theorem,

$$F(s)R(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_i = (s - p_i)F(s)\Big|_{s=p_i}$$

Applying this, we find the constants and can then write,

$$F(s)R(s) = \frac{5/6}{s} - \frac{5/2}{s+2} + \frac{5/3}{s+3}$$

The time response y(t) can be found by inverse Laplace transform from Table 3.1 as:

$$y(t) = \mathscr{L}^{-1}[F(s)R(s)] \longrightarrow \boxed{y(t) = \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t}}$$

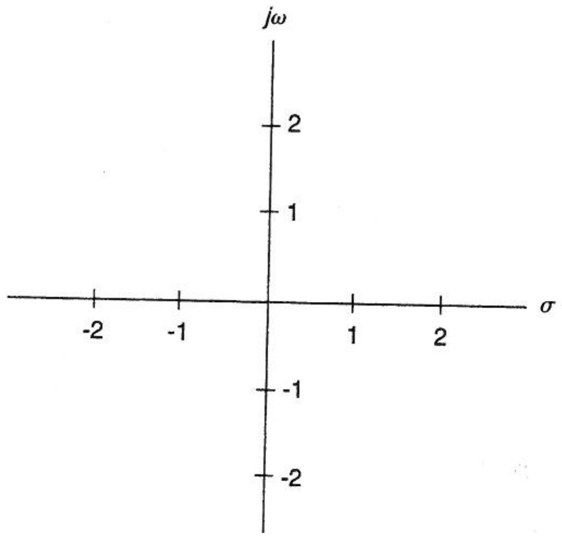
One of the major problems in a study of control systems is the inability to determine the poles and zeros of the system transfer function. Much of control system analysis consists of using approximate methods to determine the poles and zeros so that the stability and dynamic response of the system can be determined. Seldom is it necessary to actually find the time function of the system. The poles and zeros are instead used to determine essential properties of the system.

Modern computer software is of great value in finding the poles and zeros of the system transfer function and even the time response of the system for specific inputs. In addition, manual approximation techniques have now been adapted into computer software.

S-Plane Plot

- The Laplace variable **s** is a complex number: $s = \sigma + j\omega$
- To display properties of a control system, its often useful to make plots in the complex s-plane.
- In s-plane plot, the x-axis denotes the real part of s, i.e. σ and the y-axis denotes the imaginary part of s, i.e. $j\omega$.

FIGURE 4.11
Structure of an *s*-plane plot.



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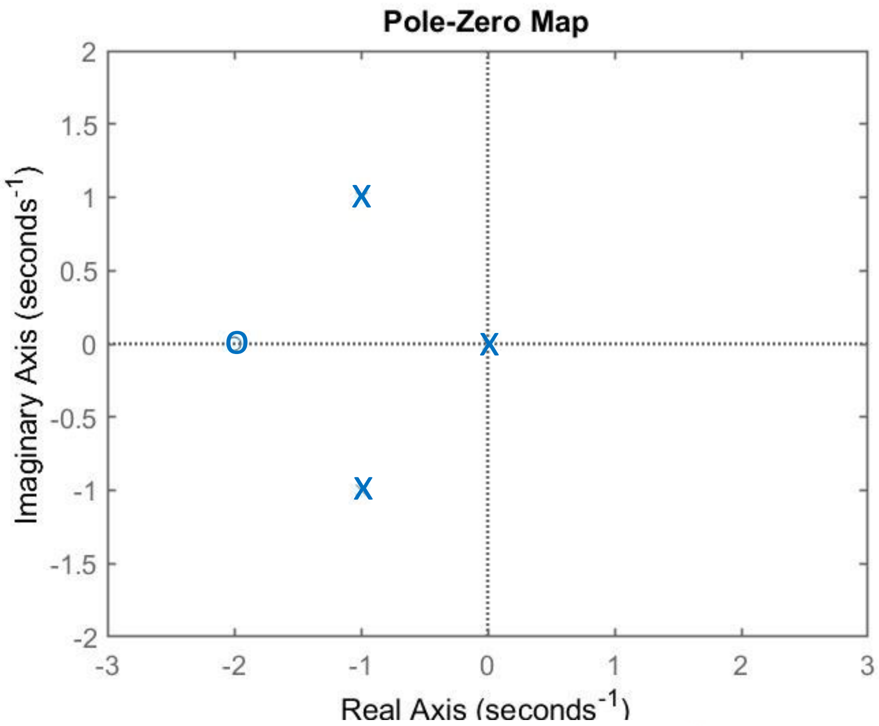
Poles and Zeros in S-Plane plot

In the s-plane plot, the **poles** are shown by **cross** (x) and **zeros** are shown by **circle** (o).

S-Plane plot for transfer function $P(s) = \dfrac{s+2}{s^3+2s^2+2s}$



$$P(s) = \frac{s+2}{s^3+2s+2s} = \frac{s+2}{s(s^2+2s+2)} = \frac{s+2}{s(s+1+j)(s+1-j)}$$

$$s^2+2s+2=0 \longrightarrow s = \frac{-2 \pm \sqrt{2^2-4(1)(2)}}{2} = -1 \pm j$$

Ex. 4.12

For the following system transfer function, plot the pole-zero in the s-plane.

$$P(s) = \frac{(s+4)}{s(s+6)(s^2+6s+34)}$$

Solution:

We have a zero at -4, poles at 0 and -6, but we must find the roots of the quadratic to see what poles it will contribute.

$$p = \frac{-6 \pm \sqrt{36 - 4(34)}}{2} = -3 \pm \frac{\sqrt{-100}}{2} = -3 \pm 5j$$

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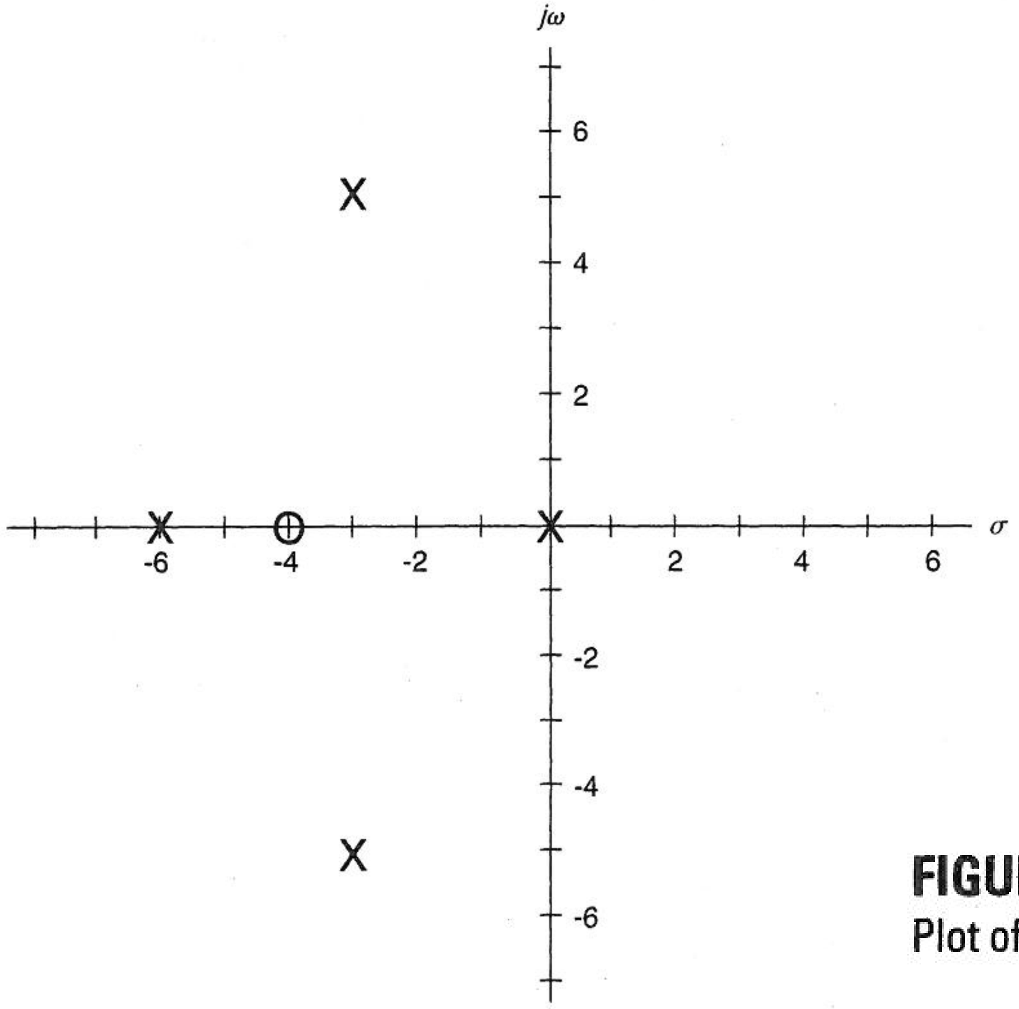

FIGURE 4.12
Plot of the poles and zeros of example 4.12.

Ex. 4.13

Determine the transfer function involved in the pole-zero pattern shown in figure 4.13.

Solution:

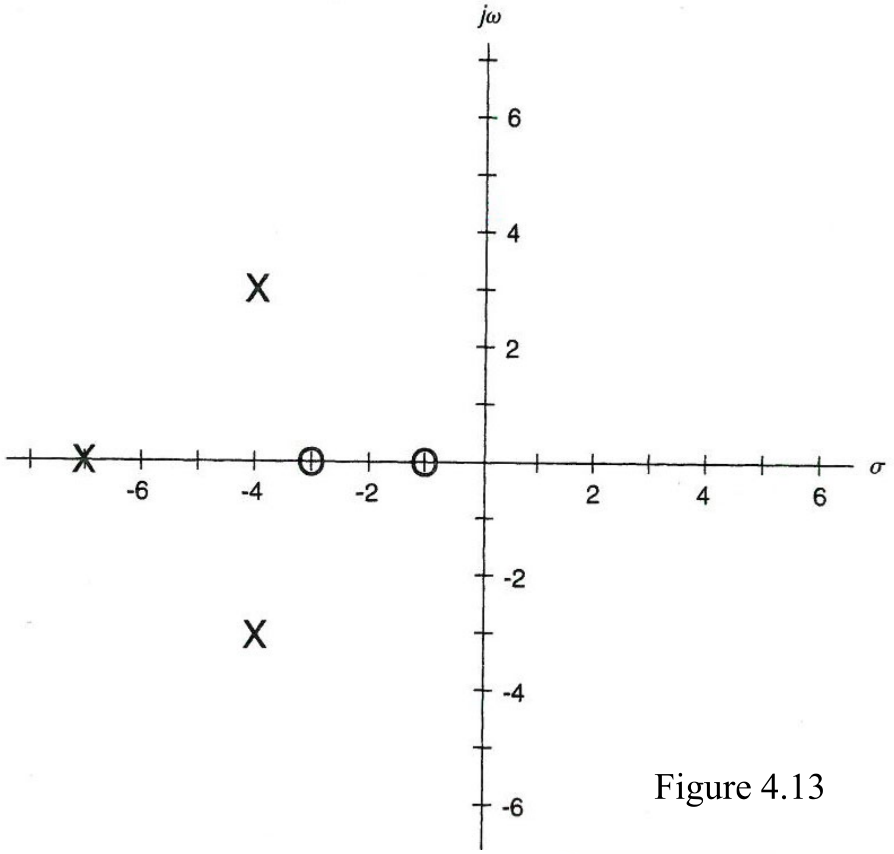

Figure 4.13

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Ex. 4.14

(MATLAB Example)

The transfer function for some system has been found to be given by,

$$F(s) = \frac{100}{s^3 + 9s^2 + 23s + 220}$$

Use MATLAB to find the poles and zeros of F(s) and draw the pole-zero plot.

Solution

To find the poles we need to find the roots of the characteristic equation, which is just the denominator set equal to zero,

$$s^3 + 9s^2 + 23s + 220 = 0$$

Find the roots of a polynomial in MATLAB using "roots" command: →

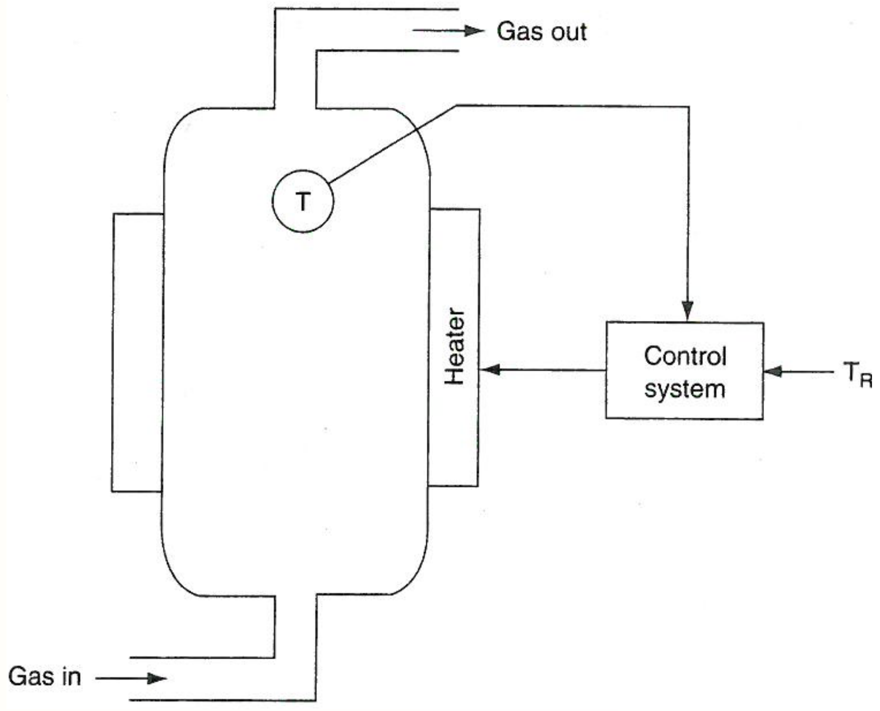
→ Draw s-plane plot in MATLAB using "pzmap" command:

Ex. 4.15 (MATLAB Example)

Figure 4.7 shows a system that consists of a holding tank into which gas is flowing in and out. The gas comes in at some variable temperature. A control system is to heat and regulate the outlet gas temperature, T, to a reference value, T_R. The transfer function of the control system has been found to be,

$$\frac{Y(s)}{T_R(s)} = \frac{4.7s^2 + 41.9s + 130}{s^3 + 4s^2 + 69s + 130}$$

where the time scale is minutes and we use $Y(s)$ to represent the Laplace transform of the temperature, T. (a) Find the poles of the system and write the transfer function in terms of these poles. (b) At $t = 0$, the control system is turned on to control the gas to a temperature of 200°C. Determine the gas temperature as a function of time and graph it over a suitable range to demonstrate transient and steady-state response.



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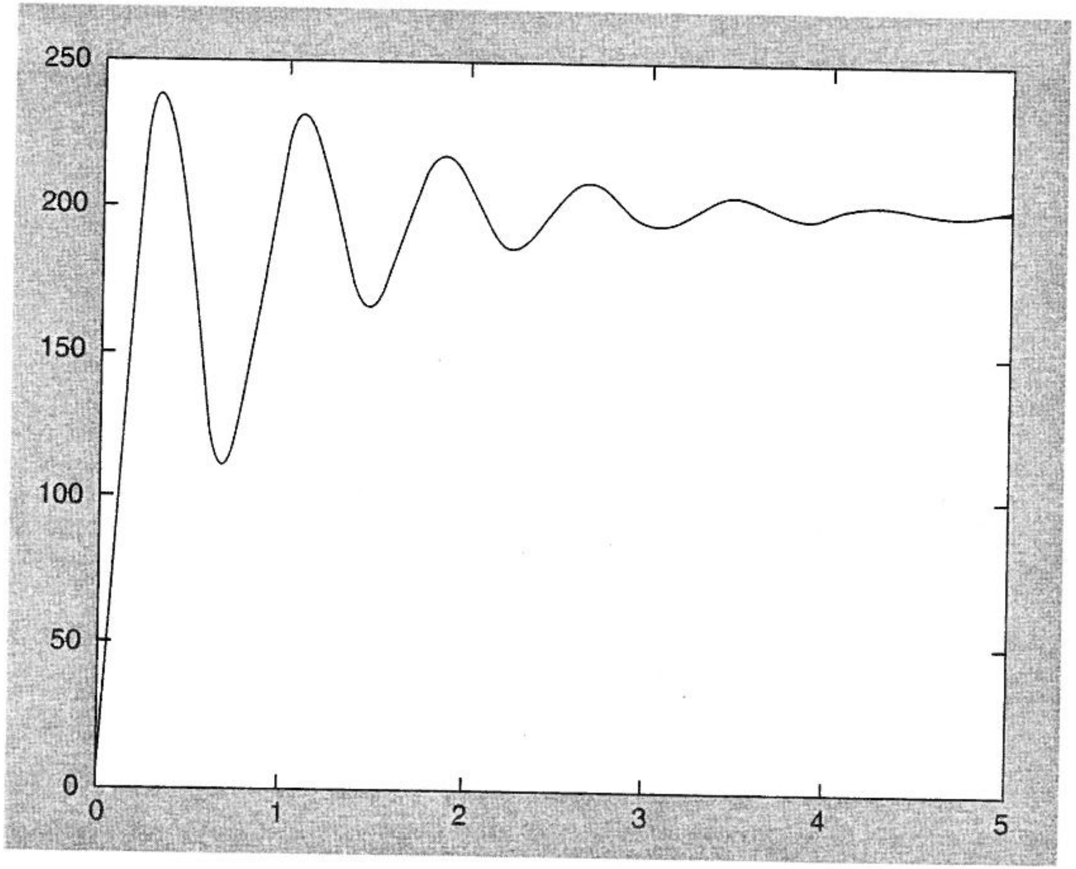
- **Plotting in MATLAB**



- **HW 4** is due on Sep. 26 at 11 AM and must be submitted on Canvas.
- There will be **no class on Sep. 17** (Reading Day).
- **Exam 1** will be held in class on Sep. 19 and will cover **Lectures 1-3.**
- Exam 1 will be closed-book; however, you are allowed to bring one page of formulas (front and back, U.S. Letter size). The Laplace Transform table (Table 3.1) will be provided in the exam.
- <u>No</u> calculators are allowed in the exam; please bring only a pen or pencil. Additionally, MATLAB is <u>NOT</u> part of the exam.