A

COMPLEX NUMBERS

This appendix provides a review of complex numbers. A complex number is a method for representation of two-dimensional (orthogonal) data that is written as follows:

$$z = x + jy \tag{A.1}$$

where x is the real part, y is the imaginary part, and $j = \sqrt{-1}$. The complex conjugate of z is denoted by z^* and simply means the imaginary part has the opposite sign,

$$z^* = x - jy \tag{A.2}$$

A complex number can also be represented in a *polar* form involving magnitude and phase angle. In this case, the number is written as,

$$z = Re^{j\theta} \tag{A.3}$$

The magnitude of the complex number z is defined as,

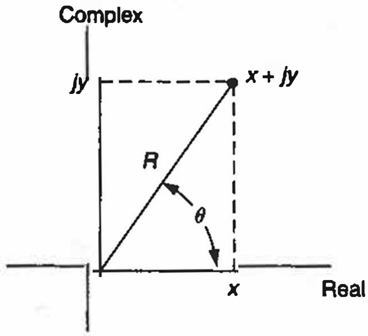
$$R = |z| = \sqrt{x^2 + y^2} \tag{A.4}$$

and the phase angle is defined by,

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) \tag{A.5}$$

Figure A.1 shows a graphical representation of a complex number. The real part is plotted on the horizontal axis and the complex part on the vertical axis. The phase angle is the angle from the positive real axis and varies from 0 to $\pm$ 180°. To get

FIGURE A.1
Rectangular and polar representation of a complex number.



the phase angle correct—in the correct quadrant—it is very important to note the signs of the real and imaginary parts.

EXAMPLE A.1

Contrast the phase angles of the following complex numbers: (a) 4 + 3j, (b) −4 + 3j,
(c) −4 − 3j, and (d) 4 − 3j.

Solution:
From equation A.5, we find,

a. $\theta_a = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$, which is in the 1st quadrant

b. $\theta_b = \tan^{-1}\left(\frac{3}{-4}\right) = \tan^{-1}\left(-\frac{3}{4}\right) = 126.9°$ because it is in the second quadrant

c. $\theta_c = \tan^{-1}\left(\frac{-3}{-4}\right) = \tan^{-1}\left(\frac{3}{4}\right) = -126.9°$ because it is in the third quadrant

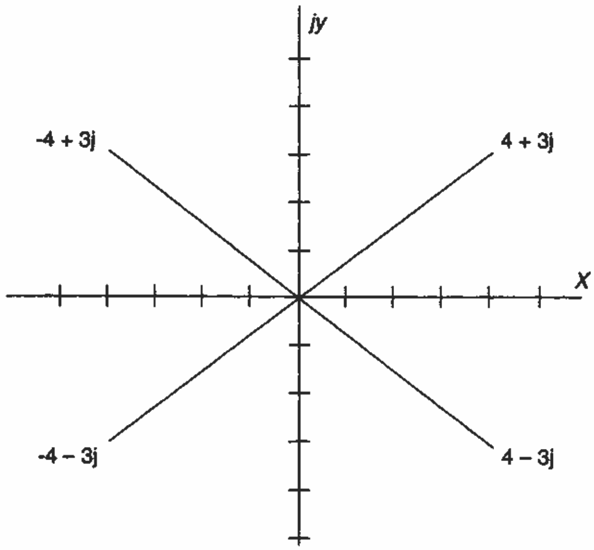
d. $\theta_d = \tan^{-1}\left(\frac{-3}{4}\right) = \tan^{-1}\left(-\frac{3}{4}\right) = -36.9°$ because it is in the fourth quadrant

Notice that the pairs (a) and (c) as well as (b) and (d) compute to the same angle if no account is taken for the signs of the real and imaginary parts. Thus, the correct angle has to be interpreted from signs of the individual real and imaginary parts to determine the correct quadrant of the angle. Figure A.2 shows the graphical view of these complex numbers.

Complex Number Operations The following operations involving complex numbers,
$z = x + jy$ and $w = u + jw$, are presented as a review of their properties:

$$\text{Product:} \quad zw = (x + jy)(u + jw) = (xu - yw) + j(xw + yu)$$

FIGURE A.2
Complex number in four quadrants.



or $\quad zw = |z|e^{j\theta}|w|e^{j\phi} = |z||w|e^{j(\theta+\phi)}$ **(A.6)**

Division: $\quad \dfrac{z}{w} = \dfrac{x + jy}{u + jw} = \dfrac{(x + jy)(u - jw)}{(u + jw)(u - jw)} = \dfrac{(xu + yw) + j(yu - xw)}{(u^2 + w^2)}$

or,

$$\frac{z}{w} = \frac{|z|e^{j\theta}}{|w|e^{j\phi}} = \frac{|z|}{|w|}e^{j(\theta-\phi)} \tag{A.7}$$

where $|z|$, $|w|$, θ, and ϕ are defined from equations A.4 and A.5. Generally, we prefer to use the polar form for multiplication and division because the mathematical processes are simpler.