Control Systems - ENGR 33041

Lecture 10: Frequency Response Method (Bode Plot)

Instructor:

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Slides prepared based on:

Modern Control Engineering, K. Ogata

The Fundamentals of Control Theory, B. Douglas

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Frequency response method for control systems

Definition:

By the term *frequency response,* we mean the steady-state response of a system to a sinusoidal input.

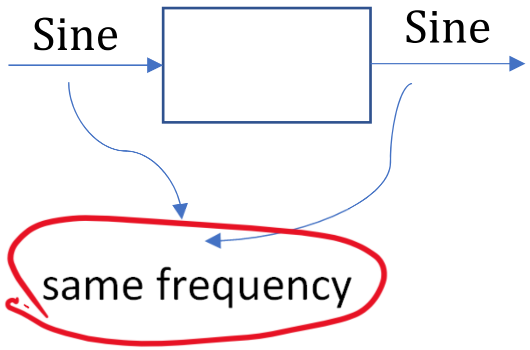
In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

The information we get from frequency response analysis is different from what we get from root-locus analysis. In fact, the frequency response and root-locus approaches complement each other.

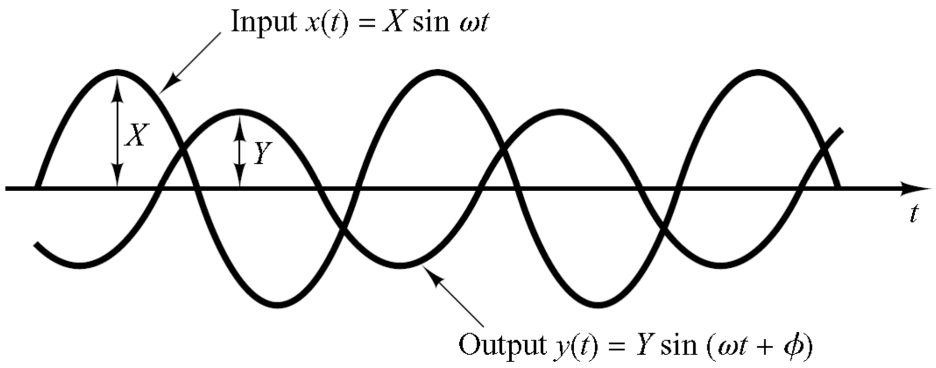
Advantage

- Frequency response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment.

In a stable linear system, a sine wave input always generates a sine wave output of the same frequency.



In fact, the only two differences possible between an input sine wave and the output sine wave that is generated are the amplitude and phase.



Amplitude = height of the sine wave
Phase = shifting sine wave in time

Definition: **Sinusoidal transfer function**

A transfer function in which s is replaced by $j\omega$, where ω is frequency.

$G(j\omega)$ is a complex number.

$$G(s) = \frac{K}{Ts + 1} \longrightarrow G(j\omega) = \frac{K}{jT\omega + 1}$$

Amplitude: $|G(j\omega)|$

phase: $\angle G(j\omega)$

Obtaining steady-state outputs to sinusoidal inputs

The steady-state output of a transfer function system to the sinusoidal input can be obtained directly from the sinusoidal transfer function (replacing s with $j\omega$).

Amplitude of output sinusoid = Amplitude of input sinusoid × Amplitude of sinusoidal transfer function

Phase of output sinusoid = Phase of input sinusoid + Phase of sinusoidal transfer function

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Math Reminder: If you have a fraction with both numerator & denominator represented by complex numbers:

$$G(jw) = \frac{a+bj}{c+dj}$$

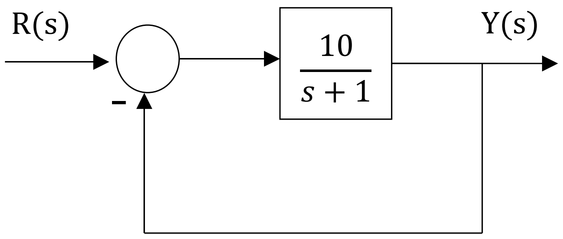
$$|G(jw)| = \frac{|num|}{|den|} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$

$$\angle G(jw) = \angle num - \angle den = \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(\frac{d}{c}\right)$$

Ex 10.1

Consider a unity-feedback system with the open-loop transfer function:

$$G(s) = \frac{10}{s+1}$$



Obtain the steady-state output of the system when it is subjected to the following input:

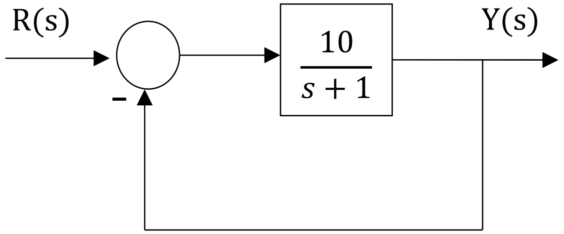
$$r(t) = \sin(t + 30^\circ) \longrightarrow y(t) = A \sin(t + \theta)$$

Solution

In a stable LTI system, a sine wave input always generates a sine wave output of the same frequency at the steady-state.

Amplitude of the output sine = Amplitude of the input sine x Amplitude of sinusoidal TF

Phase of the output sine = Phase of the input sine + Phase of sinusoidal TF



The open-loop transfer function: $G(s) = \dfrac{10}{s+1}$

The closed-loop transfer function: $F(s) = \dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{G(s)+1} = \dfrac{\frac{10}{s+1}}{\frac{10}{s+1}+1} = \dfrac{10}{s+11}$

Sinusoidal closed-loop TF: $F(j\omega) = \dfrac{10}{j\omega + 11}$

$$\rightarrow |F(j\omega)| = \frac{10}{\sqrt{\omega^2 + 121}}$$

$$\rightarrow \measuredangle F(j\omega) = -\tan^{-1}\left(\frac{\omega}{11}\right)$$

$r(t) = \sin(t + 30°) \longrightarrow \omega = 1\ \text{rad/s}$

$$\rightarrow |F(j1)| = \frac{10}{\sqrt{1+121}} = 0.905$$

$$\rightarrow \measuredangle F(j1) = -\tan^{-1}(1/11) = -5.19°$$

$y_{ss}(t) = 1 \times 0.905 \sin(t + 30° - 5.19°) \longrightarrow \boxed{y_{ss}(t) = 0.905 \sin(t + 24.8°)}$

Introduction to Bode Plot (Bode diagram)

So far, we have learned that we can find the steady-state response to a sinusoidal input, using the sinusoidal transfer function (replacing s with $j\omega$ in TF).

In fact, the steady-state response of an LTI system to a sinusoidal input is a sinusoidal output of the same frequency with possible different amplitude and phase:

$$\boxed{\begin{array}{c}\text{Amplitude of output} \;=\; \text{Amplitude of input} \;\times\; \text{Amplitude of sinusoidal transfer function} \\ \text{Phase of output} \;=\; \text{Phase of input} \;+\; \text{Phase of sinusoidal transfer function}\end{array}}$$

If you were only concerned with the response to a single frequency, the above formula would work. However, more often than not, you are going to be interested in the whole frequency spectrum. i.e. You change the sinusoidal input frequency and look at the response at each frequency. So, how can you visualize the gain and phase shift across the entire spectrum?

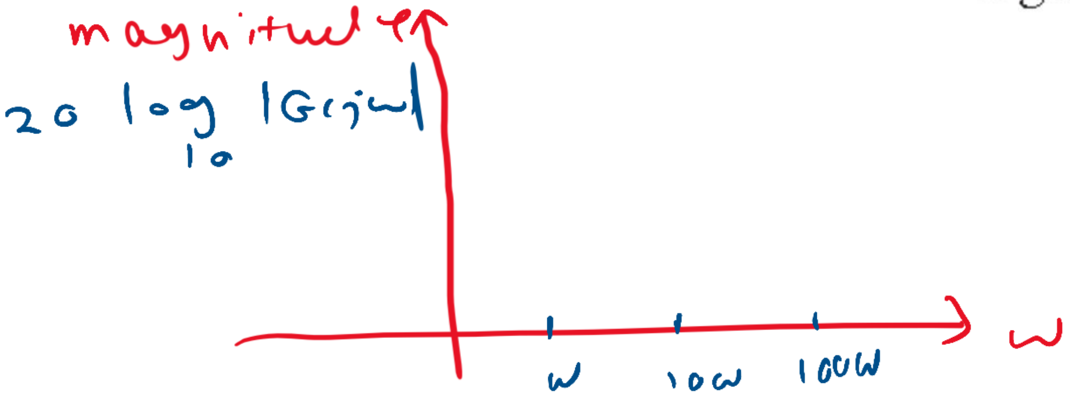
One way to do that is with "Bode Plot".

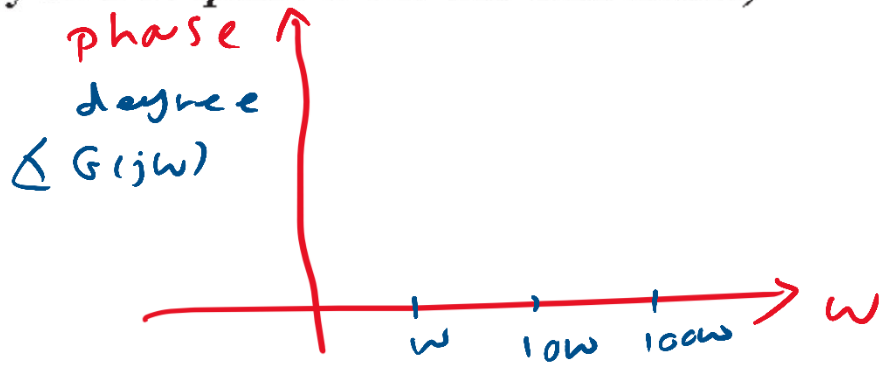
Hendrik Wade Bode
(1905-1982)

Bode Plot (diagram)

Bode Diagrams or Logarithmic Plots. A Bode diagram consists of two graphs: One is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20 \log|G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (but in decibels) or phase angle (in degrees). (The frequency range of interest determines the number of logarithmic cycles required on the abscissa.)





Bode Plot in MATLAB

The command **bode** computes magnitudes and phase angles of the system and plot it in MATLAB.

```
bode(num,den)                 % draw Bode plot in MATLAB
```

You can also customize the horizontal (semilog) axis in Bode plot. For example, the following command generates **100** points logarithmically equally spaced between **0.01** and **1000 rad/s**:

```
w=logspace(-2,3,100)          % vector w includes 100 points logarithmically equally spaced between
                              % 10^-2 rad/s and 10^3 rad/s
```

and you can draw Bode plot with these customized frequency points:

```
bode(num,den,w)               % draw Bode plot in MATLAB with user-specified frequency points w
```

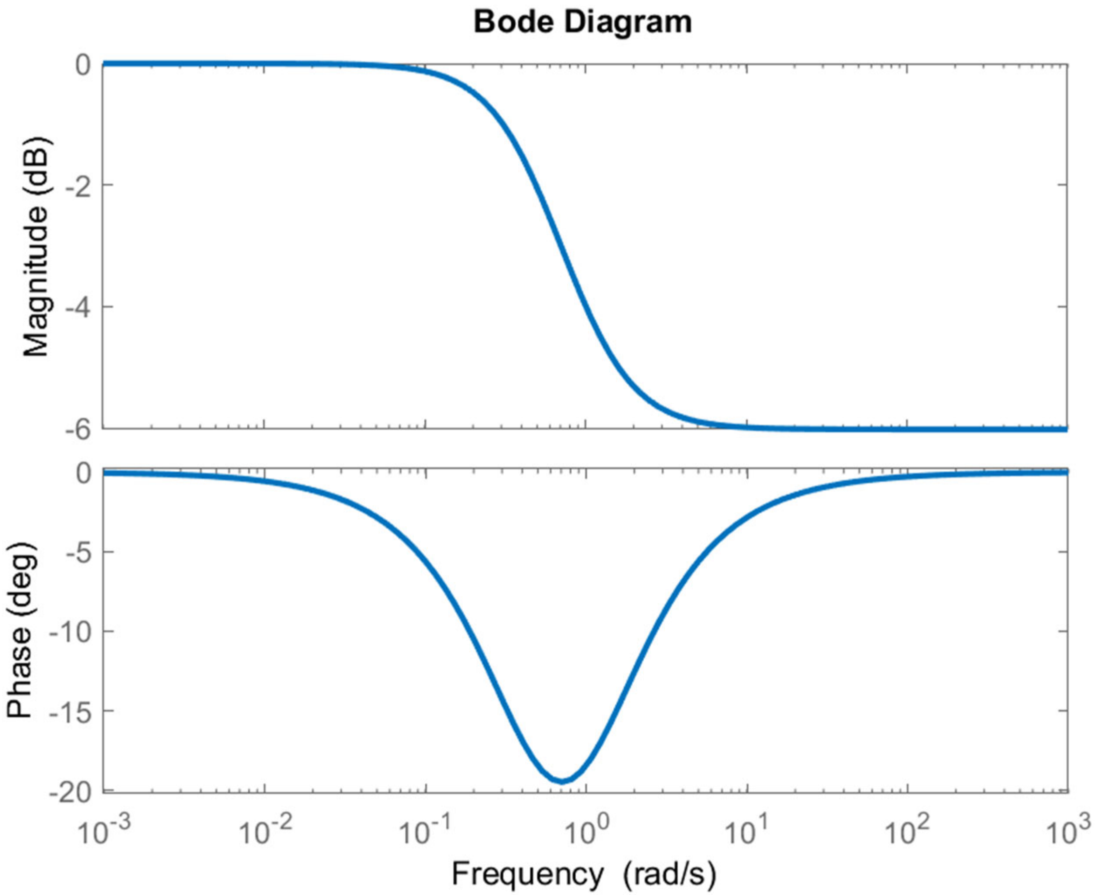
Ex 10.2

Using MATLAB, draw the Bode plot of $G_1(s)$ in the frequency range of 0.001 rad/s and 1000 rad/s:

$$G_1(s) = \frac{s+1}{2s+1}$$

Solution:

```
>> bode([1 1], [2 1], logspace(-3,3,100))
```



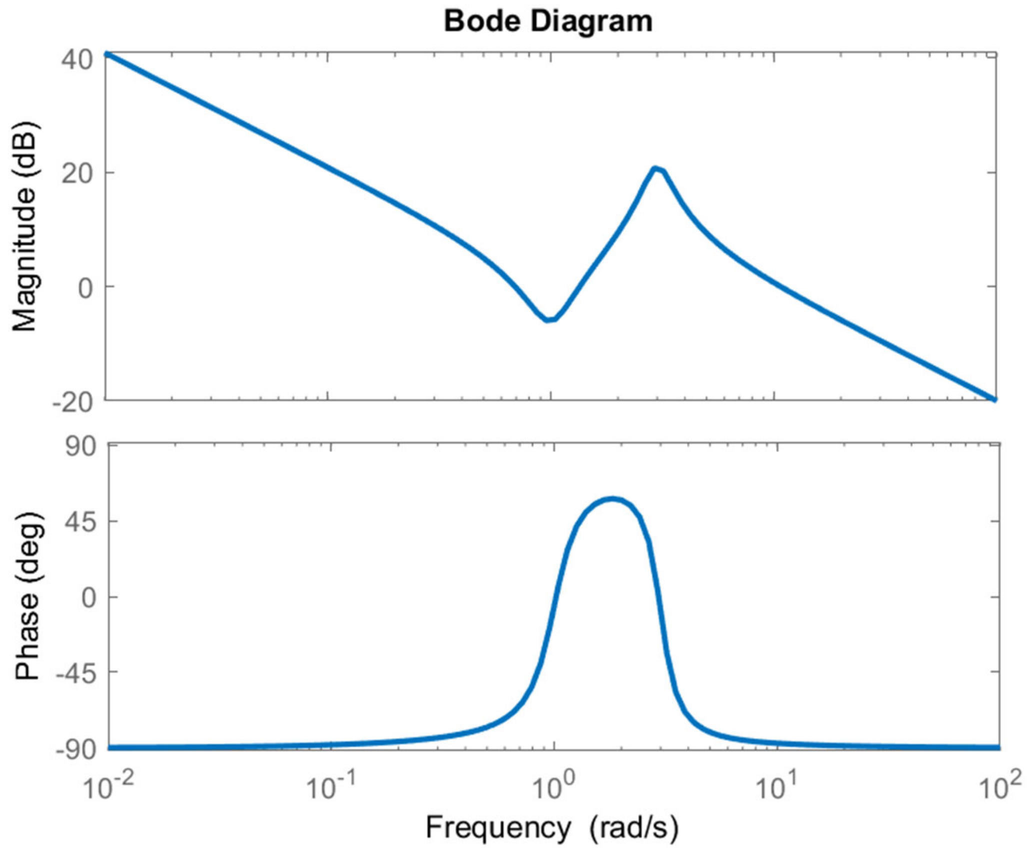
Ex 10.3

Using MATLAB, draw the bode plot of $G(s)$ in the frequency range of 0.01 and 100:

$$G(s) = \frac{10(s^2 + 0.4s + 1)}{s(s^2 + 0.8s + 9)}$$

Solution:

```
>> bode([10 4 10], [1 0.8 9 0], logspace(-2,2,100))
```

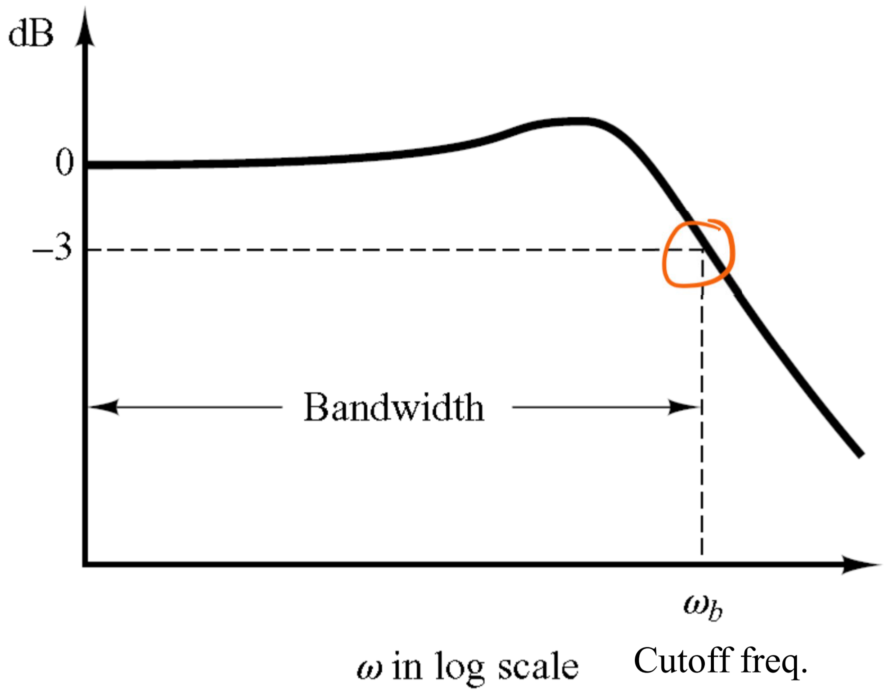


Cutoff Frequency and Bandwidth

Cutoff Frequency

The frequency ω_b at which the magnitude of the closed-loop frequency response is 3 dB below its magnitude at zero-frequency is called the *cutoff frequency*.

The closed-loop system filters out the signal components whose frequencies are greater than the cutoff frequency and transmits those signal components with frequencies lower than the cutoff frequency.



Bandwidth

The frequency range $0 \leq \omega \leq \omega_b$ in which the magnitude of the closed-loop frequency response is greater than –3 dB is called the *bandwidth* of the system.

The bandwidth indicates the frequency where the gain starts to fall off from its low-frequency value.

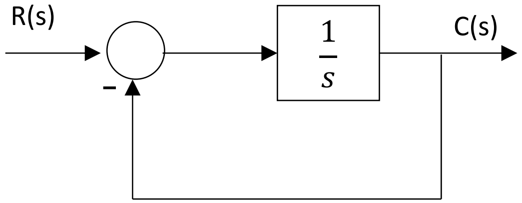
The specification of the bandwidth may be determined by the following factors:

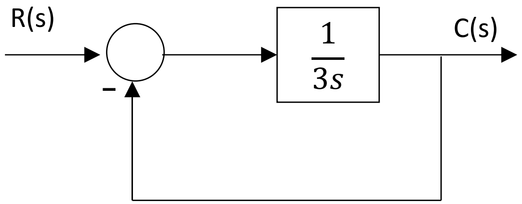
1. The ability to reproduce the input signal. A large bandwidth corresponds to a small rise time, or fast response. **Roughly speaking, we can say that the bandwidth is proportional to the speed of response.** (For example, to decrease the rise time in the step response by a factor of 2, the bandwidth must be increased by approximately a factor of 2.)

2. The necessary filtering characteristics for high-frequency noise.

For the system to follow arbitrary inputs accurately, it must have a large bandwidth. From the viewpoint of noise, however, the bandwidth should not be too large (to prevent amplifying high frequency noise). Thus, there are conflicting requirements on the bandwidth, and a compromise is usually necessary for good design. Note that a system with large bandwidth requires high-performance components, so the cost of components usually increases with the bandwidth.

Ex 10.4 Consider the following two systems:





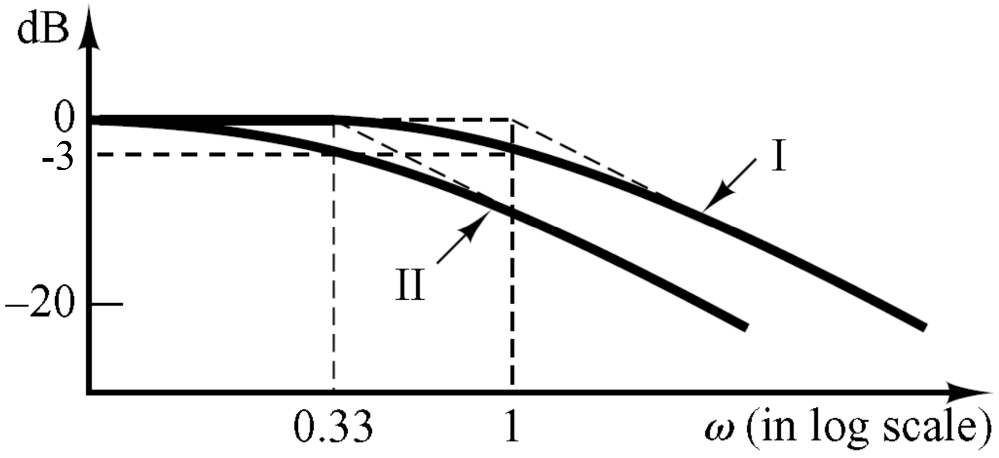
System I: $\frac{C(s)}{R(s)} = \frac{1}{s+1} = \frac{\frac{1}{s}}{1+\frac{1}{s}}$ System II: $\frac{C(s)}{R(s)} = \frac{1}{3s+1} = \frac{1/3s}{1+1/3s}$

Compare the bandwidths of these two systems and show that the system with the larger bandwidth has a faster step response and can follow the input much better than the one with the smaller bandwidth.

Solution:

The sinusoidal frequency response of systems: System I: $G(j\omega)=\frac{1}{j\omega+1}$, System II: $G(j\omega)=\frac{1}{3j\omega+1}$

Bode magnitude plot: (asymptotic curves are shown by dashed lines and exact curves by solid lines)




```
>> bode([1], [1 1)
>> hold on
>> bode([1], [3 1])
```

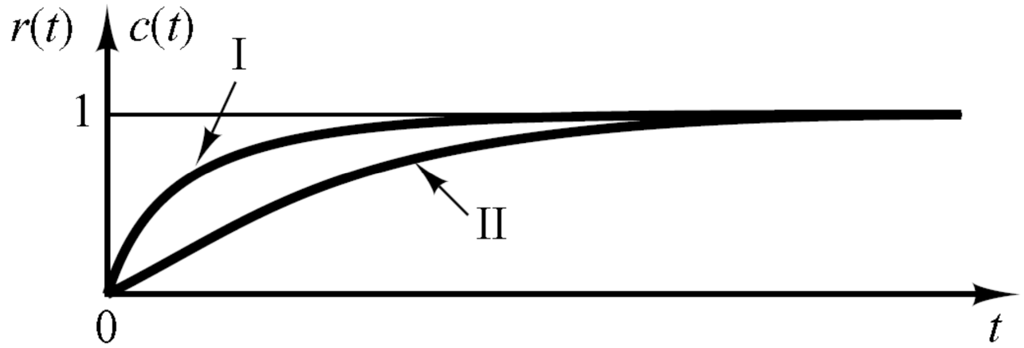
According to the closed-loop frequency-response curves, we find that the bandwidth of system I is $0 \le \omega \le 1$ rad/sec and that of system II is $0 \le \omega \le 1/3$ rad/sec.

Now, let's check their step responses:

System I: $\frac{C(s)}{R(s)} = \frac{1}{s+1} \longrightarrow C(s) = \frac{1}{s+1} \times \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1} \longrightarrow c(t) = 1 - e^{-t}$

System II: $\frac{C(s)}{R(s)} = \frac{1}{3s+1} \longrightarrow C(s) = \frac{1/3}{s+1/3} \times \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1/3} \longrightarrow c(t) = 1 - e^{-t/3}$

The step responses are shown for the two systems:

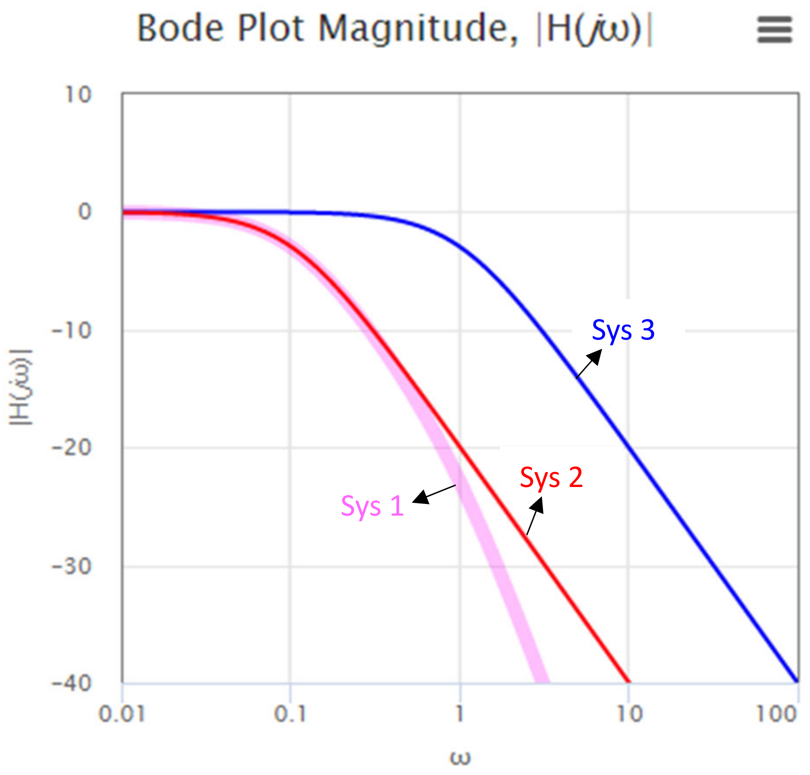


Clearly, system I, whose bandwidth is three times wider than that of system II, has a faster speed of response and can follow the input much better.

Ex 10.5 Among the three systems shown below, which one has the faster step response? Why?

Solution:

System 3, because it has the largest bandwidth among the three systems.



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HW 10 is due Nov. 21, 11:00 AM and must be submitted on Canvas.