Finite Element Modeling and Simulation with ANSYS Workbench

Chapter 7

Three-Dimensional Elasticity

Introduction

- Engineering designs often involve 3-D structures that cannot be adequately represented using 1-D or 2-D models.
- Solid elements based on 3-D elasticity theory are the most general elements for stress analysis
- In general, 3-D structural analysis is one of the most important and powerful ways of providing insight into the behavior of engineering design.

Review of Theory of Elasticity

The state of stress in a 3-D elastic body

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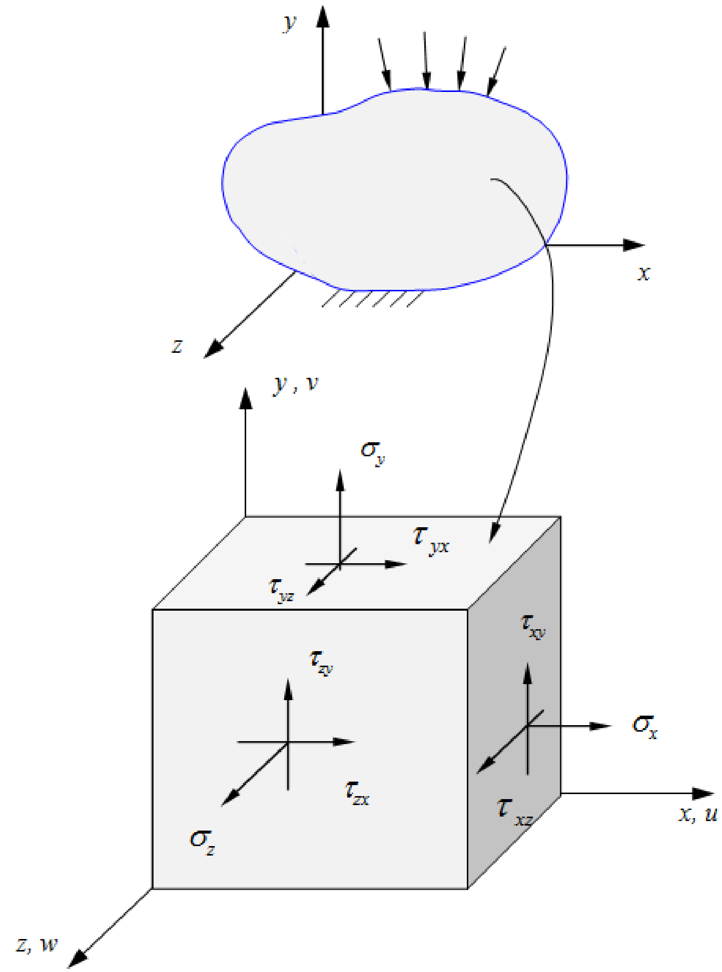
Review of Theory of Elasticity

- The state of stress at a point in a 3-D elastic body



Review of Theory of Elasticity

The six independent stress components

$$\boldsymbol{\sigma} = \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}, \quad \text{or} \quad [\sigma_{ij}].$$

The six independent strain components

$$\boldsymbol{\varepsilon} = \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \quad \text{or} \quad [\varepsilon_{ij}].$$

And there we have it… the ***von Mises stress***! Here it is in it's most recognizable form:

$$\sigma_{\mathrm{VM}} = \sqrt{\frac{1}{2}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2\right] + 3\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}$$

And if we've manipulated our stress tensor such that we know the principal stresses, we can simplify the equation down to:

$$\sigma_{\mathrm{VM}} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}}$$

Review of Theory of Elasticity

Stress-Strain Relations

The stress-strain relation in 3-D is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}.$$

Or in a matrix form: $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$

Review of Theory of Elasticity

- Displacements

The displacement field can be described as

$$\mathbf{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}.$$

Review of Theory of Elasticity

- Strain-Displacement Relations

Strain field is related to the displacement field as given below

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

These six equations can be written in the following index or tensor form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3$$

Or simply,

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \qquad \text{(tensor notation)}$$

Stress & Strain

- When a force is applied to a structural member, that member will develop both stress and strain as a result of the force.

- Stress is the force carried by the member per unit area, and typical units are lbf/in^2 (psi) for US Customary units and N/m^2 (Pa) for SI units:

$$\sigma = \frac{F}{A}$$

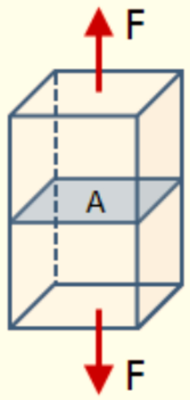
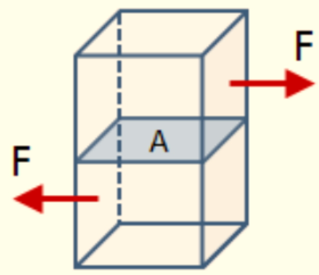
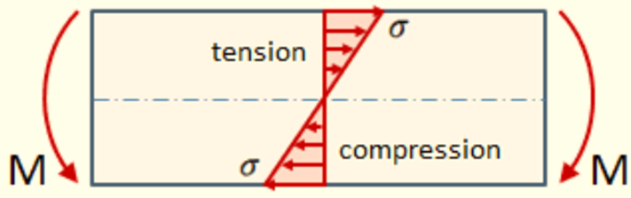
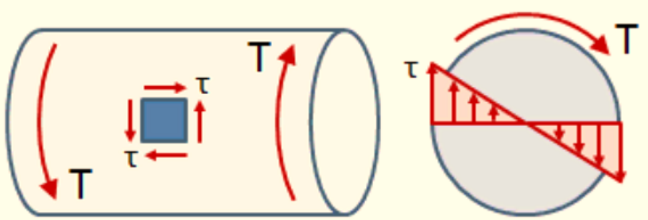
Where F is the applied force and A is the cross-sectional area over which the force acts. The applied force will cause the structural member to deform by some length, in proportion to its stiffness. Strain is the ratio of the deformation to the original length of the part:

$$\epsilon = \frac{L - L_0}{L_0} = \frac{\delta}{L_0}$$

Where L is the deformed length, L_0 is the original undeformed length, and δ is the deformation (the difference between the two).

Stress & Strain

There are different types of loading which result in different types of stress, as outlined in the table.

Loading Type	Stress Type	Illustration
Axial Force	• Axial Stress (general case) • Tensile Stress (if force is tensile) • Compressive Stress (if force is compressive)	
Shear Force	Transverse Shear Stress	
Bending Moment	Bending Stress	
Torsion	Torsional Stress	

Stress & Strain**

- Axial stress and bending stress are both forms of *normal stress*, σ, since the direction of the force is normal to the area resisting the force.
- Transverse shear stress and torsional stress are both forms of *shear stress*, τ, since the direction of the force is parallel to the area resisting the force.

Normal Stress	
Axial Stress:	$\sigma = \dfrac{F}{A}$
Bending Stress:	$\sigma_b = \dfrac{My}{I_c}$

Shear Stress	
Transverse Stress:	$\tau = \dfrac{F}{A}$
Torsional Stress:	$\tau = \dfrac{Tr}{J}$

In the equations for axial stress and transverse shear stress, F is the force and A is the cross-sectional area of the member. In the equation for bending stress, M is the bending moment, y is the distance between the centroidal axis and the outer surface, and I_c is the centroidal moment of inertia of the cross section about the appropriate axis. In the equation for torsional stress, T is the torsion, r is the radius, and J is the polar moment of inertia of the cross section.

** https://mechanicalc.com/reference/strength-of-materials#hookes-law

Hooke's Law

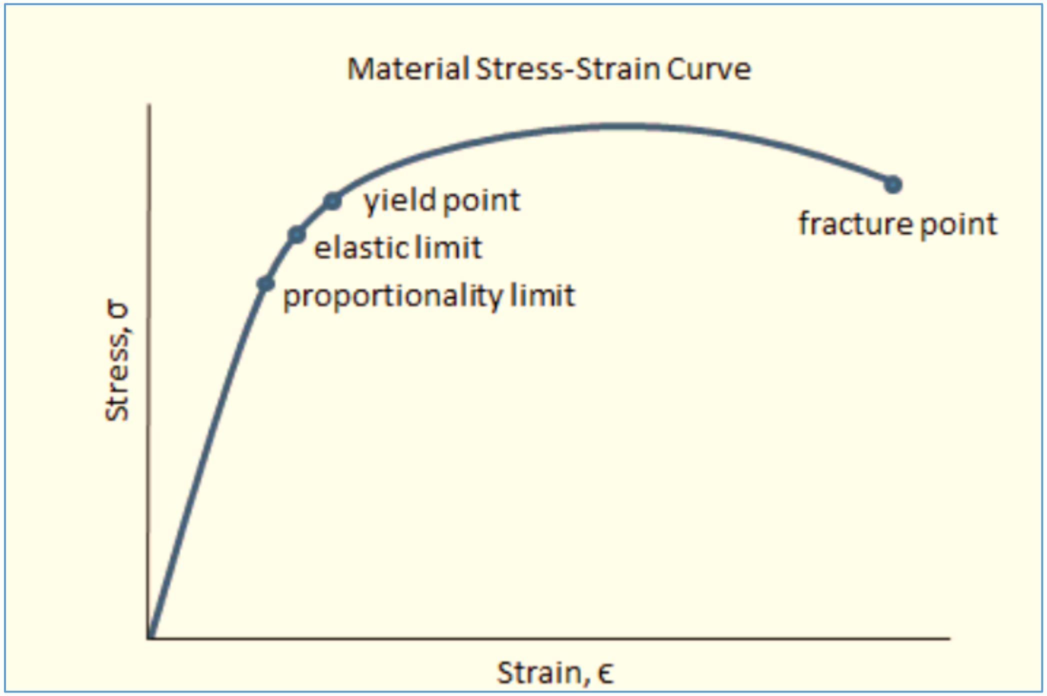
Stress & Strain

Stress is proportional to strain in the elastic region of the material's stress-strain curve (below the proportionality limit, where the curve is linear).

Normal stress and strain are related by:

$$\sigma = E\,\epsilon$$

where E is the elastic modulus of the material, σ is the normal stress, and ϵ is the normal strain.



Shear stress and strain are related by:

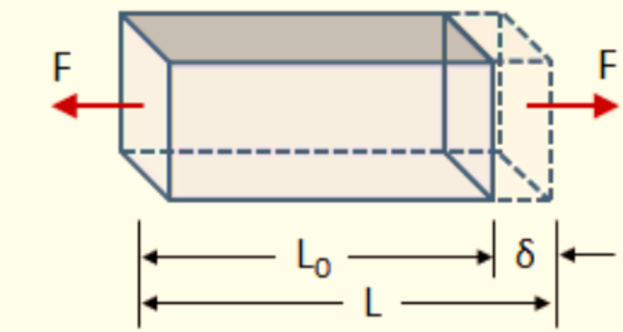
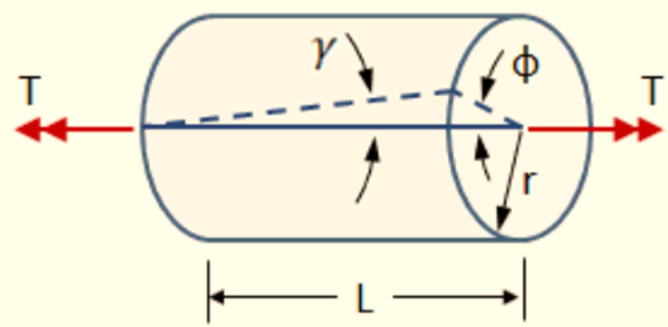
$$\tau = G\,\gamma$$

where G is the shear modulus of the material, τ is the shear stress, and γ is the shear strain. The elastic modulus and the shear modulus are related by:

$$G = \frac{E}{2(1+\nu)}$$

where ν is Poisson's ratio.

Hooke's law is analogous to the spring force equation, $F = k\,\delta$. Essentially, everything can be treated as a spring. Hooke's Law can be rearranged to give the deformation (elongation) in the material:

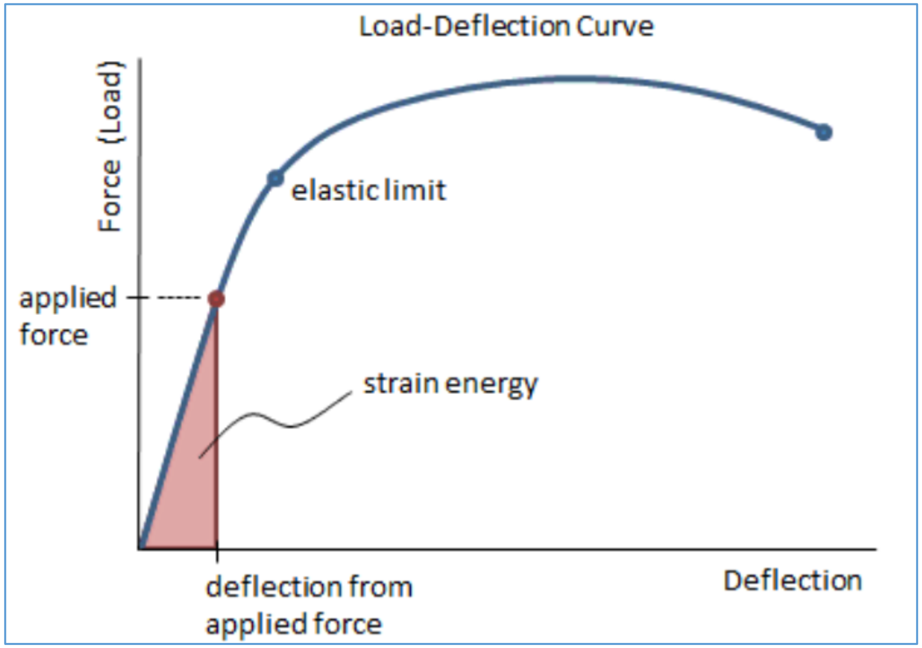
Axial Elongation (from normal stress)	$\delta = L_0 \epsilon = \frac{L_0 \sigma}{E} = \frac{F L_0}{AE}$	
Angle of Twist (from shear/torsional stress)	$\phi = \frac{\tau L}{rG} = \frac{TL}{GJ}$	

Strain Energy

- When force is applied to a structural member, that member deforms and stores potential energy, just like a spring. The strain energy (i.e. the amount of potential energy stored due to the deformation) is equal to the work expended in deforming the member.

- The total strain energy corresponds to the area under the load deflection curve and has units of in-lbf in US Customary units and N-m in SI units.

- The elastic strain energy can be recovered, so if the deformation remains within the elastic limit, then all of the strain energy can be recovered.



Strain energy is calculated as:

General Form:	$U = \text{Work} = \int F\,dL$	(area under load-deflection curve)
Within Elastic Limit:	$U = \frac{1}{2}F\delta = \frac{F^2 L_0}{2AE} = \frac{\sigma^2 L_0 A}{2E}$	(area under load-deflection curve)
	$U = \frac{1}{2}k\delta^2 = \frac{AE\delta^2}{2L_0}$	(spring potential energy)

Note that there are two equations for strain energy within the elastic limit. The first equation is based on the area under the load deflection curve. The second equation is based on the equation for the potential energy stored in a spring. Both equations give the same result, they are just derived somewhat differently.

Strain Energy

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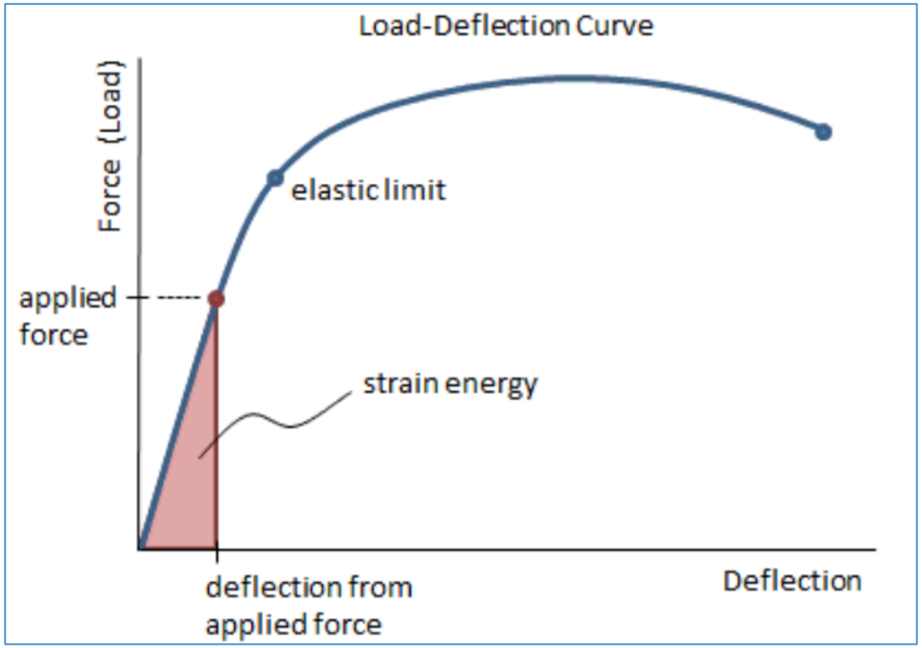
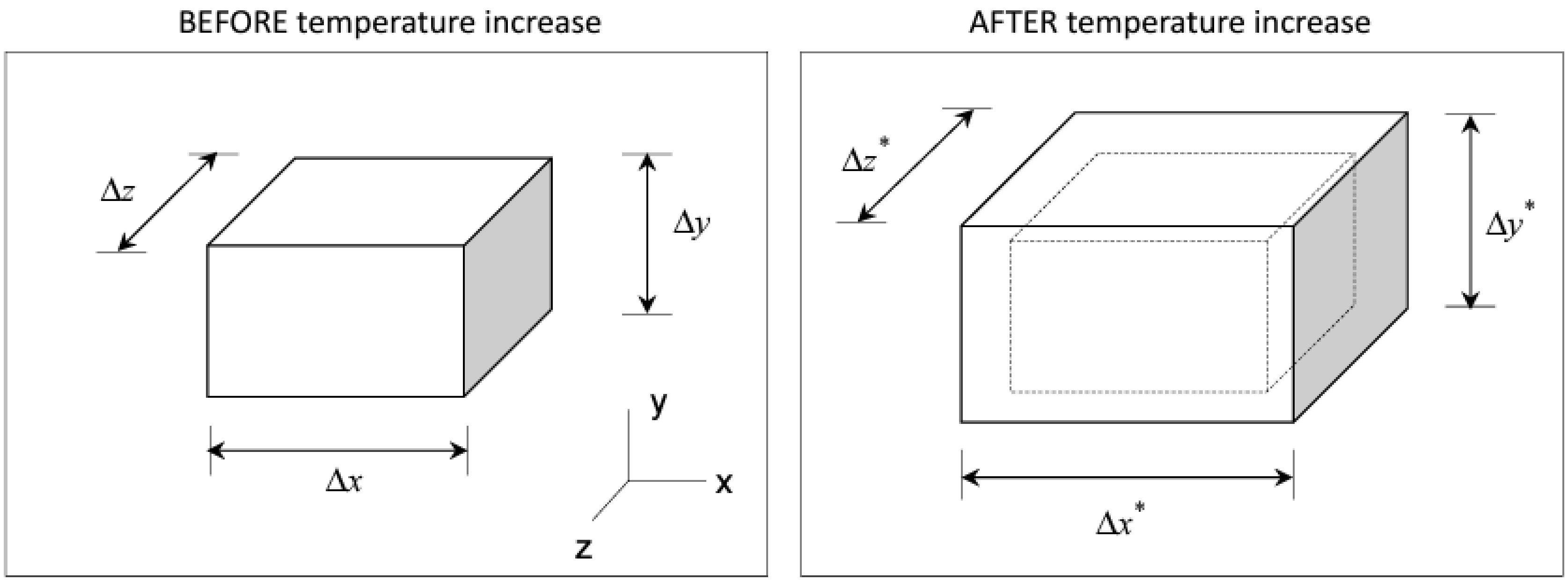


Strain energy is calculated as:

		(area under load-deflection curve)
Within Elastic Limit:	$U = \frac{1}{2}F\delta = \frac{F^2 L_0}{2AE} = \frac{\sigma^2 L_0 A}{2E}$	(area under load-deflection curve)
	$U = \frac{1}{2}k\delta^2 = \frac{AE\delta^2}{2L_0}$	(spring potential energy)

Thermal strains As a result of a uniform increase in temperature , most engineering materials will experience a uniform extensional strain in all three directions. This extensional strain is proportional to the temperature increase .
:

Consider the cube shown below that is given a uniform temperature increase



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The thermal strains induced by the temperature increase are found from the usual definitions:

$$\varepsilon_{x,T} = \lim_{\Delta x \to 0} \left(\frac{\Delta x^{*} - \Delta x}{\Delta x} \right)$$

$$\varepsilon_{y,T} = \lim_{\Delta y \to 0} \left(\frac{\Delta y^{*} - \Delta y}{\Delta y} \right)$$

$$\varepsilon_{z,T} = \lim_{\Delta z \to 0} \left(\frac{\Delta z^{*} - \Delta z}{\Delta z} \right)$$

Since this thermal strain is uniform and is proportional to ΔT , we can write these as:

$$\varepsilon_{x,T} = \varepsilon_{y,T} = \varepsilon_{z,T} = \alpha \Delta T$$

where α is the coefficient of thermal expansion (having units of 1/°F, or 1/°C). Note that temperature changes produce only extensional strains (no shear strains).

Generalized Hooke's law for normal stresses/strains . Recall that for uni-axial loading along the x-axis, the normal strains in the x, y and z directions in the body were found to be:

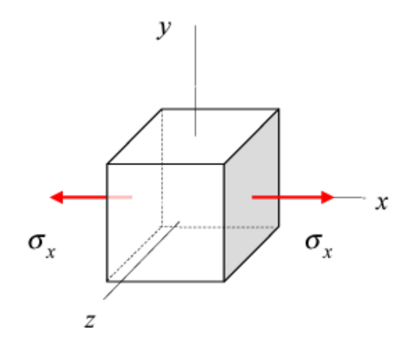
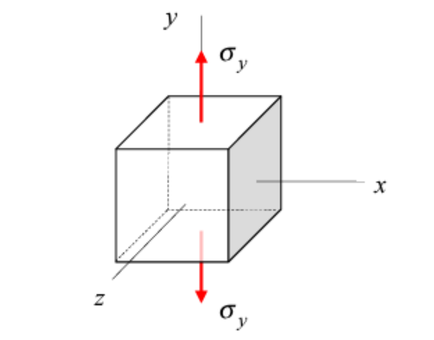
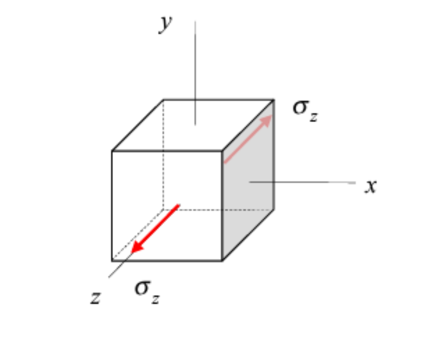
$$\varepsilon_x = \sigma_x / E$$

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$$

where E and ν are the Young's modulus and Poisson's ratio for the material. For a 3-D loading of a body, we have three normal stress components σ_x , σ_y and σ_z acting simultaneously.

For this case, we will consider the strains due to each normal component of stress individually and add these together using linear superposition (along with the thermal strains) to determine the resulting three components of strain ε_x , ε_y and ε_z .

Consider the individual contributions of the three components of stress shown in this slide.

Strains due to mechanical loading in the x-direction	Strains due to mechanical loading in the y-direction	Strains due to mechanical loading in the z-direction
		
$\varepsilon_x = \sigma_x / E$	$\varepsilon_x = -\nu\varepsilon_y = -\nu\sigma_y / E$	$\varepsilon_x = -\nu\varepsilon_z = -\nu\sigma_z / E$
$\varepsilon_y = -\nu\varepsilon_x = -\nu\sigma_x / E$	$\varepsilon_y = \sigma_y / E$	$\varepsilon_y = -\nu\varepsilon_z = -\nu\sigma_z / E$
$\varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$	$\varepsilon_z = -\nu\varepsilon_y = -\nu\sigma_y / E$	$\varepsilon_z = \sigma_z / E$

The total strain in each direction is found through superposition of the individual strains along with the thermal strains. Adding together these components (across each row of the above table) gives:

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha\Delta T$$

The above are known as the generalized Hooke's law equations for normal stresses/strains due to 3-D loadings on a body.

Review of Theory of Elasticity

Equilibrium Equations

The stresses and body force vector ***f*** at each point satisfy the following three equilibrium equations for elastostatic problems

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0,$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0.$$

Or in index or tensor notation

$$\sigma_{ij,j} + f_i = 0\,.$$

Review of Theory of Elasticity

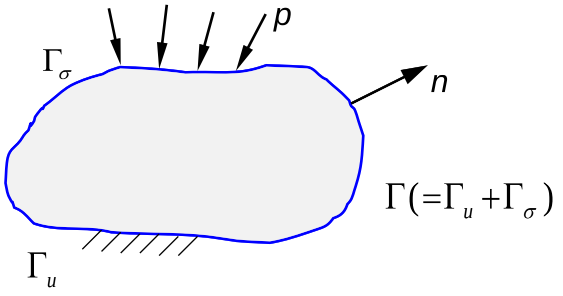
❑ Boundary Conditions (BCs)

At each point on the boundary Γ and in each direction, either displacement or traction (stress on the boundary) should be given.

$$u_i = \bar{u}_i, \qquad on \;\; \Gamma_u \; (specified \;\; displacement);$$
$$t_i = \bar{t}_i, \qquad on \;\; \Gamma_\sigma \; (specified \;\; traction);$$

The traction (stress on a surface) is defined by

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$



with n being the normal.

Review of Theory of Elasticity

- Stress Analysis

For 3-D stress analysis, one needs to solve the introduced equations under given BCs in order to obtain the stress, strain and displacement fields.

Analytical solutions are often difficult to find and thus numerical methods such as the FEA is often applied in 3-D stress analysis.

Modeling of 3-D Elastic Structures

3-D stress analysis is
one of the most challenging tasks in FEA …

- Mesh Discretization

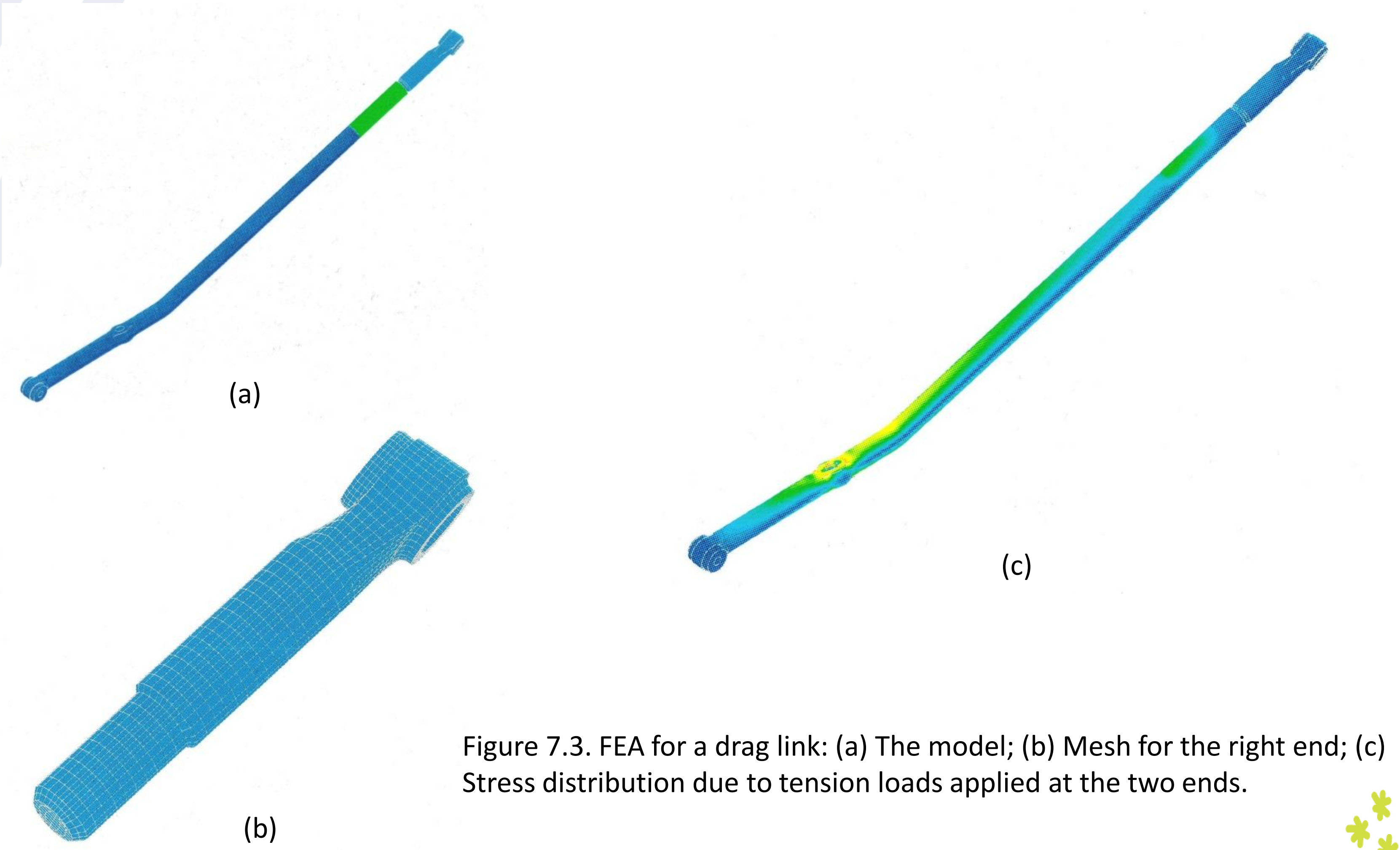
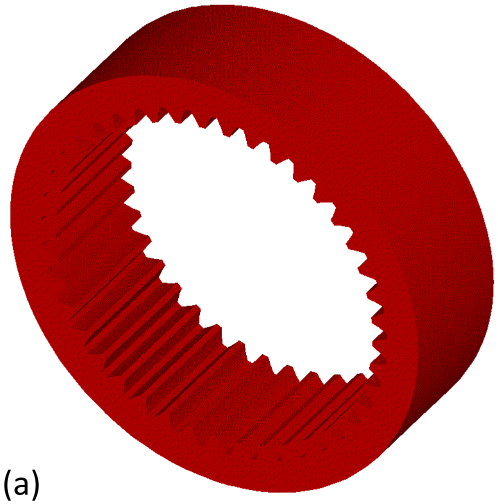
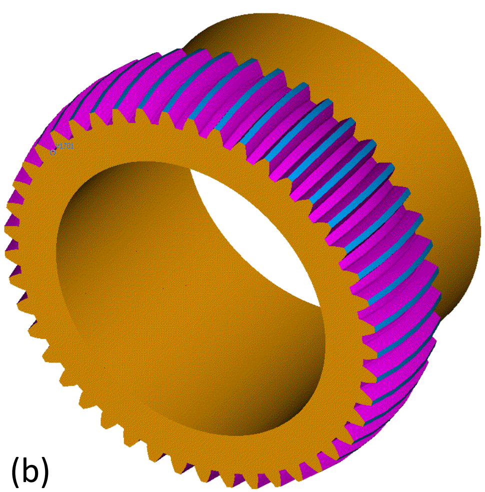


Figure 7.3. FEA for a drag link: (a) The model; (b) Mesh for the right end; (c) Stress distribution due to tension loads applied at the two ends.

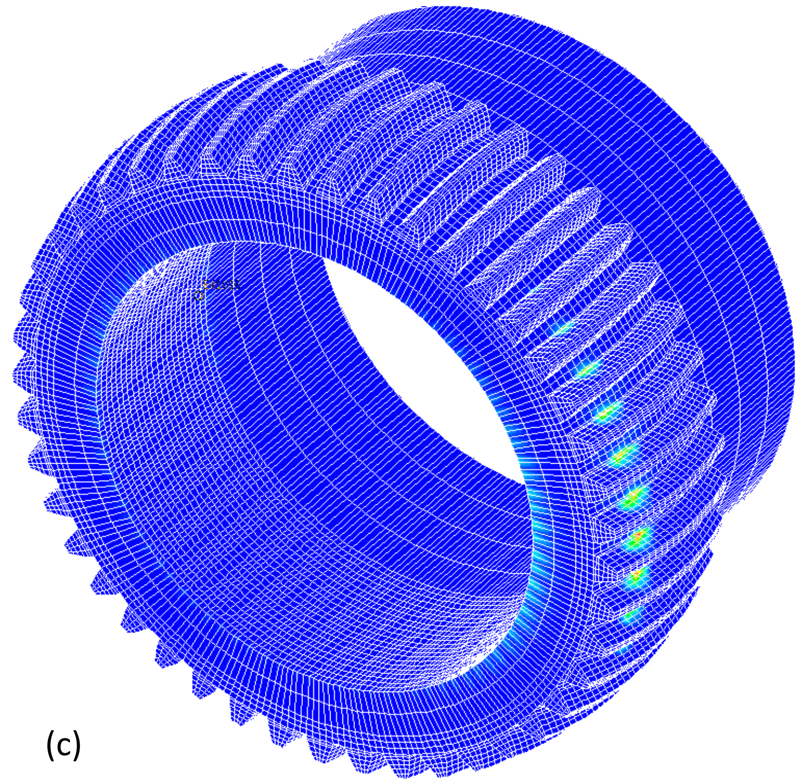
Modeling of 3-D Elastic Structures



(a)



(b)



(c)

Figure 7.4. Analysis of a gear coupling: (a) Ring gear; (b) Hub gear; (c) High contact stresses in the gear teeth obtained using nonlinear FEA.

Modeling of 3-D Elastic Structures

- Boundary Conditions: Supports

- Fixed support: prevent the geometry entity from moving or deforming.
- Frictionless support: prevent the geometry entity from moving or deforming in the normal direction.

- Cylindrical Support: prevent the cylindrical face from moving radially, axially or tangentially, or deforming in combinations of these directions.

Modeling of 3-D Elastic Structures

- Boundary Conditions: Loads

  - Force load
  - Moment load
  - Pressure load
  - Bearing load

Inertia loads such as acceleration, standard earth gravity or rotational velocity may have nontrivial effect on structures' stress behaviors as well.

Other loading types such as thermal, electric or magnetic loads can also be involved, but are less common.

Modeling of 3-D Elastic Structures

❑ Assemble Analysis: Contacts

- Bonded: considered as glued together, allowing no sliding or separation between the contacting regions.
- No separation: Frictionless sliding is allowed along the contact faces, but separation of faces in contact is not allowed.
- Frictionless: This allows free sliding, assuming a zero coefficient of friction. Gap can form in between regions in contact.
- Rough: It assumes an infinite friction coefficient between the bodies in contact. No sliding can occur.
- Frictional: This model allows bodies in contact to slide relative to each other, once an equivalent shear stress up to a certain magnitude is exceeded.

FEA formulation for 3-D elasticity …

Formulation of Solid Elements

- General Formulation

Interpolate the displacements using shape functions N_i

$$u = \sum_{i=1}^{N} N_i u_i, \quad v = \sum_{i=1}^{N} N_i v_i, \quad w = \sum_{i=1}^{N} N_i w_i,$$

u_i, v_i, and w_i are nodal values of the displacement on the element, and N is the number of nodes on that element.

In matrix form

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(3\times1)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix}_{(3\times3N)} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}_{(3N\times1)}$$

Or $\mathbf{u}=\mathbf{N}\mathbf{d}$

Formulation of Solid Elements

The strain vector

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

where **B** is the matrix relating the nodal displacement vector **d** to the strain vector **ε**.

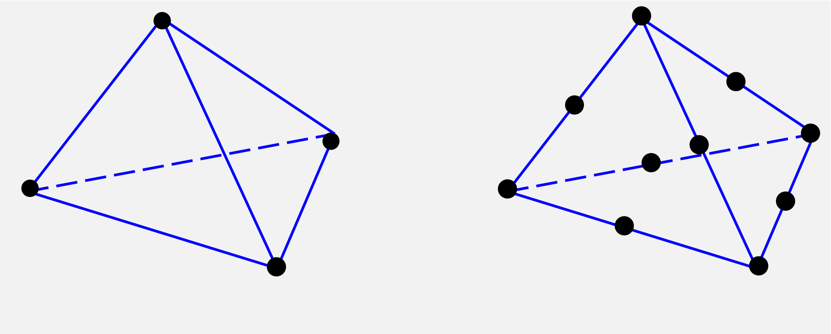
We can apply the following expression to determine the stiffness matrix for the element

$$\mathbf{k} = \int_{v} \mathbf{B}^{T}\mathbf{E}\,\mathbf{B}\,dv.$$

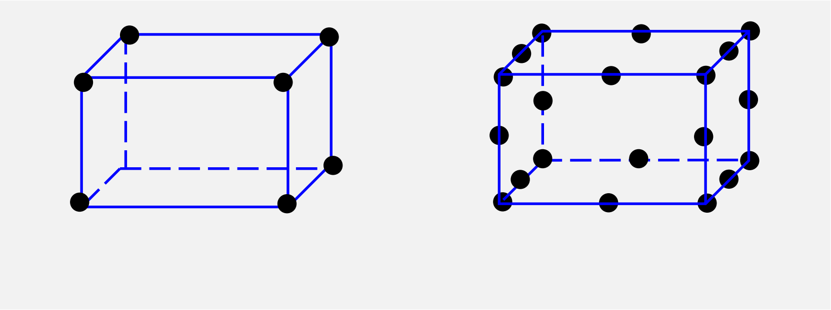
Formulation of Solid Elements

- Typical Solid Element Types

Tetrahedron:

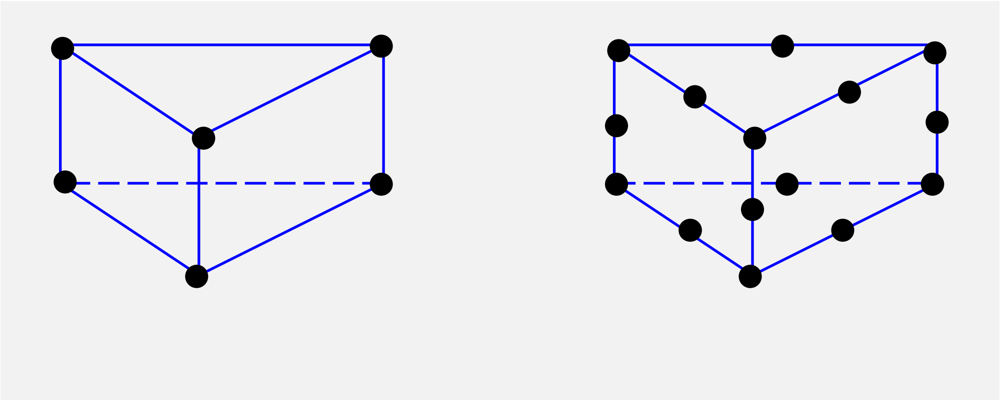


Hexahedron (brick):



Formulation of Solid Elements

Penta:



- Whenever possible, one should try to apply higher-order (quadratic) elements for 3-D stress analysis. Avoid using the linear, especially the 4-node tetrahedron elements in 3-D stress analysis.
- However, it is fine to use them for deformation analysis or in vibration analysis.

Formulation of Solid Elements

- Formulation of a Linear Hexahedral Element Type

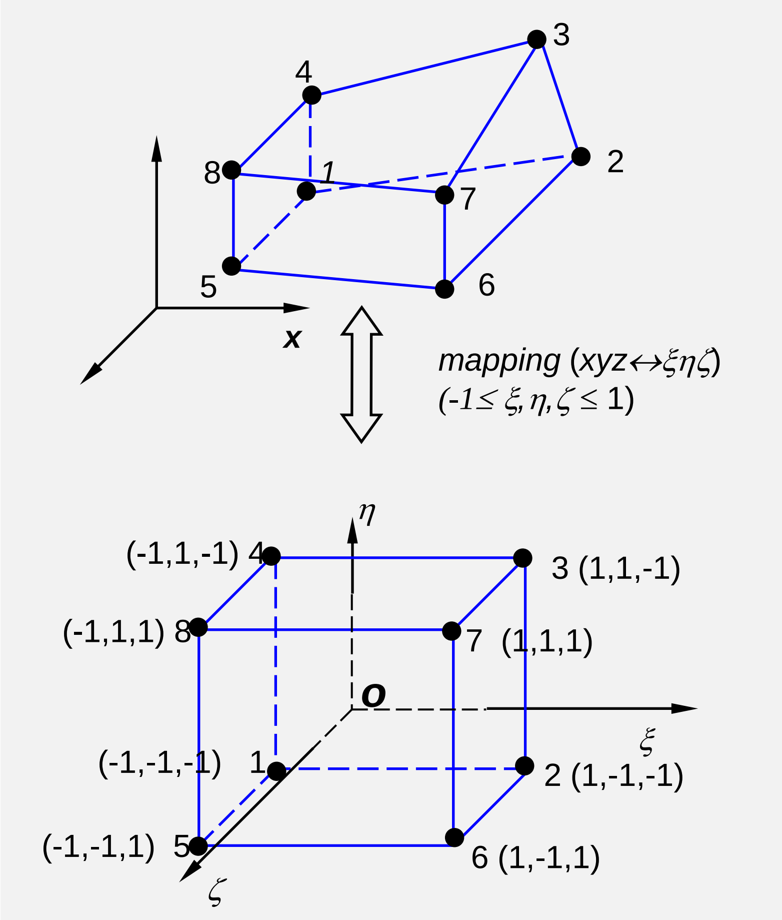


Figure 7.6. Mapping an element to the natural coordinate system

Formulation of Solid Elements

Displacement Field in the Element

$$u = \sum_{i=1}^{8} N_i u_i, \qquad v = \sum_{i=1}^{8} N_i v_i, \qquad w = \sum_{i=1}^{8} N_i w_i.$$

Shape Functions

$$N_1(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \quad ,$$

$$N_2(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \quad ,$$

$$N_3(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta) \quad ,$$

$$\vdots \qquad\qquad \vdots$$

$$N_8(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta) \quad .$$

What are the shapes of elements in FEA?

FEA models are composed of discrete elements that approximate the geometry and behavior of the real structure. There are different types of elements, such as bars, beams, plates, shells, solids, and tetrahedra, that have different shapes, dimensions, degrees of freedom, and interpolation functions

Formulation

Note that we have the following relations for the shape functions

$$N_i(\xi_j, \eta_j, \zeta_j) = \delta_{ij}, \quad i, j = 1, 2, \cdots, 8.$$

$$\sum_{i=1}^{8} N_i(\xi, \eta, \zeta) = 1.$$

Coordinate Transformation (Mapping)

$$x = \sum_{i=1}^{8} N_i x_i, \qquad y = \sum_{i=1}^{8} N_i y_i, \qquad z = \sum_{i=1}^{8} N_i z_i.$$

Formulation of Solid Elements

Jacobian Matrix

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}}_{\equiv\ \mathbf{J}\quad \text{Jacobian matrix}} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix}.$$

Inverting this relation, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix}, \quad \text{with } \frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \text{ and so on.}$$

The steps are similar for *v* and *w.*

What is Jacobian matrix explain?

Jacobian matrix is **a matrix of partial derivatives**. Jacobian is the determinant of the Jacobian matrix. The matrix will contain all partial derivatives of a vector function. The main use of Jacobian is found in the transformation of coordinates.

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Formulation of Solid Elements

These relations lead to the following expression for the strain

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial x}{\partial x} + \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \cdots_{use(6.15)} = \mathbf{B}\,\mathbf{d}$$

where **d** is the nodal displacement vector, that is: $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}.$

Formulation of Solid Elements

Strain energy is evaluated as

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V (\mathbf{E}\boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon}\, dV$$

$$= \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{E}\,\boldsymbol{\varepsilon}\, dV$$

$$= \frac{1}{2}\mathbf{d}^T \left[\int_V \mathbf{B}^T \mathbf{E}\,\mathbf{B}\, dV \right] \mathbf{d}.$$

The element stiffness matrix is

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E}\,\mathbf{B}\; dV.$$

In $\xi\eta\zeta$ coordinates

$$dV = (\det \mathbf{J})\, d\xi\, d\eta\, d\zeta$$

Therefore

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{E}\,\mathbf{B}\,(\det \mathbf{J})\, d\xi\, d\eta\, d\zeta.$$

It is easy to verify that the dimensions of this stiffness matrix is 24x24.

What Is von Mises Stress?
For reading only

Most FEA codes spit it out as if it's the holy grail of field outputs - but what is von Mises stress?

Where Did It Originate?

- It is believed that the first mention of the idea that would later be termed 'von Mises stress' was contained within a letter from James Clerk Maxwell to William Thomson in 1865, although this only described the general conditions.
- Tytus Maksymillian Huber (1) then anticipated the criterion in some more detail in 1904, in a paper that suggested separation of hydrostatic and distortion strain energy, rather than relying on total strain energy as his predecessors had.

What Is von Mises Stress?

Hydrostatic and Deviatoric Components of Stress

Before we get into the yield criterion and von Mises stress, first we must do some homework and remind ourselves the difference between hydrostatic and deviatoric stress.

Simply put, hydrostatic stress is the stress that causes change in volume of the material and deviatoric stress is that which causes change in shape. Any stress tensor can be decomposed as follows:

$$\sigma_{ij} = \frac{1}{3}\delta_{ij}\sigma_{kk} + \sigma'_{ij}$$

What Is von Mises Stress?

Where the first part of the equation is the hydrostatic stress (simply the average of the three normal stresses), and deviatoric stress, σ'_{ij}, makes up the rest of the tensor.

The same can be done to the strain tensor as shown below:

$$\epsilon_{ij} = \frac{1}{3}\delta_{ij}\epsilon_{kk} + \epsilon'_{ij}$$

What Is von Mises Stress?

Yield Criteria

- The goal of the yield criterion was to develop a method whereby the ductile behavior of materials could be predicted for any complex, 3D loading condition, rather than just for typical nominal laboratory test loading.
- To do this, we must transfer the stress state into a single scalar value that is compared to a material's yield strength and can be measured by a simple tensile test.

What Is von Mises Stress?

The yield criterion must, therefore, relate the full stress tensor to the deformation strain energy density in some way. We can do this by first noting that total strain energy, which has units of *Energy/Volume*, *W*, is:

$$W = \int \boldsymbol{\sigma} : d\boldsymbol{\epsilon}$$

And for linear elastic materials, this equals:

$$W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon} = \frac{1}{2}\left[\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + 2(\sigma_{xy}\epsilon_{xy} + \sigma_{yz}\epsilon_{yz} + \sigma_{xz}\epsilon_{xz})\right]$$

Formulation of Solid Elements

Stresses

To compute the stresses within an element, one uses the following relation once the nodal displacement vector is known for that element.

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{EBd}$$

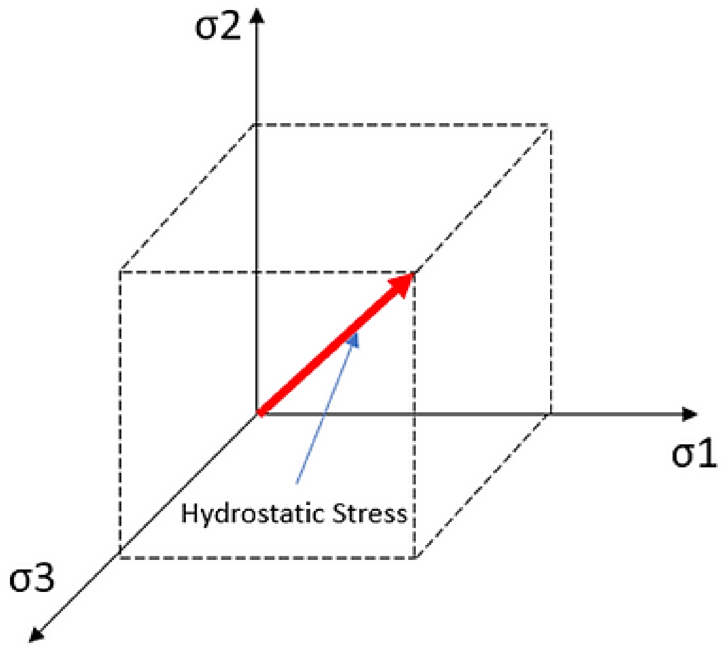
The von Mises stress for 3-D problems is given by

$$\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Where $\sigma_1, \sigma_2 \text{ and } \sigma_3$ are the three principal stresses.

The Yield Surface

If we take a second to think about what hydrostatic stress (stresses equal in all three principal directions) is and how it looks in 3D, we can visualize a line emanating from zero and extending equidistantly from all of the principal axes to form the $\sigma_1 = \sigma_2 = \sigma_3$ axis as shown in the image below.

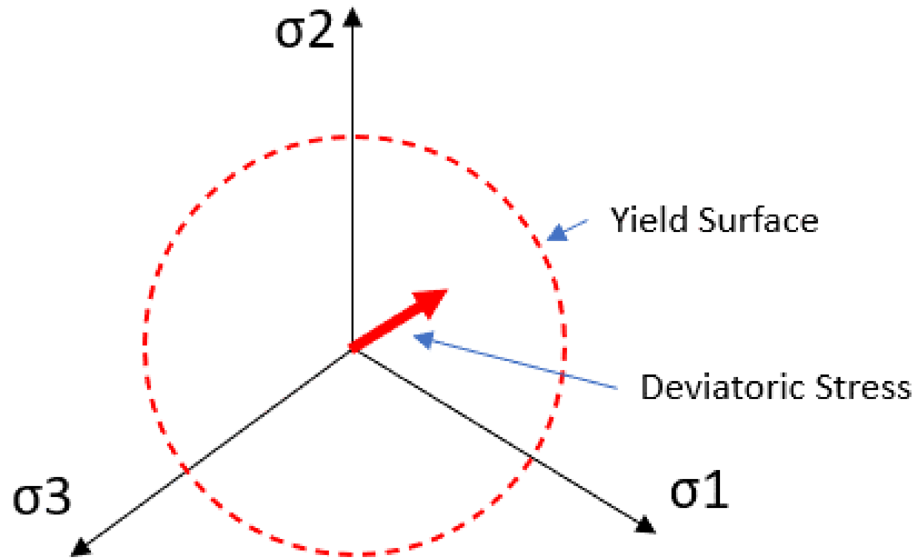


The Yield Surface

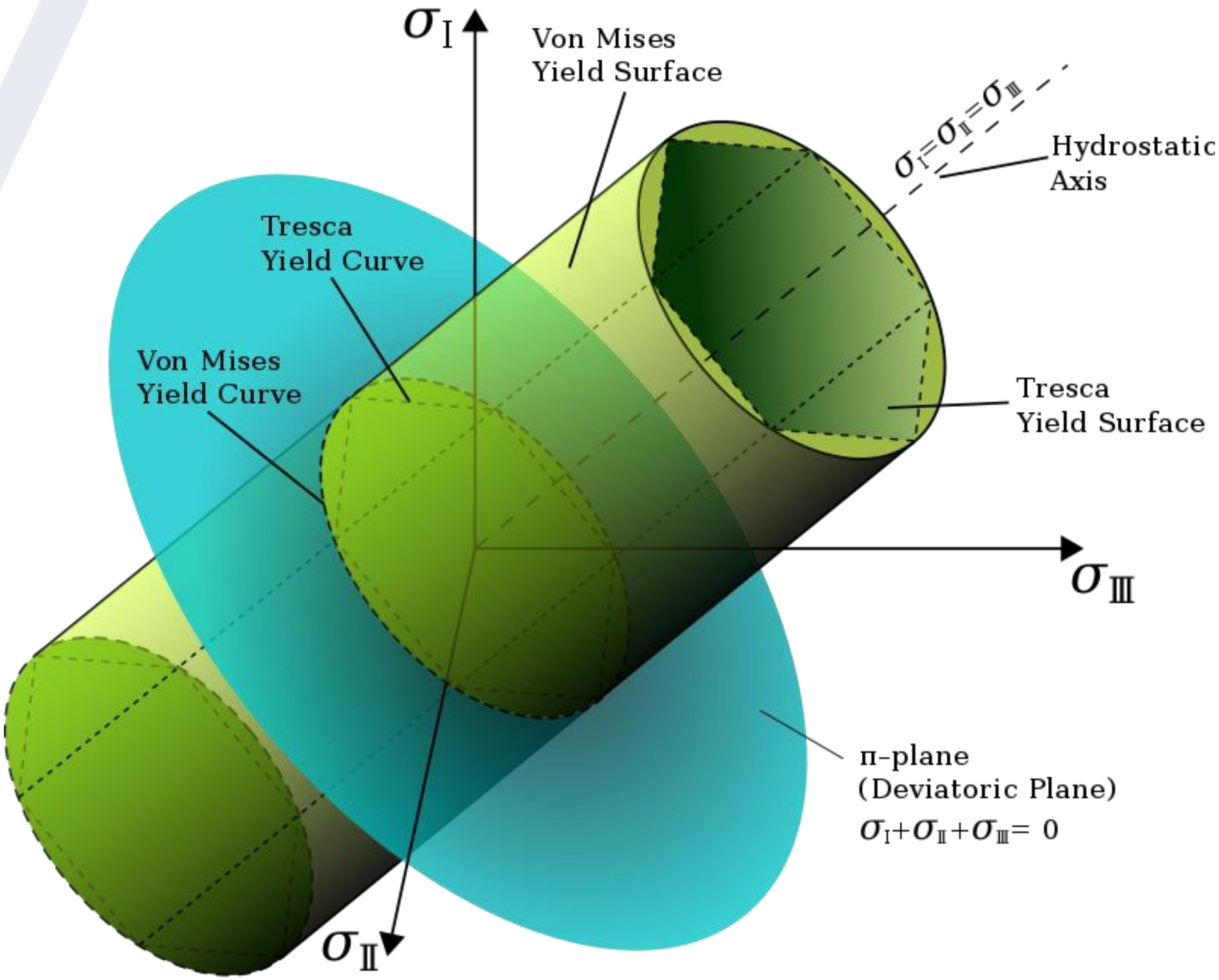
- Now, remember that when we talk about deviatoric stress we're really talking about the distance from this line in a plane perpendicular to it. With that in mind, we can draw the above figure a little differently, as if we were looking down the line of hydrostatic stress ($\sigma_1 = \sigma_2 = \sigma_3$ axis) and

- in this way, we only see deviatoric stresses on our plot; the magnitude of hydrostatic stress is irrelevant (remember, this is really only true in metals). This might seem like a pointless exercise until we acknowledge that it is deviatoric stresses alone that result in yielding, as discussed prior.

The Yield Surface



- If we were to put a point on each principal axis where yielding occurs in a tensile test, and then join them up with a circle, we now have a nice 2D visualization of our von Mises yield surface.



In 3D, this looks like the famous cylinder that is concentric with the hydrostatic stress condition. This excellent schematic is provided by [Wikipedia].

Formulation of Solid Elements

❑ Treatment of Distributed Loads

Distributed loads need to be converted into nodal forces using the equivalent energy concept as discussed in earlier chapters.

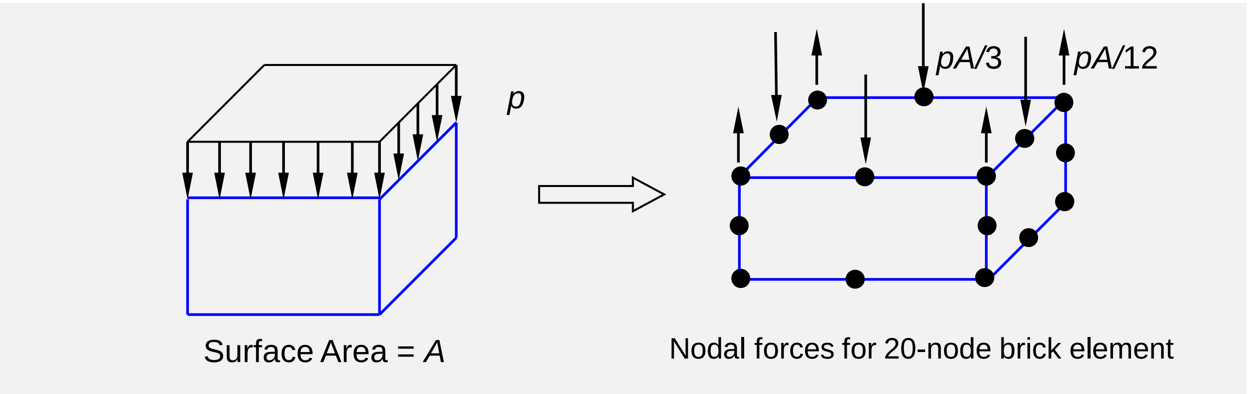


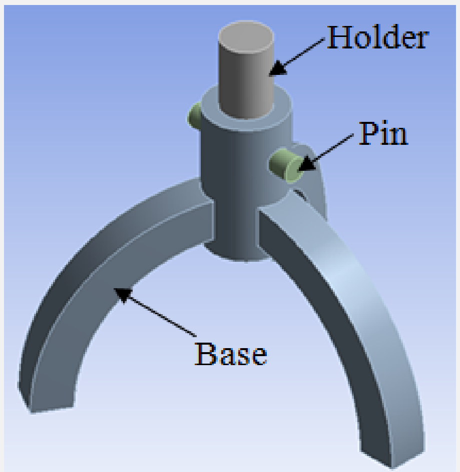
Figure 7.7. Equivalent nodal forces on a 20-node brick element for a pressure load *p*.

Case Study with ANSYS Workbench

Analysis of a base stand assembly …

Case Study with ANSYS Workbench

<Problem Description> A base stand assembly includes a base, a holder and a pin, as shown in the following figure. The stand assembly is made of structural steel. Assume a no-separation condition for all contact regions. Determine the deformation and von Mises stress distributions of the assembly under the given load and boundary conditions.



<u>Material</u>: Structural Steel (E = 200 GPa, ν = 0.3)

<u>Boundary Conditions</u>:

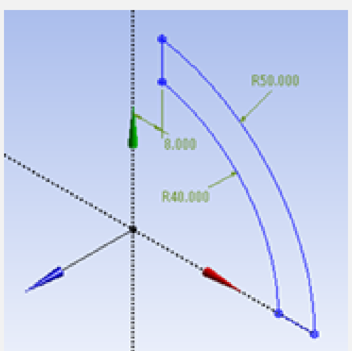
The bottom faces of the leg-base are fixed.

A downward force of 1kN is applied to the holder's top face.

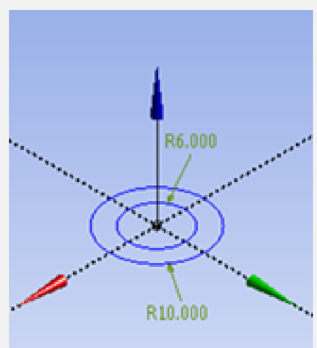
<u>Geometry Construction</u>:

The bottom of the hub-base is 35 mm above the ground level.

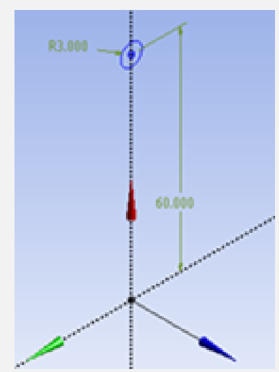
The holder is 36 mm tall, 18 mm of which is in contact with the hub-base.



Sketch of the leg-base (extrude 5mm on both sides)



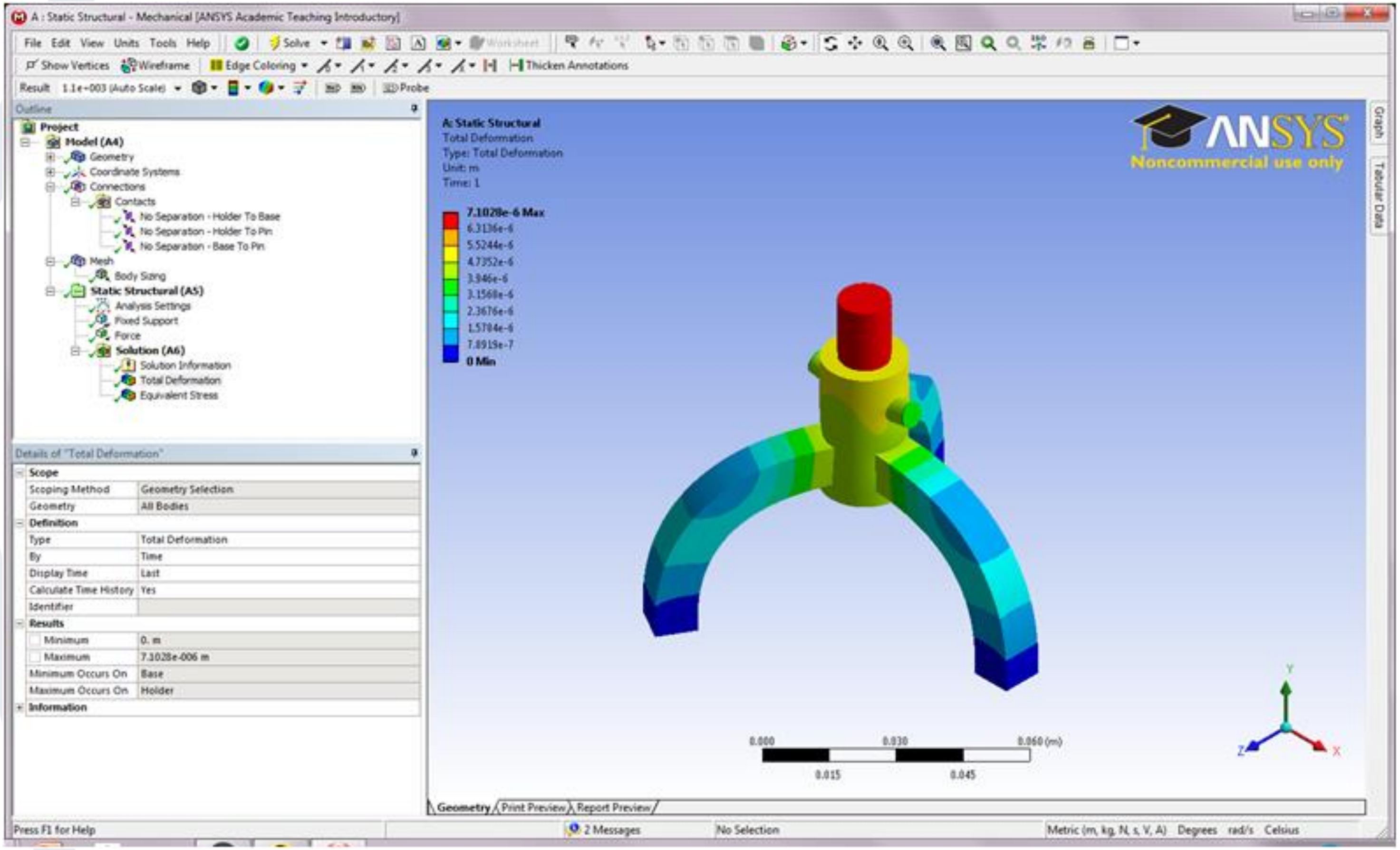
Sketch of the hub-base (extrude 35mm on one side)



Sketch of the pin (extrude 15mm on both sides)

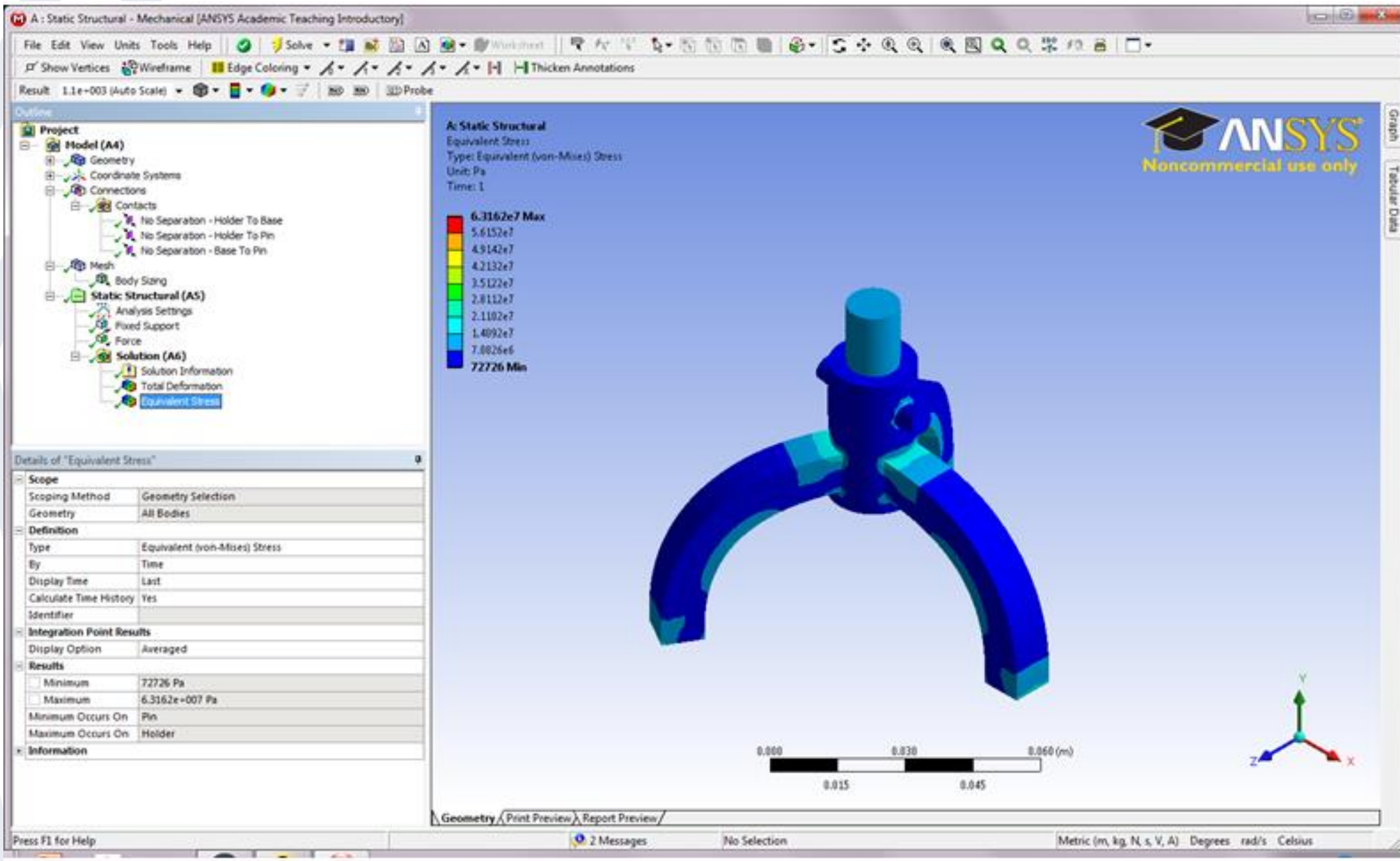
Case Study with ANSYS Workbench

Run a ***Static Structural Analysis*** to review the assembly deformation results.



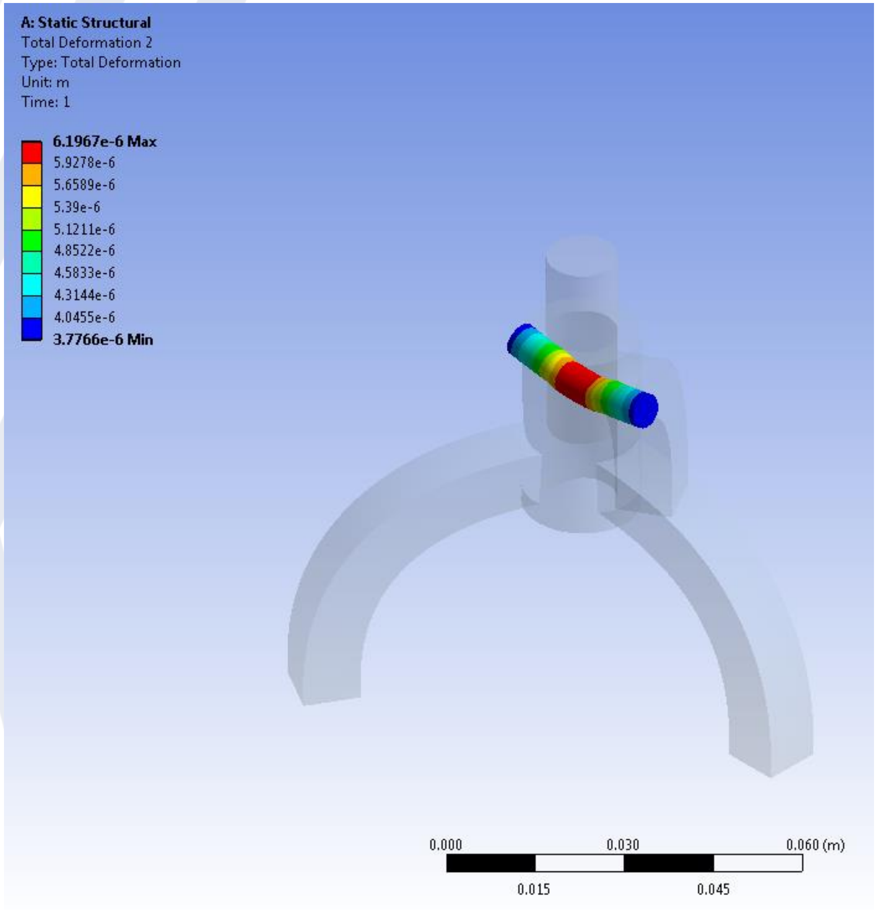
Case Study with ANSYS Workbench

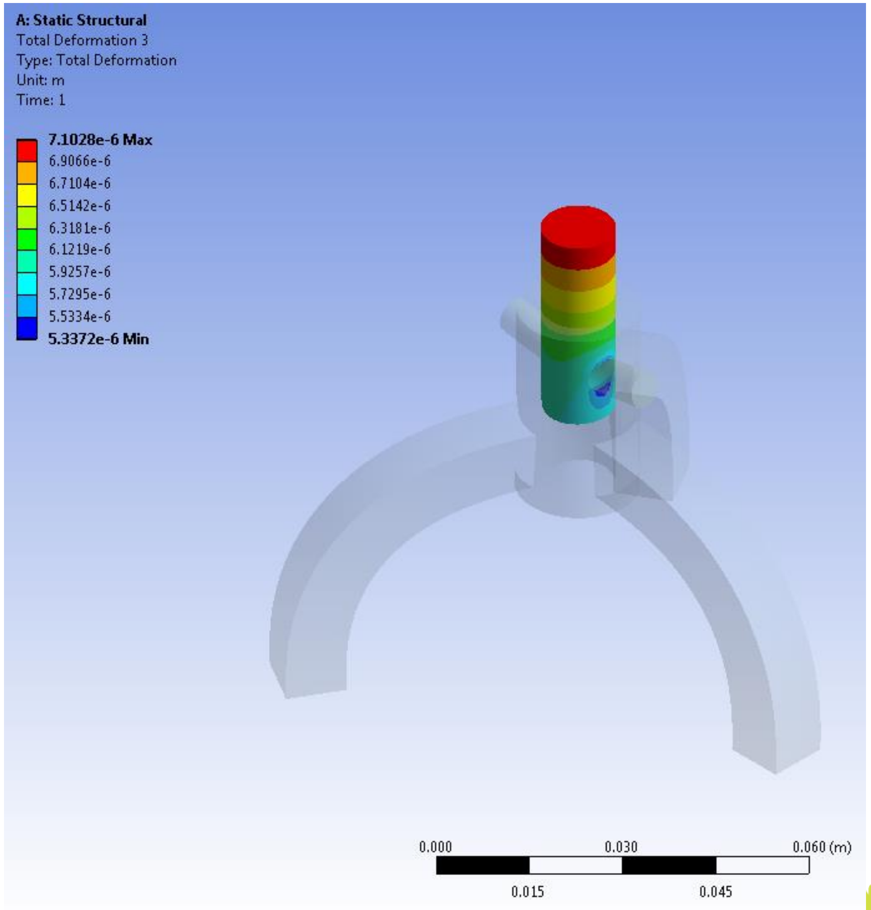
Click on ***Equivalent Stress*** in the ***Outline*** to review the stress results.



Case Study with ANSYS Workbench

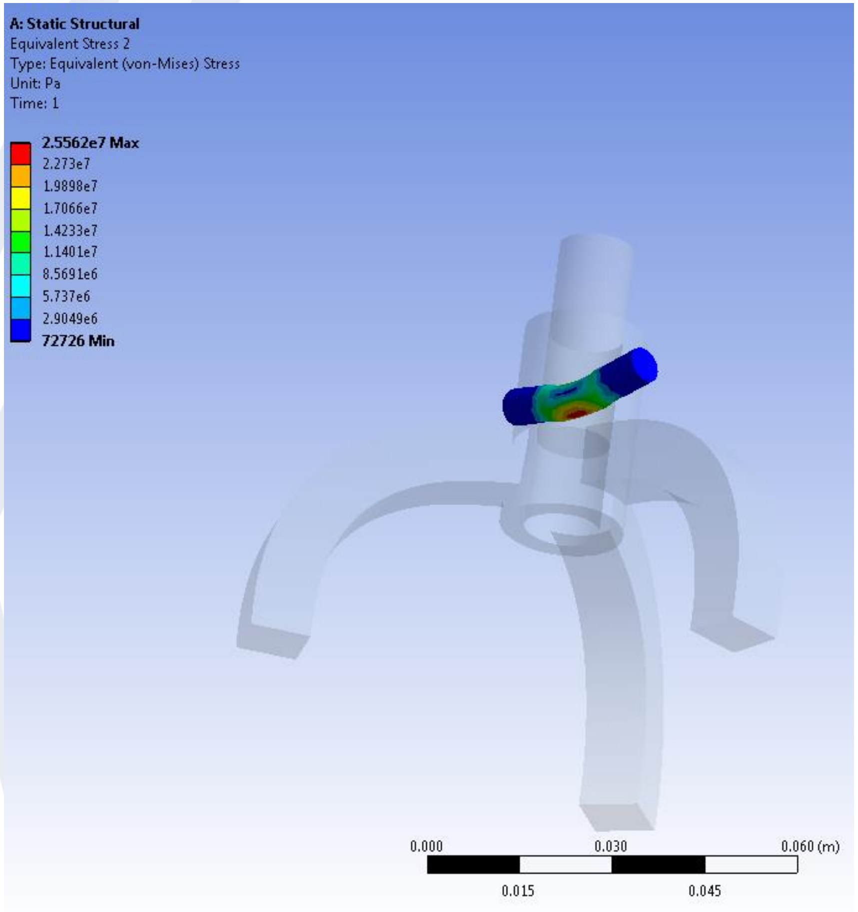
Review the deformation of the pin and the holder, respectively.

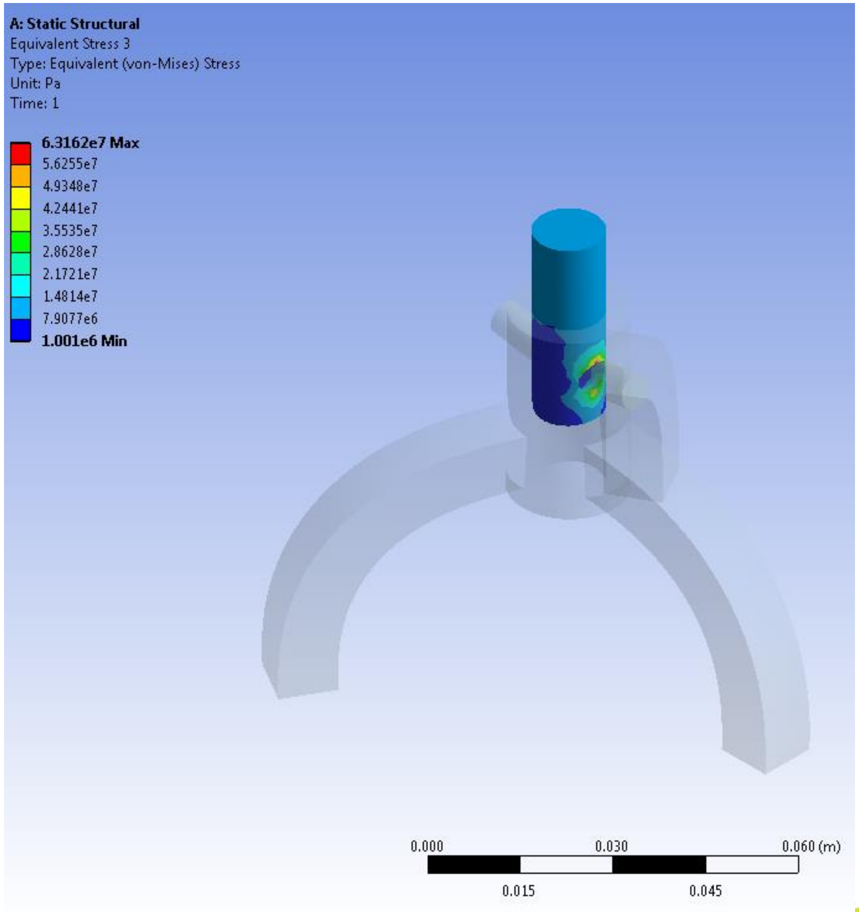




Case Study with ANSYS Workbench

Review the von Mises stress of the pin and the holder, respectively.





Summary

In this chapter,

- General 3-D deformation and stress analyses are discussed.
- Solid elements are the most accurate elements and should be applied to when the bar, beam, plane stress/strain, plate/shell elements are no longer valid or accurate.
- For stress concentration problems, higher-order solid elements such as 10-node tetrahedron or 20-node hexahedron (brick) elements should be employed in the FEA.
- For solids having symmetrical features, symmetric FEA models can be more effective and efficient.