Control Systems - ENGR 33041

Lecture 6: Types of Controllers and Static Response

Instructor:

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Slides prepared based on Textbook

Control Systems Technology, C. Johnson and H. Malki

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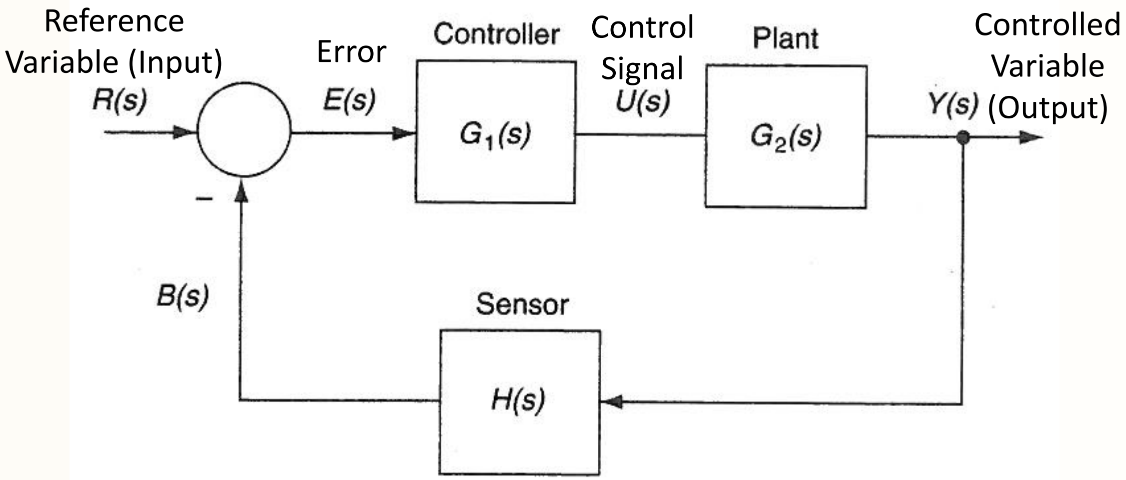
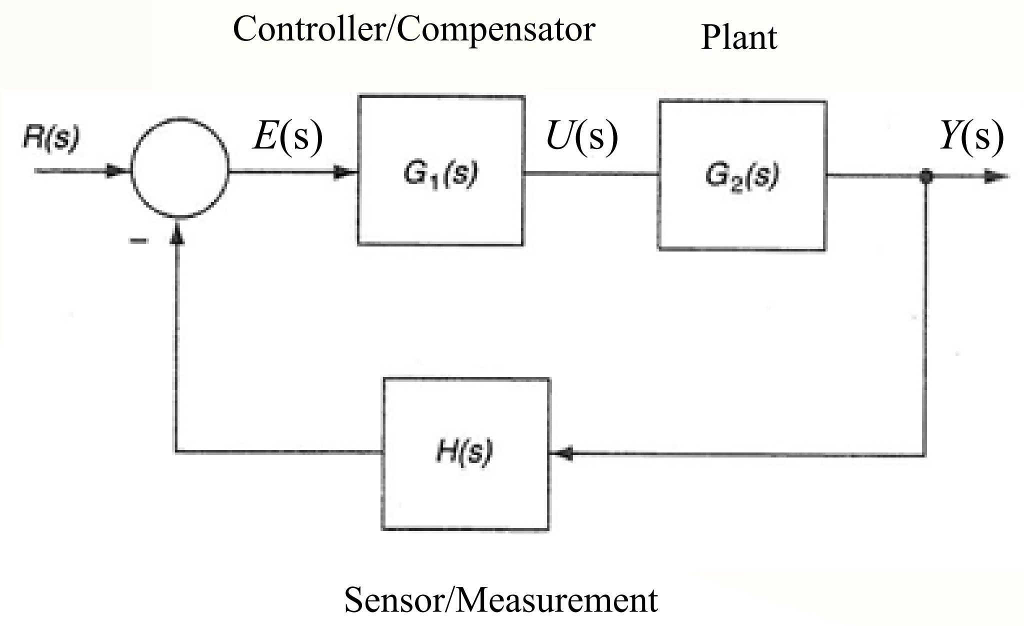
Introduction



Schematic of a typical feedback control system

The objective of a control system is to regulate the value of a dynamic variable. This variable is part of a physical facility or system, called the plant. Example would be regulating the room temperature of house, in which *the house is the plant*. Another example is controlling the attitude of an aircraft, in which the *aircraft is the plant*. The plant has a transfer function that describes how its output, which is the controlled variable *Y(s),* varies in relation to its input, which is the control signal or controlling variable *U(s)*.

Types of Controllers/Compensators



CONTROLLER/COMPENSATOR TRANSFER FUNCTIONS

4.5.1 Proportional, Integral, and Derivative Controllers

$E \rightarrow \boxed{K_P} \rightarrow U_P$

Proportional $u_P(t) = K_P\, e(t)$ $\quad G_P(s) = \dfrac{U_P(s)}{E(s)} = K_P$

$U_P(s) = K_P E(s)$

$E \rightarrow \boxed{\dfrac{K_I}{s}} \rightarrow U_I$

Integral $u_I(t) = K_I \int e(t)d(t)$ $\quad G_I(s) = \dfrac{K_I}{s}$

$U_I(s) = K_I \dfrac{E(s)}{s}$ $\quad G_I(s) = \dfrac{U_I(s)}{E(s)} = \dfrac{K_I}{s}$

$E \rightarrow \boxed{sK_D} \rightarrow U_D$

Derivative $u_D(t) = K_D \dfrac{de(t)}{dt}$ $\quad G_D(s) = sK_D$

$U_D(s) = K_D\, s\, E(s)$ $\quad G_D(s) = \dfrac{U_D}{E} = sK_D$

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$$E_{PI}(s) \longrightarrow \boxed{k_p + \frac{k_I}{s}} \longrightarrow U_{PI}(s)$$

Proportional-Integral

$$G_{PI}(s) = K_P + \frac{K_I}{s} = \frac{sK_P + K_I}{s}$$

The PI mode reduces steady-state error.

Proportional-Derivative

$$G_{PD}(s) = K_P + sK_D$$

The PD mode reduces overshoot.

help with stability

Proportional-Integral-Derivative

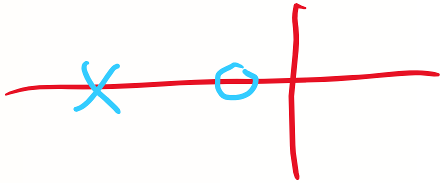
$$G_{PID}(s) = K_P + sK_D + \frac{K_I}{s} = \frac{sK_P + s^2K_D + K_I}{s}$$

Lead and Lag Compensation

$$E \longrightarrow \boxed{K\frac{s+a}{s+b}} \longrightarrow U$$

Lead Compensation

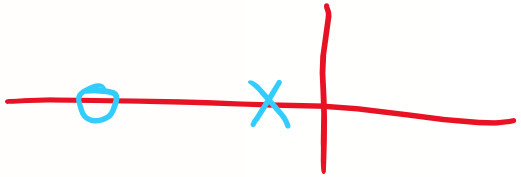
$$G_{lead}(s) = K\frac{s+a}{s+b} \qquad b > a$$



$a \to 0$ similar effect as PD

Lag Compensation

$$G_{lag}(s) = K\frac{s+a}{s+b} \qquad b < a$$



$b \to 0$ similar effect as PI

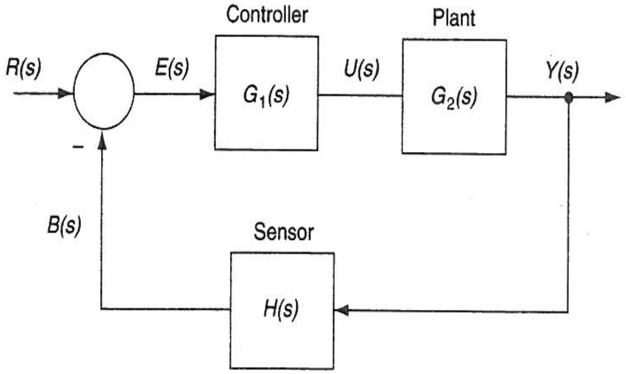
Lead-lag Compensation

$$G_{ll}(s) = K\frac{(s+a_1)(s+a_2)}{(s+b_1)(s+b_2)} \qquad a_1 < b_1,\ a_2 > b_2$$

$$a_1 a_2 = b_1 b_2$$

Similar effect as PID controller

5.1 Purpose



Assume the control system is stable. We'd like to know:

- **Static Response:**
  - **Steady-state error between the input and output**: error between the controlled variable and reference variable after a relatively long time ($t \rightarrow \infty$)
    - Process control (regulation): Input is a constant value (step function), and we want the output to be a fixed value in time.
    - Servomechanism control: Input is changing. For example, a ramp input or a parabolic (quadratic) input and the output should track the changing input.
  - **Disturbance Error**
    - Disturbance is another input to the system apart form the reference. What steady-state error, if any, would result from such a disturbance. This is known as the Disturbance Error.

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5.2 Static Response
5.2.1 Steady-State Error

Steady-state error: Difference between the desired (reference) variable and controlled variable under nominal conditions with no disturbances when $t \rightarrow \infty$

If the system is stable, it can be shown that the two FVT conditions hold: Thus, we can use FVT to find the steady-state value of a function:

Steady-state value

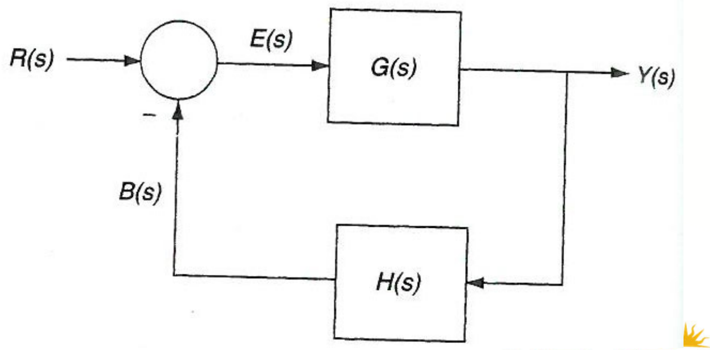
$$f_{ss} \equiv \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

We can use the same principle to find the steady state value of *error*, *ess:*

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

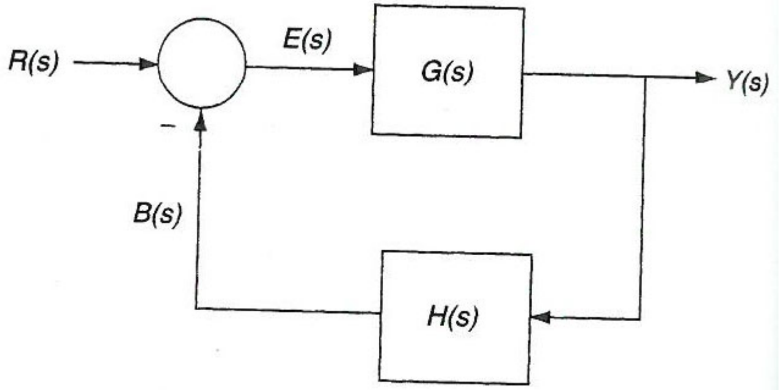
$$E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

URE 4.15
canonical form of a control
em block diagram.



- **Canonical Form**

FIGURE 4.15
The canonical form of a control system block diagram.



$$E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \longrightarrow Y(s) = \frac{G(s)R(s)}{1 + G(s)H(s)}$$

$$\Rightarrow \quad E(s) = R(s) - H(s)\frac{G(s)R(s)}{1 + G(s)H(s)}$$

$$\Rightarrow \quad E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

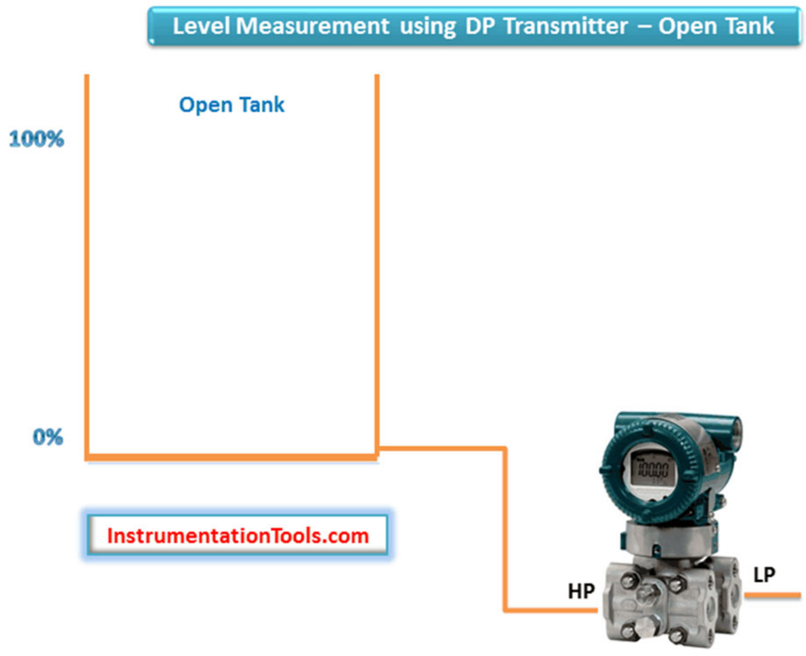
Steady-state error

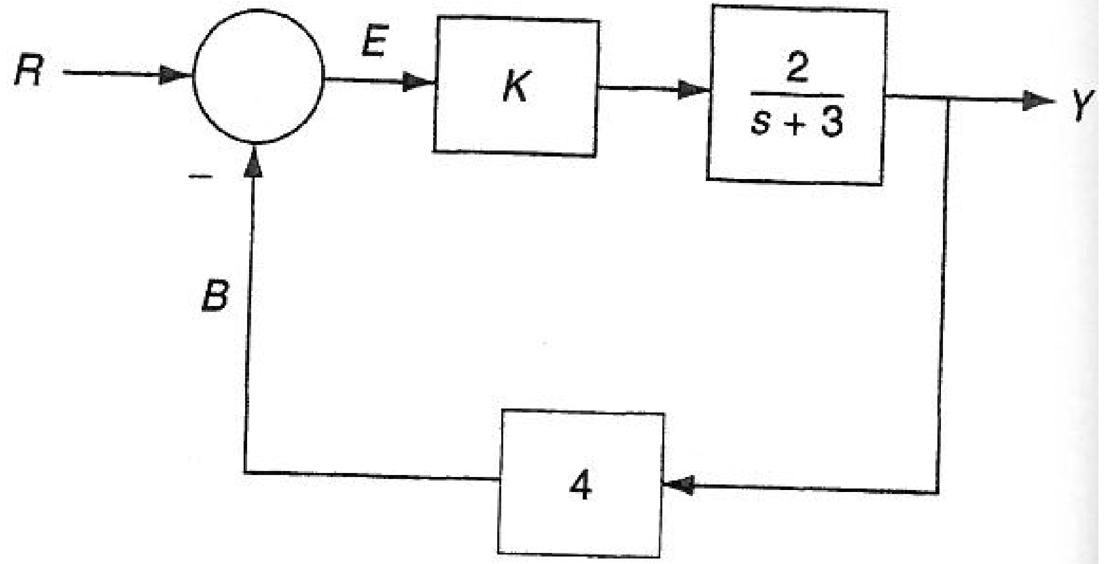
$$\Rightarrow \quad \boxed{e_{ss} = \lim_{s \to 0}\left[\frac{sR(s)}{1 + G(s)H(s)}\right]}$$

This equation tells us the steady-state error, *ess,* depends on:
(1) the nature of the reference variable *R(s),* and
(2) the characteristic of the open-loop transfer function, *G(s) H(s).*

Ex. 5.1

A level control system uses a sensor that converts level to voltage with a transfer function of 4 V/m. So, the level between 0.0 and 0.5 m becomes 0.0 to 2.0 volts. Figure 5.1 shows the system block diagram with a proportional controller of gain, K. Find an expression for the steady-state level error for a fixed level reference of $L = 0.3$ m. Specify the level error for a gain of 10.





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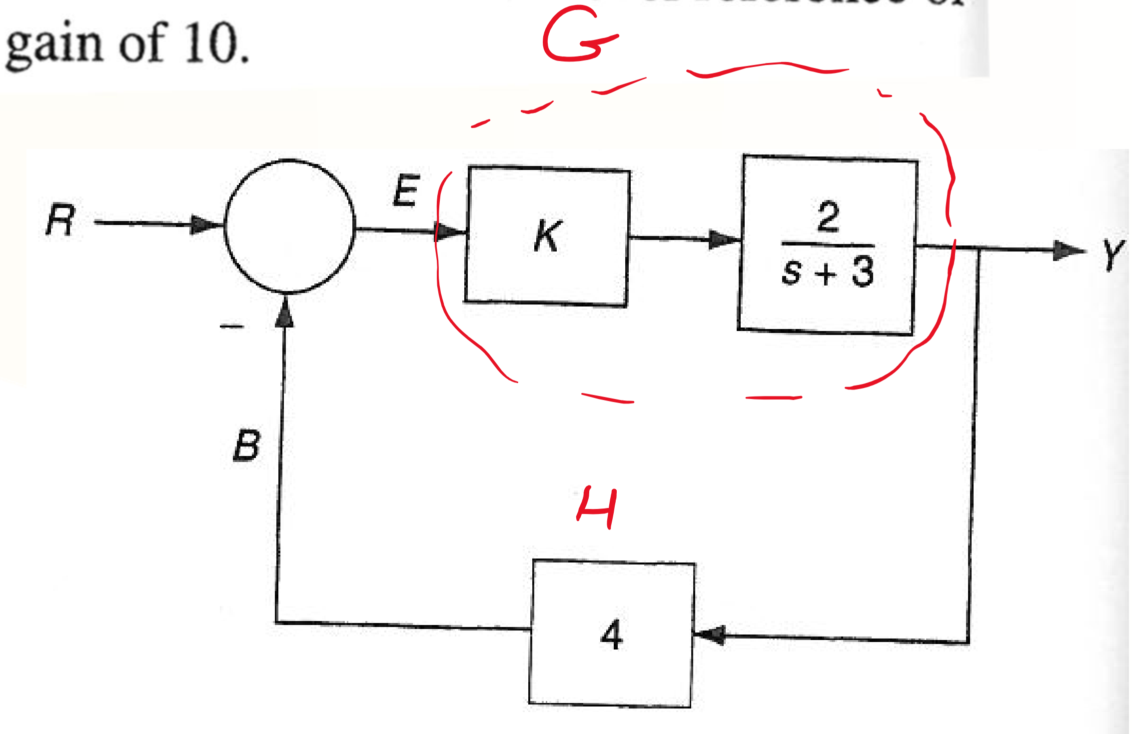
Solution:



Fixed-level reference 0.3m implies:

r(t) = 0.3, or

R(s) = 0.3/*s*

Since B is in voltage, R should be in voltage. Converting the reference input to voltage:

$$R(s) = 4 \times (\frac{0.3}{s}) = \frac{1.2}{s}$$

$$e_{ss} = \lim_{s\to 0}\left[\frac{sR(s)}{1+G(s)H(s)}\right] \rightarrow e_{ss} = \lim_{s\to 0}\left[\frac{s\left(\frac{1.2}{s}\right)}{1+\frac{8K}{s+3}}\right] = \frac{1.2}{1+8K/3} = \frac{3.6}{3+8K} \text{ volts}$$

But remember, this is the error in voltage. To find the error in position, we must divide by 4. Therefore, the level steady-state error is:

$$e_{ss}(level) = \frac{3.6/4}{3+8K} = \frac{0.9}{3+8K} \text{ meters}$$

For a gain of 10 (K = 10), the steady-state error of level is 0.9/(3+8(10)) = 0.0108 m

NOTE:

As you can see, the larger the proportional gain, the smaller the error, but for the proportional control system shown here, the error will never be zero.

System Type

$f(t) \Rightarrow F(s)$

$\int f(t) \Rightarrow F(s)/s$

- A pure integrator has a Laplace transform of 1/*s*. For "n" integrators, we have a transform with $(1/s)^n$. To specifically note the presence of integrators, they are factored out of the open-loop transfer function, which is:

$$G(s)H(s) = \frac{N(s)}{s^n Q(s)} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{s^n (s+p_1)(s+p_2)\cdots(s+p_l)}$$

where N(s) is numerator polynomial in s and Q(s) is denominator polynomial in s, after the n integrators have been factored out. In this representation, *N(s)* and *Q(s)* have no roots at zero.

The system types are now defined as:

Type 0 system: has $n = 0$ and thus no integrators

Type 1 system: has $n = 1$ and thus one integrator

Type 2 system: has $n = 2$ and two integrators

and so on.

Constant Reference

- The reference is a constant. For simplicity, lets assume the constant value is unity, *r(t) = 1* or a unit step. The Laplace transform is R(*s*) = 1/*s*. The steady-state error is

$$e_{ss} = \lim_{s \to 0}\left[\frac{sR(s)}{1 + G(s)H(s)}\right] \longrightarrow e_{ss} = \lim_{s \to 0}\left[\frac{s(1/s)}{1 + \frac{N(s)}{s^n Q(s)}}\right] = \lim_{s \to 0}\left[\frac{s^n}{s^n + \frac{N(s)}{Q(s)}}\right]$$

- For type 0 system: $$e_{ss} = \frac{1}{1 + N(0)/Q(0)} = \frac{1}{1 + K_p}$$

where $K_p = N(0)/Q(0)|_{\text{Type 0}}$ is called the *positional error constant*. We can also find *Kp* using this formula:

$$\boxed{K_p = \lim_{s \to 0}[\mathrm{G(s)H(s)}]}$$

- Type 0 system will always have some error.
- Type 1 or above will have e_{ss} = 0 because there is an s left in the numerator. So, those systems will have a zero steady-state error.

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Ramp Reference

- Tracking control system is common in servomechanism and robotics. The simplest tracking control is to track a linear ramp with unit slope, *r(t) = t* or *R(s) = 1/s²*.

$$e_{ss} = \lim_{s\to 0}\left[\frac{sR(s)}{1+G(s)H(s)}\right] \longrightarrow e_{ss} = \lim_{s\to 0}\left[\frac{s(1/s^2)}{1+\dfrac{N(s)}{s^n Q(s)}}\right] = \lim_{s\to 0}\left[\frac{s^{n-1}}{s^n + \dfrac{N(s)}{Q(s)}}\right]$$

- Type 0 system will have 1/s in the numerator and a steady-state error that increases without limit (error growing in time).
- Type 1 system will have a steady-state error given by a constant value as

$$e_{ss} = \frac{1}{N(0)/Q(0)} = \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{N(0)/Q(0)} = \frac{1}{K_v}$$

where $K_v = N(0)/Q(0)|_{\text{Type 1}}$

Kv is called the *velocity error constant* and can be calculated using this formula:

$$\boxed{K_v = \lim_{s \to 0}[sG(s)H(s)]}$$

- With a single integrator (Type 1), the system would track a ramping reference but with some constant error, $e_{ss}=1/Kv$.
- Systems of Type 2 or above have no error, $e_{ss}=0$ for a ramp input.

$e_{ss} = \frac{1}{K_v}$

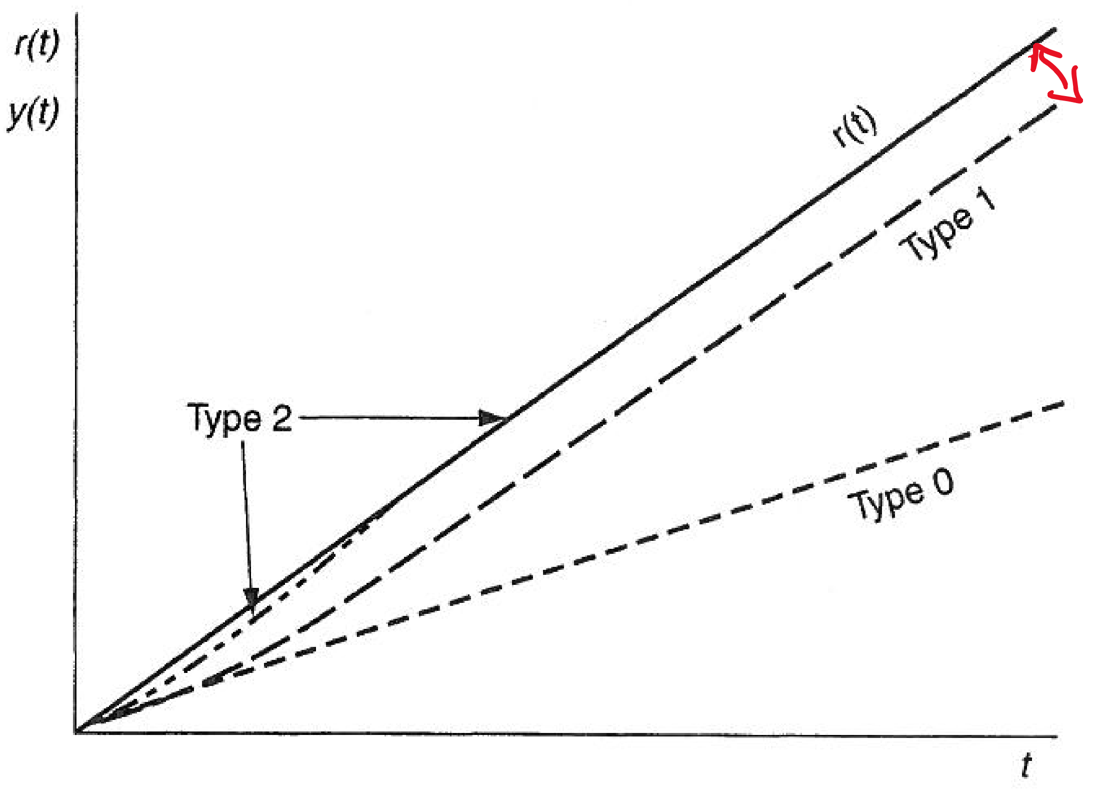

FIGURE 5.2
Response of Types 0, 1 and 2 systems to a ramp input.

Parabolic Reference

- A parabolic (quadratic) input has the form $r(t) = (t^2/2)$ or $R(s) = 1/s^3$
- The steady-state error in this case is

$$e_{ss} = \lim_{s \to 0} \left[\frac{s\left(\frac{1}{s^3}\right)}{1 + \frac{N(s)}{s^n Q(s)}} \right] = \lim_{s \to 0} \left[\frac{s^{n-2}}{s^n + \frac{N(s)}{Q(s)}} \right]$$

- Type 0 and 1 systems will have an error that increases without limit.
- Type 2 system will have a constant error

$$e_{ss} = \frac{1}{N(0)/Q(0)} = \frac{1}{K_a}$$

where $K_a = N(0)/Q(0)|_{\text{Type 2}}$ is called the *acceleration error constant.*

K_a can be calculated using this formula:

$$\boxed{K_a = \lim_{s \to 0} [s^2 G(s) H(s)]}$$

Note:

For the systems of Type 3 and above, the steady-state error to the parabolic reference is zero.

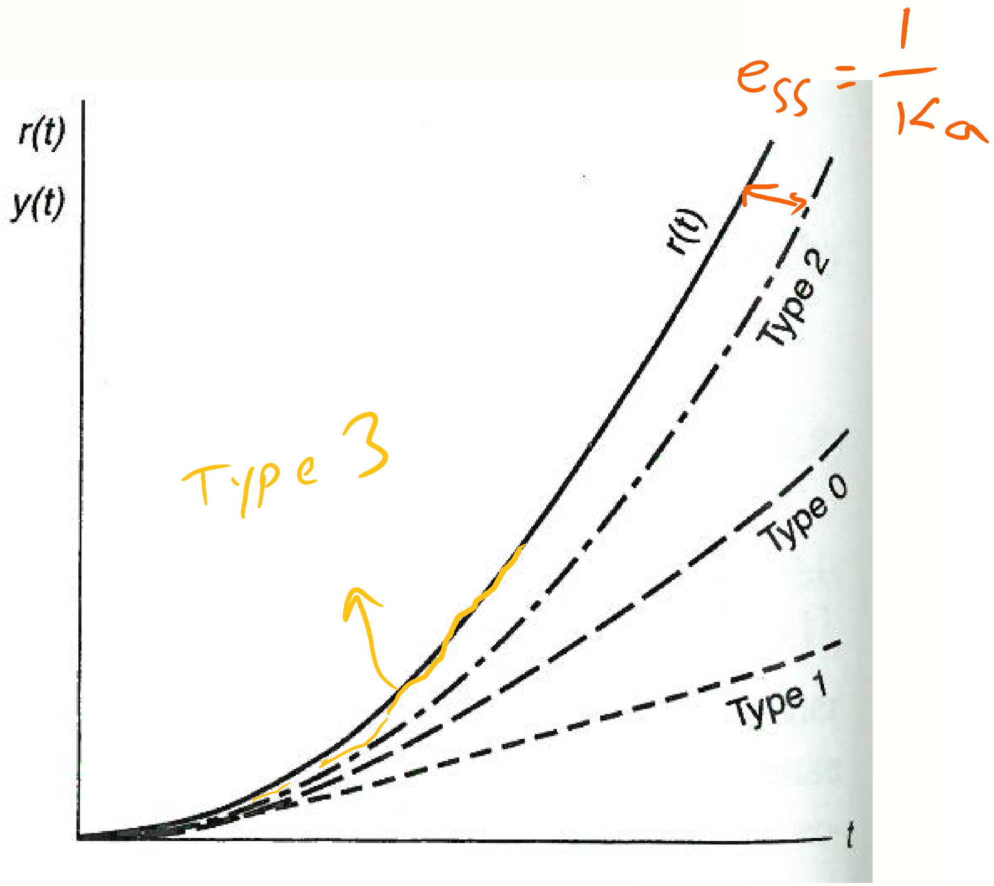

FIGURE 5.3

Response of Types 0, 1 and 2 systems to a parabolic input.

Table 5.1 Summary of steady-state error for normalized input signals for stable systems

Reference Input / Open-loop system	Constant $r(t) = 1$	Ramp $r(t) = t$	Parabola $r(t) = t^2/2$
Type 0	$\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$\dfrac{1}{K_v}$	∞
Type 2	0	0	$\dfrac{1}{K_a}$

$$K_p = \lim_{s\to 0}[G(s)H(s)]$$

$$K_v = \lim_{s\to 0}[sG(s)H(s)]$$

$$K_a = \lim_{s\to 0}[s^2G(s)H(s)]$$

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Table 5.2 Steady-state error for general input signals for stable systems

Type of Open-loop System \ Reference Input	Constant $r = r_0$	Ramp $r = v_0 t$	Parabolic $r = \frac{1}{2} a_0 t^2$
Type 0	$\frac{r_0}{1+K_p}$	∞	∞
Type 1	0	$\frac{v_0}{K_v}$	∞
Type 2	0	0	$\frac{a_0}{K_a}$

$$K_p = \lim_{s\to 0}[G(s)H(s)]$$

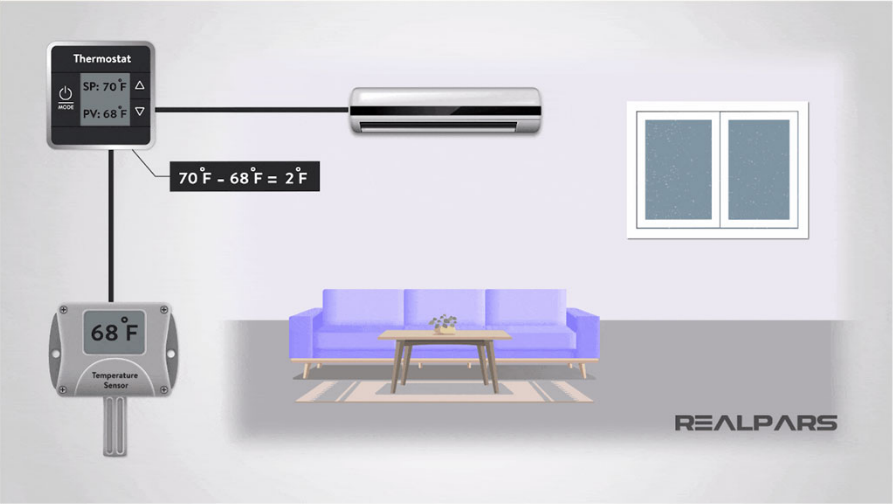
$$K_v = \lim_{s\to 0}[sG(s)H(s)]$$

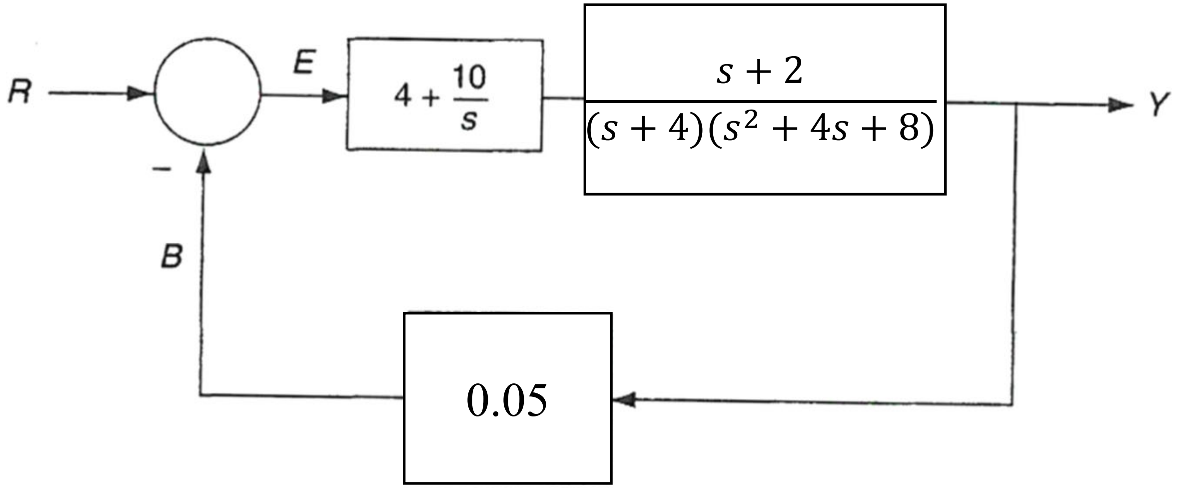
$$K_a = \lim_{s\to 0}[s^2G(s)H(s)]$$

Ex. 5.2

A proportional-integral (PI) temperature control system is shown below. The sensor converts the temperature to the voltage with the transfer function of $0.05\ \mathrm{V/^{\circ}C}$. Determine the steady-state temperature error for the following reference inputs:

Fixed 4 V; Ramp at $0.02\mathrm{V}/s$; and Parabola at $0.01\mathrm{V}/s^2$





Solution:

The PI controller block can be written in the form:

$$4 + \frac{10}{s} = \frac{4s + 10}{s}$$

Open-Loop TF →

$$G(s)H(s) = \frac{N(s)}{s^n Q(s)}$$

$$G(s)H(s) = \frac{0.05(4s+10)(s+2)}{s(s+4)(s^2+4s+8)}$$

Type 1 system

1. Since this is a Type 1 system, there will be no error for a constant reference input of 4.0 Volts, according to Table 5.1. Thus, The temperature will be T=4.0 V/[0.05 V/^{o}C] = 80 ^{o}C

$e_{ss} = 0$

2. Ramp input of 0.02 V/s: According to Table 5.2, there will be a steady-state error of V_0/K_v and

$$\lim_{s \to 0} \cancel{s} \frac{0.05(4s+10)(s+2)}{\cancel{s}(s+4)(s^2+4s+8)} = \frac{1}{32}$$

$$K_v = \lim_{s \to 0}[sG(s)H(s)] \longrightarrow K_v=1/32=0.03125 \longrightarrow e_{ss}= V_0/K_v =0.02/0.03125=0.64 \text{ volts}$$

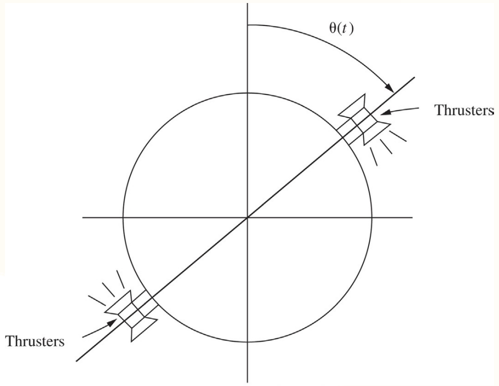
$$\longrightarrow e_{ss}(\text{temp})= 0.64/0.05 =12.8\ ^{o}\text{C}$$

This is the voltage error, however, so to find the temperature error we need to divide by the scale factor of 0.05 V/°C. So there will be a constant lagging error of 12.8°C between the ramping reference temperature and the actual temperature. In other words, the system tracks the input ramp by ramping up the output, but the output is always 12.8°C behind the input.

3. Quadratic input of 0.01 V/s^2: Table 5.1 shows that for this Type 1 system, the error to a quadratic input grows without limit, meaning that this system cannot track quadratic changes.

Ex. 5.3

Consider the satellite attitude (position) control system, as shown in the Figure 5.5. The satellite was stabilized with a PD controller. What is the steady-state error of satellite attitude (position) for the following reference inputs: Constant input of 5, ramp input of 8t, and parabola input of $2t^2$



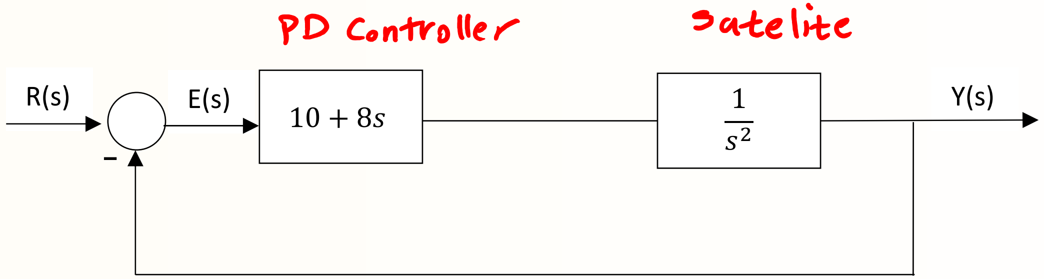


Figure 5.5
satellite attitude control system

Solution:

open-loop TF = $G(s)\,H(s)$, $H(s) = 1$ $\qquad G(s) = (10+8s)\frac{1}{s^2} = \frac{10+8s}{s^2}$

Type 2 system
- steady-state error for Constant input = 0
- Steady-state error for ramp input = 0
- Steady-state error fo parabola $2t^2$ = 0.4

$r = \frac{1}{2}\underbrace{(4)}_{a_0}t^2$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \cancel{s^2}\left(\frac{10+8s}{\cancel{s^2}}\right) = 10 \Rightarrow e_{ss} = \frac{a_0}{K_a} = \frac{4}{10}$$

5.2.2 Disturbance Error

a) Finding the Output

When, there is more than one input, we can use the superposition principle to find the output: $y(t) = y_r(t) + y_d(t)$

$$Y(s) = Y_r(s) + Y_d(s)$$

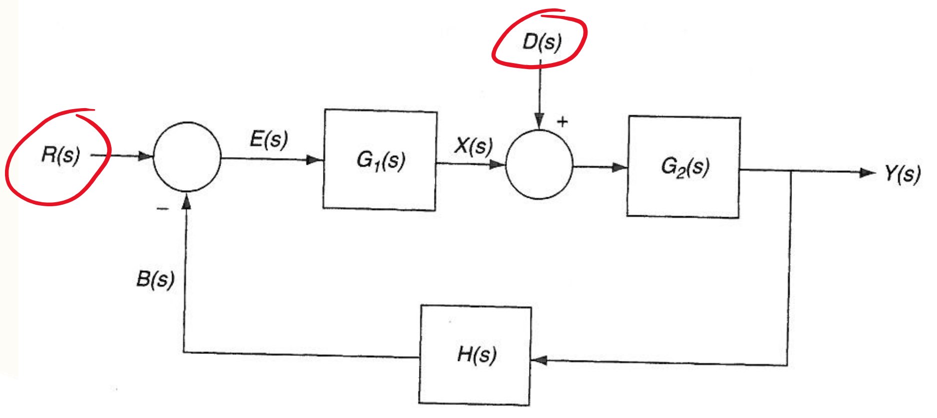
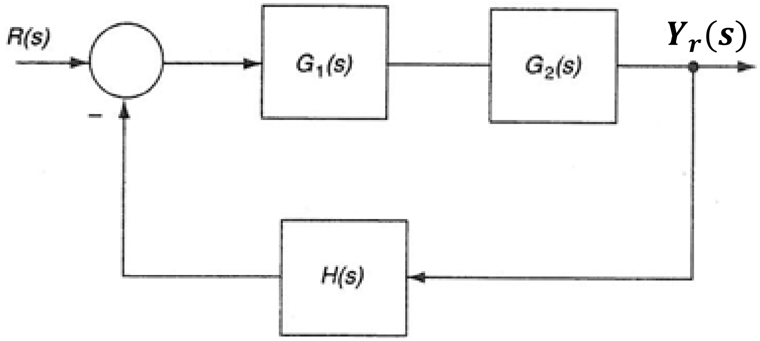


FIGURE 4.25
Canonical control system with a disturbance.

1) Response to *R*(*s*)

Assume the other input is zero: $D(s) = 0$

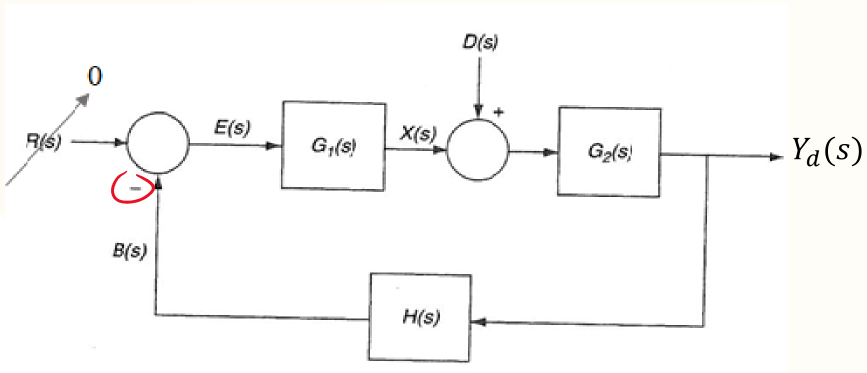


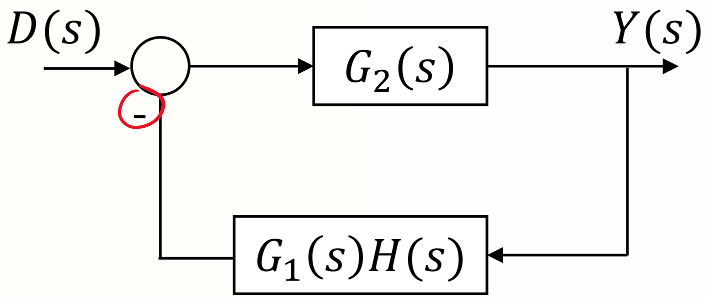
$$G = G_1G_2$$

$$\frac{Y_r(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \rightarrow \boxed{Y_r(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}R(s)}$$

2) Response to *D*(*s*)

Assume the other input is zero: $R(s) = 0$





$$\frac{Y_d(s)}{D(s)} = \frac{G_2(s)}{1 + G_2(s) \times G_1(s)H(s)} \longrightarrow \boxed{Y_d(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} D(s)}$$

Superposition principle:

$$Y(s) = Y_r(s) + Y_d(s) \longrightarrow \boxed{Y(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} R(s) + \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} D(s)}$$

5.2.2 Disturbance Error

b) Steady-state response to disturbance

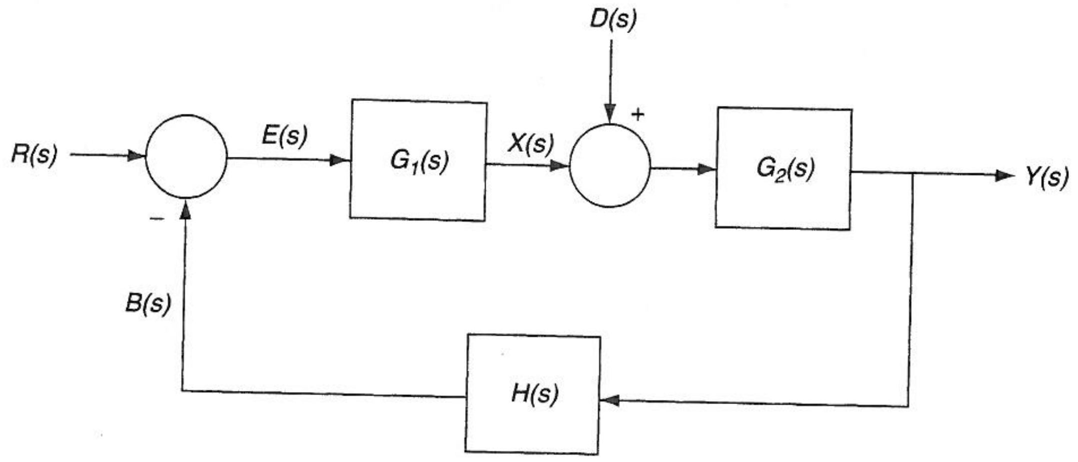


FIGURE 4.25
Canonical control system with a disturbance.

From previous slide, we have:

$$\frac{Y_d(s)}{D(s)} = \frac{G2(s)}{1 + H(s)G1(s)G2(s)}$$

$$\Rightarrow Y_d(s) = \frac{G_2(s)\,D(s)}{1 + H(s)\,G_1(s)\,G_2(s)}$$

steady state response to disturbance

$$\boxed{y_{dss} = \lim_{s\to 0}\left[\frac{sG2(s)D(s)}{1 + H(s)G1(s)G2(s)}\right]}$$

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Ex. 5.4

Figure 5.5 shows a disturbance applied to the PI temperature control system between the controller and the plant in the feed-forward part of the loop. Evaluate the steady-state response to a disturbance with a constant value of 1.

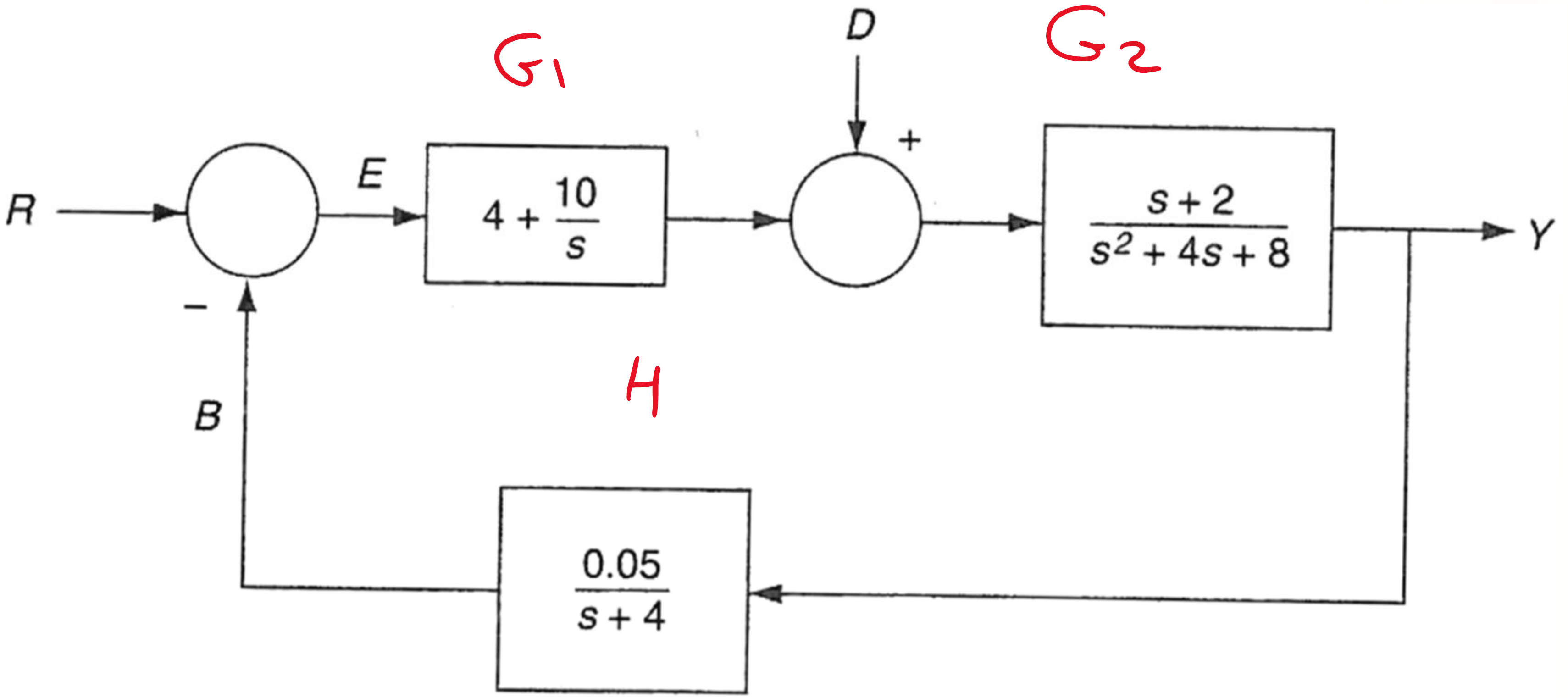

FIGURE 5.5
PI control system with a disturbance

Solution to Ex. 5.4

$$G1(s) = 4 + 10/s = (4s + 10)/s$$

$$G2(s) = \frac{(s+2)}{s^2 + 4s + 8}$$

$$H(s) = \frac{0.05}{s+4}$$

From the Slide 5.2.2 Disturbance Error:

$$y_{dss} = \lim_{s\to 0}\left[\frac{sG2(s)D(s)}{1 + H(s)G1(s)G2(s)}\right] \rightarrow y_{dss} = \lim_{s\to 0}\left[\frac{s\dfrac{(s+2)}{s^2+4s+8}D(s)}{1 + \dfrac{0.05(4s+10)(s+2)}{s(s+4)(s^2+4s+8)}}\right]$$

For a constant disturbance, $D(s) = 1/s$,

$$y_{dss} = \left[\frac{(2/8)}{1+\infty}\right] = 0$$

So, for this system, a constant disturbance will create <u>no</u> steady-state error in the output.

- Homework 6 is **due October 17, 11 am** and must be submitted on Canvas.

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