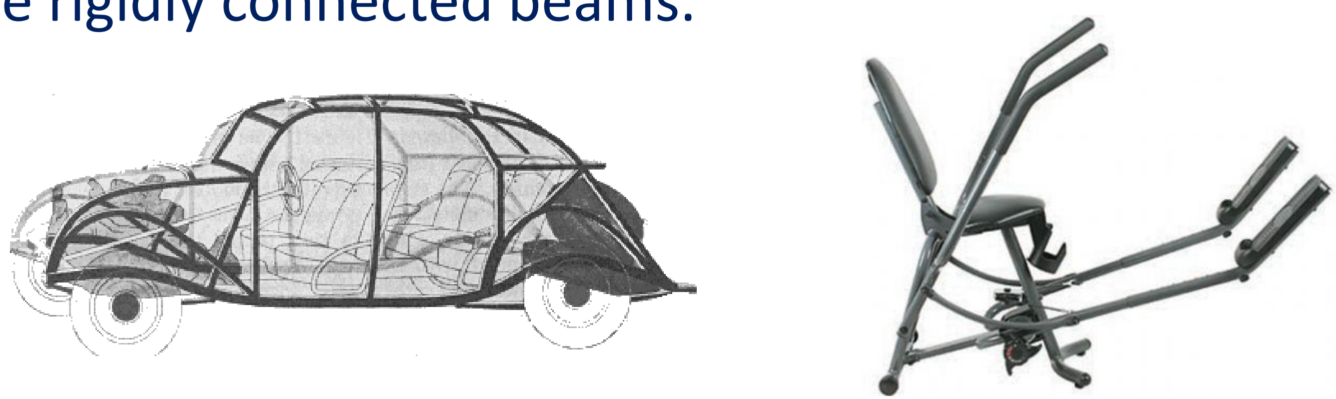
Finite Element Modeling and Simulation with ANSYS Workbench

Chapter 3

Beams and Frames

Introduction

- Beams and frames are frequently used in constructions, in engineering equipment and in everyday life.
- Beams are slender structural members subjected primarily to transverse loads.
- The term ‘frame’ is used for structures constructed of two or more rigidly connected beams.



(a) (b)

Figure 3.1. Beams and Frames used in a car and an exercise machine

Essential features of the two well-known beam models …

Review of the Beam Theory.

Review of the Beam Theory

- Euler-Bernoulli beam and Timoshenko beam

Both models have at their core the assumption of small deformation and linear elastic isotropic material behavior. They are applicable to beams with uniform cross sections.

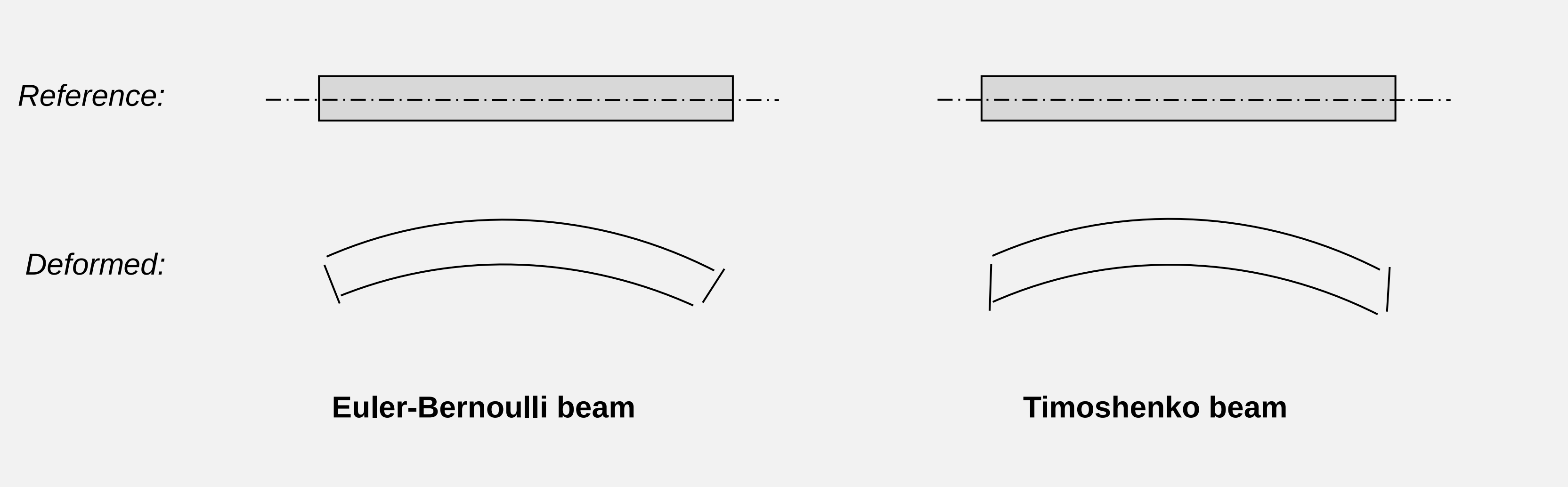


Figure 3.2. Two common beam models.

Review of the Beam Theory

❑ Euler-Bernoulli beam and Timoshenko beam

For an Euler-Bernoulli beam, the forces on a beam only cause it to bend. A planar cross section remains plane and perpendicular to the neutral axis after deformation.

The Timoshenko model accounts for both transverse shear and bending deformation. A planar cross section remains plane but does not remain normal to the neutral axis after deformation.

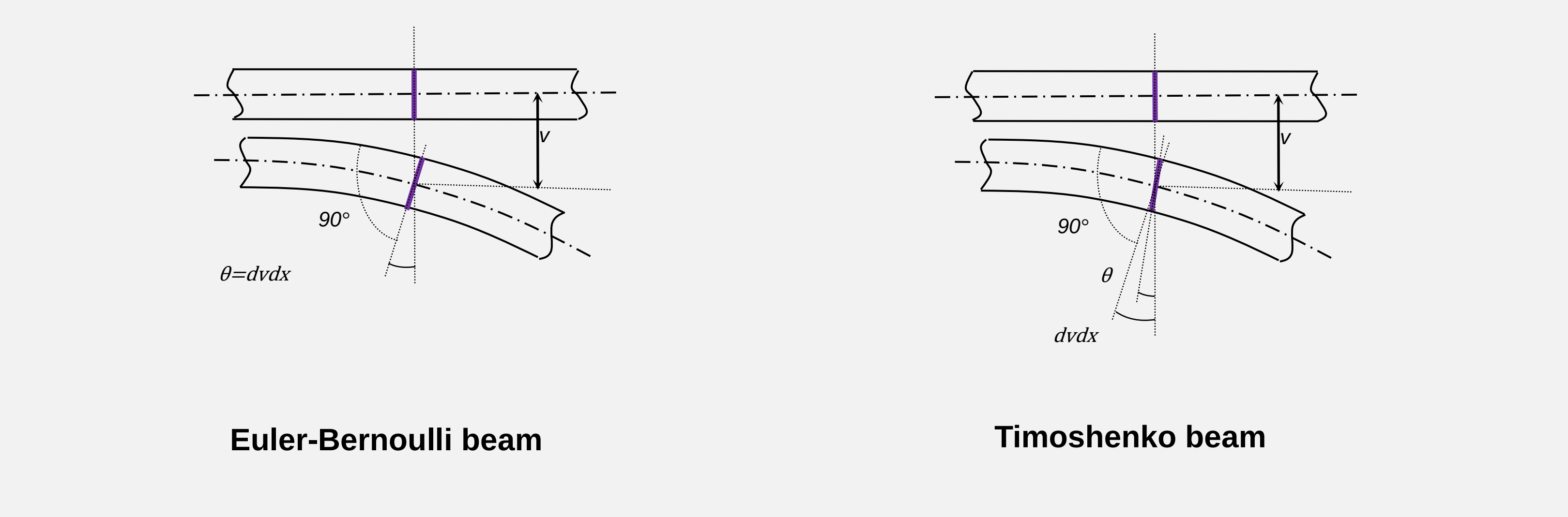


Figure 3.3. Deflection of Euler-Bernoulli beam and Timoshenko beam.

Euler–Bernoulli beam theory (also known as **engineer's beam theory** or **classical beam theory**) is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. It covers the case corresponding to small deflections of a beam that is subjected to lateral loads only. By ignoring the effects of shear deformation and rotatory inertia, it is thus a special case of Timoshenko beam theory.

The **Timoshenko–Ehrenfest beam theory** was developed by Stephen Timoshenko and Paul Ehrenfest early in the 20th century.[4][5] The model takes into account shear deformation and rotational bending effects, making it suitable for describing the behaviour of thick beams, sandwich composite beams, or beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam.

Review of the Beam Theory

❑ Stress, Strain, Deflection and Their Relations

Simple beam bending is often analyzed using the Euler-Bernoulli beam theory, which is popular in engineering owing to its simplicity.

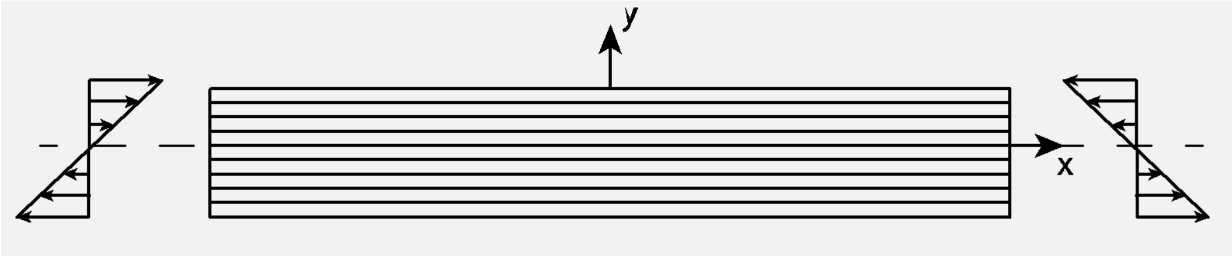


Figure 3.4. Bending produces axial stress $\sigma(x)$ along thin material layers in a beam.

The bending strain $$\varepsilon(x) = -y\kappa(x) = -y\frac{d^2v}{dx^2}$$

The bending moment

$$M(x) = \int -y\sigma(x)dA = \int Ey^2\kappa(x)\,dA = EI\kappa(x) = EI\frac{d^2v}{dx^2}$$

The bending stress $$\sigma(x) = -\frac{M(x)y}{I}$$

where y is the vertical distance of a thin layer from the neutral axis and v the beam deflection. The internal resisting bending moment M(x) is a function of the axial bending stress

Review of the Beam Theory

The simple beam theory is analogous to the uniaxial Hooke's law. As shown in Table 3.1, the bending moment is linearly proportional to the deflected beam curvature through a bending stiffness (EI) constant. This resembles the linear stress-strain relationship described by the Hooke's law.

Table 3.1. Analogy between the constitutive equations for bars and beams.

	Stress measurement	Strain measurement	Constitutive equation
Bar	Axial stress: $\sigma(x)$	Axial strain: $\varepsilon(x)$	$\sigma(x) = E\varepsilon(x)$
Beam	Bending moment: $M(x)$	Curvature: $\frac{d^2v}{dx^2}$	$M(x) = EI\frac{d^2v}{dx^2}$

Hooke's law, law of elasticity discovered by the English scientist Robert Hooke in 1660, which states that, for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load. Under these conditions the object returns to its original shape and size upon removal of the load.

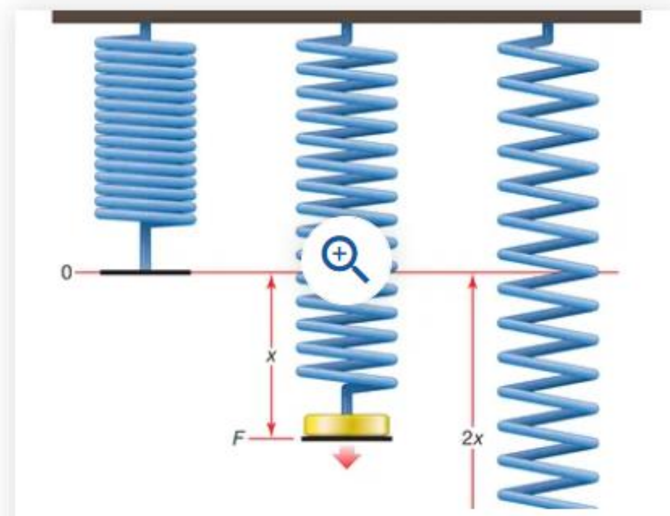
Formula

$$F_s = -kx$$

F_s = spring force

k = spring constant

x = spring stretch or compression



Hooke's law

The **negative sign in Hooke's law** shows that the restoring force exerted by the spring is in the opposite direction to the force that causes the displacement

Review of the Beam Theory

The basic equations that govern the problems of simple beam bending are summarized in Figure 3.5.

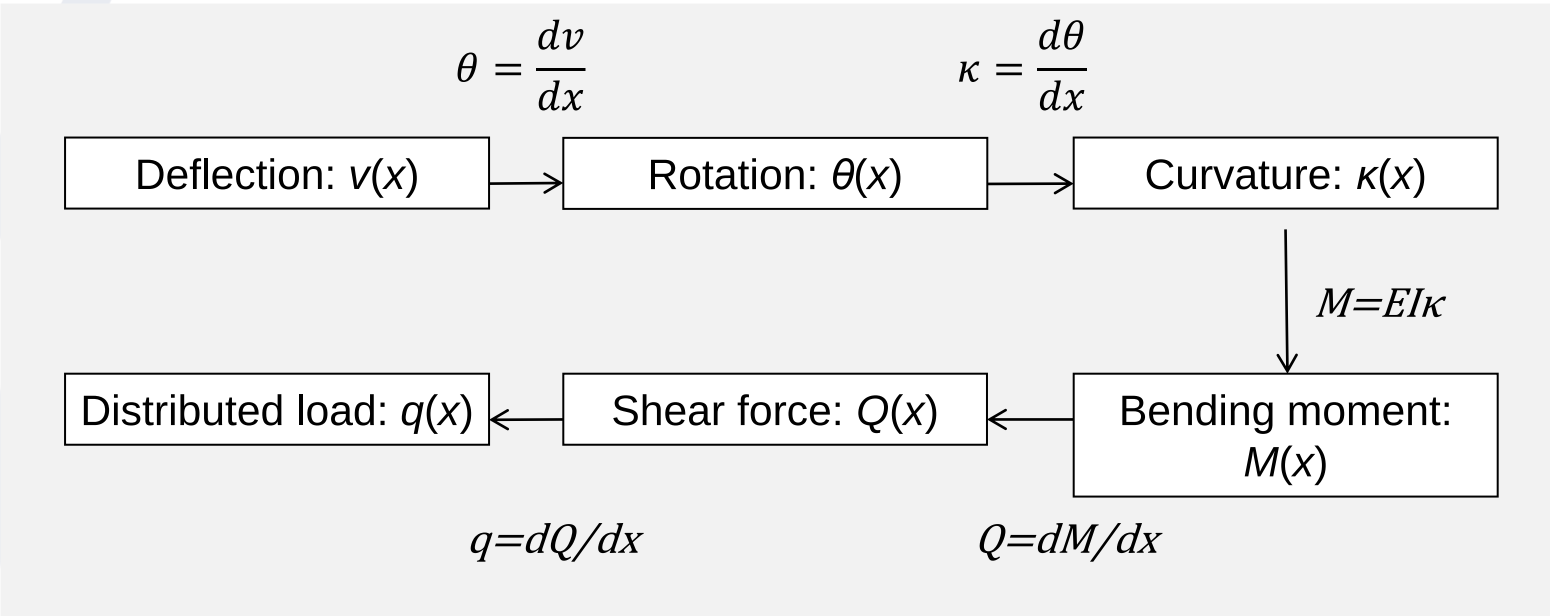
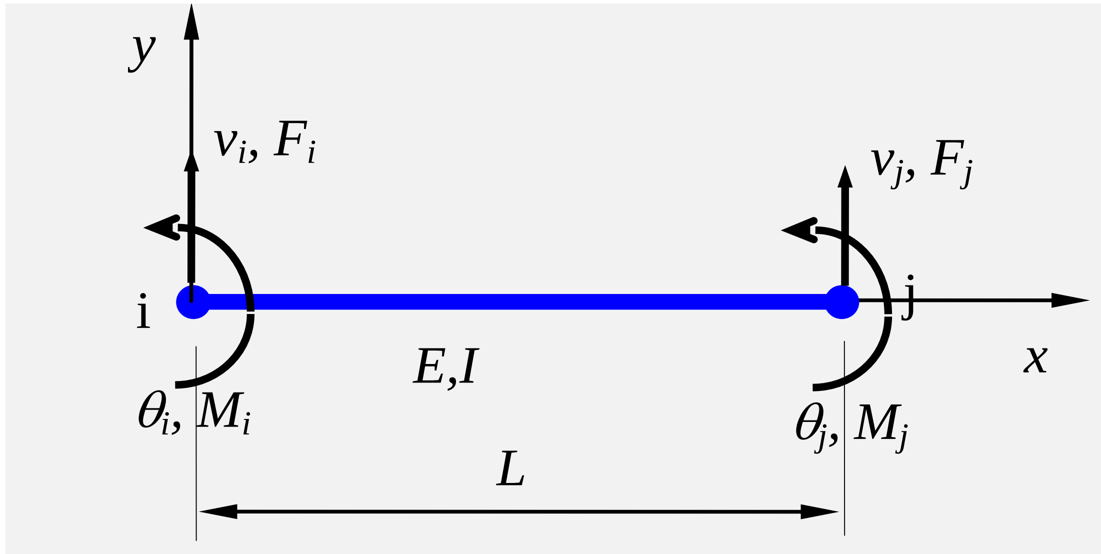


Figure 3.5. The governing equations for a simple beam.

The setup of a simple beam element:



L, I, E	length, moment of inertia of the cross-sectional area and elastic modulus of the beam, respectively
$v = v(x)$	deflection (lateral displacement) of the neutral axis of the beam
$\theta = \dfrac{dv}{dx}$	rotation of the beam about the z-axis
$Q = Q(x)$	(internal) shear force
$M = M(x)$	(internal) bending moment about z-axis
F_i, M_i, F_j, M_j	applied (external) lateral forces and moments at node i and j, respectively

Modeling is an idealization process…

Modeling of Beams and Frames.

Modeling of Beams and Frames

❑ Cross Sections and Strong/Weak Axis

Beams are available in various cross-sectional shapes.

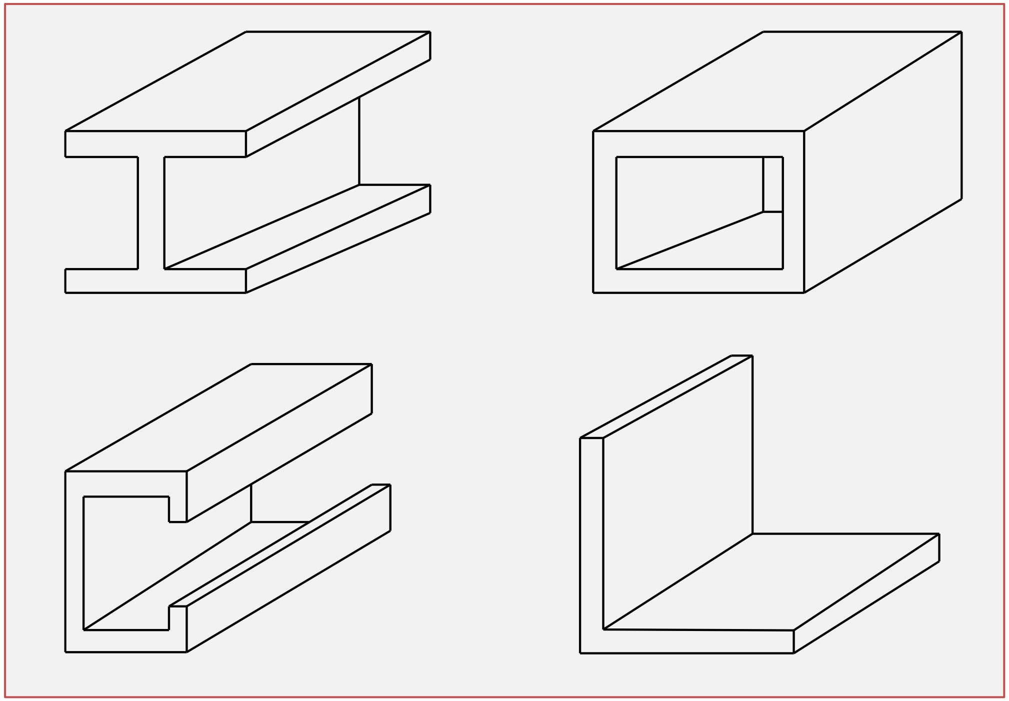


Figure 3.6. Common beam cross section profiles.

Modeling of Beams and Frames

The bending stiffness (EI) measures a beam's ability to resist bending.

It is more difficult to bend a flat ruler with its flat (wide) surface facing forward rather than facing up.

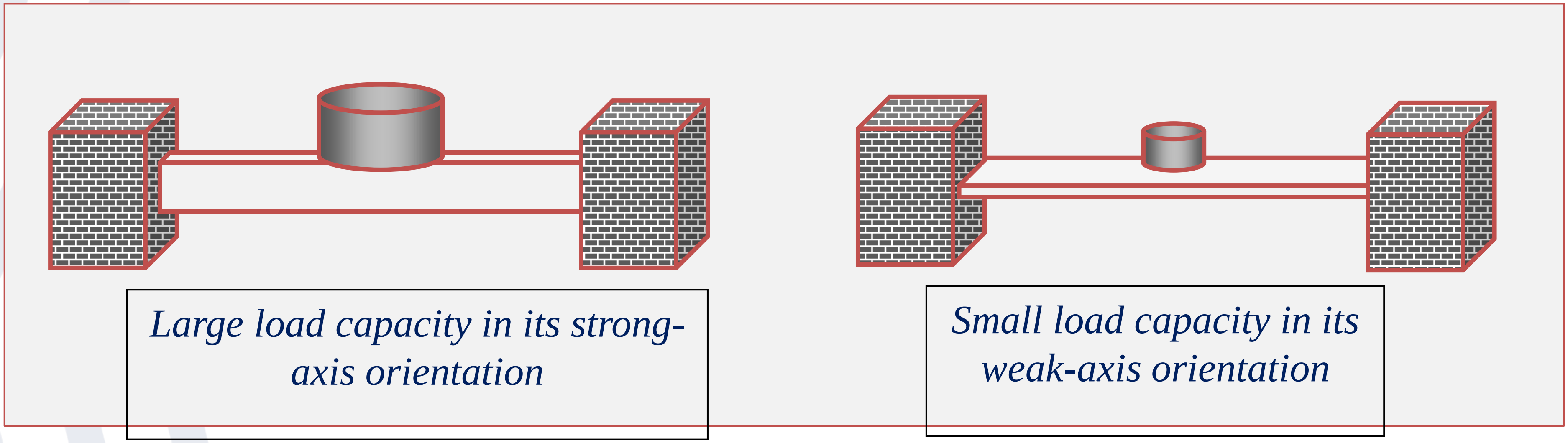


Figure 3.7. A beam loaded on its strong and weak axes.

Support Conditions

Beams have three types of end support conditions: fixed support, pinned support and roller support.

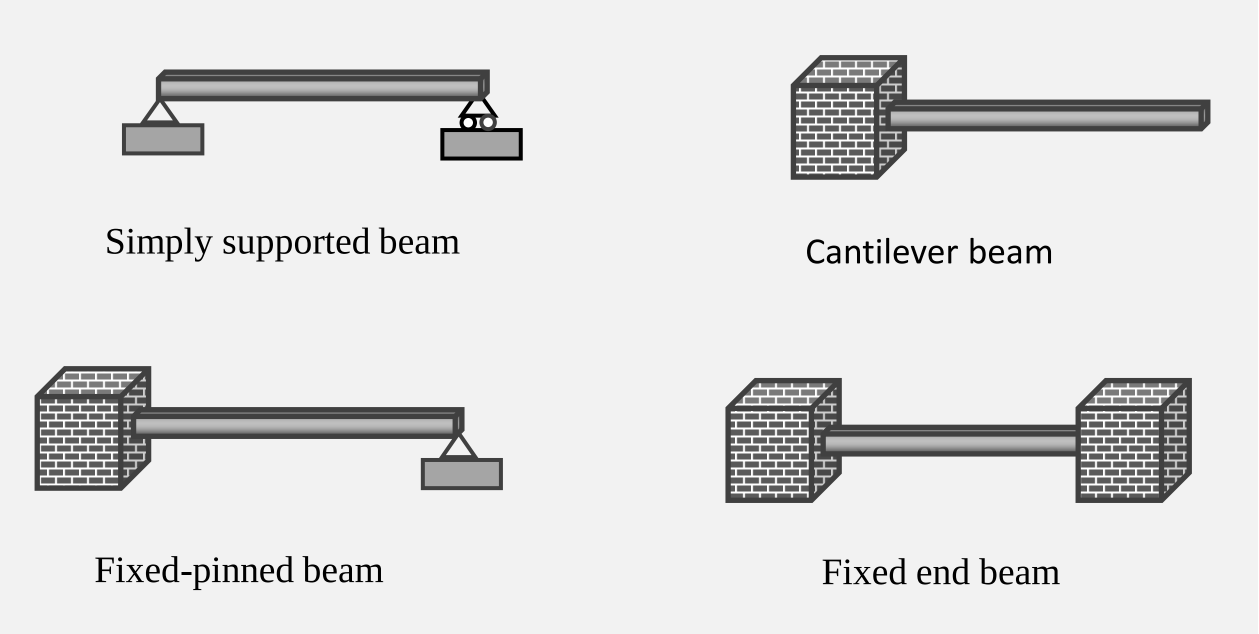


Figure 3.8. Beam supports and beam types.

Modeling of Beams and Frames

- Conversion of a physical model to a line model

Under the uniform cross section assumption, it is apparent that a beam needs only to be modeled at the center axis (neutral axis) of the actual 3-D beam structure.

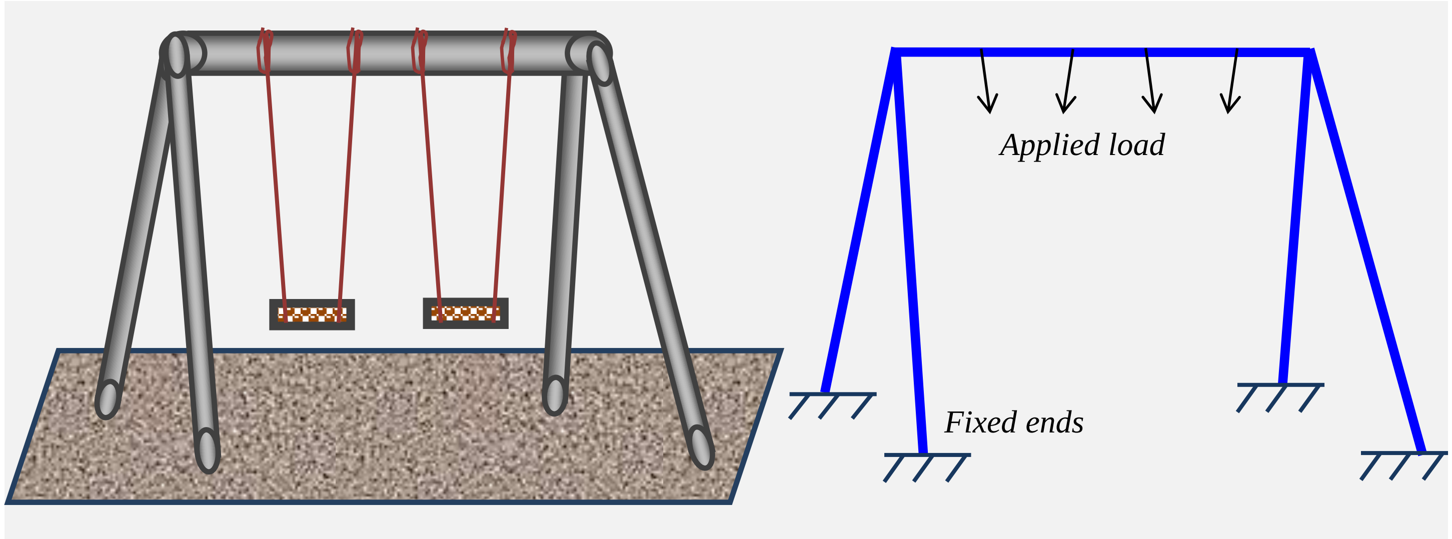


Figure 3.9. A swing set and its simplified line model.

Finite element formulation based on the simple beam theory …

Formulation of the Bar Element Chapter 2 slide

Stiffness Matrix – Direct Method

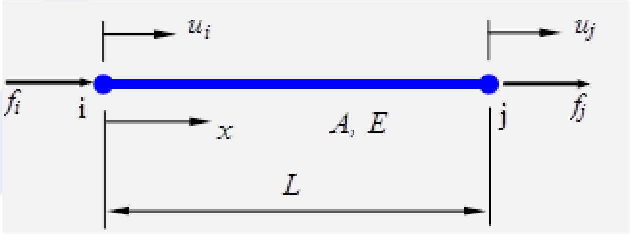


Figure 2.4. Notation for a bar element.

We have
$$u(x) = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j$$

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L}$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L} \quad \text{and} \quad \sigma = \frac{F}{A}$$

Therefore
$$F = \frac{EA}{L}\Delta = k\Delta$$

We conclude that the bar behaves like a spring. The element stiffness matrix is:

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

Element equilibrium equation is:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Degree of Freedom (DOF): Number of components of the displacement vector at a node. For 1-D bar element along the *x*-axis, we have one DOF at each node.

Formulation of the Beam Element

Element Stiffness Equation – the Direct Approach

The FE equation for a beam takes the form

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}$$

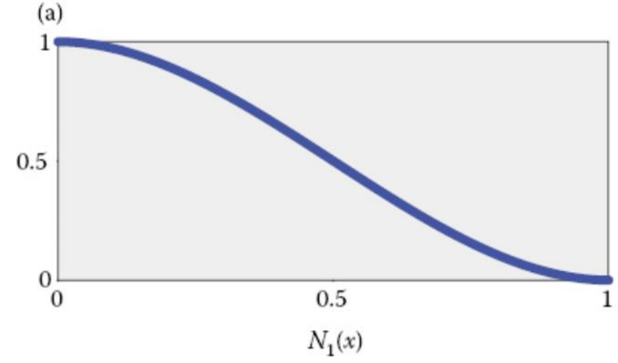
E.g., the first column represents the forces/moments to keep the shape with

$$v_i = 1,\ \theta_i = v_j = \theta_j = 0$$

Using the results from strength of materials for a cantilever beam with a force k_{11} and moment k_{21} applied at the free end, we have

$$v_i = \frac{k_{11}L^3}{3EI} - \frac{k_{21}L^2}{2EI} = 1 \qquad \text{and} \qquad \theta_i = -\frac{k_{11}L^2}{2EI} + \frac{k_{21}L}{EI} = 0$$

Figure 3.11a



Each column in the stiffness matrix represents the forces needed to keep the element in a special deformed shape. For example, the first column represents the forces/moments to keep the shape with vi = 1, θi = vj = θj = 0 as shown in Figure 3.11a. Thus, using the results from strength of materials for a cantilever beam with a force k11 and moment k21 applied at the free end,

$$v_i = \frac{k_{11}L^3}{3EI} - \frac{k_{21}L^2}{2EI} = 1 \quad \text{and} \quad \theta_i = -\frac{k_{11}L^2}{2EI} + \frac{k_{21}L}{EI} = 0$$

$v = v(x)$ deflection (lateral displacement) of the neutral axis of the beam

$\theta = \frac{dv}{dx}$ rotation of the beam about the z-axis

Formulation of the Beam Element

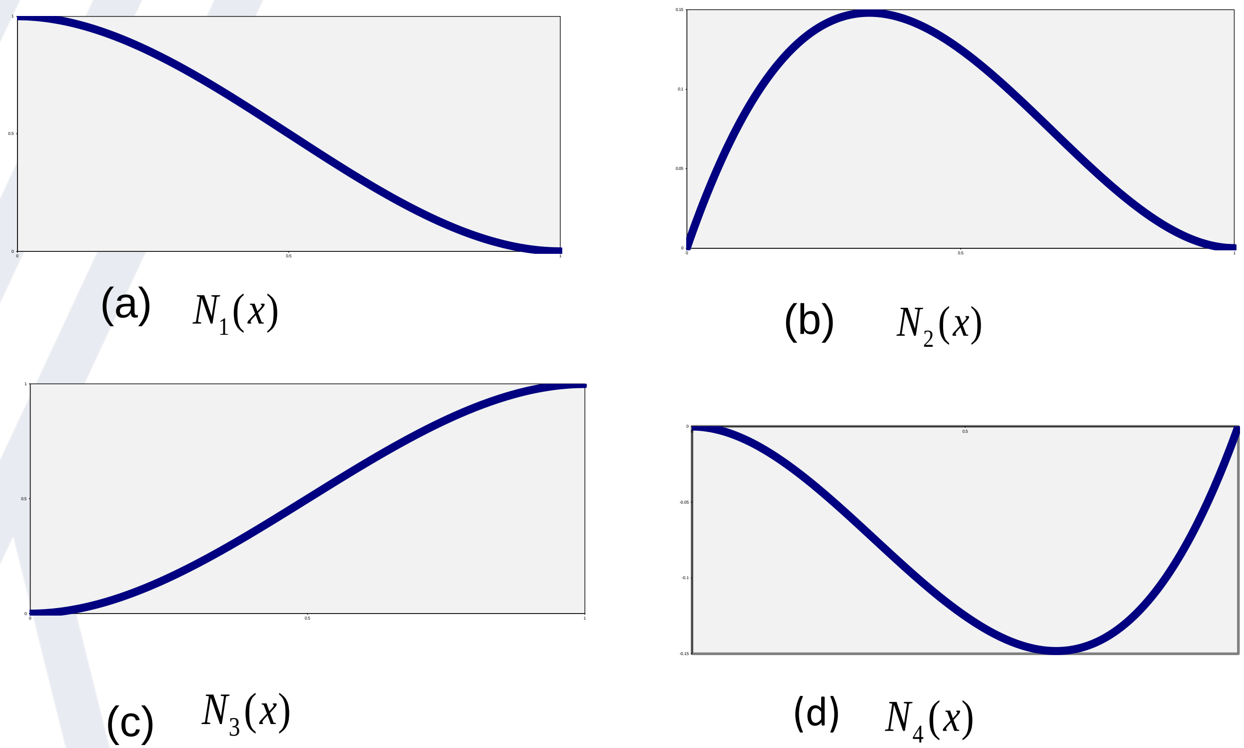


Figure 3.11. Four configurations or shapes for the simple beam element.

eg: **u(x)** = N1 u1 + N2 u2 The function that relates the variation of field variable with the nodal value of the field variable is called the "SHAPE FUNCTION". The number of shape functions will depend upon the number of nodes and the number of variables per node.

- A beam element is **a slender structural member that offers resistance to forces and bending under applied loads**.
- A beam element differs from a truss element in that a beam resists moments (twisting and bending) at the connections. These three node elements are formulated in three-dimensional space.

2.2.1 Displacement function

A typical finite element, e, is defined by nodes, i, j, m, etc., and straight line boundaries. Let the displacements $\mathbf{u}$ at any point within the element be approximated as a column vector, $\hat{\mathbf{u}}$:

$$\boxed{\mathbf{u} \approx \hat{\mathbf{u}} = \sum_k \mathbf{N}_k \mathbf{a}_k^e = [\mathbf{N}_i, \mathbf{N}_j, \ldots] \begin{Bmatrix} \mathbf{a}_i \\ \mathbf{a}_j \\ \vdots \end{Bmatrix}^e = \mathbf{N}\mathbf{a}^e} \qquad (2.1)$$

in which the components of $\mathbf{N}$ are prescribed functions of position and $\mathbf{a}^e$ represents a listing of nodal displacements for a particular element.

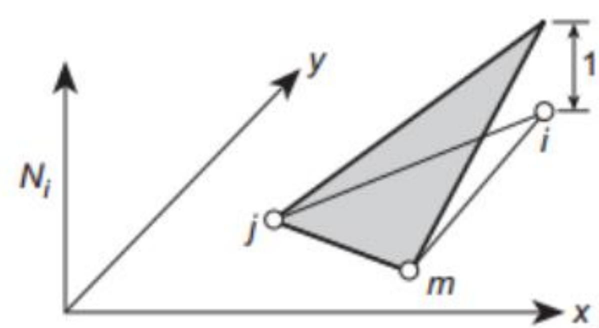


Fig. 2.2 Shape function N_i for one element.

In the case of plane stress, for instance,

$$\mathbf{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix}$$

represents horizontal and vertical movements of a typical point within the element and

$$\mathbf{a}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

the corresponding displacements of a node i.

The functions $\mathbf{N}_i$, $\mathbf{N}_j$, $\mathbf{N}_m$ have to be chosen so as to give appropriate nodal displacements when the coordinates of the corresponding nodes are inserted in Eq. (2.1). Clearly, in general,

$$\mathbf{N}_i(x_i, y_i) = \mathbf{I} \quad \text{(identity matrix)}$$

while

$$\mathbf{N}_i(x_j, y_j) = \mathbf{N}_i(x_m, y_m) = \mathbf{0}, \qquad \text{etc.}$$

which is simply satisfied by suitable linear functions of x and y.

If both the components of displacement are specified in an identical manner then we can write

$$\mathbf{N}_i = N_i \mathbf{I}$$

and obtain N_i from Eq. (2.1) by noting that $N_i = 1$ at x_i, y_i but zero at other vertices.

The most obvious linear function in the case of a triangle will yield the shape of N_i of the form shown in Fig. 2.2. Detailed expressions for such a linear interpolation are given in Chapter 4, but at this stage can be readily derived by the reader.

The functions $\mathbf{N}$ will be called *shape functions* and will be seen later to play a paramount role in finite element analysis.

Formulation of the Beam Element

Recall that each column in the stiffness matrix represents the forces needed to keep the element in a special deformed shape. Solving the system equations, we obtain

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}$$

Formulation of the Beam Element

Element Stiffness Equation – the Energy Approach

Introduce four shape functions

$$N_1(x) = 1 - 3x^2/L^2 + 2x^3/L^3$$

$$N_2(x) = x - 2x^2/L + x^3/L^2$$

$$N_3(x) = 3x^2/L^2 - 2x^3/L^3$$

$$N_4(x) = -x^2/L + x^3/L^2$$

On each element, the beam deflection (v) is written as

$$v(x) = \mathbf{Nu} = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$$

Formulation of the Beam Element

Consider the curvature of the beam

$$\frac{d^2 v}{dx^2} = \frac{d^2}{dx^2}\mathbf{N}\mathbf{u} = \mathbf{B}\mathbf{u}$$

where the strain-displacement matrix **B** is given by

$$\mathbf{B} = \frac{d^2}{dx^2}\mathbf{N} = \begin{bmatrix} N_1^{''}(x) & N_2^{''}(x) & N_3^{''}(x) & N_4^{''}(x) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix}$$

Strain energy stored in the beam element is

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon dV$$

Formulation of the Beam Element

Applying the basic equations in the simple beam theory, we have

$$U = \frac{1}{2}\int_0^L\int_A\left(-\frac{My}{I}\right)^T\frac{1}{E}\left(-\frac{My}{I}\right)dAdx = \frac{1}{2}\int_0^L M^T\frac{1}{EI}Mdx$$

$$= \frac{1}{2}\int_0^L\left(\frac{d^2v}{dx^2}\right)^T EI\left(\frac{d^2v}{dx^2}\right)dx = \frac{1}{2}\int_0^L(\mathbf{Bu})^T EI(\mathbf{Bu})dx$$

$$= \frac{1}{2}\mathbf{u}^T\left(\int_0^L\mathbf{B}^T EI\mathbf{B}dx\right)\mathbf{u}$$

We conclude that the stiffness matrix for the simple beam element is

$$\mathbf{k} = \int_0^L\mathbf{B}^T EI\mathbf{B}dx$$

Carry out the integration, then we arrive at the same stiffness matrix as the one given by the Direct Approach.

Formulation of the Beam Element

- Treatment of distributed Loads

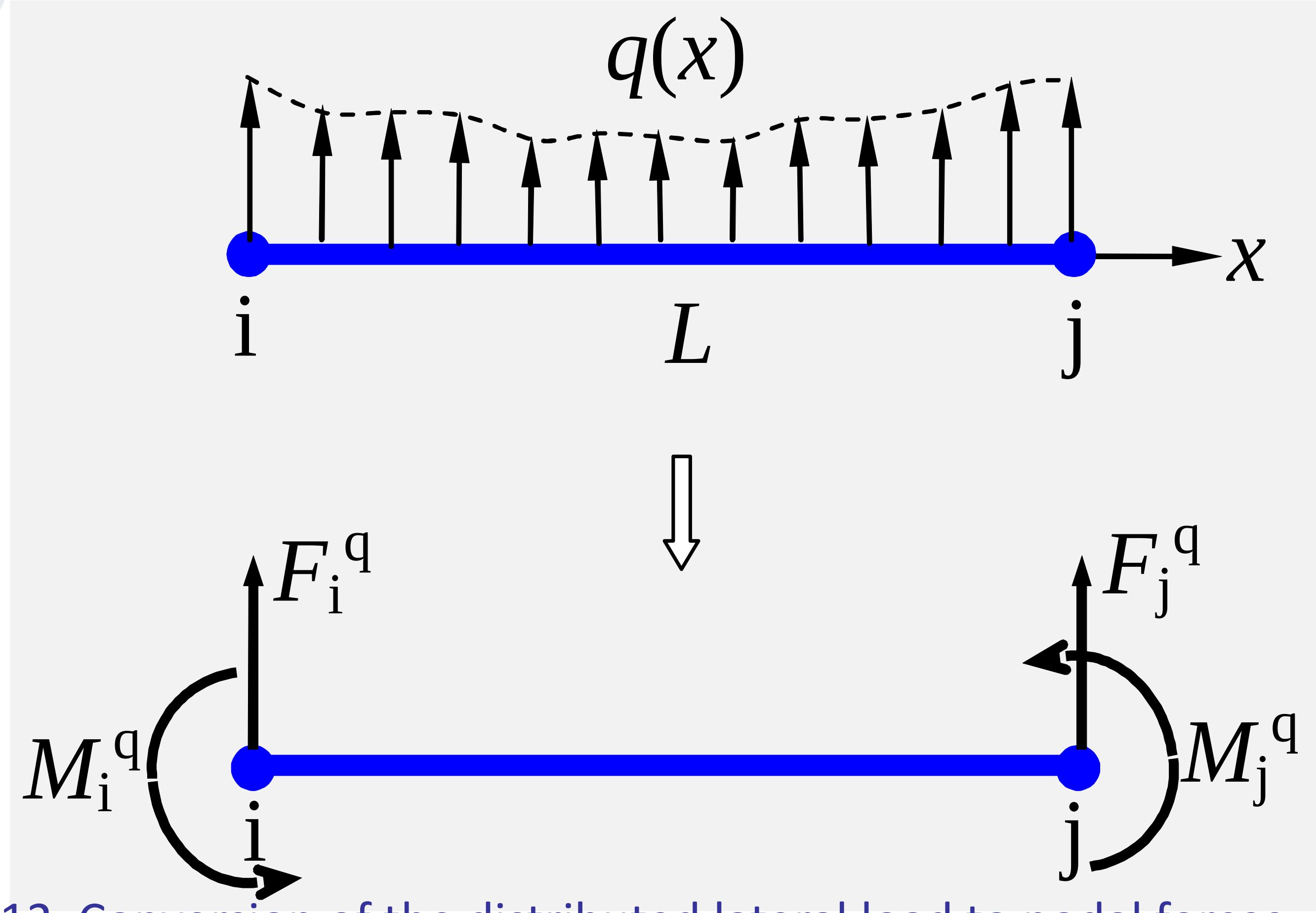


Figure 3.12. Conversion of the distributed lateral load to nodal forces and moments.

Formulation of the Beam Element

Consider the work done by the distributed load *q*

$$W_q = \frac{1}{2}\int_0^L v(x)q(x)dx = \frac{1}{2}\int_0^L (\mathbf{Nu})^T q(x)dx = \frac{1}{2}\mathbf{u}^T \int_0^L \mathbf{N}^T q(x)dx$$

The work done by the equivalent nodal forces (and moments) is

$$W_{f_q} = \frac{1}{2}\begin{bmatrix} v_i & \theta_i & v_j & \theta_j \end{bmatrix} \begin{Bmatrix} F_i^q \\ M_i^q \\ F_j^q \\ M_j^q \end{Bmatrix} = \frac{1}{2}\mathbf{u}^T \mathbf{f}_q$$

By equating $W_q = W_{f_q}$, we obtain the equivalent nodal force vector as

$$\mathbf{f}_q = \int_0^L \mathbf{N}^T q(x)dx$$

Formulation of the Beam Element

For constant q, we have the results shown in Figure 3.13. An example of this result is given in Figure 3.14.

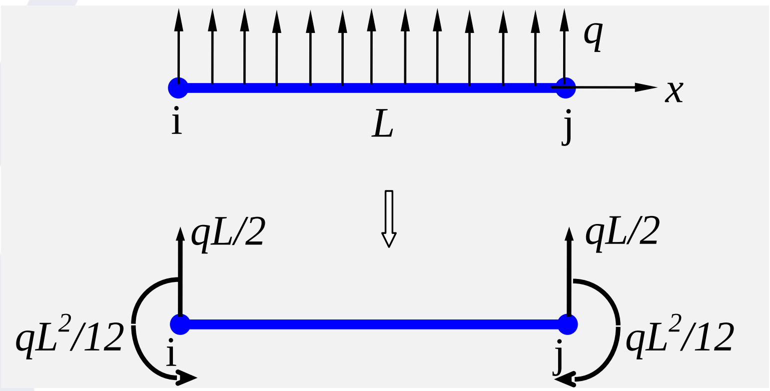


Figure 3.13. Conversion of a constant distributed lateral load to nodal forces and moments.

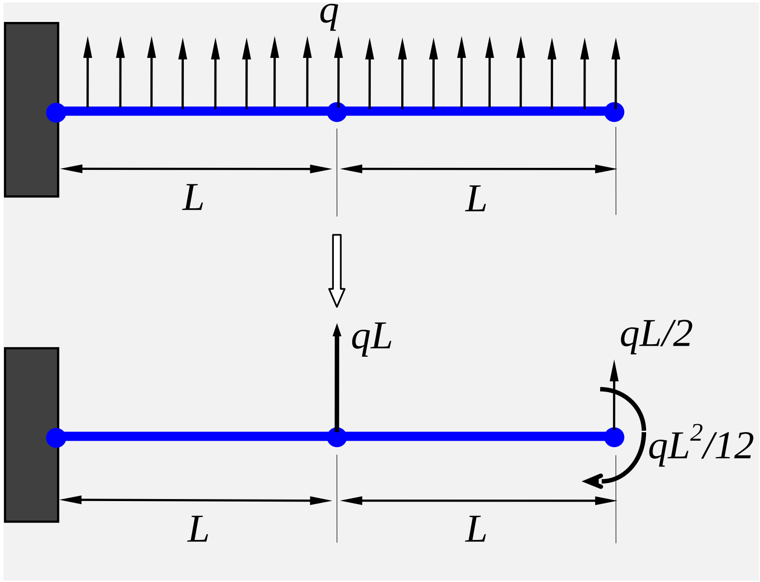


Figure 3.14. Conversion of a constant distributed lateral load on two beam elements.

Formulation of the Bar Element
Chapter 1 Slide

❑ Stiffness Matrix – Direct Method

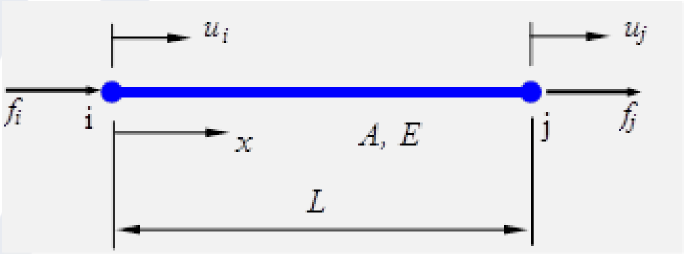


Figure 2.4. Notation for a bar element.

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Formulation of the Beam Element

Element Stiffness Equation – the Direct Approach

The FE equation for a beam takes the form

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}$$

E.g., the first column represents the forces/moments to keep the shape with

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Formulation of the Beam Element

- Stiffness matrix for a general beam element

Stiffness matrix for a *general 2-D beam element:*

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_i & v_i & \theta_i & u_j & v_j & \theta_j \end{array} \\
\mathbf{k} = & \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
$$

We now look at problems formulated with beam elements ....

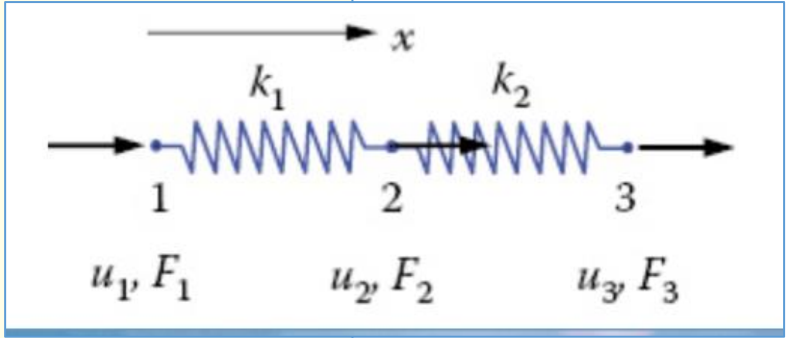
1.2.3 Boundary and Load Conditions

Assuming that node 1 is fixed, and same force P is applied at node 2 and node 3, that is

$$u_1 = 0 \text{ and } F_2 = F_3 = P$$

we have from Equation 1.6

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ P \end{Bmatrix}$$



which reduces to

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \end{Bmatrix}$$

and

$$F_1 = -k_1 u_2$$

Unknowns are

$$\mathbf{u} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

and the reaction force F_1 (if desired).

Solving the equations, we obtain the displacements

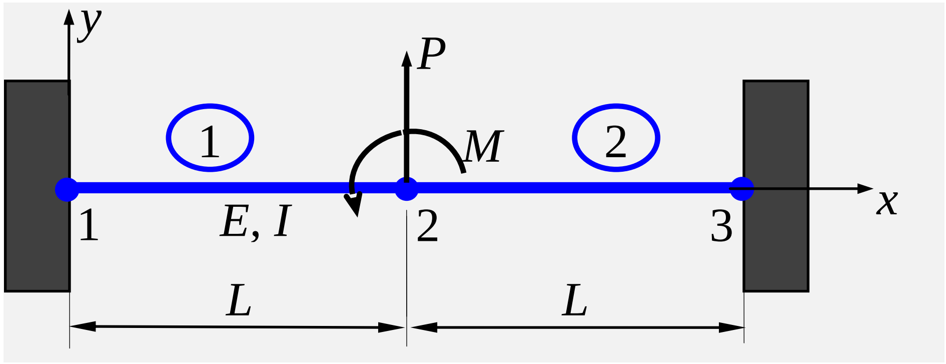
$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{Bmatrix}$$

and the reaction force

$$F_1 = -2P$$

❑ Example Problems

Example 3.1



Given: The beam shown is clamped at the two ends and acted upon by the force *P* and moment *M* in the mid-span.

Find: The deflection and rotation at the center node and the reaction forces and moments at the two ends.

Examples with Beam Element

Solution: Element stiffness matrices are

$$
\mathbf{k}_1 = \frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}
\qquad
\mathbf{k}_2 = \frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_2 & \theta_2 & v_3 & \theta_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}
$$

Global FE equation is

$$
\frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{matrix} \\ \left[\begin{array}{cc|cc|cc} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ \hline -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ \hline 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \end{array}
\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \\ F_{3Y} \\ M_3 \end{Bmatrix}
$$

Examples with Beam Element

Loads and constraints (BC's) are

$$F_{2Y} = -P, \qquad M_2 = M, \qquad v_1 = v_3 = \theta_1 = \theta_3 = 0$$

Reduced FE equation

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ M \end{Bmatrix}$$

Solving this, we obtain

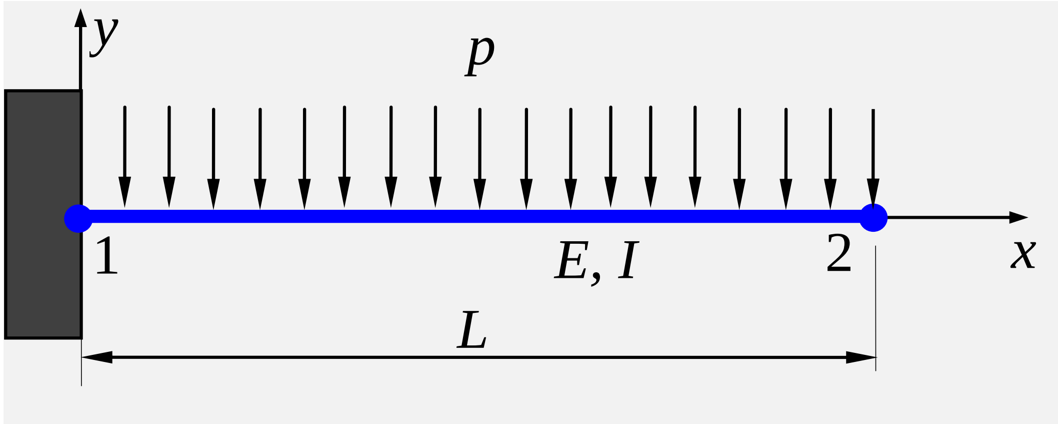
$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{24EI}\begin{Bmatrix} -PL^2 \\ 3M \end{Bmatrix}$$

From the global FE equation, we obtain the reaction forces and moments

$$\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{3Y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \\ -12 & -6L \\ 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} 2P + 3M/L \\ PL + M \\ 2P - 3M/L \\ -PL + M \end{Bmatrix}$$

Examples with Beam Element

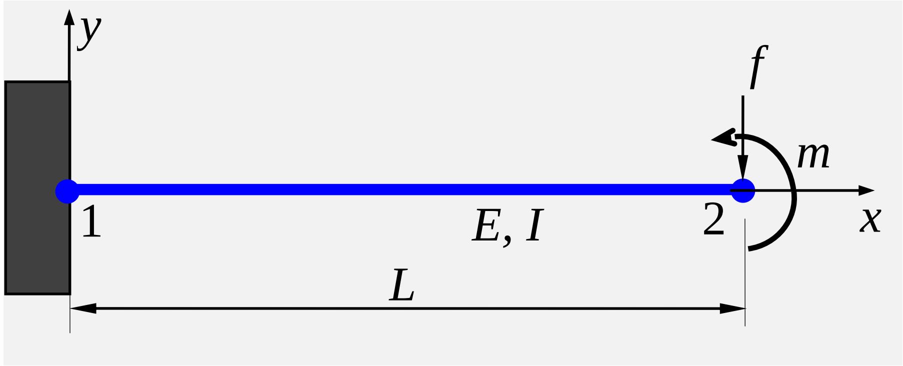
Example 3.2



Given: A cantilever beam with distributed **lateral load *p* as** shown above.

Find: The deflection and rotation at the right end, the reaction force and moment at the left end.

Solution: The work-equivalent nodal loads are shown below



where $f = pL/2, \quad m = pL^2/12$

Applying the FE equation, we have

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \end{Bmatrix}$$

Load and constraints (BCs) are

$$F_{2Y} = -f, \qquad M_2 = m$$

$$v_1 = \theta_1 = 0$$

Examples with Beam Element

Reduced equation is

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -f \\ m \end{Bmatrix}$$

Solving this, we obtain

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{6EI}\begin{Bmatrix} -2L^2 f + 3Lm \\ -3Lf + 6m \end{Bmatrix} = \begin{Bmatrix} -pL^4/8EI \\ -pL^3/6EI \end{Bmatrix}$$

If the equivalent moment *m* is ignored, we have

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{6EI}\begin{Bmatrix} -2L^2 f \\ -3Lf \end{Bmatrix} = \begin{Bmatrix} -pL^4/6EI \\ -pL^3/4EI \end{Bmatrix}$$

Examples with Beam Element

From the FE equation, we can calculate the reaction force and moment as

$$\begin{Bmatrix} F_{1Y} \\ M_1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} -12 & 6L \\ -6L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ 5pL^2/12 \end{Bmatrix}$$

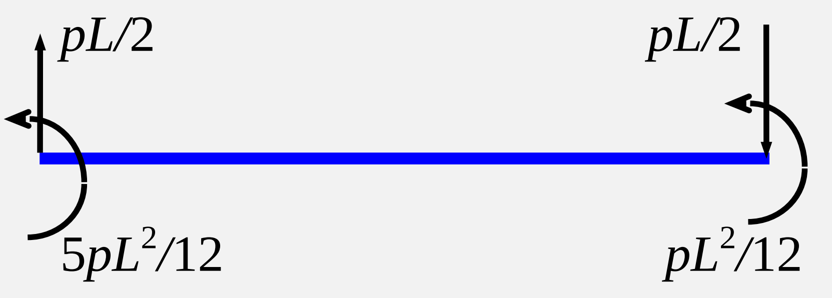
The equivalent nodal forces for the distributed lateral load *p*

$$\begin{Bmatrix} -pL/2 \\ -pL^2/12 \end{Bmatrix}$$

The correct *reaction forces* can be obtained as follows

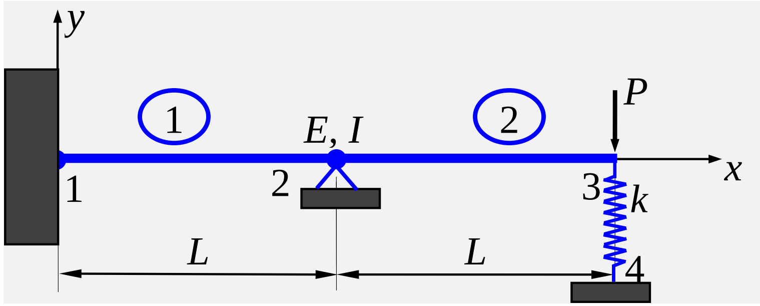
$$\begin{Bmatrix} F_{1Y} \\ M_1 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ 5pL^2/12 \end{Bmatrix} - \begin{Bmatrix} -pL/2 \\ -pL^2/12 \end{Bmatrix} = \begin{Bmatrix} pL \\ pL^2/2 \end{Bmatrix}$$

Check the results:



Examples with Beam Element

Example 3.3



Given: P = 50 kN, k = 200 kN/m, L = 3 m, E = 210 GPa, $I = 2\times10^{-4}$ m^4.

Find: Deflections, rotations and reaction forces.

Examples with Beam Element

Solution:

The beam has a roller (or hinge) support at node 2 and a spring support at node 3. We use two beam elements and one spring element to solve this problem.

The spring stiffness matrix is given by

$$\begin{array}{cc} & \begin{array}{cc} v_3 & v_4 \end{array} \\ \mathbf{k}_s = & \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array}$$

Adding this stiffness matrix to the global FE equation, we have

$$\begin{array}{c} \begin{array}{ccccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \end{array} \\ \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ & & 24 & 0 & -12 & 6L & 0 \\ & & & 8L^2 & -6L & 2L^2 & 0 \\ & & & & 12+k' & -6L & -k' \\ & & & & & 4L^2 & 0 \\ & \textit{Symmetry} & & & & & k' \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \\ F_{3Y} \\ M_3 \\ F_{4Y} \end{Bmatrix} \end{array}$$

where $$k' = \frac{L^3}{EI}k$$

Examples with Beam Element

Apply the boundary conditions

$$v_1 = \theta_1 = v_2 = v_4 = 0,$$
$$M_2 = M_3 = 0, \qquad F_{3Y} = -P$$

The reduced equation

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving this equation, we obtain the deflection and rotations at node 2 and node 3

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = -\frac{PL^2}{EI(12+7k')}\begin{Bmatrix} 3 \\ 7L \\ 9 \end{Bmatrix}$$

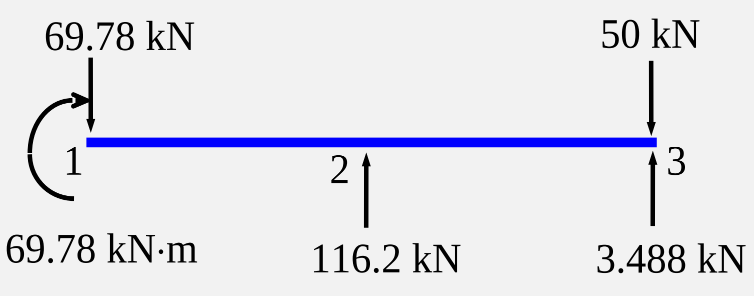
Plugging in the given numbers, we can calculate

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -0.002492\text{rad} \\ -0.01744\text{m} \\ -0.007475\text{rad} \end{Bmatrix}$$

The nodal reaction forces

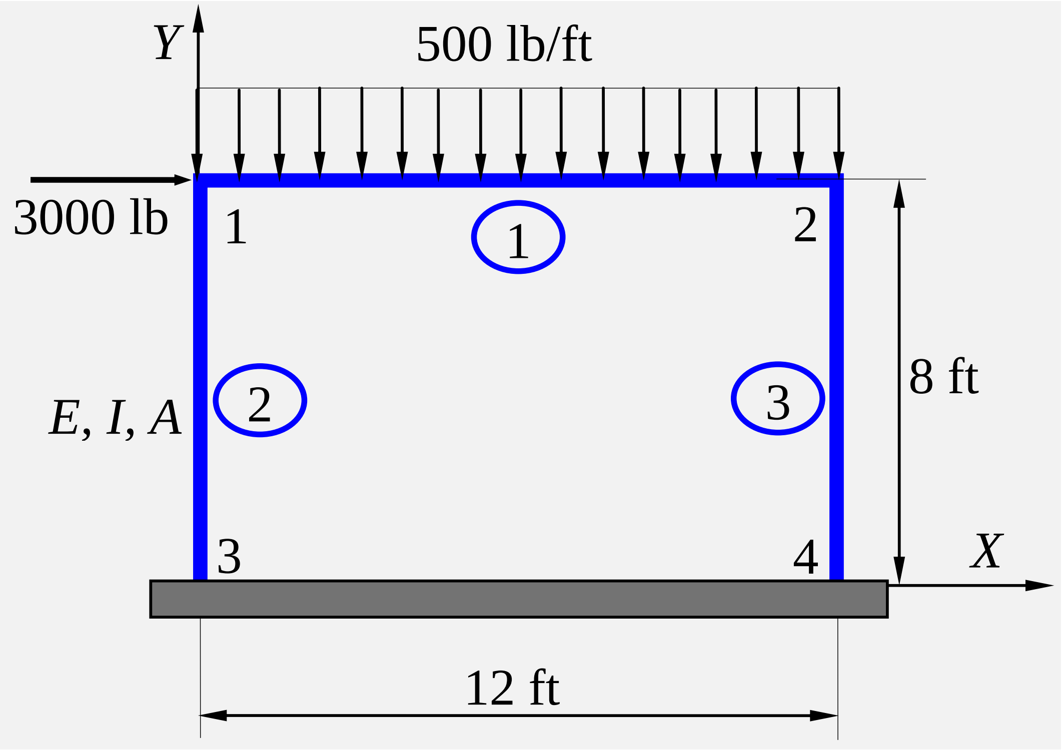
$$\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ F_{4Y} \end{Bmatrix} = \begin{Bmatrix} -69.78\,\text{kN} \\ -69.78\,\text{kN}\cdot\text{m} \\ 116.2\,\text{kN} \\ 3.488\,\text{kN} \end{Bmatrix}$$

Checking the results:



Examples with Beam Element

Example 3.4



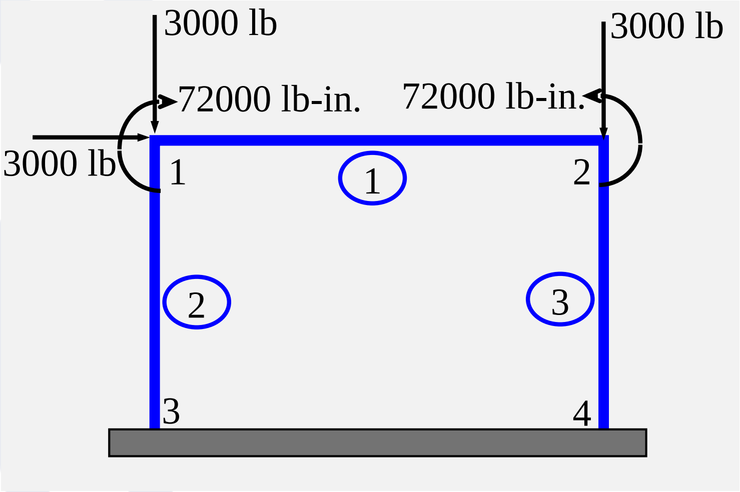
Given: $E = 30 \times 10^6$ psi, $I = 65$ in.^{4}, $A = 6.8$ in.^{2}

Find: Displacements and rotations of the two joints 1 and 2.

Examples with Beam Element

Solution:

Convert the distributed load to its equivalent nodal loads to obtain the following FE mode.



500x12x12=72000 Lb-in

In *local coordinate system*, the stiffness matrix for a general 2-D beam element is

$$
\begin{array}{cccccc}
u_i & v_i & \theta_i & u_j & v_j & \theta_j
\end{array}
$$

$$
\mathbf{k} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
$$

Examples with Beam Element

Element Connectivity Table

Element	Node i (1)	Node j (2)
1	1	2
2	3	1
3	4	2

For element 1, we have

$$\mathbf{k}_1 = \mathbf{k}_1' = 10^4 \times \begin{array}{c} \begin{matrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{matrix} \\ \begin{bmatrix} 141.7 & 0 & 0 & -141.7 & 0 & 0 \\ 0 & 0.784 & 56.4 & 0 & -0.784 & 56.4 \\ 0 & 56.4 & 5417 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 141.7 & 0 & 0 \\ 0 & -0.784 & -56.4 & 0 & 0.784 & -56.4 \\ 0 & 56.4 & 2708 & 0 & -56.4 & 5417 \end{bmatrix} \end{array}$$

Examples with Beam Element

For elements 2 and 3, the stiffness matrix in *local system* is

$$
\mathbf{k}_2' = \mathbf{k}_3' = 10^4 \times
\begin{array}{c}
\begin{array}{cccccc} u_i' & v_i' & \theta_i' & u_j' & v_j' & \theta_j' \end{array} \\
\begin{bmatrix}
212.5 & 0 & 0 & -212.5 & 0 & 0 \\
0 & 2.65 & 127 & 0 & -2.65 & 127 \\
0 & 127 & 8125 & 0 & -127 & 4063 \\
-212.5 & 0 & 0 & 212.5 & 0 & 0 \\
0 & -2.65 & -127 & 0 & 2.65 & -127 \\
0 & 127 & 4063 & 0 & -127 & 8125
\end{bmatrix}
\end{array}
$$

where i = 3, j = 1 for element 2, and i = 4, j = 2 for element 3.

Examples with Beam Element

The transformation matrix **T** is

$$\mathbf{T} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $l = 0$, $m = 1$ for both elements 2 and 3. Thus,

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples with Beam Element

Using the transformation relation

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

We obtain the stiffness matrices in the *global coordinate system* for elements 2 and 3

$$\mathbf{k}_2 = 10^4 \times \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & \theta_3 & u_1 & v_1 & \theta_1 \end{array} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array}$$

$$\mathbf{k}_3 = 10^4 \times \begin{array}{c} \begin{array}{cccccc} u_4 & v_4 & \theta_4 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix} \end{array}$$

Examples with Beam Element

Assembling the global FE equation and noticing the following boundary conditions

$$u_3 = v_3 = \theta_3 = u_4 = v_4 = \theta_4 = 0$$
$$F_{1X} = 3000\,\text{lb}, \quad F_{2X} = 0, \quad F_{1Y} = F_{2Y} = -3000\,\text{lb},$$
$$M_1 = -72000\,\text{lb}\cdot\text{in.}, \quad M_2 = 72000\,\text{lb}\cdot\text{in.}$$

We obtain the condensed FE equation

$$10^4 \times \begin{bmatrix} 144.3 & 0 & 127 & -141.7 & 0 & 0 \\ 0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\ 127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 144.3 & 0 & 127 \\ 0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\ 0 & 56.4 & 2708 & 127 & -56.4 & 13542 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ -3000 \\ -72000 \\ 0 \\ -3000 \\ 72000 \end{Bmatrix}$$

Examples with Beam Element

Solving that equation, we obtain

$$\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.092\text{in.} \\ -0.00104\text{in.} \\ -0.00139\text{rad} \\ 0.0901\text{in.} \\ -0.0018\text{in.} \\ -3.88\times 10^{-5}\text{rad} \end{Bmatrix}$$

To calculate the reaction forces and moments at the two ends, we employ the element FE equations for element 2 and element 3 with known nodal displacement vectors. We obtain

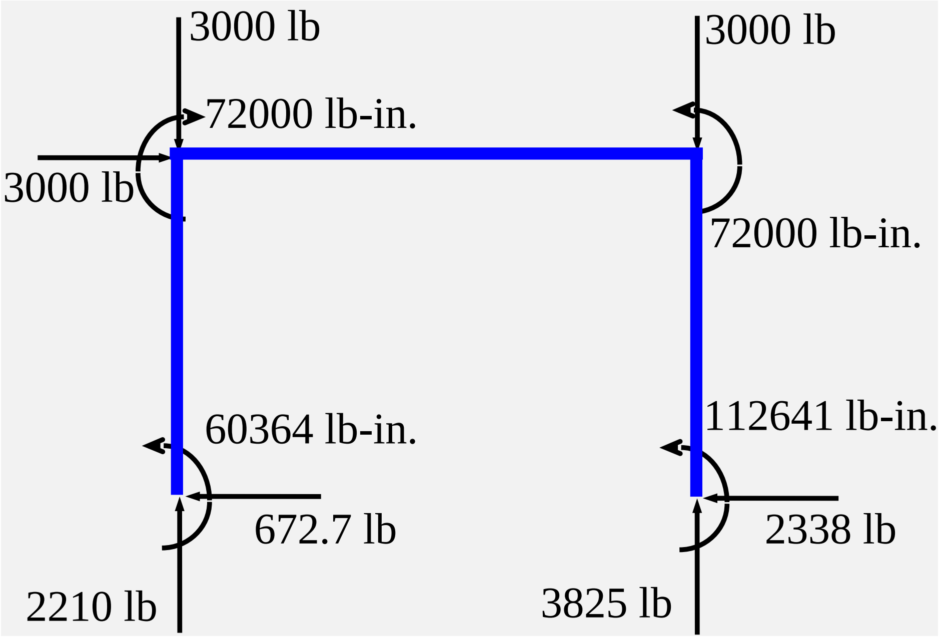
$$\begin{Bmatrix} F_{3X} \\ F_{3Y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} -672.7\text{lb} \\ 2210\text{lb} \\ 60364\text{lb}\cdot\text{in.} \end{Bmatrix}$$

and

$$\begin{Bmatrix} F_{4X} \\ F_{4Y} \\ M_4 \end{Bmatrix} = \begin{Bmatrix} -2338\text{lb} \\ 3825\text{lb} \\ 112641\text{lb}\cdot\text{in.} \end{Bmatrix}$$

Examples with Beam Element

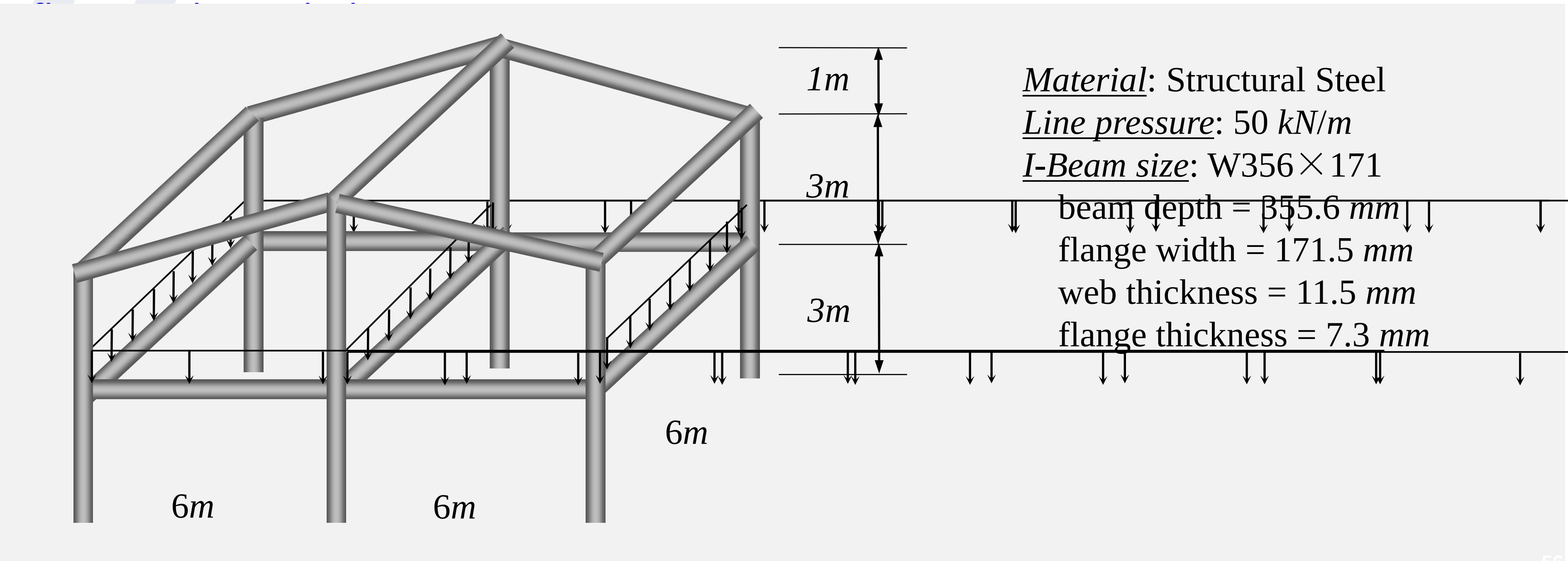
Check the results:



Analysis of a two-story building…

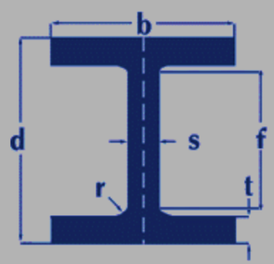
Case Study with ANSYS Workbench

<Problem Description> Steel framing systems provides cost-effective solutions for low-rise buildings. They have high strength-to-weight ratios and can be pre-fabricated and custom-designed. Consider the following two-story building constructed with structural steel I-beams. Determine the deformations and the stresses in the frame when a uniform load of 50 *kN/m* is applied on the second



Sample Beam Cross section

356 X 171 X 51KG S355J0 Yel Universal Beam

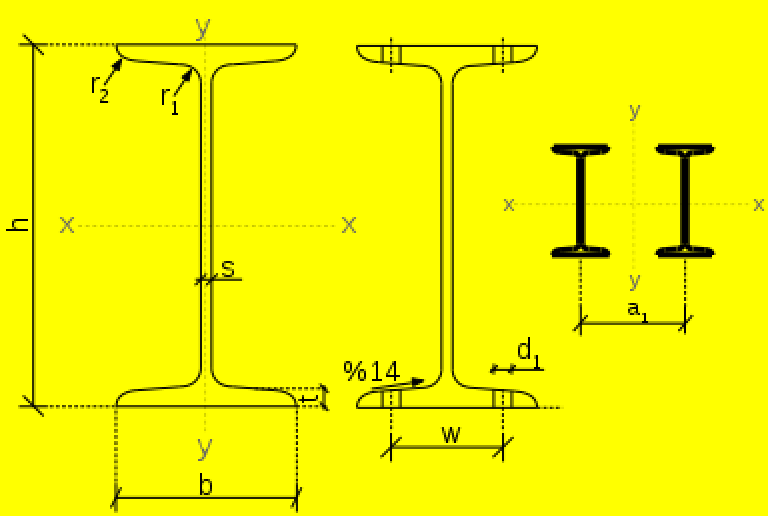


Dimensions	Specifications	Description

Measurements from diagram

d	355.6	Depth d mm	b	171.5	Width b mm
s	7.3	Web s mm	t	11.5	Flange t mm
r	10.2	Root Radius r mm	f	311.6	Depth Between Fillets f mm
	51	kg/m		20	Approx. Metres per Tonne
	64.9	Area of Section cm²		1.36	Surface Area per Metre m²
	26.7	Surface Area per Tonne m²		0.013	Surface Area Two End Faces m²

IPN steel beam
DIN1025/1



h - Beam height

b - Flange width

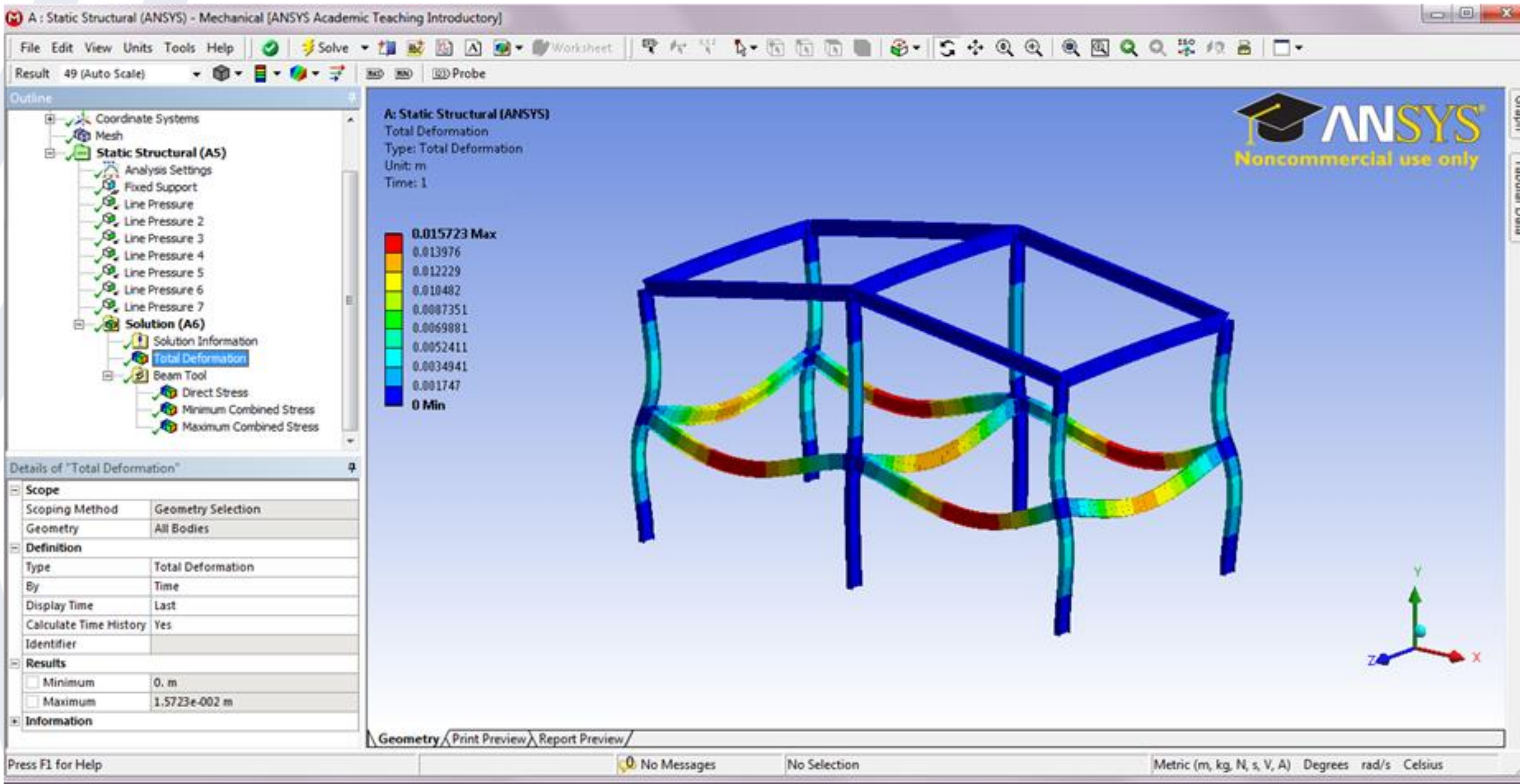
s = r_1 - Web thickness

t - Flange thickness

a_1 - Distance between beam axes so that $I_x = I_y$ (see chart)

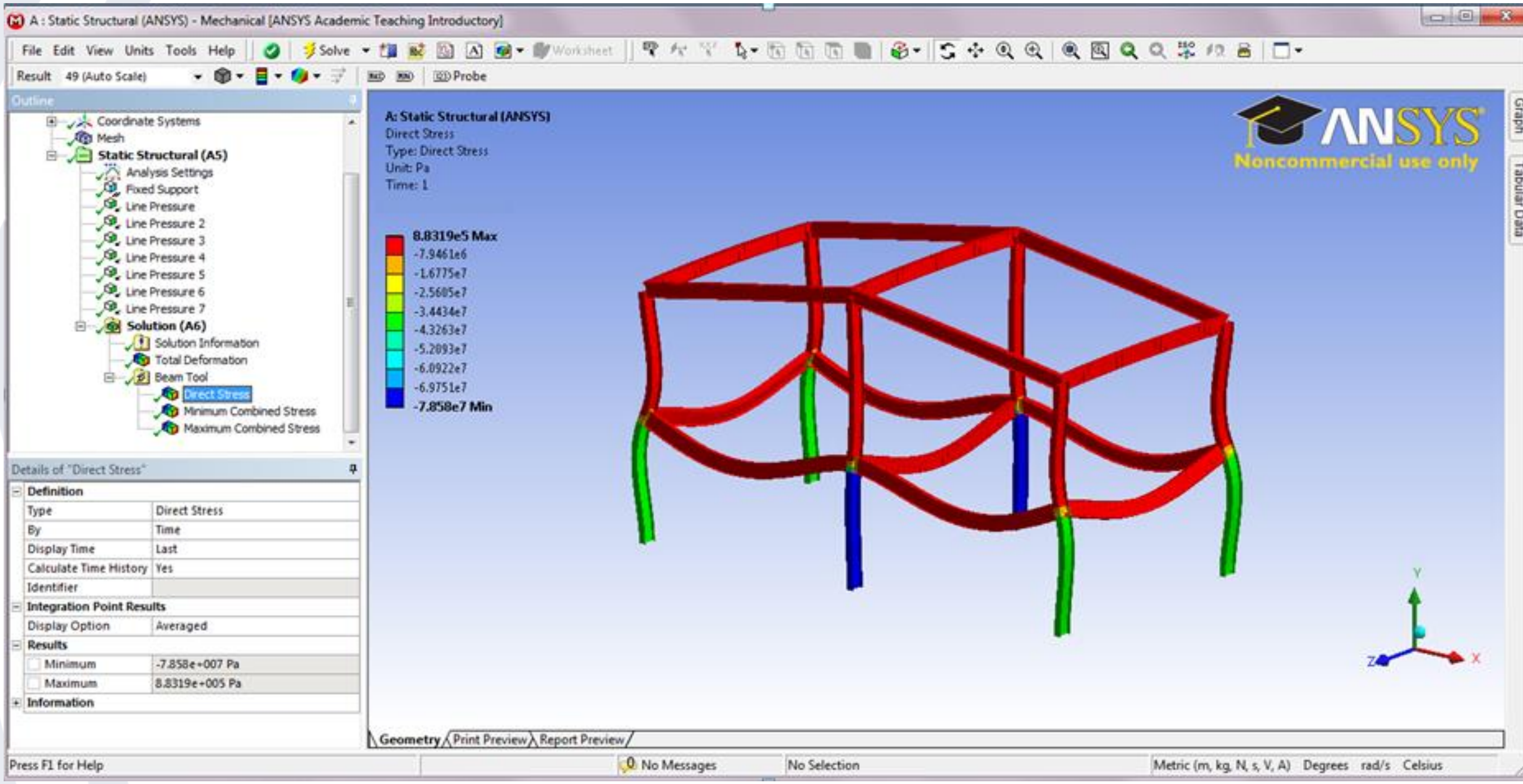
Case Study with ANSYS Workbench

Run a ***Static Structural Analysis*** to review the frame deformation results.



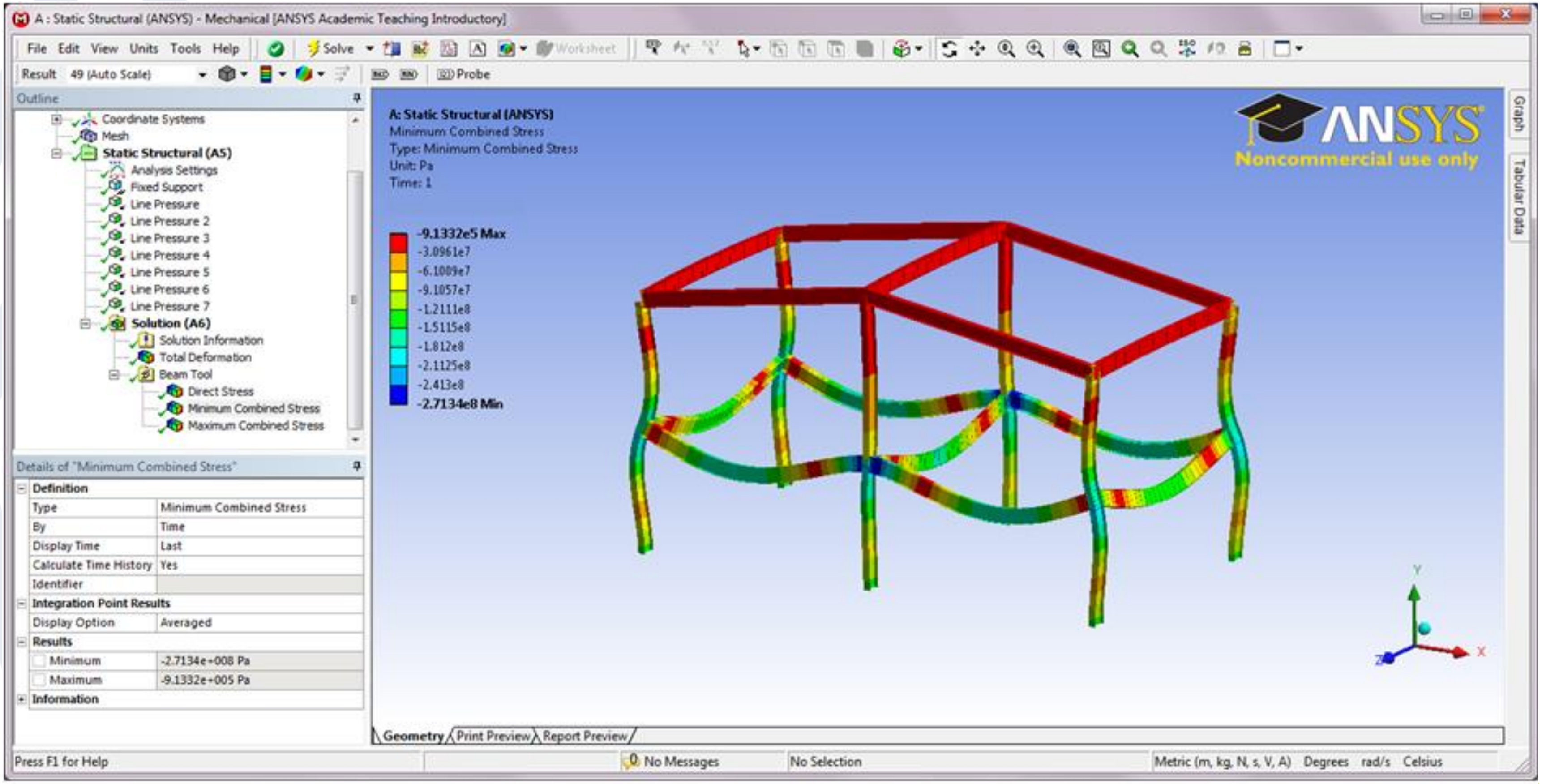
Case Study with ANSYS Workbench

Select ***Direct Stress*** under ***Beam Tool*** to review the axial stress results in beams.



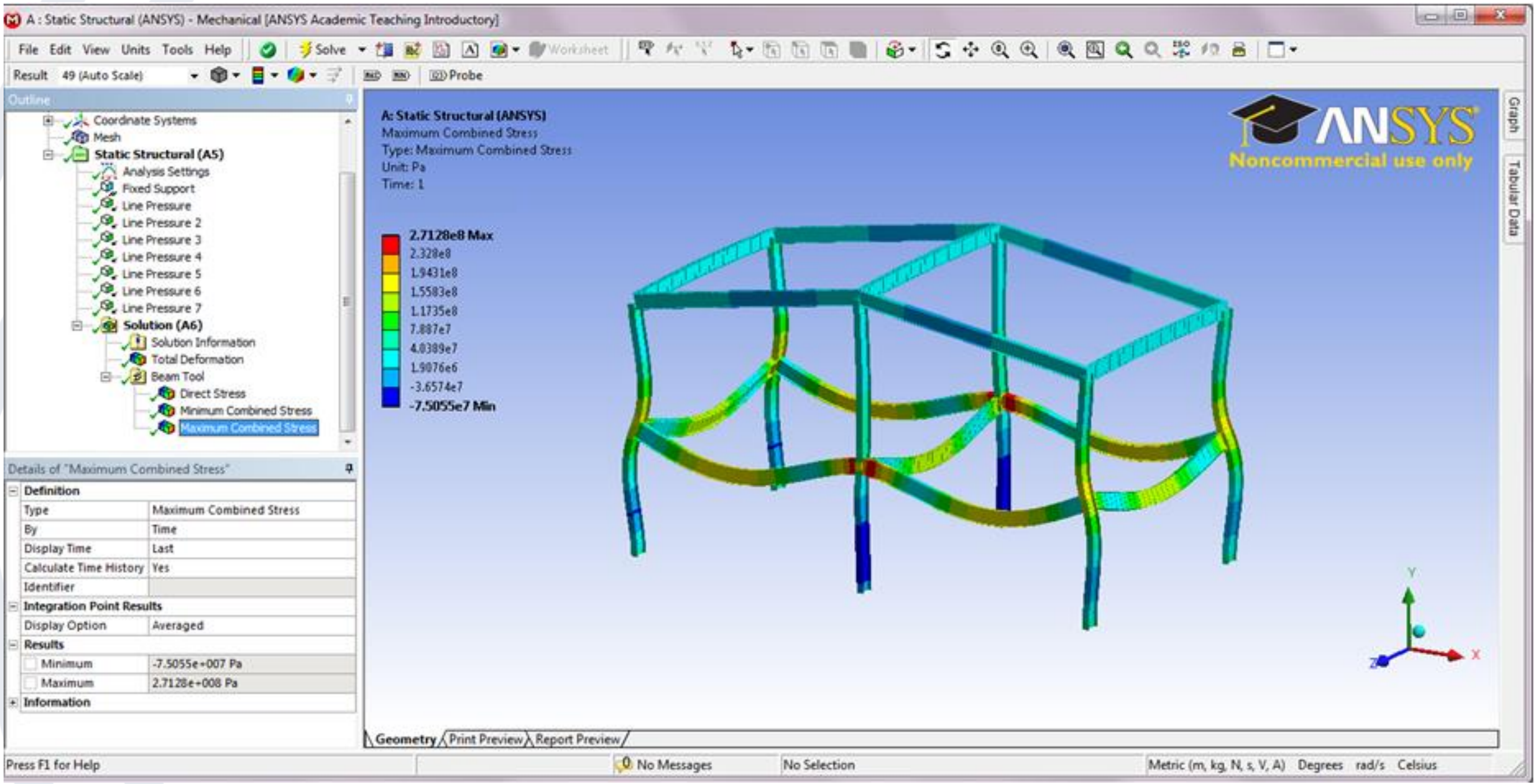
Case Study with ANSYS Workbench

Select ***Minimum Combined Stress*** under ***Beam Tool*** to retrieve the linear combination of the ***Direct Stress*** and the ***Minimum Bending Stress*** results in beams.



Case Study with ANSYS Workbench

Select ***Maximum Combined Stress*** under ***Beam Tool*** to retrieve the linear combination of the ***Direct Stress*** and the ***Maximum Bending Stress*** results in beams.



Summary

In this chapter,

- Beam element which can be used in frame analysis are studied.
- The concept of the shape functions is further explored and the derivations of the stiffness matrix using the energy approach are emphasized.
- Treatment of distributed loads is discussed and several examples are studied.
- A two-story building structure with I-beams is analyzed using ANSYS Workbench.

Review of Learning Objectives

Now that you have finished this chapter, you should be able to

1. Set up simplified finite element models for beams and frames.
2. Derive the element stiffness matrix for plane beams using direct/energy approach.
3. Explain the concept of shape functions and their characteristics for beam elements.
4. Find the equivalent nodal forces of distributed loads on beams.
5. Determine the deflection and rotation at a point of a beam using hand calculation to verify the finite element solutions.
6. Apply the general beam element stiffness matrix to the analysis of simple frames.
7. Create line models from concept points, sketches, or by body translation in Workbench.
8. Perform static structural analyses on beams and frames using Workbench.