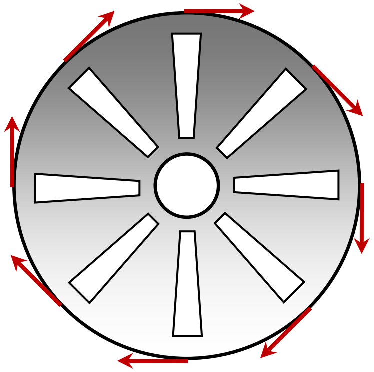
Finite Element Modeling and Simulation with ANSYS Workbench

Chapter 4

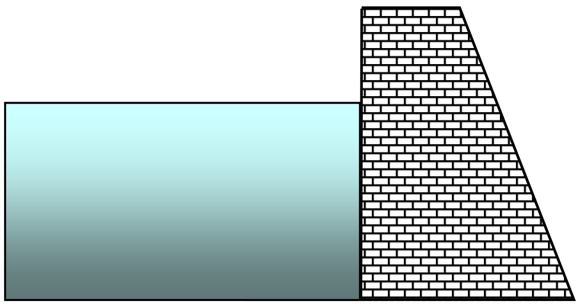
Two-Dimensional Elasticity

Introduction

- Many structures that are three-dimensions can be satisfactorily treated as two-dimensional problems.



(a) A rotating disk

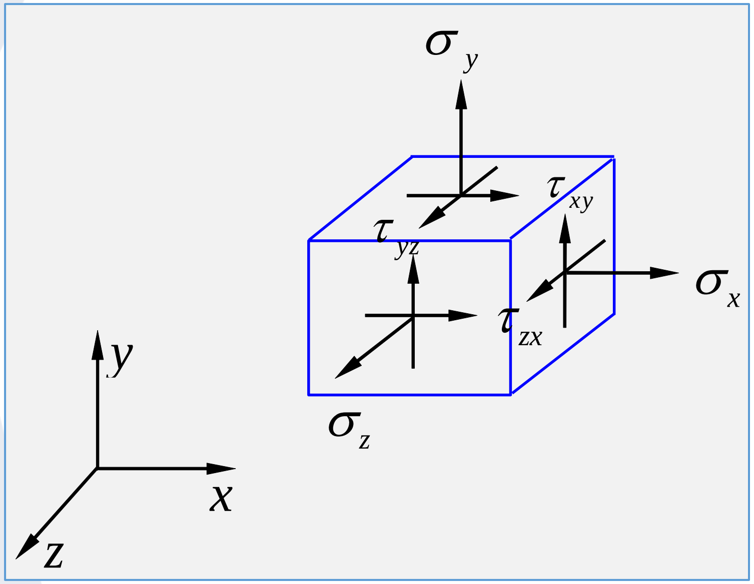


(b) A reservoir dam

Figure 4.1. Examples of 2-D elasticity problems.

Review of 2-D Elasticity Theory

- In general, the stresses and strains in a structure consist of six independent components. Stresses, Strains, Shear Stresses and Strains.



$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$$

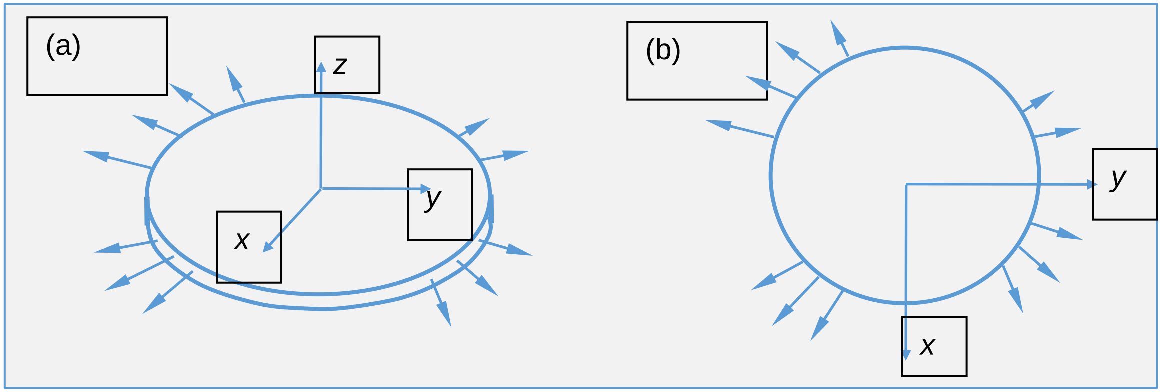
$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$$

Figure 4.2. Stress components at a point in a structure.

Review of 2-D Elasticity Theory

- Under certain conditions, a 3-D stress analysis can be reduced to a 2-D analysis.

  - Plane Stress



$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \qquad (\varepsilon_z \neq 0)$$

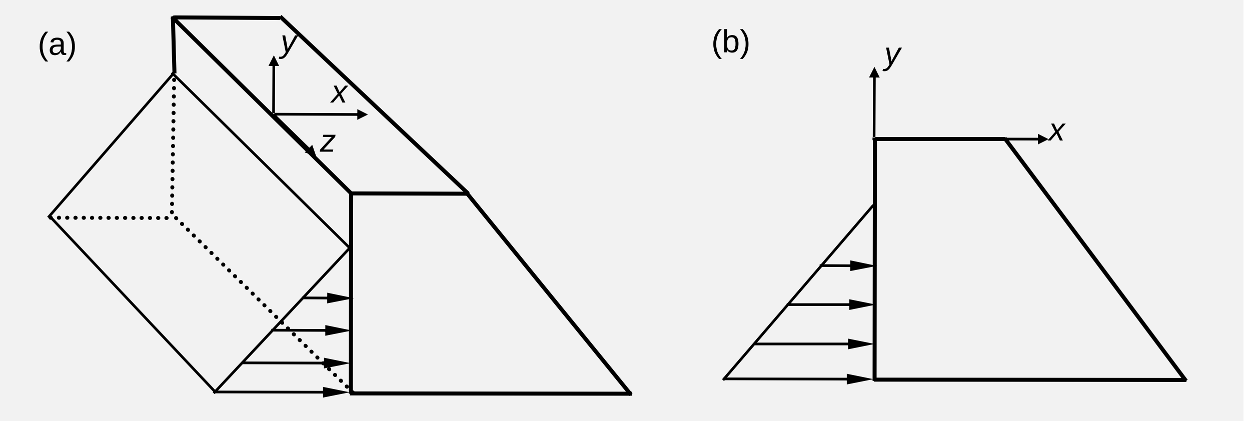
Figure 4.3Plane stress condition: (a) structure with in-plane loading; and (b) 2-D model.

Plane stress is approximation, as opposed to plane strain, which is an exact solution. Plane strain is a particular solution of the whole three-dimensional elasticity equations, whereas plane stress is only attained in the limit when the thickness of the loaded body approaches zero.

Review of 2-D Elasticity Theory

- Under certain conditions, a 3-D stress analysis can be reduced to a 2-D analysis.

Figure 4.4Plane strain condition: (a) structure with uniform transverse load; and (b) 2-D model.



$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad (\sigma_z \neq 0)$$

in-plane strain field by three components

- Plane Strain

Plane stress is approximation, as opposed to plane strain, which is an exact solution. Plane strain is a particular solution of the whole three-dimensional elasticity equations, whereas plane stress is only attained in the limit when the thickness of the loaded body approaches zero.

A long structure with a uniform cross section and transverse loading along its thickness (z-direction), such as a tunnel or a dam, can be regarded a plane strain case (Figure 4.4).

Review of 2-D Elasticity Theory

❑ Stress-Strain (constitutive) Equations

- Plane Stress

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \quad \text{or} \quad \varepsilon = \mathbf{E}^{-1}\sigma + \varepsilon_0$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right) \quad \text{or} \quad \sigma = \mathbf{E}\varepsilon + \sigma_0$$

where ε_0 is the initial strain (e.g., due to a temperature change), E Young's modulus, ν Poisson's ratio, and G the shear modulus. Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

Review of 2-D Elasticity Theory

Stress-Strain (constitutive) Equations

- Plane Strain $E \rightarrow \dfrac{E}{1-\nu^2}; \qquad \nu \rightarrow \dfrac{\nu}{1-\nu}; \qquad G \rightarrow G$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)$$

or, $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_0$

where $\boldsymbol{\sigma}_0 = -\mathbf{E}\boldsymbol{\varepsilon}_0$ is the initial stress.

where ε_0 is the initial strain (e.g., due to a temperature change), E Young's modulus, ν Poisson's ratio, and G the shear modulus. Note that

$$G = \frac{E}{2(1+\nu)} \qquad (4.4)$$

for homogeneous and isotropic materials.

Stress-Strain (constitutive) Equations

We can also express stresses in terms of strains by solving the above equation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}\right) \tag{4.5}$$

or,

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_0$$

where $\boldsymbol{\sigma}_0 = -\mathbf{E}\boldsymbol{\varepsilon}_0$ is the initial stress.

For *plane strain* case, we need to replace the material constants in the above equations in the following fashion:

$$E \to \frac{E}{1-\nu^2}; \qquad \nu \to \frac{\nu}{1-\nu}; \qquad G \to G \tag{4.6}$$

For example, the stress is related to strain by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}\left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}\right)$$

❑ Stress-Strain (constitutive) Equations

For example, the stress is related to strain by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)$$

in the *plane strain* case.

Initial strain due to a *temperature change* (thermal loading) is given by the following for the plane stress case

$$\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \qquad (4.7)$$

where α is the coefficient of thermal expansion, ΔT the change of temperature. For the plane strain case, α should be replaced by $(1+\nu)\alpha$ in Equation 4.7. Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

Review of 2-D Elasticity Theory

Strain and Displacement Relations

For small strains and small rotations, we have

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form, we write

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \mathbf{Du}$$

From this relation, we know that, if the displacements are represented by polynomials, the strains (and thus stresses) will be polynomials of an order that is one order lower than the displacements.

Equilibrium Equations

In plane elasticity, the stresses satisfy the following equilibrium equation.

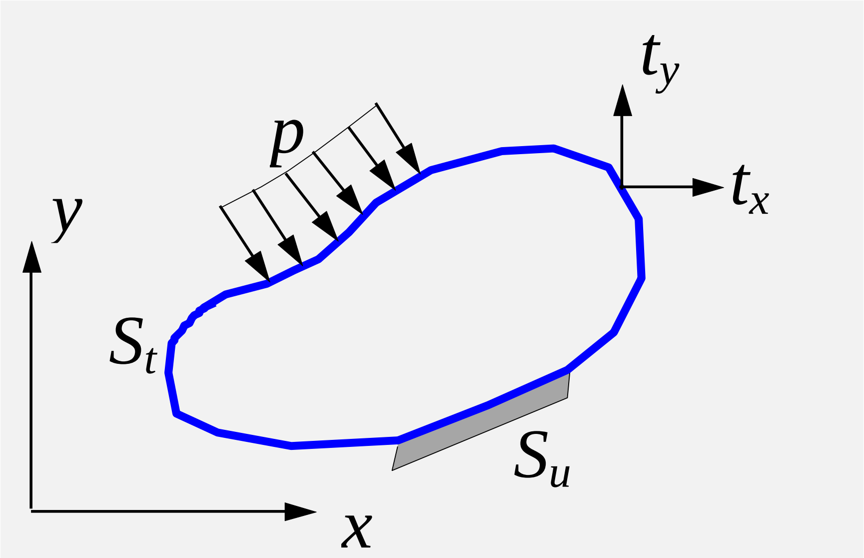
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

where f_x and f_y are body forces (forces per unit volume, such as gravity forces). In the FEM, these equilibrium conditions are satisfied in an approximate sense.

Boundary Conditions

The boundary S of the 2-D region can be divided into two parts, Su and St (Figure 4.5). The boundary conditions (BCs) can be described as:



$$u = \bar{u},\ v = \bar{v}, \qquad \text{on } S_u$$

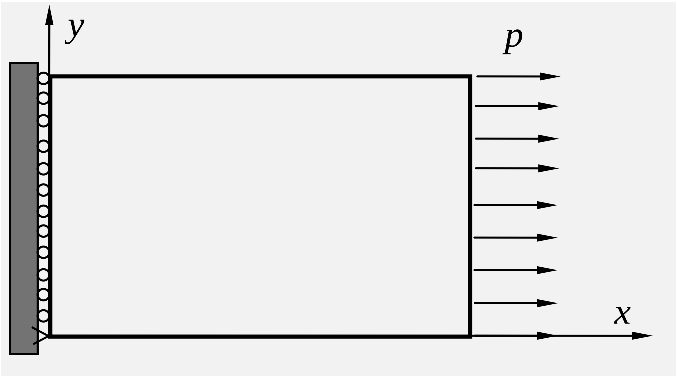
$$t_x = \bar{t}_x,\ t_y = \bar{t}_y, \qquad \text{on } S_t$$

Figure 4.5 Boundary conditions for a structure.

in which t_x and t_y are tractions (stresses on the boundary) and the barred quantities are those with known values.

Exact Elasticity Solution

Example 4.1



The exact solution for this simple problem is as follows

Displacement: $u = \frac{p}{E}x,\qquad v = -\nu\frac{p}{E}y$

Strain: $\varepsilon_x = \frac{p}{E},\qquad \varepsilon_y = -\nu\frac{p}{E},\qquad \gamma_{xy} = 0$

Stress: $\sigma_x = p,\qquad \sigma_y = 0,\qquad \tau_{xy} = 0$

Exact (or analytical) solutions for simple problems are numbered. That is why we need the FEM for solutions of 2-D elasticity problems in general.

For plane stress or plane strain, we model only a 2-D region…

Modeling/Formulation of 2-D Elasticity Problems

Finite element discretization of a planar surface

- For a plane stress or plane strain analysis, we model only a 2-D region, that is, a planar surface or cross section of the original 3-D structure.
- The region then needs to be divided into an element discretization made of triangles, quadrilaterals, or a mixture of both.
- The element behaviors need to be specified to set up the problem type as either plane stress or plane strain.
- The discretization can be structured (mapped mesh on a three-sided or four-sided surface region with equal numbers of element divisions for the opposite sides) or unstructured (free mesh),
- As shown in Figure 4.6. A structured surface mesh has regular connectivity that can be described as a 2-D array, and it generally has better solution reliability compared to a free mesh

Modeling of 2-D Elasticity Problems

- Finite element discretization of a planar surface

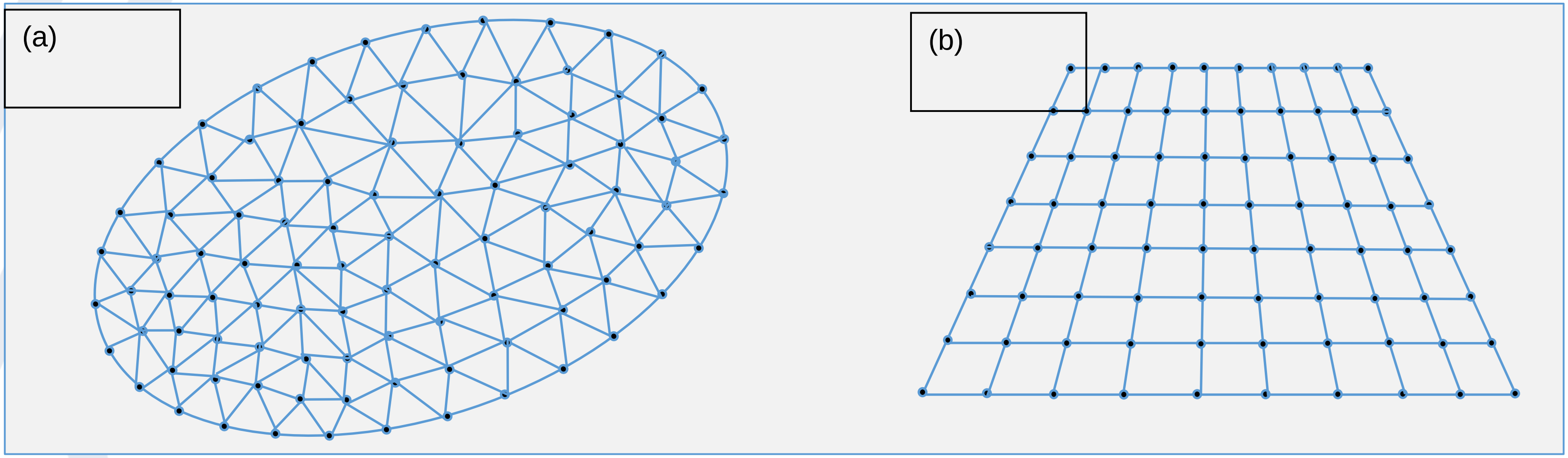


Figure 4.6 Finite element discretization: (a) Unstructured (free) mesh; and (b) Structured (mapped) mesh.

- Support conditions need to suppress rigid-body motion.

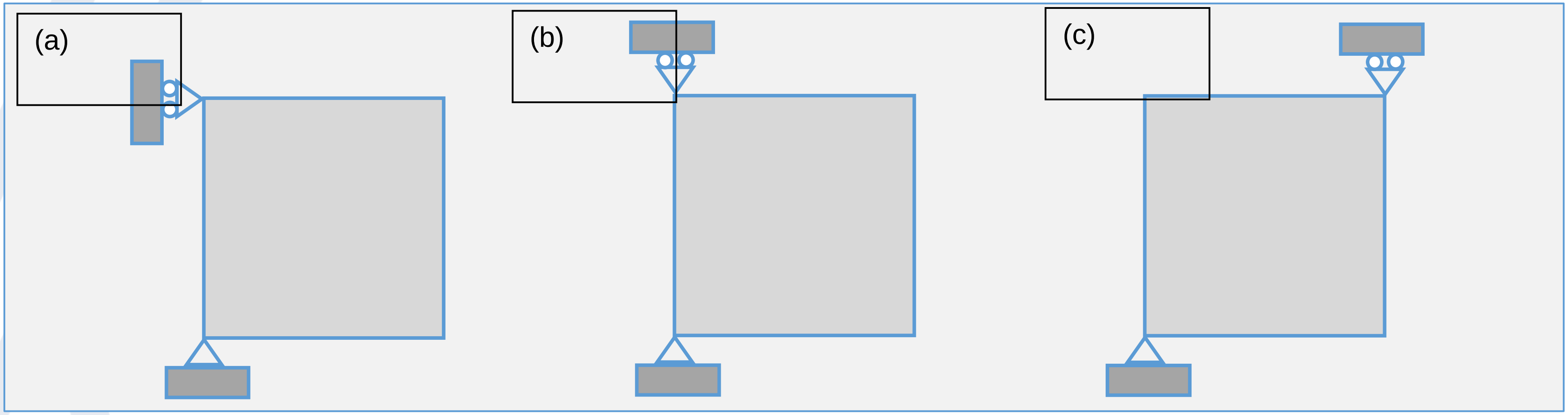


Figure 4.7 Support conditions in 2-D: (a) and (c) Effective constraints; (b) Ineffective constraints.

Formulation of Plane Stress/Strain Element

A General Formula for the Stiffness Matrix

Displacements (*u*, *v*) in a plane element can be interpolated from nodal displacements (u_i, v_i) using shape functions N_i :

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \qquad \mathbf{u} = \mathbf{N}\mathbf{d}$$

where **N** is the *shape function matrix*, **u** the displacement vector and **d** the *nodal* displacement vector.

The strain vector is $\varepsilon = \mathbf{D}\mathbf{u} = \mathbf{D}\mathbf{N}\mathbf{d},$ or $\varepsilon = \mathbf{B}\mathbf{d}$

where **B** = **DN** is the *strain-displacement matrix*.

Formulation of Plane Stress/Strain Element

Consider the strain energy stored in an element

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon dV = \frac{1}{2}\int_V \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right) dV$$

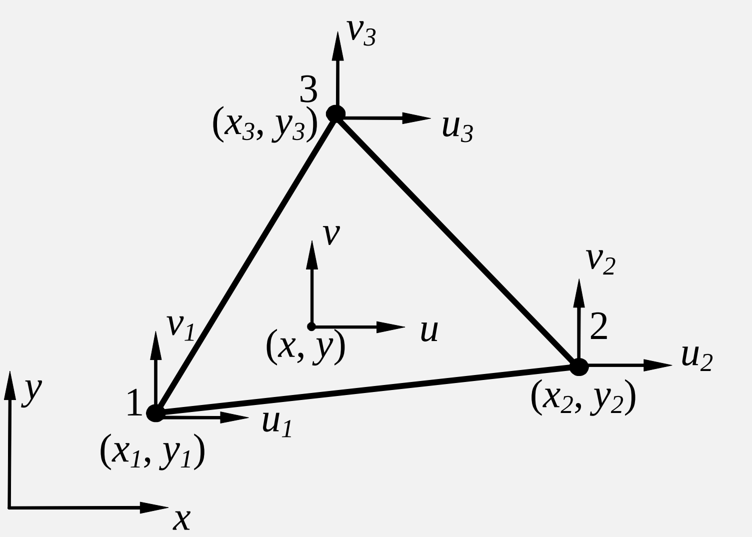
$$= \frac{1}{2}\int_V (\mathbf{E}\varepsilon)^T \varepsilon dV = \frac{1}{2}\int_V \varepsilon^T \mathbf{E}\varepsilon dV = \frac{1}{2}\mathbf{d}^T \int_V \mathbf{B}^T \mathbf{E}\mathbf{B} dV \mathbf{d}$$

$$= \frac{1}{2}\mathbf{d}^T \mathbf{k}\mathbf{d}$$

Thus, the *element stiffness matrix* is

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E}\mathbf{B} dV$$

Constant Strain Triangle (CST or T3)



$$u = b_1 + b_2 x + b_3 y, \qquad v = b_4 + b_5 x + b_6 y$$

$$\varepsilon_x = b_2, \qquad \varepsilon_y = b_6, \qquad \gamma_{xy} = b_3 + b_5$$

Figure 4.8. Linear triangular element (T3).

Formulation of Plane Stress/Strain Element

We write

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$N_1 = \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

$$N_2 = \frac{1}{2A}\{(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$$

$$N_3 = \frac{1}{2A}\{(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$$

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

(The area of the triangle)

Using the strain-displacement relation , we have

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{Bd} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ (i, j = 1, 2, 3).

The element stiffness matrix for the CST

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV = tA(\mathbf{B}^T \mathbf{E} \mathbf{B})$$

Formulation of Plane Stress/Strain Element

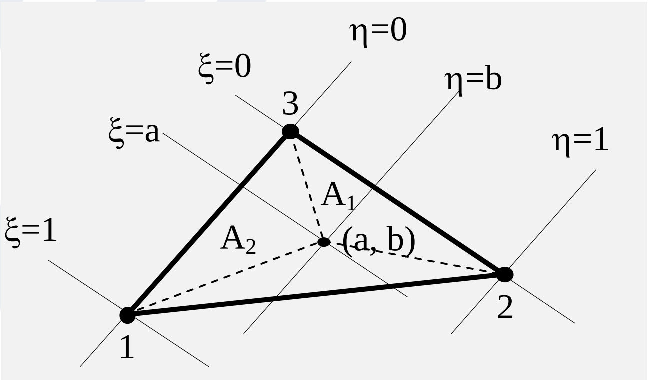


Figure 4.9. The natural coordinate system defined on the triangle.

Introduce the *natural coordinates* (ξ,η)

$$\xi = A_1 / A, \eta = A_2 / A$$

The *shape functions* are given by

$$N_1 = \xi, \qquad N_2 = \eta, \qquad N_3 = 1 - \xi - \eta$$

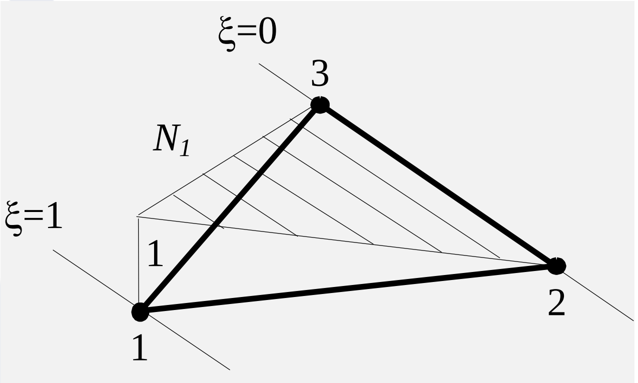


Figure 4.10. Plot of the shape function N_1.

Note that $N_1 + N_2 + N_3 = 1$

$$N_i = \begin{cases} 1, & \text{at node i;} \\ 0, & \text{at other nodes} \end{cases}$$

Formulation of Plane Stress/Strain Element

Two coordinate systems : global coordinates (*x*, *y*) & natural (local) coordinates (ξ, η)

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

or

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ (i, j = 1, 2, 3)

Displacement *u* or *v* on the element can be viewed as functions of (*x*, *y*) or (ξ, η)

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

where **J** is called the *Jacobian matrix* of the transformation.

We can calculate

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det \mathbf{J} = x_{13} y_{23} - x_{23} y_{13} = 2A$$

Formulation of Plane Stress/Strain Element

Using $$\varepsilon = \mathbf{Du} = \mathbf{DNd} = \mathbf{Bd}$$

We obtain the following strain-displacement matrix

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

which is the same as what we derived earlier.

Notes about the CST element

- Use in areas where the strain gradient is small.
- Use in mesh transition areas (fine mesh to coarse mesh).
- Avoid using CST in stress concentration or other crucial areas in the structure, such as edges of holes and corners.
- Recommended only for quick and preliminary FE analysis of 2-D problems.

Formulation of Plane Stress/Strain Element

Quadratic Triangular Element (LST or T6)

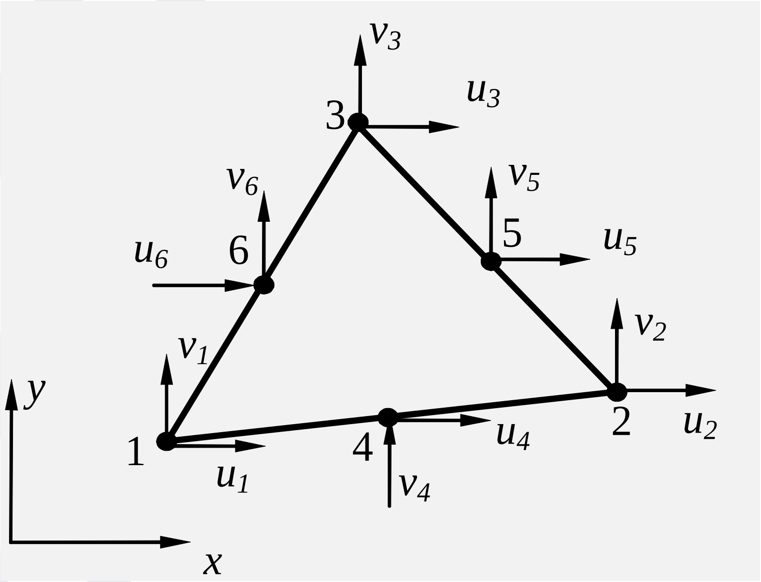


Figure 4.11. Quadratic triangular element (T6).

$$u = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2$$
$$v = b_7 + b_8 x + b_9 y + b_{10} x^2 + b_{11} xy + b_{12} y^2$$

$$\varepsilon_x = b_2 + 2b_4 x + b_5 y$$
$$\varepsilon_y = b_9 + b_{11} x + 2b_{12} y$$
$$\gamma_{xy} = (b_3 + b_8) + (b_5 + 2b_{10})x + (2b_6 + b_{11})y$$

Formulation of Plane Stress/Strain Element

Shape functions in natural coordinate system

$$N_1 = \xi(2\xi - 1)$$
$$N_2 = \eta(2\eta - 1)$$
$$N_3 = \zeta(2\zeta - 1)$$
$$N_4 = 4\xi\eta$$
$$N_5 = 4\eta\zeta$$
$$N_6 = 4\zeta\xi$$

where $\zeta = 1 - \xi - \eta$

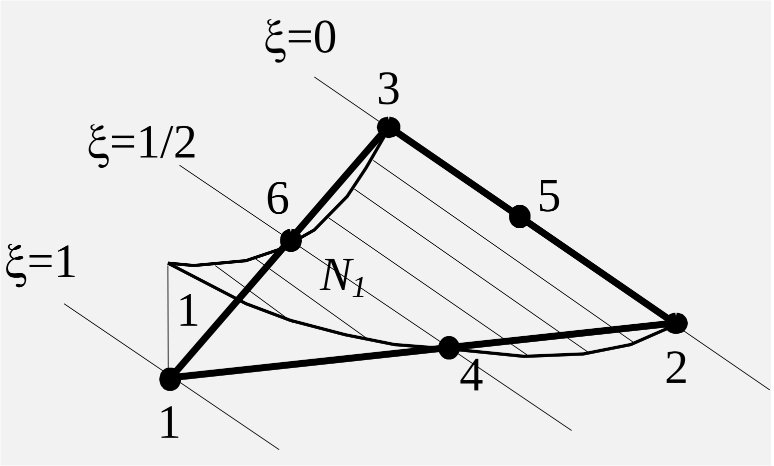


Figure 4.12. Plot of the shape function N_1

Displacements can be written as $u = \sum_{i=1}^{6} N_i u_i, \quad v = \sum_{i=1}^{6} N_i v_i$

The element stiffness matrix is $\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} dV$

Formulation of Plane Stress/Strain Element

❑ Linear Quadrilateral Element (Q4)

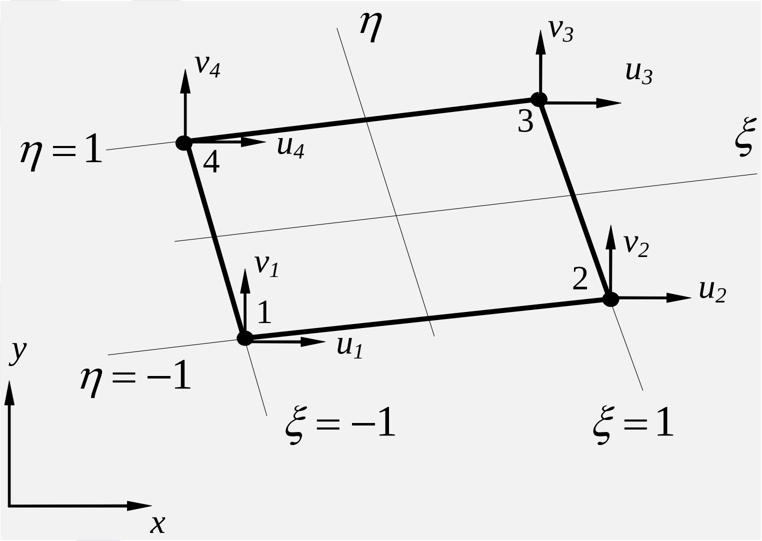


Figure 4.13. Linear quadrilateral element (Q4).

Four shape functions

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\sum_{i=1}^{4} N_i = 1$$

The displacement field

$$u = \sum_{i=1}^{4} N_i u_i, \qquad v = \sum_{i=1}^{4} N_i v_i$$

Formulation of Plane Stress/Strain Element

Quadratic Triangular Element (LST or T6)

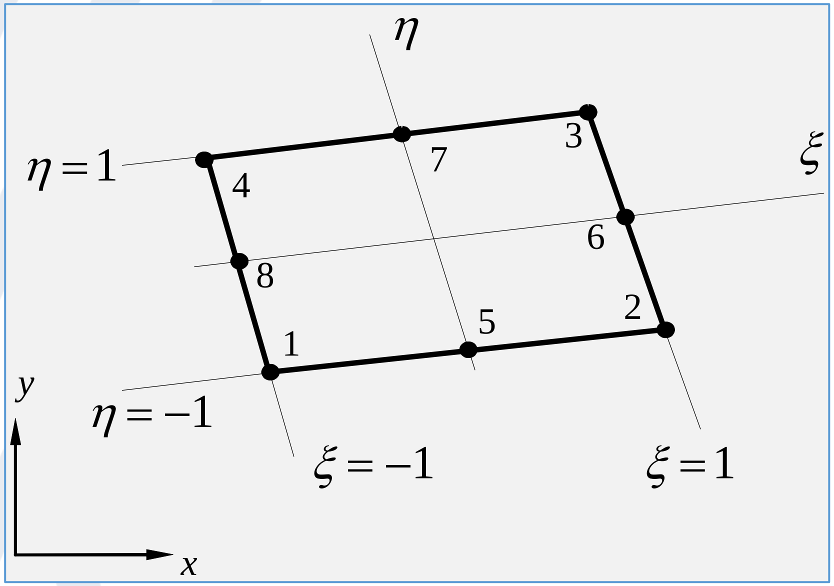


Figure 4.14. Quadratic quadrilateral element (Q8).

The displacement field

$$u = \sum_{i=1}^{8} N_i u_i, \qquad v = \sum_{i=1}^{8} N_i v_i$$

Eight shape functions

$$N_1 = \frac{1}{4}(1-\xi)(\eta-1)(\xi+\eta+1)$$

$$N_2 = \frac{1}{4}(1+\xi)(\eta-1)(\eta-\xi+1)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1)$$

$$N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1+\eta)(1-\xi^2)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

$$\sum_{i=1}^{8} N_i = 1$$

Formulation of Plane Stress/Strain Element

Note the following when applying the 2-D elements

- Q4 and T3 are usually used together in a mesh with linear elements.
- Q8 and T6 are usually applied in a mesh composed of quadratic elements.
- Linear elements are good for deformation analysis, i.e., when global responses need to be determined.
- Quadratic elements are preferred for stress analysis, their high accuracy and flexibility in modeling complex geometry, such as curved boundaries.

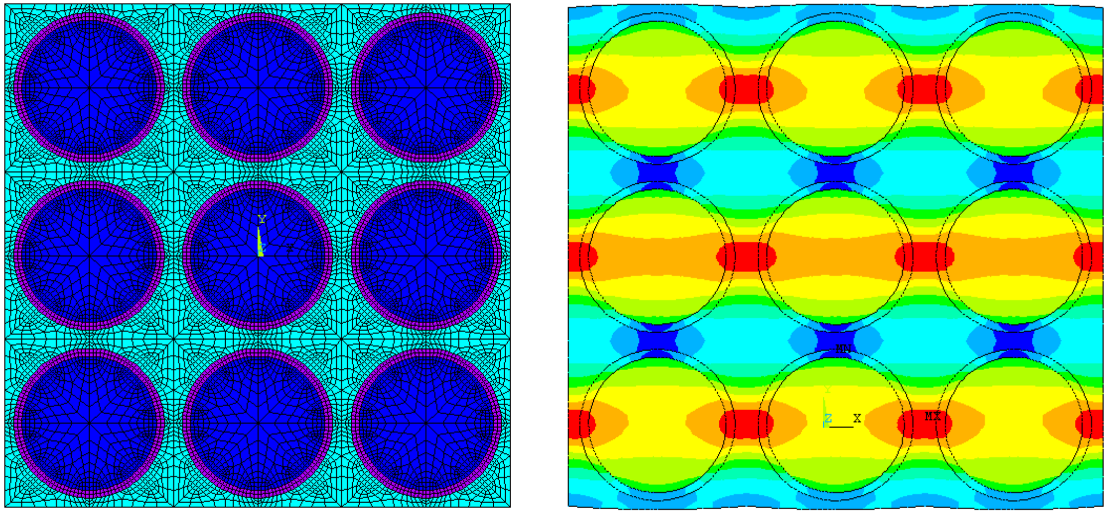
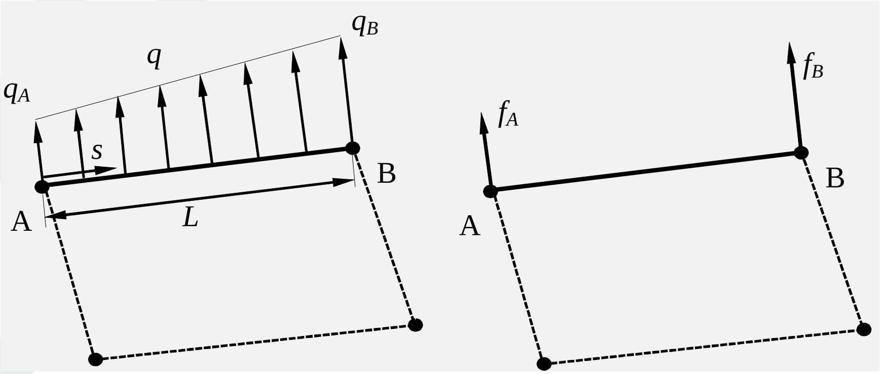


Figure 4.15. Analysis of composite materials (mesh and stress contour plots).

Formulation of Plane Stress/Strain Element

Transformation of Loads



Linear displacement and traction

$$u_n(s) = (1 - s/L)u_{nA} + (s/L)u_{nB}$$

$$q(s) = (1 - s/L)q_A + (s/L)q_B$$

The work done by the traction q applied on edge of a Q4 element

$$W_q = \frac{1}{2} t \int_0^L \left(\begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} \begin{bmatrix} 1 - s/L \\ s/L \end{bmatrix} \right) \left(\begin{bmatrix} 1 - s/L & s/L \end{bmatrix} \begin{bmatrix} q_A \\ q_B \end{bmatrix} \right) ds$$

$$= \frac{1}{2} \begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} t \int_0^L \begin{bmatrix} (1 - s/L)^2 & (s/L)(1 - s/L) \\ (s/L)(1 - s/L) & (s/L)^2 \end{bmatrix} ds \begin{bmatrix} q_A \\ q_B \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} \frac{tL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_A \\ q_B \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} \begin{Bmatrix} f_A \\ f_B \end{Bmatrix}$$

The equivalent nodal force vector

$$\begin{Bmatrix} f_A \\ f_B \end{Bmatrix} = \frac{tL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \end{Bmatrix}$$

For constant q, we have

$$\begin{Bmatrix} f_A \\ f_B \end{Bmatrix} = \frac{qtL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

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- Stress Calculation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \mathbf{E} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{EBd}$$

The von Mises Stress: $\sigma_e \le \sigma_Y$

where $$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

For 2-D problems, the two principle stresses in the plane are determined by

$$\sigma_1^{\,P} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2^{\,P} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

For plane stress conditions $$\sigma_e = \sqrt{(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)}$$

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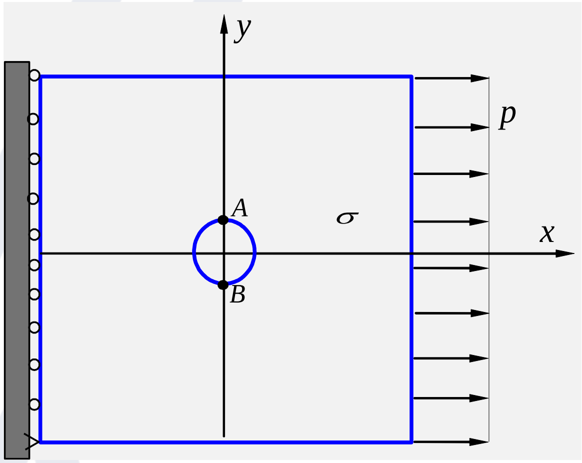
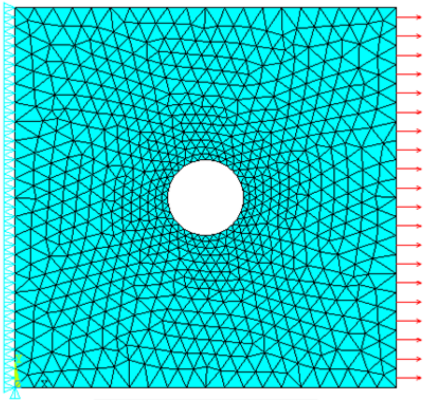
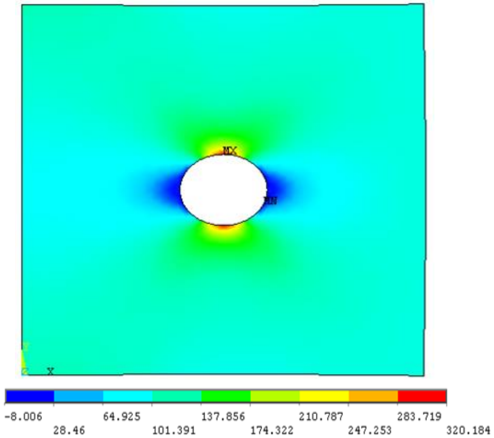
Example 4.2

A square plate with a hole at the center is under a tension load *p* in *x* direction.





FE mesh (T6, 1518 elements)



stress plot and deformed shape

Elem. Type	No. of Elem.	Total DOFs	Max stress (psi)
Q4	506	1102	312.42
Q4	3352	7014	322.64
Q4	31349	64106	322.38
...	...	...	...
T6	1518	6254	320.18
T6	2562	10494	321.23
T6	24516	100702	322.24
...	...	...	...
Q8	501	3188	320.58
Q8	2167	13376	321.70
Q8	14333	88636	322.24

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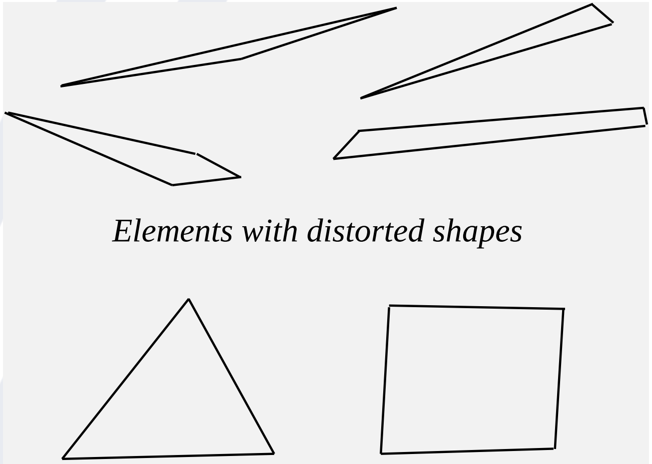
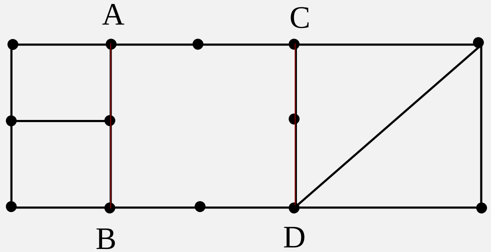
General Comments on the 2-D Elements



Elements with distorted shapes

Elements with normal shapes



Improper connections (gaps along AB and CD)

- *Know the attributes of each type of elements*:

T3 & Q4: linear displacement, constant strain/stress;
T6 & Q8: quadratic displacement, linear strain/stress.

- *Choose the right type of elements for a given problem;* when in doubt, use higher order elements (T6 or Q8) or a finer mesh.

- *Avoid elements with large aspect ratios and corner angles*.

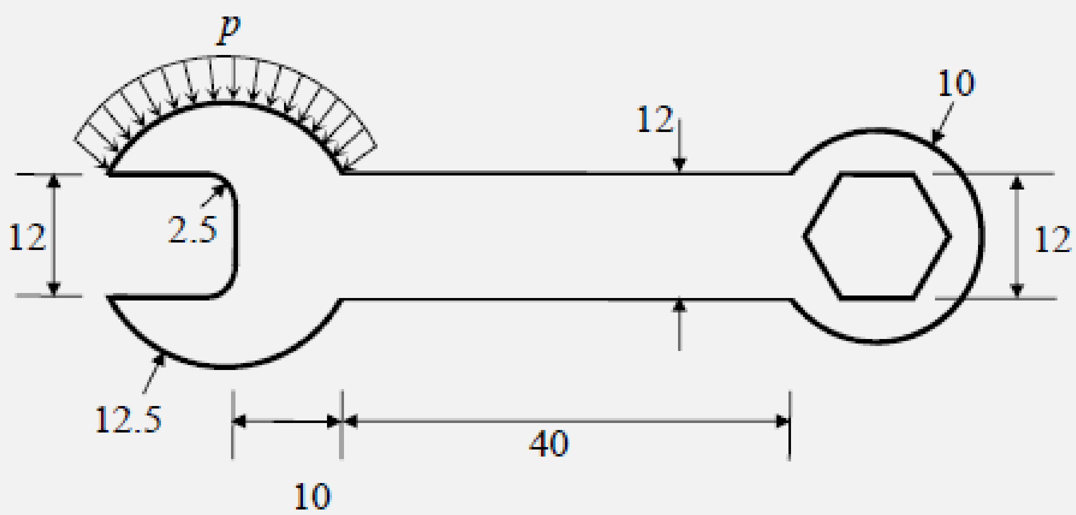
- *Make sure the elements are connected properly.*

Deformation and stress analysis of a combination wrench…

Case Study with ANSYS Workbench

Case Study with ANSYS Workbench

<Problem Description> A combination wrench is a convenient tool that is used to apply torque to loosen or tighten a fastener. The wrench shown below is made of stainless steel and has a thickness of 3*mm*. Determine the maximum deformation and the distribution of von Mises stresses under the given distributed load and boundary conditions.



Material: Stainless Steel

$E = 193$ GPa, $\nu = 0.27$

Boundary Conditions:

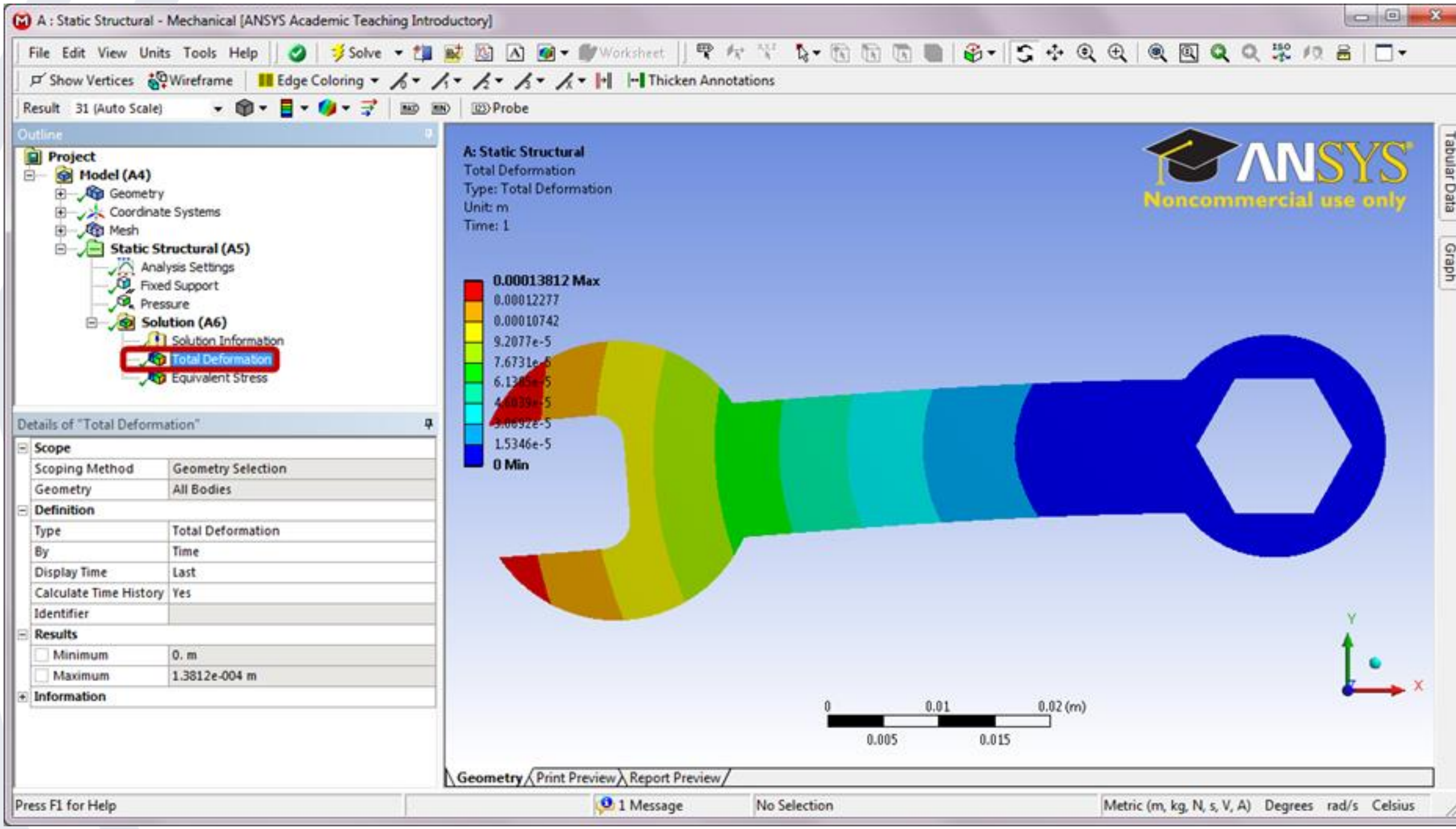
The hexagon on the right is fixed on all sides.

Pressure $p = 2$ MPa

(All units are in millimeters.)

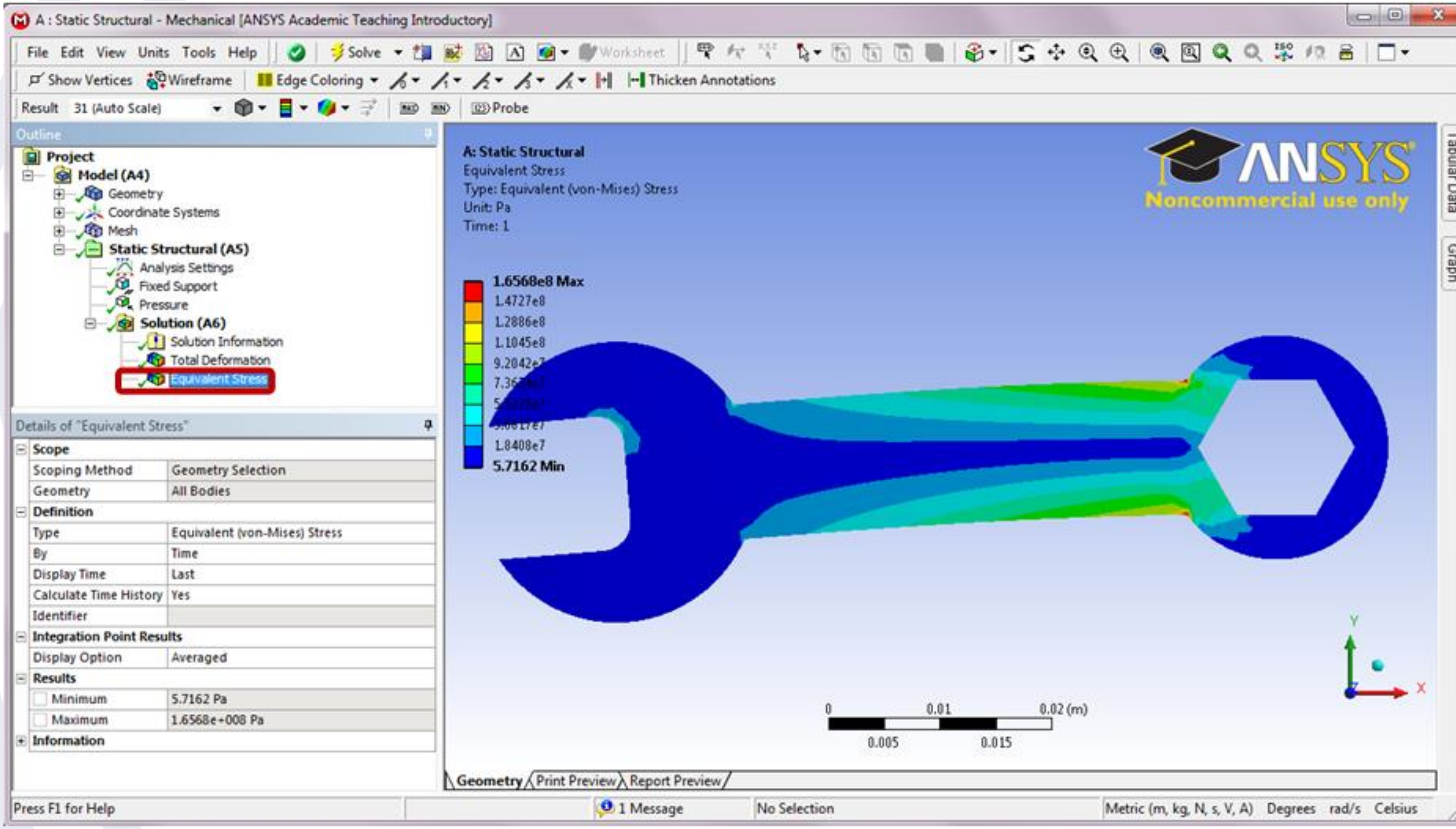
Case Study with ANSYS Workbench

Run a ***Static Structural Analysis*** to review the wrench deformation results.



Case Study with ANSYS Workbench

Select ***Equivalent Stress*** in the ***Outline*** to review the von-Mises stress distribution.



Summary

In this chapter,

- The 2-D elasticity equations are reviewed and 2-D elements for analyzing plane stress and plane strain problems are discussed.
- FE formulations for 2-D stress analysis is introduced. It is emphasized that linear triangular (T3) and linear quadrilateral (Q4) elements are good for deformation analysis and not accurate for stress analysis.
- For stress analysis, quadratic triangular (T6) and quadratic quadrilateral (Q8) elements are recommended. Bad shaped elements with large aspect ratios and large or small angles should be avoided in an FE mesh.
- A wrench model is built and the stress is analyzed in *ANSYS Workbench* to show how to conduct a 2-D FEA.