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Accelerated methods of order $2p$ for systems of nonlinear equations☆

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ABSTRACT

We present a new iterative method of order of convergence 5, for solving nonlinear systems, by composing the Midpoint method with Newton's method and using an approximation for the Jacobian matrix in order to reduce the required number of functional evaluations per iteration. In addition, we compare the efficiency index of these methods with that of Newton's method and present several numerical tests, which confirm the theoretical results and allow us to compare these variants with Newton's method.

1. Introduction

In this paper, a general procedure is designed in order to accelerate the convergence of iterative methods for solving systems of nonlinear equations, from order p to $2p$. Some methods existing in the literature are based on the use of interpolation quadrature formulas (see [1–3]), or include the second partial derivative of the nonlinear function, or its estimation (see [4]). On the other hand, another known acceleration technique consists of the composition of two iterative methods of orders p_1 and p_2, respectively, to obtain a method of order p_1p_2 [5]. Usually, new evaluations of the Jacobian matrix and the nonlinear function are needed in order to increase the order of convergence. However, the existence of an extensive literature on higher order methods (see, for example, [6–8]) reveals that they are only limited by the nature of the problem to be solved: in particular, the numerical solution of quadratic equations and nonlinear integral equations are needed in the study of dynamical models of chemical reactors [9], or in radioactive transfer [10]. Moreover, many of these numerical applications use high precision in their computations; the results of these numerical experiments show that the high order methods associated with a multiprecision arithmetic floating point is very useful, because it yields a clear reduction in iterations.

By composing any iterative method of order p with Newton's method we obtain a new method for solving nonlinear systems of order $2p$. As a particular case, we present a new method of order 5 by composing the Midpoint method with Newton's method and using an adequate approximation for the Jacobian matrix in order to reduce the required number of functional evaluations per step and improve the efficiency index.

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a real solution $\bar{x}$ of the nonlinear system $F(x) = 0$, with n equations and n unknowns. This solution can be obtained as a fixed point of some function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of the fixed point iteration method

$$x^{(k+1)} = G(x^{(k)}), \quad k = 0, 1, \ldots,$$

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where $x^{(0)}$ is the initial estimation. The best-known fixed point method is the classical Newton's method (N), given by

$$x^{(k+1)} = x^{(k)} - J_F(x^{(k)})^{-1}F(x^{(k)}), \quad k = 0, 1, \ldots,$$

where $J_F(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration $x^{(k)}$. In the following, we remember some notions about the convergence of an iterative method.

Definition 1.1. Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ convergent to $\bar{x}$. Then, convergence is called

(a) *linear*, if there exists M, $0 < M < 1$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|, \quad \forall k \geq k_0$$

(b) *of order* p, $p > 1$, if there exists M, $M > 0$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|^p, \quad \forall k \geq k_0.$$

Definition 1.2 (*See [11]*). Let $\bar{x}$ be a zero of function F and suppose that $x^{(k-1)}$, $x^{(k)}$ and $x^{(k+1)}$ are three consecutive iterations close to $\bar{x}$. Then, the computational order of convergence p can be approximated using the formula

$$p \approx \frac{\ln\left(\|x^{(k+1)} - x^{(k)}\| / \|x^{(k)} - x^{(k-1)}\|\right)}{\ln\left(\|x^{(k)} - x^{(k-1)}\| / \|x^{(k-1)} - x^{(k-2)}\|\right)}. \quad (1)$$

In addition, in order to compare different methods, we use the efficiency index, $p^{1/d}$ (see [12]), where p is the order of convergence and d is the total number of new functional evaluations (per iteration) required by the method.

Since the methods presented in this paper can be considered as iterative fixed point methods, we study their convergence by using the following result.

Theorem 1.1 (*See [5]*). *Let $G(x)$ be a fixed point function with continuous partial derivatives of order p with respect to all components of x. The iterative method $x^{(k+1)} = G(x^{(k)})$ is of order p if*

$$G(\bar{x}) = \bar{x};$$

$$\frac{\partial^k g_i(\bar{x})}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}} = 0, \quad \text{for all } 1 \leq k \leq p-1,\ 1 \leq i, j_1, \ldots, j_k \leq n;$$

and

$$\frac{\partial^p g_i(\bar{x})}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_p}} \neq 0, \quad \text{for at least one value of } i, j_1, \ldots, j_p,$$

where g_i are the component functions of G.

The rest of this paper is organized as follows: Section 2 describes the general method of order $2p$ obtained by composing any iterative method of order p with Newton's method. As a particular case, we present the Newton–Midpoint composed method. It is a three-step iterative method with sixth order convergence. A reduced Newton–Midpoint method is obtained by approximating the Jacobian matrix in order to reduce the number of functional evaluations per iteration. This reduced method has convergence order 5.

Section 3 is devoted to numerical results obtained by applying some of the obtained methods to several systems of nonlinear equations. From these results, we compare the different methods, confirming the theoretical results.

2. Description of the method and convergence analysis

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function and let $\bar{x}$ be a zero of the nonlinear system $F(x) = 0$.

Let us consider a fixed point function $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that the iterative method $y^{(k+1)} = M(y^{(k)})$ converges to the solution $\bar{x}$ of $F(x) = 0$, with convergence order p. We define a new iteration function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by applying Newton's iteration function to $M(x)$:

$$G(x) = M(x) - J_F^{-1}(M(x))F(M(x)). \quad (2)$$

Then, the iterative process is:

$$x^{(k+1)} = M(x^{(k)}) - J_F^{-1}(M(x^{(k)}))F(M(x^{(k)})). \quad (3)$$

The proof of the following theorem can be deduced very easily by the well-known result of the convergence property of Newton's method.

Theorem 2.1. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{x} \in \mathbb{R}^n$ that is a solution of the system $F(x) = 0$. Let us suppose that $J_F(x)$ is continuous and nonsingular in $\bar{x}$. Let $M(x)$ be the iteration function of a method with convergence order $p > 1$. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using the iterative expression (3) converges to $\bar{x}$ with convergence order $2p$.*

It is necessary to analyze the set of methods described in Theorem 2.1 in order to establish the optimal relation between the order of convergence and the functional evaluations involved. To get this aim, we make a composition of Newton's method with the Midpoint method, obtaining a new one (NM), with sixth order convergence and efficiency index $6^{\frac{1}{2n+3n^2}}$, by applying Theorem 2.1 and whose iterative expression is:

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{1}{2} J_F(x^{(k)})^{-1} F(x^{(k)}), \\
z^{(k)} &= x^{(k)} - J_F(y^{(k)})^{-1} F(x^{(k)}), \\
x^{(k+1)} &= z^{(k)} - J_F(z^{(k)})^{-1} F(z^{(k)}).
\end{aligned}
$$

Let us note that it is possible to define higher order iterative formulas by using the general result of Theorem 2.1, but the computational effort can make them less efficient than the previous one.

Now, we reduce the number of functional evaluations of the described methods, estimating $J_F(M(x))$ by a linear combination of the Jacobian matrices used in the initial method $M(x)$. Each method must be analyzed in order to find the optimal coefficients of the linear combination. In the case of the Newton–Midpoint composed method (NM), we find an estimation of the Jacobian matrix, which suppose an improvement of the convergence order of the Midpoint method (MP), with few additional functional evaluations.

In the following, we establish some basic results that are needed to prove the convergence of these reduced methods.

Let us consider $x \in \mathbb{R}^n$, $n > 1$, and denote by $H_{qi}(x)$ the (q, i)-entry of the inverse of the Jacobian matrix of $F(x)$. It can be easily proved that:

$$\sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l} \frac{\partial f_i(x)}{\partial x_r} = -\sum_{i=1}^{n} H_{ji}(x) \frac{\partial^2 f_i(x)}{\partial x_l \partial x_r} \tag{4}$$

and

$$\sum_{j=1}^{n} H_{ij}(x) J_{jm}(x) = \delta_{im}, \tag{5}$$

where δ_{im} is the Kronecker symbol.

Lemma 2.1. *Let $\lambda(x)$ be an iteration function whose coordinates are:*

$$\lambda_q(x) = x_q - \frac{1}{2} \sum_{j=1}^{n} H_{qj}(x) f_j(x), \tag{6}$$

for $q = 1, \ldots, n$. Then,

$$\frac{\partial \lambda_q(\bar{x})}{\partial x_l} = \frac{1}{2} \delta_{ql}, \tag{7}$$

$$\frac{\partial^2 \lambda_q(\bar{x})}{\partial x_r \partial x_l} = \frac{1}{2} \sum_{j=1}^{n} H_{qj}(\bar{x}) \frac{\partial^2 f_j(\bar{x})}{\partial x_r \partial x_l}, \tag{8}$$

for $q, l, r \in \{1, 2, \ldots, n\}$.

Proof. Let us note that by direct differentiation, if q and l are arbitrary and fixed,

$$\frac{\partial \lambda_q(x)}{\partial x_l} = \delta_{ql} - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial H_{qj}(x)}{\partial x_l} f_j(x) - \frac{1}{2} \sum_{j=1}^{n} H_{qj}(x) J_{jl}(x),$$

and by applying (5):

$$\frac{\partial \lambda_q(x)}{\partial x_l} = \frac{1}{2} \delta_{ql} - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial H_{qj}(x)}{\partial x_l} f_j(x).$$

We set now $x = \bar{x}$. Hence,

$$\frac{\partial \lambda_q(\bar{x})}{\partial x_l} = \frac{1}{2} \delta_{ql},$$

since $f_j(\bar{x}) = 0$. If the second derivative of $\lambda_q(x)$ is analyzed for q, l and r arbitrary and fixed:

$$\frac{\partial^2 \lambda_q(x)}{\partial x_r \partial x_l} = -\frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 H_{qj}(x)}{\partial x_r \partial x_l} f_j(x) - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial H_{qj}(x)}{\partial x_l}\frac{\partial f_j(x)}{\partial x_r}. \tag{9}$$

Setting $x = \bar{x}$ in (9) and by using (4), we have

$$\frac{\partial^2 \lambda_q(\bar{x})}{\partial x_r \partial x_l} = \frac{1}{2}\sum_{j=1}^{n} H_{qj}(\bar{x})\frac{\partial^2 f_j(\bar{x})}{\partial x_r \partial x_l}. \quad \square \tag{10}$$

Theorem 2.2. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{x} \in \mathbb{R}^n$ that is a solution of the system $F(x) = 0$. Let us suppose that $J_F(x)$ is continuous and nonsingular in $\bar{x}$. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by:*

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{1}{2}J_F(x^{(k)})^{-1}F(x^{(k)}) \\ z^{(k)} &= x^{(k)} - J_F(y^{(k)})^{-1}F(x^{(k)}) \\ x^{(k+1)} &= z^{(k)} - (\alpha J_F(x^{(k)}) + \beta J_F(y^{(k)}))^{-1}F(z^{(k)}), \end{aligned} \tag{11}$$

converges to $\bar{x}$ with order 4 for the family of methods verifying $\alpha + \beta = 1$. Moreover, the reduced method with $\alpha = -1$ and $\beta = 2$, denoted by RNM, has convergence order 5.

Proof. Let us consider the iteration function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$G(x) = \mu(x) - (\alpha J_F(x) + \beta J_F(\lambda(x)))^{-1}F(\mu(x)) = 0, \tag{12}$$

where $\lambda(x) = x - \frac{1}{2}H(x)F(x)$, $H(x) = J_F(x)^{-1}$ and $\mu(x) = x - H(\lambda(x))F(x)$.

This expression is equivalent to:

$$\alpha J_F(x)(G(x) - \mu(x)) + \beta J_F(\lambda(x))(G(x) - \mu(x)) + F(\mu(x)) = 0. \tag{13}$$

Let us denote by g_i, λ_i and μ_i, $i = 1, 2, \ldots, n$, the coordinate functions of $G(x)$, $\lambda(x)$ and $\mu(x)$, respectively. Expanding $g_i(x)$, $x \in \mathbb{R}^n$, in a Taylor series about $\bar{x}$ yields

$$g_i(x) = g_i(\bar{x}) + \sum_{a_1=1}^{n} \frac{\partial g_i(\bar{x})}{\partial x_{a_1}} e_{a_1} + \frac{1}{2}\sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \frac{\partial^2 g_i(\bar{x})}{\partial x_{a_1}\partial x_{a_2}} e_{a_1}e_{a_2} + \frac{1}{6}\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\sum_{a_3=1}^{n} \frac{\partial^3 g_i(\bar{x})}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} e_{a_1}e_{a_2}e_{a_3} + \cdots, \tag{14}$$

where $e_{a_k} = x_{a_k} - \bar{x}_{a_k}$, $1 \leq a_1, \ldots, a_k \leq n$.

The ith component of (13) is

$$\alpha \sum_{j=1}^{n} J_{ij}(x)(g_j(x) - \mu_j(x)) + \beta \sum_{j=1}^{n} J_{ij}(\lambda(x))(g_j(x) - \mu_j(x)) + f_i(\mu(x)) = 0 \tag{15}$$

and, by direct differentiation of (15), being i and l arbitrary and fixed,

$$\begin{aligned} &\alpha \sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_l}\left(g_j(x) - \mu_j(x)\right) + \beta \sum_{j=1}^{n}\left(\sum_{q=1}^{n} \frac{\partial J_{ij}(\lambda(x))}{\partial \lambda_q(x)}\frac{\partial \lambda_q(x)}{\partial x_l}\right)\left(g_j(x) - \mu_j(x)\right) \\ &\quad + \sum_{j=1}^{n}\left(\alpha J_{ij}(x) + \beta J_{ij}(\lambda(x))\right)\left(\frac{\partial g_j(x)}{\partial x_l} - \frac{\partial \mu_j(x)}{\partial x_l}\right) + \sum_{q=1}^{n} \frac{\partial f_i(\mu(x))}{\partial \mu_q(x)}\frac{\partial \mu_q(x)}{\partial x_l} = 0. \end{aligned} \tag{16}$$

When $x = \bar{x}$, we take into account that $g_j(\bar{x}) = \bar{x}$, $\mu_j(\bar{x}) = \bar{x}$ and $f_i(\bar{x}) = 0$. Moreover, by Theorem 1.1, $\frac{\partial \mu_j(\bar{x})}{\partial x_l} = \frac{\partial^2 \mu_j(\bar{x})}{\partial x_r \partial x_l} = 0$, since $\mu(x)$ is the iteration function of the Midpoint method. Then, we have

$$(\alpha + \beta)\sum_{j=1}^{n} J_{ij}(\bar{x})\frac{\partial g_j(x)}{\partial x_l} = 0.$$

Moreover, it is known that i and l are arbitrary and $J_F(\bar{x})$ is nonsingular. Then, if $\alpha + \beta \neq 0$,

$$\frac{\partial g_j(\bar{x})}{\partial x_l} = 0. \tag{17}$$

Now, by direct differentiation of (16), being r arbitrary and fixed,

$$\alpha \sum_{j=1}^{n} \frac{\partial^2 J_{ij}(x)}{\partial x_r \partial x_l} \left(g_j(x) - \mu_j(x)\right) + \alpha \sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_l} \left(\frac{\partial g_j(x)}{\partial x_r} - \frac{\partial \mu_j(x)}{\partial x_r}\right) + \alpha \sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_r} \left(\frac{\partial g_j(x)}{\partial x_l} - \frac{\partial \mu_j(x)}{\partial x_l}\right)$$
$$+ \sum_{j=1}^{n} (\alpha J_{ij}(x) + \beta J_{ij}(\lambda(x))) \left(\frac{\partial^2 g_j(x)}{\partial x_r \partial x_l} - \frac{\partial^2 \mu_j(x)}{\partial x_r \partial x_l}\right) + \beta \sum_{j=1}^{n} \left[\sum_{q=1}^{n} \left(\sum_{p=1}^{n} \frac{\partial^2 J_{ij}(\lambda(x))}{\partial \lambda_p(x) \partial \lambda_q(x)} \frac{\partial \lambda_p(x)}{\partial x_r}\right) \frac{\partial \lambda_q(x)}{\partial x_l}\right]$$
$$\times \left(g_j(x) - \mu_j(x)\right) + \beta \sum_{j=1}^{n} \left(\sum_{q=1}^{n} \frac{\partial J_{ij}(\lambda(x))}{\partial \lambda_q(x)} \frac{\partial^2 \lambda_q(x)}{\partial x_r \partial x_l}\right) \left(g_j(x) - \mu_j(x)\right) + \beta \sum_{j=1}^{n} \left(\sum_{q=1}^{n} \frac{\partial J_{ij}(\lambda(x))}{\partial \lambda_q(x)} \frac{\partial \lambda_q(x)}{\partial x_l}\right)$$
$$\times \left(\frac{\partial g_j(x)}{\partial x_r} - \frac{\partial \mu_j(x)}{\partial x_r}\right) + \beta \sum_{j=1}^{n} \left(\sum_{q=1}^{n} \frac{\partial J_{ij}(\lambda(x))}{\partial \lambda_q(x)} \frac{\partial \lambda_q(x)}{\partial x_r}\right) \left(\frac{\partial g_j(x)}{\partial x_l} - \frac{\partial \mu_j(x)}{\partial x_l}\right)$$
$$+ \sum_{q=1}^{n} \left(\sum_{p=1}^{n} \frac{\partial^2 f_i(\mu(x))}{\partial \mu_p(x) \partial \mu_q(x)} \frac{\partial \mu_p(x)}{\partial x_r}\right) \frac{\partial \mu_q(x)}{\partial x_l} + \sum_{q=1}^{n} \frac{\partial f_i(\mu(x))}{\partial \mu_q(x)} \frac{\partial^2 \mu_q(x)}{\partial x_r \partial x_l} = 0. \tag{18}$$

Let us substitute $x = \bar{x}$. Then,

$$(\alpha + \beta) \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial^2 g_j(\bar{x})}{\partial x_r \partial x_l} = 0.$$

So, if $\alpha + \beta \neq 0$, it can be concluded that

$$\frac{\partial^2 g_j(\bar{x})}{\partial x_r \partial x_l} = 0. \tag{19}$$

Now, we prove the fourth order of convergence. To get this aim, it is necessary to differentiate (18) with respect to x_s, being s arbitrary and fixed, and evaluate the resulting expression in $x = \bar{x}$. Then, by using (4), (5) and Lemma 2.1, and by taking into account that $\frac{\partial \mu_j(\bar{x})}{\partial x_l} = \frac{\partial^2 \mu_j(\bar{x})}{\partial x_r \partial x_l} = 0$, it is proved that:

$$(\alpha + \beta) \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial^3 g_j(\bar{x})}{\partial x_s \partial x_r \partial x_l} - (\alpha + \beta - 1) \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial^3 \mu_j(\bar{x})}{\partial x_s \partial x_r \partial x_l} = 0.$$

Therefore, as $J_F(\bar{x})$ is assumed to be nonsingular, it can be concluded that the order of the family of methods verifying $\alpha + \beta = 1$ is, at least, 4.

By differentiating again respect to x_t, being t arbitrary and fixed, evaluating the resulting expression in $x = \bar{x}$ and using (4), Lemma 2.1, we find that:

$$-\left(\alpha + \frac{\beta}{2}\right) \sum_{j=1}^{n} \frac{\partial J_{ij}(\bar{x})}{\partial x_l(x)} \frac{\partial^3 \mu_j(\bar{x})}{\partial x_t \partial x_s \partial x_r} - \left(\alpha + \frac{\beta}{2}\right) \sum_{j=1}^{n} \frac{\partial J_{ij}(\bar{x})}{\partial x_r(x)} \frac{\partial^3 \mu_j(\bar{x})}{\partial x_t \partial x_s \partial x_l} - \left(\alpha + \frac{\beta}{2}\right) \sum_{j=1}^{n} \frac{\partial J_{ij}(\bar{x})}{\partial x_t(x)} \frac{\partial^3 \mu_j(\bar{x})}{\partial x_s \partial x_r \partial x_l}$$
$$- \left(\alpha + \frac{\beta}{2}\right) \sum_{j=1}^{n} \frac{\partial J_{ij}(\bar{x})}{\partial x_s(x)} \frac{\partial^3 \mu_j(\bar{x})}{\partial x_t \partial x_r \partial x_l} - (\alpha + \beta) \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial^4 g_j(\bar{x})}{\partial x_t \partial x_s \partial x_r \partial x_l}$$
$$- (\alpha + \beta - 1) \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial^4 \mu_j(\bar{x})}{\partial x_t \partial x_s \partial x_r \partial x_l} = 0. \tag{20}$$

Then if $\alpha + \beta = 1$ and $\alpha + \frac{\beta}{2} = 0$, or equivalently if $\alpha = -1$ and $\beta = 2$, by applying (14) and Theorem 1.1, we conclude that the iterative expression (11) converges to $\bar{x}$ with convergence order 5. □

Let us note that the kind of combination of expression (11) cannot be efficient by using other iterative methods different from the Midpoint method. Although formula (11) needs to compute inversions of three matrices, the key of this expression is the good relation between its order of convergence and the number of functional evaluations per iteration required, that is, its efficiency index, widely used in order to compare different iterative methods.

We observe that the efficiency index of RNM method is $5^{\frac{1}{2n+2n^2}}$. In Fig. 1 we show the efficiency indices of MP, NM, RNM and Newton's methods, for different values of the size of the nonlinear system. We note that the RNM method has the highest efficiency index. For higher values of n, the efficiency indices hold this tendency, although the difference between them is small.

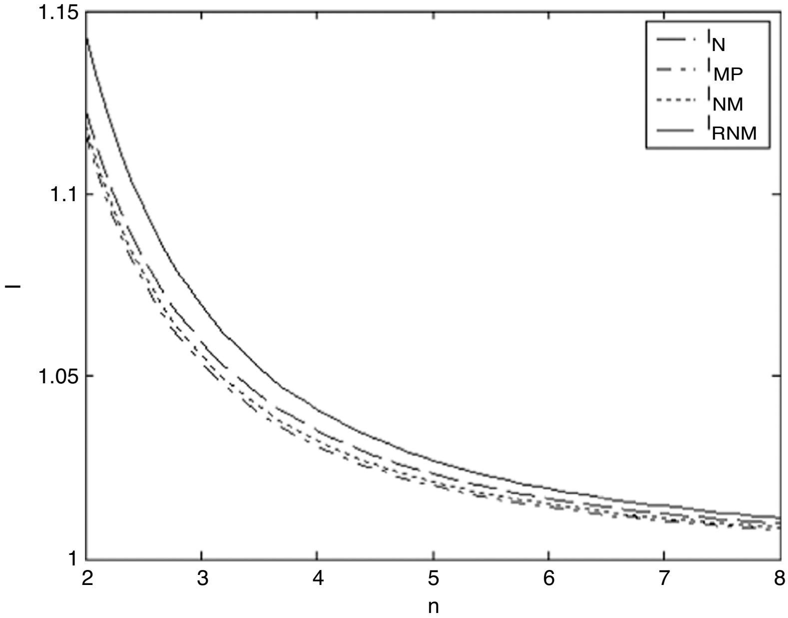

Fig. 1. Efficiency indices for $n > 1$.

3. Numerical results

In this section we will check the effectiveness of some numerical methods in order to estimate the zeros of several nonlinear functions:

(a) $F(x_1, x_2) = (\sin(x_1) + x_2\cos(x_1), x_1 - x_2), \bar{x} = (0, 0)^T$.
(b) $F(x_1, x_2) = (\exp(x_1^2) - \exp(\sqrt{2}x_1), x_1 - x_2), \bar{x} = (\sqrt{2}, \sqrt{2})^T$.
(c) $F(x_1, x_2) = \left(x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 + \frac{1}{2}\right), \bar{x} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T$.
(d) $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, n$, such that

$$f_i(x) = x_i x_{i+1} - 1, \quad i = 1, 2, \ldots, n-1$$
$$f_n(x) = x_n x_1 - 1.$$

When n is odd, the exact zeros of $F(x)$ are $\bar{x}_1 = (1, 1, \ldots, 1)$ and $\bar{x}_2 = (-1, -1, \ldots, -1)$. Results appearing in Table 1 are obtained for $n = 99$ and all the methods converge to $\bar{x}_1$.

(e) $F(x_1, x_2) = (x_1 + \exp(x_2) - \cos(x_2), 3x_1 - x_2 - \sin(x_2)), \bar{x} = (0, 0)^T$.
(f) $F(x_1, x_2) = (3x_1^2 + \exp(x_2)\sin(x_2) - 3, x_1x_2 - \sin(x_2)), \bar{x} = (1, 0)^T$.
(g) Let be a nonlinear boundary value problem

$$y''(x) = y(x)^3 + \sin(y'(x)^2), \quad x \in [0, 1]$$
$$y(0) = 0, \qquad y(1) = 1$$

taken from [13]. By using the second order finite differences method, we take the nodes $x_i = ih$, $i = 0, 1, \ldots, n$ where $h = \frac{1}{n}$, and use second order approximations for $y'(x_i)$ and $y''(x_i)$. Denoting the unknowns values $y(x_i)$ by y_i, $i = 0, 1, 2, \ldots, n$ the solution of the following nonlinear system provides us an estimation of the solution of the boundary value problem:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i^3 - \sin\left(\left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2\right) = 0, \quad i = 1, 2, 3, \ldots, n-1.$$

These nonlinear functions have been chosen in order to have different points of view: the second partial derivatives of function (a) is null at the solution, so that the convergence order of the methods increase; the case of function (d) is that of a big-sized system and the Jacobian matrix of the functions (e) and (f) are singular at some points. Finally, a practical problem is consider in (g) taking $n = 100$.

Numerical computations have been carried out in MATLAB, with variable precision arithmetic that uses floating point representation of 200 decimal digits of mantissa. Every iterate $x^{(k+1)}$ is obtained from the previous one, $x^{(k)}$, by adding one

Table 1
Numerical results.

$F(x)$	$x^{(0)}$	Iterations				p			
		N	MP	NM	RNM	N	MP	NM	RNM
(a)	$(0.8, 0.8)^T$	9	6	4	5	3.0	3.0	9.0	5.0
(b)	$(3, 3)^T$	17	11	7	8	2.0	3.0	6.0	5.0
(c)	$(2, 3)^T$	10	7	5	5	2.0	3.0	5.8	5.0
(d)	$(2, \dots, 2)^T$	12	10	6	5	2.0	3.0	6.0	5.0
(e)	$(1.5, -1.5)^T$	171	44	7	12	–	–	6.0	5.0
(f)	$(0, 1)^T$	331	211	130	156	1.0	1.0	1.0	1.0
(g)	$(1, \dots, 1)^T$	9	6	4	5	2.0	3.0	5.9	4.48

or more terms of the form $A^{-1}b$ where $x^{(k)} \in \mathbb{R}^n$, A is a real $n \times n$ matrix and $b \in \mathbb{R}^n$. The matrix A and the vector b are different according to the method used, but in any case the inverse calculation $-A^{-1}b$ is carried out solving the linear system $Ay = -b$, using Gaussian elimination with partial pivoting.

The stopping criterion used is $\|x^{(k+1)} - x^{(k)}\| + \|F(x^{(k)})\| < 10^{-100}$. Therefore, we check that iterates converge to a limit and moreover that this limit is a solution of the system of nonlinear equations.

Table 1 shows several results obtained by using the previously described methods and the classical Newton's method, in order to estimate the zeros of functions from (a) to (g). For every method, we specify the initial estimation $x^{(0)}$ and analyze the number of iterations needed to converge to the solution and the computational order of convergence, p, estimated by (1).

The value of p that appears in Table 1 is the last coordinate of vector p when the variation between its coordinates is small. When this does not happen, the value of p is said to be not conclusive, and is denoted by "–" in the mentioned table.

In practice, it is observed that in case of $\frac{\partial^2 f_i(\bar{x})}{\partial x_a \partial x_b} = 0$ for all i, a, b, as in example (a), the convergence of Newton's and Midpoint methods is of order $p = 3$, while NM has 9 as computational order and RNM holds 5th order. If this condition is not verified, the computational order of convergence of the modified methods coincides with the theoretical values, although the RNM method shows to be quite efficient to find the solution in a half, or less indeed, of the iterations of Newton's method.

Nevertheless, it can be observed in (d) that the efficiency of the methods remains although the system of equations is big-sized. Indeed, in cases (e) and (f), where the convergence is affected by the behavior of the Jacobian matrix at some points, the new modified methods show a good stability and more effectiveness than classical Newton's method.

Another way to study the efficiency of the methods is to analyze their running time. Nowadays, the speed of calculations with computers is very high for small and well conditioned systems. Nevertheless, when the system is close to singularity as in cases (e) and (f), it can be appreciated a better behavior in the new methods presented in this work, than in classical methods such as Newton's method or the Midpoint method. The new methods are more stable and the running time is better. For example, in system (e) we have calculated the mean running times of the algorithms and we have obtained 1.5438 s for getting convergence with Newton's method, 2.6563 with the Midpoint method, 0.6078 with NM method and 1.1016 with RNM method. These results confirm the good behavior of the presented methods.

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