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Iterative methods of order four and five for systems of nonlinear equations☆

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ABSTRACT

The Adomian decomposition is used in order to obtain a family of methods to solve systems of nonlinear equations. The order of convergence of these methods is proved to be $p \geq 2$, under the same conditions as the classical Newton method. Also, numerical examples will confirm the theoretical results.

1. Introduction

The main goal of this work is to obtain new iterative formulas in order to solve systems of nonlinear equations. They are proved to be modifications on the classical Newton method which accelerate the convergence of the iterative process.

In previous works, the authors have obtained variants on the Newton method based on quadrature formulas whose truncation error was up to $O(h^5)$ (see [1,2]). Indeed, a general interpolatory quadrature formula is used in [3] in order to obtain a family of modified Newton methods with order of convergence up to $2d + 1$, when the partial derivatives of each coordinate function in the solution, from order two until d, are zero. Moreover, in [3], we modify the general method from [4] getting a collection of multipoint iterative methods obtained from the Newton method by replacing $F(x^{(k)})$ by a linear combination of values of $F(x)$ in different points.

Nevertheless, the approach used in this paper to solve a nonlinear system is different: by using Adomian polynomials, we obtain a family of multipoint iterative formulas, which include the Newton and Traub (see [5]) methods in the simplest cases.

The decomposition method using Adomian polynomials is used to solve different problems on applied mathematics in [6]. Indeed, Babolian et al. (see [7]) apply this general method to a concrete nonlinear system. Nevertheless, with a different system, it is necessary to repeat all the process. In [8], Adomian decomposition method is applied to construct some numerical algorithms for solving systems of two nonlinear equations.

We deduce in Section 2, by means of Adomian decomposition, a family of iterative formulas that can be applied to solve any nonlinear system without knowledge about Adomian polynomials. These iterative formulas involve classical methods, like those of Newton (order $p = 2$) and Traub (order $p = 3$), and also new methods whose convergence order is proved to be higher.

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Now, we remember the most common notions about nonlinear systems and the convergence of an iterative method.

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution $\bar{x} \in D$ of the nonlinear system $F(x) = 0$, of n equations with n variables. This solution can be obtained as a fixed point of some function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of the fixed point iteration method

$$x^{(k+1)} = G(x^{(k)}), \quad k = 0, 1, \dots, \tag{1}$$

where $x^{(0)}$ is the initial estimation. The best known fixed point method is the classical Newton method, given by

$$x^{(k+1)} = x^{(k)} - J_F(x^{(k)})^{-1}F(x^{(k)}), \quad k = 0, 1, \dots, \tag{2}$$

where $J_F(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration $x^{(k)}$.

Definition 1. Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ convergent to $\bar{x}$. Then, convergence is called

(a) *linear*, if there exist M, $0 < M < 1$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|, \quad \forall k \geq k_0. \tag{3}$$

(b) *of order* p, $p \geq 2$, if there exist M, $M > 0$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|^p \quad \forall k \geq k_0. \tag{4}$$

Definition 2 (*See [9]*). Let $\bar{x}$ be a zero of the function F and suppose that $x^{(k-1)}$, $x^{(k)}$ and $x^{(k+1)}$ are three consecutive iterations close to $\bar{x}$. Then, the computational order of convergence ρ can be approximated using the formula

$$\rho \approx \frac{\ln\left(\left\|x^{(k+1)} - \bar{x}\right\| / \left\|x^{(k)} - \bar{x}\right\|\right)}{\ln\left(\left\|x^{(k)} - \bar{x}\right\| / \left\|x^{(k-1)} - \bar{x}\right\|\right)}. \tag{5}$$

In addition, in order to compare different methods, we use the efficiency index, $p^{1/d}$ (see [10]), where p is the order of convergence and d is the total number of new functional evaluations (per iteration) required by the method.

In Section 3, we study the convergence of the different methods by using the following result.

Theorem 1 (*See [5]*). *Let $G(x)$ be a fixed point function with continuous partial derivatives of order p with respect to all components of x. The iterative method $x^{(k+1)} = G(x^{(k)})$ is of order p if*

$$G(\bar{x}) = \bar{x};$$

$$\frac{\partial^k g_i(\bar{x})}{\partial x_{j_1}\partial x_{j_2}\dots\partial x_{j_k}} = 0, \quad \textit{for all } 1 \leq k \leq p-1,\ 1 \leq i, j_1, \dots, j_k \leq n;$$

$$\frac{\partial^p g_i(\bar{x})}{\partial x_{j_1}\partial x_{j_2}\dots\partial x_{j_p}} \neq 0, \quad \textit{for at least one value of } i, j_1, \dots, j_p$$

where g_i are the component functions of G.

Finally, numerical tests are made in Section 4 comparing the classical and new methods, confirming the theoretical results.

2. Description of the methods

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n > 1$, be a sufficiently differentiable function whose coordinate functions are $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, 2, \dots, n$. Let $\bar{x}$ be a zero of the nonlinear system $F(x) = 0$ and $\alpha \in \mathbb{R}^n$ an estimation of $\bar{x}$. Then, this system is equivalent to:

$$F(\alpha) + J_F(\alpha)(x-\alpha)^{\mathrm{T}} + K(x) = 0,$$

where $J_F(\alpha)$ is the Jacobian matrix of the function F evaluated in the estimation α and $K : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ verifies:

$$K(x) = F(x) - F(\alpha) - J_F(\alpha)(x-\alpha)^{\mathrm{T}}.$$

Then,

$$x = \alpha - J_F^{-1}(\alpha)F(\alpha) - J_F^{-1}(\alpha)K(x).$$

Let us denote the linear component as $c \equiv \alpha - J_F^{-1}(\alpha)F(\alpha) \in \mathbb{R}^n$, and by $P(x)$ the nonlinear one, $P(x) = -J_F^{-1}(\alpha)K(x)$, $P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with coordinate functions $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. So,

$$x = c + P(x).$$

Let us suppose that each one of the respective ith component of the approximation x of the solution $\bar{x}$ and also of $N(x)$ can be written as $x_i = \sum_{l=0}^{\infty} x_l^i$ and $P_i(x) = \sum_{l=0}^{\infty} A_l^i$, $i = 1, 2, \dots, n$, where $A_l^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are Adomian polynomials.

Subsequently, a first estimation of $\bar{x}$ is $x^0 \equiv (x_0^1, x_0^2, \dots, x_0^n)^{\mathrm{T}}$, where

$$x_0^i = c_i = \alpha_i - \sum_{j=1}^{n} H_{ij}(\alpha) f_j(\alpha),$$

being $H_{ij}(\alpha)$ the (i, j)-entry of the inverse matrix of the Jacobian one, whose (i, j) entry is $J_{ij}(x)$. So,

$$x^0 = \alpha - J_F^{-1}(\alpha) F(\alpha) \tag{6}$$

and $x \simeq x^0$ which corresponds to the classical Newton method (CN).

If a better approximation is needed, a new term in the series development of x is used, $x_i \simeq x_0^i + x_1^i$, where

$$x_1^i = A_0^i = P_i(x_0^i) = -\sum_{j=1}^{n} H_{ij}(\alpha) k_j(x_0^i) \tag{7}$$

and $k_j(x)$ is the jth coordinate function of $K(x)$.

Then,

$$x^1 = -J_F^{-1}(\alpha) K(x^0) \tag{8}$$

and

$$\begin{aligned} x &\simeq x^0 + x^1 \\ &= \alpha - J_F^{-1}(\alpha) F(\alpha) - J_F^{-1}(\alpha) K(x^0) \\ &= \alpha - J_F^{-1}(\alpha) F(\alpha) - J_F^{-1}(\alpha) \left(F(x^0) - F(\alpha) - J_F(\alpha)(x^0 - \alpha)^{\mathrm{T}} \right) \\ &= \alpha - J_F^{-1}(\alpha) \left(F(\alpha) + F(x^0) \right). \end{aligned} \tag{9}$$

This method, whose convergence order is 3, was described in [5], and has been generalized (to linear combinations of F evaluated on different iterations) in [3].

The term x_2^i in the series development of x_i is obtained as

$$\begin{aligned} x_2^i &= A_1^i = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(P_i \left(\sum_{l=0}^{\infty} \lambda^l x_l^1, \dots, \sum_{l=0}^{\infty} \lambda^l x_l^n \right) \right)_{\lambda=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(-\sum_{j=1}^{n} H_{ij}(\alpha) k_j \left(\sum_{l=0}^{\infty} \lambda^l x_l^1, \dots, \sum_{l=0}^{\infty} \lambda^l x_l^n \right) \right)_{\lambda=0} \\ &= -\sum_{j=1}^{n} H_{ij}(\alpha) \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(f_j \left(\sum_{l=0}^{\infty} \lambda^l x_l^1, \dots, \sum_{l=0}^{\infty} \lambda^l x_l^n \right) \right)_{\lambda=0} \\ &\quad + \sum_{j=1}^{n} H_{ij}(\alpha) \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(f_j(\alpha) + \sum_{m=1}^{n} \frac{\partial f_j(\alpha)}{\partial x_m} \left(\sum_{l=0}^{\infty} \lambda^l x_l^m - \alpha_m \right) \right)_{\lambda=0}. \end{aligned} \tag{10}$$

Let us denote $\left(\sum_{l=0}^{\infty} \lambda^l x_l^1, \dots, \sum_{l=0}^{\infty} \lambda^l x_l^n \right)$ by $\mu(\lambda, x)$, whose mth component is denoted by $\mu_m(\lambda, x)$, and note that $\mu(0, x) = (x_0^1, \dots, x_0^n)^{\mathrm{T}} = x^0$. So,

$$x_2^i = -\sum_{j=1}^{n} H_{ij}(\alpha) \left(\sum_{m=1}^{n} \frac{\partial f_j(\mu(\lambda, x))}{\partial \mu_m(\lambda, x)} \frac{\partial \mu_m(\lambda, x)}{\partial \lambda} \right)_{\lambda=0} + \sum_{j=1}^{n} H_{ij}(\alpha) \left(\sum_{m=1}^{n} \frac{\partial f_j(\alpha)}{\partial x_m} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\mu_m(\lambda, x) - \alpha_m \right) \right)_{\lambda=0} \tag{11}$$

and, taking into account that $\frac{\partial \mu_m(\lambda, x)}{\partial \lambda} = \sum_{l=1}^{\infty} l \lambda^{l-1} x_l^m$,

$$\begin{aligned} x_2^i &= -\sum_{j=1}^{n} H_{ij}(\alpha) \left(\sum_{m=1}^{n} \frac{\partial f_j(x^0)}{\partial x_m} x_1^m \right) + \sum_{j=1}^{n} H_{ij}(\alpha) \left(\sum_{m=1}^{n} \frac{\partial f_j(\alpha)}{\partial x_m} x_1^m \right) \\ &= -\sum_{j=1}^{n} H_{ij}(\alpha) \left(\sum_{m=1}^{n} \frac{\partial f_j(x^0)}{\partial x_m} x_1^m \right) + \sum_{m=1}^{n} \delta_{im} x_1^m \\ &= x_1^i - \sum_{j=1}^{n} \sum_{m=1}^{n} H_{ij}(\alpha) \frac{\partial f_j(x^0)}{\partial x_m} x_1^m, \end{aligned} \tag{12}$$

by using

$$\sum_{j=1}^{n} H_{ij}(x)J_{jm}(x) = \delta_{im}, \tag{13}$$

where δ_{im} is the Kronecker symbol.

Then, in vectorial notation:

$$x^2 = x^1 - J_F^{-1}(\alpha)J_F(x^0)x^1 \tag{14}$$

and, using (6), (8) and (14), a new estimation of the solution $\bar{x}$ is:

$$\begin{aligned} x &\simeq x^0 + x^1 + x^2 \\ &= \alpha - J_F^{-1}(\alpha)F(\alpha) - \left[2J_F^{-1}(\alpha) - J_F^{-1}(\alpha)J_F(x^0)J_F^{-1}(\alpha)\right]F(x^0). \end{aligned} \tag{15}$$

This expression corresponds to a new method which involves only a new function evaluation with respect to the previously described methods and whose convergence order will be proved to be 4. We call this new method NAd1, as it is a variant of the Newton method that use Adomian polynomials of sub-index 1. Nevertheless, an iterative expression of NAd1 can be used with no knowledge of Adomian polynomials, only in terms of the previous estimations and the Newton approximation. So, being $x^{(0)}$ the initial estimation of the iterative process and being

$$\bar{x}^{(k+1)} = x^{(k)} - J_F^{-1}(x^{(k)})F(x^{(k)})$$

the $(k+1)$th approximation of the Newton method, a new estimation $x^{(k+1)}$ can be obtained by means of the following expression:

$$x^{(k+1)} = \bar{x}^{(k+1)} - \left[2J_F^{-1}(x^{(k)}) - J_F^{-1}(x^{(k)})J_F(\bar{x}^{(k+1)})J_F^{-1}(x^{(k)})\right]F(\bar{x}^{(k+1)}). \tag{16}$$

Indeed, other new methods can be obtained if more terms are added in the truncation of the theoretical series developments of each ith component of the estimation x, $x_i = \sum_{l=0}^{\infty} x_l^i$ and $N_i(x) = \sum_{l=0}^{\infty} A_l^i$, $i = 1, 2, \ldots, n$. In particular, calculating the components of $x^3 = (x_3^1, \ldots, x_3^n)^T$ by means of the respective Adomian polynomial A_2^i, $i = 1, 2, \ldots, n$.

$$\begin{aligned} x_3^i &= A_2^i = \frac{1}{2}\frac{d^2}{d\lambda^2}\left(N_i\left(\mu(\lambda, x)\right)\right)_{\lambda=0} \\ &= \frac{1}{2}\frac{d^2}{d\lambda^2}\left(-\sum_{j=1}^{n} H_{ij}(\alpha)k_j\left(\mu(\lambda, x)\right)\right)_{\lambda=0} \\ &= -\frac{1}{2}\sum_{j=1}^{n} H_{ij}(\alpha)\frac{d}{d\lambda}\left(\sum_{m=1}^{n}\frac{\partial f_j(\mu(\lambda, x))}{\partial \mu_m(\lambda, x)}\frac{\partial \mu_m(\lambda, x)}{\partial \lambda}\right)_{\lambda=0} - \frac{d}{d\lambda}\left(\sum_{m=1}^{n}\frac{\partial f_j(\alpha)}{\partial x_m}\frac{\partial \mu_m(\lambda, x)}{\partial \lambda}\right)_{\lambda=0} \\ &= -\frac{1}{2}\sum_{j=1}^{n} H_{ij}(\alpha)\sum_{m=1}^{n}\left\{\sum_{a=1}^{n}\frac{\partial^2 f_j(\mu(\lambda, x))}{\partial \mu_m(\lambda, x)\partial \mu_a(\lambda, x)}\frac{\partial \mu_m(\lambda, x)}{\partial \lambda}\frac{\partial \mu_a(\lambda, x)}{\partial \lambda}\right\}_{\lambda=0} \\ &\quad - \frac{1}{2}\sum_{j=1}^{n} H_{ij}(\alpha)\sum_{m=1}^{n}\left\{\frac{\partial f_j(\mu)}{\partial \mu_m(\lambda, x)}\frac{\partial \mu_m(\lambda, x)}{\partial \lambda}\right\}_{\lambda=0} - \sum_{m=1}^{n}\left\{\frac{\partial f_j(\alpha)}{\partial x_m}\frac{\partial^2 \mu_m(\lambda, x)}{\partial \lambda^2}\right\}_{\lambda=0}. \end{aligned}$$

As $\frac{d^2\mu_m(\lambda,x)}{d\lambda^2} = \sum_{l=2}^{\infty}(l-1)l\lambda^{l-2}x_l^m$, by evaluating the above expressions in $\lambda = 0$ and using (13), each component x_3^i is defined by:

$$x_3^i = -\frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{n}\sum_{a=1}^{n} H_{ij}(\alpha)\frac{\partial^2 f_j(x^0)}{\partial x_m \partial x_a}x_1^m x_1^a - \sum_{j=1}^{n} H_{ij}(\alpha)\sum_{m=1}^{n}\frac{\partial f_j(x^0)}{\partial x_m}x_2^m + \sum_{m=1}^{n}\delta_{im}x_2^m. \tag{17}$$

Then, by using vectorial notation, x^3 can be expressed as follows:

$$x^3 = x^2 - \frac{1}{2}J_F^{-1}(\alpha)B - J_F^{-1}(\alpha)J_F(x^0)x^2, \tag{18}$$

where B is an $(n, 1)$-matrix whose jth component is

$$\begin{aligned} B_j &= \sum_{m=1}^{n}\sum_{a=1}^{n}\frac{\partial^2 f_j(x^0)}{\partial x_m \partial x_a}x_1^m x_1^a \\ &= \sum_{m=1}^{n}\sum_{a=1}^{n}\frac{\partial^2 f_j(x^0)}{\partial x_m \partial x_a}\left(\sum_{p=1}^{n} H_{mp}(\alpha)f_p(x^0)\right)\left(\sum_{p=1}^{n} H_{ap}(\alpha)f_p(x^0)\right). \end{aligned} \tag{19}$$

A new approximation of the solution is obtained:

$$\begin{aligned} x &\simeq x^0 + x^1 + x^2 + x^3 \\ &= \alpha - J_F^{-1}(\alpha)F(\alpha) - 3J_F^{-1}(\alpha)F(x^0) + 3J_F^{-1}(\alpha)J_F(x^0)J_F^{-1}(\alpha)F(x^0) \\ &\quad - J_F^{-1}(\alpha)J_F(x^0)J_F^{-1}(\alpha)J_F(x^0)J_F^{-1}(\alpha)F(x^0) - \frac{1}{2}J_F^{-1}(\alpha)B. \end{aligned} \tag{20}$$

This is a new method which involves the functional evaluation of vector B including second-order partial derivatives of $f_j, j = 1, \ldots, n$ whose convergence order is 5. We call this new method NAd2, as it use Adomian polynomials of sub-index 2. The iterative expression of NAd2, as it happens with NAd1, can be used with no knowledge of Adomian polynomials. So, being $x^{(0)}$ the initial estimation and being $\bar{x}^{(k+1)} = x^{(k)} - J_F^{-1}(x^{(k)})F(x^{(k)})$ the $(k+1)$th approximation of the Newton method, a new estimation $x^{(k+1)}$ can be obtained by means of:

$$\begin{aligned} x^{(k+1)} &= \bar{x}^{(k+1)} - 3J_F^{-1}(x^{(k)})F(\bar{x}^{(k+1)}) + 3\left(J_F^{-1}(x^{(k)})J_F(\bar{x}^{(k+1)})J_F^{-1}(x^{(k)})\right)F(\bar{x}^{(k+1)}) \\ &\quad - J_F^{-1}(x^{(k)})J_F(\bar{x}^{(k+1)})J_F^{-1}(x^{(k)})J_F(\bar{x}^{(k+1)})J_F^{-1}(x^{(k)})F(\bar{x}^{(k+1)}) - \frac{1}{2}J_F^{-1}(x^{(k)})B^{(k)}. \end{aligned} \tag{21}$$

where the jth component of $B^{(k)}$ is:

$$B_j^{(k)} = \sum_{m=1}^{n}\sum_{a=1}^{n} \frac{\partial^2 f_j(\bar{x}^{(k+1)})}{\partial x_m \partial x_a} x_1^m x_1^a,$$

being

$$x_1^m = \sum_{p=1}^{n} H_{mp}(x^{(k)}) f_p(\bar{x}^{(k+1)})$$

and

$$x_1^a = \sum_{p=1}^{n} H_{ap}(x^{(k)}) f_p(\bar{x}^{(k+1)}).$$

3. Convergence analysis

We consider $x \in \mathbb{R}^n$, $n > 1$, and denote by $H_{qi}(x)$ the (q, i) entry of the inverse of the Jacobian matrix of $F(x)$. Moreover, let $\bar{x}$ be a zero of the nonlinear system $F(x) = 0$. It can be easily proved that:

$$\sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l}\frac{\partial f_i(x)}{\partial x_r} = -\sum_{i=1}^{n} H_{ji}(x)\frac{\partial^2 f_i(x)}{\partial x_l \partial x_r}, \tag{22}$$

$$\sum_{i=1}^{n} \frac{\partial^2 H_{ji}(x)}{\partial x_s \partial x_l}\frac{\partial f_i(x)}{\partial x_r} = -\sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l}\frac{\partial^2 f_i(x)}{\partial x_s \partial x_r} - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_s}\frac{\partial^2 f_i(x)}{\partial x_r \partial x_l} - \sum_{i=1}^{n} H_{ji}(x)\frac{\partial^3 f_i(x)}{\partial x_s \partial x_r \partial x_l}, \tag{23}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial^3 H_{ji}(x)}{\partial x_u \partial x_s \partial x_l}\frac{\partial f_i(x)}{\partial x_r} &= -\sum_{i=1}^{n} \frac{\partial^2 H_{ji}(x)}{\partial x_s \partial x_l}\frac{\partial^2 f_i(x)}{\partial x_u \partial x_r} - \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(x)}{\partial x_u \partial x_l}\frac{\partial^2 f_i(x)}{\partial x_s \partial x_r} \\ &\quad - \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(x)}{\partial x_u \partial x_s}\frac{\partial^2 f_i(x)}{\partial x_l \partial x_r} - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l}\frac{\partial^3 f_i(x)}{\partial x_u \partial x_s \partial x_r} \\ &\quad - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_s}\frac{\partial^3 f_i(x)}{\partial x_u \partial x_r \partial x_l} - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_u}\frac{\partial^3 f_i(x)}{\partial x_s \partial x_r \partial x_l} - \sum_{i=1}^{n} H_{ji}(x)\frac{\partial^4 f_i(x)}{\partial x_u \partial x_s \partial x_r \partial x_l}. \end{aligned} \tag{24}$$

The following result, partially proved in [5], will be useful in the proof of the main theorem.

Lemma 1. *Let $\lambda(x)$ be the iteration function of the classical Newton method, whose coordinates are:*

$$\lambda_j(x) = x_j - \sum_{i=1}^{n} H_{ji}(x) f_i(x), \tag{25}$$

for $j = 1, \dots, n$. *Then,*

$$\frac{\partial \lambda_j(\bar{x})}{\partial x_l} = 0, \tag{26}$$

$$\frac{\partial^2 \lambda_j(\bar{x})}{\partial x_r \partial x_l} = \sum_{i=1}^{n} H_{ji}(\bar{x}) \frac{\partial^2 f_i(\bar{x})}{\partial x_r \partial x_l}, \tag{27}$$

$$\frac{\partial^3 \lambda_j(\bar{x})}{\partial x_s \partial x_r \partial x_l} = \sum_{i=1}^{n} \left[\frac{\partial H_{ji}(\bar{x})}{\partial x_r} \frac{\partial^2 f_i(\bar{x})}{\partial x_s \partial x_l} + \frac{\partial H_{ji}(\bar{x})}{\partial x_s} \frac{\partial^2 f_i(\bar{x})}{\partial x_r \partial x_l} + \frac{\partial H_{ji}(\bar{x})}{\partial x_l} \frac{\partial^2 f_i(\bar{x})}{\partial x_s \partial x_r} \right] + 2 \sum_{i=1}^{n} H_{ji}(\bar{x}) \frac{\partial^3 f_i(\bar{x})}{\partial x_s \partial x_r \partial x_l}, \tag{28}$$

and

$$\begin{aligned} \frac{\partial^4 \lambda_j(\bar{x})}{\partial x_u \partial x_s \partial x_r \partial x_l} = {} & \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_s \partial x_l} \frac{\partial^2 f_i(\bar{x})}{\partial x_r \partial x_u} + \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_r \partial x_l} \frac{\partial^2 f_i(\bar{x})}{\partial x_s \partial x_u} + \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_r \partial x_s} \frac{\partial^2 f_i(\bar{x})}{\partial x_l \partial x_u} \\ & + \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_u \partial x_r} \frac{\partial^2 f_i(\bar{x})}{\partial x_s \partial x_l} + \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_u \partial x_l} \frac{\partial^2 f_i(\bar{x})}{\partial x_s \partial x_r} + \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(\bar{x})}{\partial x_u \partial x_s} \frac{\partial^2 f_i(\bar{x})}{\partial x_l \partial x_r} \\ & + 2 \sum_{i=1}^{n} \frac{\partial H_{ji}(\bar{x})}{\partial x_r} \frac{\partial^3 f_i(\bar{x})}{\partial x_u \partial x_s \partial x_l} + 2 \sum_{i=1}^{n} \frac{\partial H_{ji}(\bar{x})}{\partial x_s} \frac{\partial^3 f_i(\bar{x})}{\partial x_u \partial x_r \partial x_l} + 2 \sum_{i=1}^{n} \frac{\partial H_{ji}(\bar{x})}{\partial x_l} \frac{\partial^3 f_i(\bar{x})}{\partial x_u \partial x_s \partial x_r} \\ & + 2 \sum_{i=1}^{n} \frac{\partial H_{ji}(\bar{x})}{\partial x_u} \frac{\partial^3 f_i(\bar{x})}{\partial x_s \partial x_r \partial x_l} + 3 \sum_{i=1}^{n} H_{ji}(\bar{x}) \frac{\partial^4 f_i(\bar{x})}{\partial x_u \partial x_s \partial x_r \partial x_l}, \end{aligned} \tag{29}$$

for $i, j, l, r, s, u \in \{1, 2, \dots, n\}$.

Proof. Let us note that by direct differentiation with respect to x_l, if j and l are arbitrary and fixed,

$$\frac{\partial \lambda_j(x)}{\partial x_l} = \delta_{jl} - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l} f_i(x) - \sum_{i=1}^{n} H_{ji}(x) J_{il}(x), \tag{30}$$

and applying (13):

$$\frac{\partial \lambda_j(x)}{\partial x_l} = - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l} f_i(x). \tag{31}$$

We set now $x = \bar{x}$. Hence,

$$\frac{\partial \lambda_j(\bar{x})}{\partial x_k} = - \sum_{i=1}^{n} \frac{\partial H_{ji}(\bar{x})}{\partial x_l} f_i(\bar{x}) = 0, \tag{32}$$

since $f_i(\bar{x}) = 0$. If the second derivative of $\lambda_j(x)$, with respect to x_r is analyzed for j, l and r arbitrary and fixed:

$$\frac{\partial^2 \lambda_j(x)}{\partial x_r \partial x_l} = - \sum_{i=1}^{n} \frac{\partial^2 H_{ji}(x)}{\partial x_r \partial x_l} f_i(x) - \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l} \frac{\partial f_i(x)}{\partial x_r}. \tag{33}$$

Setting $x = \bar{x}$ in (33) and using (22), we have

$$\frac{\partial^2 \lambda_j(\bar{x})}{\partial x_r \partial x_l} = \sum_{i=1}^{n} H_{ji}(\bar{x}) \frac{\partial^2 f_i(\bar{x})}{\partial x_r \partial x_l}. \tag{34}$$

Again, by direct differentiation on (33) with respect to x_s, being s arbitrary and fixed, and using expression (23), we have:

$$\begin{aligned} \frac{\partial^3 \lambda_j(x)}{\partial x_s \partial x_r \partial x_l} = {} & - \sum_{i=1}^{n} \frac{\partial^3 H_{ji}(x)}{\partial x_s \partial x_r \partial x_l} f_i(x) + \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_l} \frac{\partial^2 f_i(x)}{\partial x_r \partial x_s} \\ & + \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_r} \frac{\partial^2 f_i(x)}{\partial x_s \partial x_l} + \sum_{i=1}^{n} \frac{\partial H_{ji}(x)}{\partial x_s} \frac{\partial^2 f_i(x)}{\partial x_r \partial x_l} + 2 \sum_{i=1}^{n} H_{ji}(x) \frac{\partial^3 f_i(x)}{\partial x_s \partial x_r \partial x_l}. \end{aligned} \tag{35}$$

And, evaluating in $x = \bar{x}$, relation (28) is obtained.

Finally, by direct differentiation on (35) with respect to x_u, with u arbitrary and fixed, and using expression (24), we obtain:

$$\begin{aligned}\frac{\partial^4\lambda_j(x)}{\partial x_u\partial x_s\partial x_r\partial x_l} &= -\sum_{i=1}^{n}\frac{\partial^4 H_{ji}(x)}{\partial x_u\partial x_s\partial x_r\partial x_l}f_i(x)+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_s\partial x_l}\frac{\partial^2 f_i(x)}{\partial x_r\partial x_u}+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_r\partial x_l}\frac{\partial^2 f_i(x)}{\partial x_s\partial x_u}\\ &\quad+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_r\partial x_s}\frac{\partial^2 f_i(x)}{\partial x_u\partial x_l}+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_u\partial x_r}\frac{\partial^2 f_i(x)}{\partial x_s\partial x_l}+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_u\partial x_s}\frac{\partial^2 f_i(x)}{\partial x_r\partial x_l}\\ &\quad+\sum_{i=1}^{n}\frac{\partial^2 H_{ji}(x)}{\partial x_u\partial x_l}\frac{\partial^2 f_i(x)}{\partial x_r\partial x_s}+2\sum_{i=1}^{n}\frac{\partial H_{ji}(x)}{\partial x_r}\frac{\partial^3 f_i(x)}{\partial x_u\partial x_s\partial x_l}+2\sum_{i=1}^{n}\frac{\partial H_{ji}(x)}{\partial x_s}\frac{\partial^3 f_i(x)}{\partial x_u\partial x_r\partial x_l}\\ &\quad+2\sum_{i=1}^{n}\frac{\partial H_{ji}(x)}{\partial x_u}\frac{\partial^3 f_i(x)}{\partial x_s\partial x_r\partial x_l}+2\sum_{i=1}^{n}\frac{\partial H_{ji}(x)}{\partial x_l}\frac{\partial^3 f_i(x)}{\partial x_u\partial x_r\partial x_s}+3\sum_{i=1}^{n}H_{ji}(x)\frac{\partial^4 f_i(x)}{\partial x_u\partial x_s\partial x_r\partial x_l}.\end{aligned}\tag{36}$$

Then, relation (29) is obtained, by evaluating $x=\bar{x}$ in (36). □

Let us note that, by applying Theorem 1 and using expressions (26) and (27) in Lemma 1, it can be concluded that the convergence order of the Newton method is $p=2$.

Lemma 2. *Let $\lambda(x)$ be the iteration function of the classical Newton method. Moreover, let us denote by $N_{ij}(x)$ the (i,j) entry of the matrix $N(x)=J_F(\lambda(x))J_F^{-1}(x)$,*

$$N_{ij}(x)=\sum_{q=1}^{n}J_{iq}(\lambda(x))H_{qj}(x),$$

where $H_{qj}(x)$ is the (q,j) entry of the inverse of the Jacobian matrix. Then,

$$N_{ij}(\bar{x})=\delta_{ij},\tag{37}$$

$$\frac{\partial N_{ij}(\bar{x})}{\partial x_l}=-\sum_{q=1}^{n}H_{qj}(\bar{x})\frac{\partial^2 f_i(\bar{x})}{\partial x_q\partial x_l},\tag{38}$$

and

$$\frac{\partial^2 N_{ij}(\bar{x})}{\partial x_l\partial x_r}=\sum_{q=1}^{n}\sum_{p=1}^{n}H_{qj}(\bar{x})\frac{\partial^2 f_i(\bar{x})}{\partial x_q\partial x_p}\frac{\partial^2 f_i(\bar{x})}{\partial x_l\partial x_r}-\sum_{q=1}^{n}\frac{\partial H_{qj}(\bar{x})}{\partial x_r}\frac{\partial^2 f_i(\bar{x})}{\partial x_q\partial x_l}-\sum_{q=1}^{n}H_{qj}(\bar{x})\frac{\partial^3 f_i(\bar{x})}{\partial x_q\partial x_l\partial x_r},\tag{39}$$

for $i,j,l,r\in\{1,2,\ldots,n\}$.

Proof. As $\lambda(x)$ is a fixed point function, and by definition of matrix $N(x)$

$$N_{ij}(\bar{x})=\sum_{q=1}^{n}J_{iq}(\bar{x})H_{qj}(\bar{x}),$$

and by using (13), expression (37) is obtained.

Moreover, by direct differentiation with respect to x_l, if i and l are arbitrary and fixed

$$\frac{\partial N_{ij}(x)}{\partial x_l}=\sum_{q=1}^{n}\sum_{p=1}^{n}\frac{\partial J_{jq}(\lambda(x))}{\partial\lambda_p(x)}\frac{\partial\lambda_p(x)}{\partial x_l}H_{qi}(x)-\sum_{q=1}^{n}H_{qi}(x)\frac{\partial^2 f_j(x)}{\partial x_q\partial x_l}.\tag{40}$$

We set now $x=\bar{x}$ and apply (26). Hence,

$$\frac{\partial N_{ij}(\bar{x})}{\partial x_l}=-\sum_{q=1}^{n}H_{qi}(\bar{x})\frac{\partial^2 f_j(\bar{x})}{\partial x_q\partial x_l}.$$

Now, we analyze the second derivative of $N_{ij}(x)$ with respect to x_r, for i, l and r arbitrary and fixed:

$$\begin{aligned}\frac{\partial^2 N_{ij}(x)}{\partial x_r\partial x_l} &= \sum_{q=1}^{n}\sum_{p=1}^{n}\frac{\partial J_{iq}(\lambda(x))}{\partial\lambda_p(x)}\frac{\partial\lambda_p(x)}{\partial x_l}\frac{H_{qj}(x)}{\partial x_r}+\sum_{q=1}^{n}\sum_{p=1}^{n}\sum_{a=1}^{n}\frac{\partial^2 J_{iq}(\lambda(x))}{\partial\lambda_p(x)\partial\lambda_a(x)}\frac{\partial\lambda_p(x)}{\partial x_l}\frac{\partial\lambda_a(x)}{\partial x_r}H_{qj}(x)\\ &\quad+\sum_{q=1}^{n}\sum_{p=1}^{n}\frac{\partial J_{iq}(\lambda(x))}{\partial\lambda_p(x)}\frac{\partial^2\lambda_p(x)}{\partial x_l\partial x_r}H_{qj}(x)-\sum_{q=1}^{n}\frac{H_{qj}(x)}{\partial x_r}\frac{\partial^2 f_i(x)}{\partial x_q\partial x_l}-\sum_{q=1}^{n}H_{qj}(x)\frac{\partial^3 f_i(x)}{\partial x_q\partial x_l\partial x_r}.\end{aligned}\tag{41}$$

Then, setting $x=\bar{x}$ in (41) and using (26) and (27), expression (39) is obtained. □

Theorem 2. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{x} \in \mathbb{R}^n$, that is a solution of the system $F(x) = 0$. Let us suppose that $J_F(x)$ is continuous and nonsingular in $\bar{x}$. Then, the sequence $\{x^{(k)}\}_{k\geq 0}(x^{(0)} \in D)$ obtained by using the iterative expressions of methods NAd1, (16), and NAd2, (21), converges to $\bar{x}$ with convergence order 4 and 5, respectively.*

Proof. Let us consider a solution $\bar{x} \in \mathbb{R}^n$ of the nonlinear system $F(x) = 0$ as a fixed point of the iteration function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ described in (16). Let us denote by $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, 2, \dots, n$, the coordinate functions of G.

Expanding $g_i(x)$, $x \in \mathbb{R}^n$, in a Taylor series about $\bar{x}$ yields

$$g_i(x) = g_i(\bar{x}) + \sum_{a_1=1}^{n} \frac{\partial g_i(\bar{x})}{\partial x_{a_1}} e_{a_1} + \frac{1}{2}\sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \frac{\partial^2 g_i(\bar{x})}{\partial x_{a_1}\partial x_{a_2}} e_{a_1}e_{a_2} + \frac{1}{6}\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\sum_{a_3=1}^{n} \frac{\partial^3 g_i(\bar{x})}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} e_{a_1}e_{a_2}e_{a_3} + \cdots \tag{42}$$

where $e_{a_k} = x_{a_k} - \bar{x}_{a_k}$, $1 \leq a_1, \dots, a_k \leq n$.

We denote by $M_{ij}(x)$ the (i, j) entry of the matrix

$$M(x) = J_F^{-1}(x) J_F(\lambda(x)) J_F^{-1}(x)$$

where $\lambda(x) = x - H(x)F(x)$, by $H_{ij}(x)$ the (i, j) entry of $H(x) = J_F^{-1}(x)$ and by $N_{ii}(x)$ the (i, i) entry of $N(x) = J_F(x)M(x) = J_F(\lambda(x))J_F^{-1}(x)$. Thus, the jth component of the iteration function corresponding to method NAd1 is

$$g_j(x) = \lambda_j(x) - 2\sum_{i=1}^{n} H_{ji}(x) f_i(\lambda(x)) + \sum_{i=1}^{n} M_{ji}(x) f_i(\lambda(x)). \tag{43}$$

Since $H_{ji}(x)$ and $J_{ij}(x)$ are the elements of inverse matrices, (43) can be rewritten as

$$\sum_{j=1}^{n} J_{ij}(x)\left(g_j(x) - \lambda_j(x)\right) + 2f_i(\lambda(x)) - \sum_{j=1}^{n} N_{ij} f_j(\lambda(x)) = 0. \tag{44}$$

Now, by direct differentiation of (44) with respect to x_l, being j and l arbitrary and fixed,

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_l}\left(g_j(x) - \lambda_j(x)\right) + \sum_{j=1}^{n} J_{ij}(x)\left(\frac{\partial g_j(x)}{\partial x_l} - \frac{\partial \lambda_j(x)}{\partial x_l}\right) \\
&\quad + 2\sum_{q=1}^{n} \frac{\partial f_i(\lambda(x))}{\partial \lambda_q(x)} \frac{\partial \lambda_q(x)}{\partial x_l} - \sum_{j=1}^{n} \frac{\partial N_{ij}(x)}{\partial x_l} f_j(\lambda(x)) - \sum_{j=1}^{n} N_{ij}(x)\left(\sum_{q=1}^{n} \frac{\partial f_j(\lambda(x))}{\partial \lambda_q(x)} \frac{\partial \lambda_q(x)}{\partial x_l}\right) = 0.
\end{aligned} \tag{45}$$

When $x = \bar{x}$, by applying Lemma 1, expression (26), and taking into account that $g(\bar{x}) = \bar{x}$, $\lambda(\bar{x}) = \bar{x}$ and $f_i(\bar{x}) = 0$, we have

$$\sum_{j=1}^{n} \frac{\partial J_{ij}(\bar{x})}{\partial x_l}\left(\bar{x}_j - \bar{x}_j\right) + \sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial g_j(\bar{x})}{\partial x_l} = 0.$$

So,

$$\sum_{j=1}^{n} J_{ij}(\bar{x}) \frac{\partial g_j(\bar{x})}{\partial x_l} = 0.$$

Then, as $J_F(\bar{x})$ is supposed to be nonsingular, and j and l are arbitrary,

$$\frac{\partial g_j(\bar{x})}{\partial x_l} = 0. \tag{46}$$

By direct differentiation of (45) with respect to x_r, being r arbitrary and fixed,

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial^2 J_{ij}(x)}{\partial x_r \partial x_l}\left(g_j(x) - \lambda_j(x)\right) + \sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_l}\left(\frac{\partial g_j(x)}{\partial x_r} - \frac{\partial \lambda_j(x)}{\partial x_r}\right) \\
&\quad + \sum_{j=1}^{n} \frac{\partial J_{ij}(x)}{\partial x_r}\left(\frac{\partial g_j(x)}{\partial x_l} - \frac{\partial \lambda_j(x)}{\partial x_l}\right) + \sum_{j=1}^{n} J_{ij}(x)\left(\frac{\partial^2 g_j(x)}{\partial x_r \partial x_l} - \frac{\partial^2 \lambda_j(x)}{\partial x_r \partial x_l}\right) \\
&\quad + 2\sum_{q=1}^{n}\sum_{p=1}^{n} \frac{\partial^2 f_i(x)}{\partial \lambda_p(x)\partial \lambda_q(x)} \frac{\partial \lambda_p(x)}{\partial x_r} \frac{\partial \lambda_q(x)}{\partial x_l} + 2\sum_{q=1}^{n} \frac{\partial f_i(x)}{\partial \lambda_q(x)} \frac{\partial^2 \lambda_q(x)}{\partial x_r \partial x_l} - \sum_{j=1}^{n} \frac{\partial^2 N_{ij}(x)}{\partial x_r \partial x_l} f_j(\lambda(x))
\end{aligned}$$

$$-\sum_{j=1}^{n}\frac{\partial N_{ij}(x)}{\partial x_l}\left(\sum_{q=1}^{n}\frac{\partial f_j(x)}{\partial\lambda_q(x)}\frac{\partial\lambda_q(x)}{\partial x_r}\right)-\sum_{j=1}^{n}\frac{\partial N_{ij}(x)}{\partial x_r}\left(\sum_{q=1}^{n}\frac{\partial f_j(x)}{\partial\lambda_q(x)}\frac{\partial\lambda_q(x)}{\partial x_l}\right)$$
$$-\sum_{j=1}^{n}N_{ij}(x)\left(\sum_{q=1}^{n}\sum_{p=1}^{n}\frac{\partial^2 f_j(x)}{\partial\lambda_p(x)\partial\lambda_q(x)}\frac{\partial\lambda_p(x)}{\partial x_r}\frac{\partial\lambda_q(x)}{\partial x_l}\right)-\sum_{j=1}^{n}N_{ij}(x)\left(\sum_{q=1}^{n}\frac{\partial f_j(x)}{\partial\lambda_q(x)}\frac{\partial^2\lambda_q(x)}{\partial x_r\partial x_l}\right)=0. \quad (47)$$

Let us substitute $x=\bar{x}$ and apply (26) and (27) from Lemma 1, (37) and (38) from Lemma 2. Then,

$$\sum_{j=1}^{n}J_{ij}(\bar{x})\frac{\partial^2 g_j(\bar{x})}{\partial x_r\partial x_l}-\sum_{j=1}^{n}\sum_{i=1}^{n}J_{ij}(\bar{x})H_{ji}(\bar{x})\frac{\partial^2 f_i(\bar{x})}{\partial x_r\partial x_l}$$
$$+2\sum_{q=1}^{n}J_{iq}(\bar{x})H_{qi}(\bar{x})\frac{\partial^2 f_i(\bar{x})}{\partial x_r\partial x_l}-\sum_{j=1}^{n}\delta_{ij}\sum_{i=1}^{n}\sum_{q=1}^{n}J_{iq}(\bar{x})H_{qi}(\bar{x})\frac{\partial^2 f_i(\bar{x})}{\partial x_r\partial x_l}=0. \quad (48)$$

Therefore,

$$\sum_{j=1}^{n}J_{ij}(\bar{x})\frac{\partial^2 g_j(\bar{x})}{\partial x_r\partial x_l}=0.$$

Again, as $J_F(\bar{x})$ is nonsingular, and j, l and r are arbitrary,

$$\frac{\partial^2 g_j(\bar{x})}{\partial x_r\partial x_l}=0. \quad (49)$$

We now analyze the fourth order of convergence. To get this aim, it is necessary to differentiate (47) with respect to x_s, being s arbitrary and fixed, and evaluate the resulting expression in $x=\bar{x}$. Then, by using (13), (22), (26)–(28) from Lemma 1, (37) and (38) from Lemma 2, and simplifying, it is proved that:

$$\frac{\partial^3 g_j(\bar{x})}{\partial x_s\partial x_r\partial x_l}=0. \quad (50)$$

Again, being u arbitrary and fixed and using results from (22) to (24), Lemma 1 (expressions from (26) to (29)) and Lemma 2 (expressions from (37) to (39)), it can be proved that:

$$J_{ij}(\bar{x})\frac{\partial^4 g_j(\bar{x})}{\partial x_u\partial x_s\partial x_r\partial x_l}+P(\bar{x})=0, \quad (51)$$

where $P(\bar{x})$ is a linear combination of partial derivatives of f_i of second order, evaluated in $\bar{x}$.

So, by (42) and Theorem 1 we conclude that the method NAd1 of iterative expression (16) converges to $\bar{x}$ with convergence order 4. The fifth-order convergence of method NAd2 can be proved in an analogous way. □

Let us remark that the efficiency index of the method NAd1 is $I_{NAd1}=4^{\frac{1}{2n+2n^2}}$, which coincides with the efficiency index of the Newton method, and $I_{NAd2}=5^{\frac{1}{3n+2n^2}}$ for NAd2. In Fig. 1 we show the efficiency indices of NAd2 and the Newton method, for different values of the size of the nonlinear system. We note that the NAd2 method has, for $n>1$, the highest efficiency index. For higher values of n, the efficiency indices hold this tendency, although the difference between them is small.

4. Numerical examples

In this section we give numerical examples and compare the effectiveness of the iterative methods obtained. In particular, the new methods NAd1 and NAd2 are analyzed, and also the Traub method (TM) and the classical Newton method, (CN) in order to estimate the zeros of several nonlinear functions.

(a) $F(x_1,x_2)=(\exp(x_1^2)-\exp(\sqrt{2}x_1),x_1-x_2)$, $\bar{x}_1=(\sqrt{2},\sqrt{2})^{\mathrm{T}}$, $\bar{x}_2=(0,0)^{\mathrm{T}}$.
(b) $F(x_1,x_2)=(x_1+\exp(x_2)-\cos(x_2),3x_1-x_2-\sin(x_2))$, $\bar{x}=(0,0)^{\mathrm{T}}$.
(c) $F(x_1,x_2)=(x_1^2-2x_1-x_2+0.5,x_1^2+4x_2^2-4)$, $\bar{x}=(1.9007,0.3112)^{\mathrm{T}}$.
(d) $F(x_1,x_2)=(x_1^2+x_2^2-1,x_1^2-x_2^2+0.5)$, $\bar{x}_1=(\frac{1}{2},\frac{\sqrt{3}}{2})^{\mathrm{T}}$, $\bar{x}_2=(-\frac{1}{2},-\frac{\sqrt{3}}{2})^{\mathrm{T}}$.
(e) $F(x_1,x_2)=(\sin(x_1)+x_2\cos(x_1),x_1-x_2)$, $\bar{x}=(0,0)^{\mathrm{T}}$.

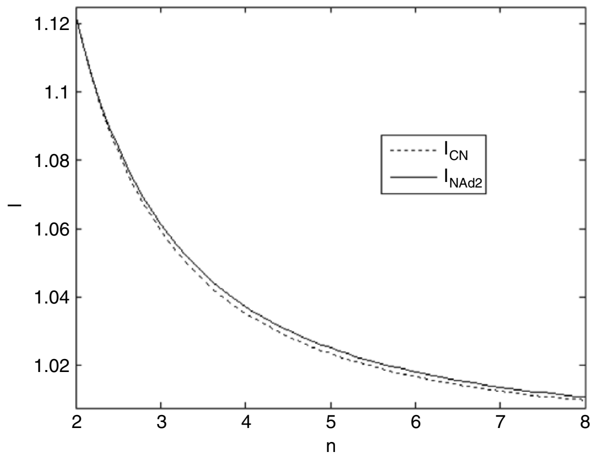

Fig. 1. Efficiency indices for $n > 1$.

Table 1
Numerical results for nonlinear systems.

$F(x)$	$x^{(0)}$	Iterations				p				Sol.
		CN	Tr	NAd1	NAd2	CN	Tr	NAd1	NAd2	
(a)	$(2.3, 2.3)^T$	10	8	7	6	2.0	3.0	3.9	3.8	$\bar{x}_1$
	$(1.8, 1.8)^T$	7	5	5	4	2.0	3.0	3.6	4.2	$\bar{x}_1$
	$(0.8, 0.8)^T$	5	4	3	3	3.0	4.3	4.6	6.7	$\bar{x}_2$
(b)	$(1.5, 2)^T$	7	6	5	4	2.0	3.0	3.6	4.6	$\bar{x}$
	$(0.3, 0.5)^T$	5	4	4	3	2.0	3.0	3.7	4.6	$\bar{x}$
(c)	$(3, 2)^T$	7	5	5	4	2.0	2.6	2.5	3.1	$\bar{x}$
	$(1.6, 0)^T$	5	4	4	4	2.1	3.8	5.0	5.3	$\bar{x}$
(d)	$(0.7, 1.2)^T$	5	4	3	3	2.0	2.5	3.7	4.7	$\bar{x}_1$
	$(-1, -2)^T$	6	4	4	4	2.0	2.9	3.0	3.7	$\bar{x}_2$
(e)	$(1.2, -1.5)^T$	6	4	4	3	2.9	3.7	5.5	7.5	$\bar{x}$
	$(-0.6, 0.6)^T$	5	3	3	3	3.0	4.3	6.4	6.6	$\bar{x}$
(f)	$(2, \dots, 2)^T$	6	5	4	4	2.0	3.0	3.3	4.1	$\bar{x}_1$
	$(-4, \dots, -4)^T$	7	6	5	4	2.0	3.0	3.7	4.7	$\bar{x}_2$
(g)	$(-1, -1, -1, -1)^T$	6	5	4	4	2.0	3.4	4.4	5.5	$\bar{x}_1$
	$(2, 2, 2, 0)^T$	7	5	5	4	2.1	3.2	4.1	4.9	$\bar{x}_2$

(f) $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$, where $x = (x_1, x_2, \dots, x_n)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, n$, such that

$$f_i(x) = x_i x_{i+1} - 1, \quad i = 1, 2, \dots, n-1$$
$$f_n(x) = x_n x_1 - 1.$$

When n is odd, the exact zeros of $F(x)$ are $\bar{x}_1 = (1, 1, \dots, 1)$ and $\bar{x}_2 = (-1, -1, \dots, -1)$. Results appearing in Table 1 are obtained for $n = 99$.

(g) $F(x) = (f_1(x), f_2(x), f_3(x), f_4(x))$, where $x = (x_1, x_2, x_3, x_4)^T$ and $f_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 1, 2, \dots, 4$, such that

$$f_1(x) = x_2x_3 + x_4(x_2 + x_3)$$
$$f_2(x) = x_1x_3 + x_4(x_1 + x_3)$$
$$f_3(x) = x_1x_2 + x_4(x_1 + x_2)$$
$$f_4(x) = x_1x_2 + x_1x_3 + x_2x_3 - 1.$$

The zeros of $F(x)$ are $\bar{x}_1 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)^T$ and $\bar{x}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)^T$.

Numerical computations have been carried out in MATLAB (MATrix LABoratory), all with a common outline: every iterate $x^{(k+1)}$ is obtained from the previous one, $x^{(k)}$, by adding one or more terms of the form $A^{-1}b$ where $x^{(k)} \in \mathbb{R}^n$, A is a real $n \times n$ matrix and $b \in \mathbb{R}^n$. The matrix A and the vector b are different according to the method used, but in any case the inverse calculation $-A^{-1}b$ is carried out solving the linear system $Ay = -b$, using Gaussian elimination with partial pivoting.

The stopping criterion used is $\left\|x^{(k+1)} - x^{(k)}\right\| + \left\|F\left(x^{(k)}\right)\right\| < 10^{-12}$. Therefore, we check that iterates converge to a limit and moreover that this limit is a solution of the system of nonlinear equations. For every method, we analyze the number of iterations needed to converge to the solution and the order of convergence will be estimated by means of the computational order of convergence p, defined in (5). The value of p that appears in Table 1 is the last coordinate of vector p when the variation between its coordinates is small.

In Table 1 it can be observed that several results obtained using the previously described methods in order to estimate the zeros of functions from (a) to (g). For every function, the following items are specified: the initial estimation $x^{(0)}$ and, for each method, the approximate solution found, the number of iterations needed and the estimated computational order of convergence p.

In practice, it is observed that in the case of $\frac{\partial^2 f_i(\alpha)}{\partial x_a \partial x_b} = 0$ for all i, a, b (cases (a) and (e)), the convergence of the classical Newton method is of order $p = 3$, as well as the Traub, NAd1 and NAd2 methods obtain a computational order near to 5, 6 and 7, respectively. If this condition is not verified, the computational order of convergence of the modified methods is about 3, 4 and 5, respectively (as it has been proved theoretically). Finally it can be concluded that, in general, the new modified methods show a good stability and less iterations than the classical Newton and Traub methods, as well as higher-order convergence. Moreover, it can be observed in case (f) that the efficiency of the methods remains although the system of equations is big-sized.

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