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A composite third order Newton–Steffensen method for solving nonlinear equations

J.R. Sharma

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, 148 106 District Sangrur, India

Abstract

In this paper, we suggest a third-order method formed by the composition of Newton and Steffensen methods for finding simple and real roots of a nonlinear equation in single variable. Per iteration the formula requires two evaluations of the function and single evaluation of the derivative. Experiments show that the method is suitable in the cases where Newton and Steffensen methods fail.

Keywords: Nonlinear equations; Rootfinding; Newton's method; Steffensen's method

1. Introduction

One of the most important occurring problems in scientific work is to locate a real root of a nonlinear equation

$$f(x) = 0. \tag{1}$$

E-mail address: jrshira@yahoo.co.in

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Analytic methods for solving such equations rarely exit, and therefore, one can hope to obtain only approximate solutions by relying on iteration methods. For a survey of the most important algorithms, some excellent textbooks are available (see, [2,3,5]). Being quadratically convergent, Newton's method (NM) is probably the best known and most widely used algorithm. Time to time the method has been derived and modified in a variety of ways. One such method derived from Newton's method by approximating the derivative with non-derivative term of difference quotient is Steffensen's (SM) method [4,6]. The method requires two evaluations of function and is quadratically convergent.

In this paper, we construct a third order method using variation in the difference quotient of Steffensen iteration formula. The method requires the information of two functions and one derivative term. For reason that becomes clear in Section 2, the method is called Newton–Steffensen method. Here, the discussion is carried out for simple and real roots.

2. The method and its convergence

The Steffensen's method for obtaining a simple root of equation (1) uses the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad n = 0, 1, 2 \ldots, \tag{2}$$

where $g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$, is the approximation to the derivative $f'(x_n)$ and is called difference quotient.

The method (2) is quadratically convergent if $f'(x) \neq 0$ i.e. $f(x_n + f(x_n)) \neq f(x_n)$ in the neighborhood of the root.

Based on the above method (2), we consider the iteration scheme of the type

$$x_{n+1} = x_n - \frac{f(x_n)}{h(x_n)}, \quad n = 0, 1, 2 \ldots, \tag{3}$$

where $h(x_n) = \frac{f(x_n + a(x_n)f(x_n)) - f(x_n)}{a(x_n)f(x_n)}$, is the new quotient for the derivative.

Here $a(x)$ is some function of x to be determined. We shall now prove the following theorem:

Theorem 1. *Assume that the function is sufficiently smooth in the neighborhood of the root (say, α), and $f'(\alpha) \neq 0$. Let $f''(x)$ be continuous in this small neighborhood, then formula (3) is cubically convergent to α if $a(x) = -1/f'(x)$.*

Proof. Let e_n be error at nth iteration, then

$$e_n = x_n - \alpha. \tag{4}$$

Using (4) in formula (3), we have

$$e_{n+1} = e_n - \frac{a(e_n+\alpha)f^2(e_n+\alpha)}{f(e_n+\alpha+a(e_n+\alpha)f(e_n+\alpha)) - f(e_n+\alpha)}. \tag{5}$$

By Taylor's expansion around $x = \alpha$, relation (5) after some simplifications gives

$$\begin{aligned} e_{n+1} = \Big[& e_n^2 a(\alpha)f'(\alpha)f''(\alpha)(1+a(\alpha)f'(\alpha)) \\ & + e_n^3 \Big\{ (1+a(\alpha)f'(\alpha))(a(\alpha)f''(\alpha) + 2a'(\alpha)f'(\alpha)) \\ & - a'(\alpha)f'(\alpha)f''(\alpha) + \frac{1}{3}(1+a(\alpha)f'(\alpha))^3 f'''(\alpha) - \frac{1}{3}(1+a(\alpha)f'(\alpha))f'''(\alpha) \\ & - \frac{1}{2}a(\alpha)(f''(\alpha))^2 \Big\} + o(e_n^4) \Big] \cdot \Big[2a(\alpha)(f'(\alpha))^2 + e_n \Big\{ a(\alpha)f'(\alpha)f''(\alpha) \\ & + 2a'(\alpha)(f'(\alpha))^2 - f''(\alpha) + (1+a(\alpha)f'(\alpha))^2 f''(\alpha) \Big\} + o(e_n^2) \Big]^{-1}. \end{aligned} \tag{6}$$

If the order is three, we have

$$1 + a(\alpha)f'(\alpha) = 0,$$

which implies

$$a(\alpha) = -\frac{1}{f'(\alpha)}. \tag{7}$$

Using (7) in (6), we get the error equation as

$$e_{n+1} = e_n^3 \left[\frac{f''(\alpha)}{2f'(\alpha)} \right]^2 + o(e_n^4). \tag{8}$$

Thus, replacing α by x in (7) we can achieve third order convergence, if $a(x) = -\frac{1}{f'(x)}$. This completes the proof. $\square$

The form of the iteration formula (3) is, therefore, given by

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)(f(x_n) - f(x_n^*))}, \tag{9}$$

where $x_n^* = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

In the formula (9) derivative f' is not replaced by any free derivative term as SM formula (2) does. The drawback of SM is that in the problems where $f(x)$ is large, the sampling function $f(x + f(x))$ becomes very large which causes to halt the iteration process and method fails to give the required root. So, it is not unreasonable to keep $f'(x)$ in play in (9). It is quite obvious that for each iteration one has to evaluate the function twice and the derivative once in iterative scheme (9). In the absence of second derivative the method may be cheaper

Table 1
Numerical Examples

$f(x)$	x_0	Root (α)	Iteration (n) by		
			NSM	NM	SM
$\tan^{-1}(x)$	2	0.00000000000000	4	Failure	Failure
$\sin(x) - x/2$	2	1.89549426703398	4	Not converges to required root	4
$10x\exp(-x^2) - 1$	1	1.67963061042845	3	5	Failure
$x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$	15	15.98287398060170	4	7	Failure
$x\log 10(x) - 1.2$	2	2.74064609597369	3	5	5

than any other method of third-order requiring the use of second derivative (see, [1]) if derivative of this order becomes lengthy to evaluate. Note that due to condition (7) we have come across the Newton iteration i.e. x_n^* in the formula. Thus, the formula (9) is the composition of NM and SM that suggests calling it Newton–Steffensen method (NSM). The sufficient condition for convergence in the neighborhood of the root is given by

$$\left|[f'(x)(f(x) - f(x^*)) - f(x)f'(x)]^2 + f^2(x)\left[f''(x)(f(x) - f(x^*)) - \frac{f(x)\bar{f}(x^*)f''(x)}{f'(x)}\right]\right| < |f'(x)(f(x) - f(x^*))|^2,$$

where $\bar{f}(x^*) = \frac{\mathrm{d}f(x^*)}{\mathrm{d}x^*}$ and x^* is same as in formula (9).

3. Numerical results

The performance of the present method NSM with NM and SM is compared. The results are summarized in Table 1.

The computations are carried out with double arithmetic precision. It can be easily seen that less number of iterations are used by NSM due to its higher order. Also, from numerical results it is very clear that in the problems where NM and SM fail, the NSM converges to root very effectively.

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