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On the improvement of the order of convergence of iterative methods for solving nonlinear systems by means of memory[☆]

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A B S T R A C T

Iterative methods with memory for solving nonlinear systems have been designed. For approximating the accelerating parameters the Kurchatov's divided difference is used as an approximation of the derivative of second order. The convergence of the proposed schemes is analyzed by means of Taylor expansions. Numerical examples are shown to compare the performance of the proposed schemes with other known ones, confirming the theoretical results.

1. Introduction

Finding a solution $\alpha \in \mathbb{R}^n$ of a system of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a sufficiently Fréchet differentiable function in an open convex set D, is a difficult problem arising in many scientific applications. The use of fixed point iterative methods is a widely used technique in order to approximate the solution of these classical problems.

One of the most used iterative schemes is Newton's method, with quadratic order of convergence and iterative expression

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

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being $F'(x^{(k)})$ the Jacobian matrix of F at the kth iteration. As each iteration of Newton's method only computes a Jacobian matrix and a functional evaluation, its computational cost is low in relation to other known iterative schemes.

Based on the iterative structure (1), many schemes have been proposed in the literature with the aim of improving the features of Newton's method. The composition, inclusion of accelerating parameters, weight functions, etc. are some of the most efficient procedures in order to design new methods with higher order of convergence, efficiency and stability. However, as there are many nonlinear functions non-differentiable, the use of derivative-free methods for solving nonlinear systems is required for these problems.

For designing derivative-free methods for solving nonlinear systems is usual to work with the divided difference operator $[\cdot,\cdot;F]:\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{L}(\mathbb{R}^n)$, such that $[x,y;F](x-y)=F(x)-F(y)$, where $\mathcal{L}(\mathbb{R}^n)$ denotes the linear mappings of $\mathbb{R}^n$.

By replacing the Jacobian matrix in the Newton's scheme by this operator, we obtain the Traub–Steffensen's family of iterative methods [1] with iterative structure

$$x^{(k+1)}=x^{(k)}-[w^{(k)},x^{(k)};F]^{-1}F(x^{(k)}),\qquad k=0,1,\ldots, \tag{2}$$

where $w^{(k)}=x^{(k)}+bF(x^{(k)})$, and b is a nonzero arbitrary parameter. Let us remark that for $b=1$ the iterative expression (2) is the well known Steffensen's method for systems, introduced by Samanski in [2].

Theorem 1. *Let $F:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D. Let us suppose that $F'(x)$ is continuous and nonsingular in $\alpha\in D$, a solution of $F(x)=0$. When the initial estimation $x^{(0)}\in\mathbb{R}^n$ is close enough to α, the family of iterative methods given by (2) converges to α with order of convergence 2 for any value of parameter b. The error equation of the family is:*

$$e^{(k+1)}=C_2(I+bF'(\alpha))(e^{(k)})^2+\mathcal{O}((e^{(k)})^3), \tag{3}$$

where $e^{(k)}=x^{(k)}-\alpha$, $k=1,2,\ldots$, $C_j=\frac{1}{j!}[F'(\alpha)]^{-1}F^{(j)}(\alpha)$, $j\geq 2$, and I denotes the identity matrix of size $n\times n$.

By means of the inclusion of accelerating parameters in the iterative expression of a derivative-free iterative method, we are going to design iterative schemes with memory for solving nonlinear systems. This type of fixed point methods has as general iterative expression $x^{(k+1)}=G(x^{(k)},x^{(k-1)},\ldots)$, and allows us to improve the order of convergence without needing new functional evaluations per iteration. In the literature, there are very few published articles that address the design of such methods. We can see, for example, the paper of Petkovic et al. [3], the work of Narang et al. [4] and the paper of Cordero et al. [5].

Along this manuscript we are going to use the notation introduced by the authors in [6], which allows us to prove the convergence results in an easy way, by using the Taylor expansion of the different elements that appear in the iterative expression. In order to obtain the Taylor series expansion of the divided difference operator, we use the Genocchi–Hermite formula [7]

$$[x+h,x;F]=\int_0^1 F'(x+th)dt,\quad \forall x,h\in\mathbb{R}^n, \tag{4}$$

so,

$$[x+h,x;F]=\int_0^1 F'(x+th)dt=F'(x)+\frac{1}{2}F''(x)h+\frac{1}{6}F'''(x)h^2+\mathcal{O}(h^3). \tag{5}$$

This work is organized as follows. Section 2 is devoted to the design of the proposed iterative families and the resulting schemes after the inclusion of more than one previous iterates on the iterative expressions. A deep analysis of the order of convergence of the designed methods is also given. Section 3 covers the numerical results for solving some test systems of nonlinear equations, including a comparison with other existing iterative methods with similar iterative structure. Finally, Section 4 collects the main conclusions of this paper.

2. Design and convergence of new methods with memory

From error equation (3) we could set $b = -[F'(\alpha)]^{-1}$, resulting in an iterative scheme with order 3. However, the value of α is unknown, so the improvement of the order of convergence of family (2) must be done in another way. One of the most efficient options, is to approximate the value of $F'(\alpha)$ without increasing the number of new functional evaluations.

In this work, the approximation of $F'(\alpha)$ is done using Kurchatov's divided difference operator $[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]$. Then we set $b := B^{(k)}$, where $B^{(k)}$ is a matrix given by:

$$B^{(k)} = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} \approx -[F'(\alpha)]^{-1}. \tag{6}$$

So, the resulting scheme is a method with memory, denoted by FM3, with iterative expression:

$$\begin{aligned} w^{(k)} &= x^{(k)} - \left[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F\right]^{-1} F(x^{(k)}), \\ x^{(k+1)} &= x^{(k)} - \left[w^{(k)}, x^{(k)}; F\right]^{-1} F(x^{(k)}), \quad k = 1, 2, \ldots \end{aligned}$$

Let us note that (6) uses functional evaluations already done by the method, so no additional functional evaluations have been added.

Theorem 2. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and let us denote by $\alpha \in D$ a solution of $F(x) = 0$, such that $F'(x)$ is continuous and nonsingular in α. Then, if $x^{(0)}$ and $x^{(1)}$ are close enough to α, the order of convergence of method FM3 is* 3.

Proof. We denote the error in each iteration of method FM3 by $e^{(k)} = x^{(k)} - \alpha$, for all k. By using the Taylor developments of $F(x^{(k-1)})$ and its derivatives around α, we get the corresponding one of Kurchatov's divided difference operator:

$$\begin{aligned} [2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F] &= F'(x^{(k-1)}) + \frac{1}{2} F''(x^{(k-1)})(2(e^{(k)} - e^{(k-1)})) \\ &\quad + \frac{1}{6} F'''(x^{(k-1)})(2(e^{(k)} - e^{(k-1)}))^2 + \mathcal{O}_3(e^{(k-1)}, e^{(k)}) \\ &= F'(\alpha)[I + 2C_2 e^{(k)} - 2C_3 e^{(k-1)} e^{(k)} + C_3 (e^{(k-1)})^2 \\ &\quad + 4C_3 (e^{(k)})^2] + \mathcal{O}_3(e^{(k-1)}, e^{(k)}), \end{aligned}$$

where $\mathcal{O}_3(e^{(k-1)}, e^{(k)})$ denotes that the sum of the exponents of $e^{(k)}$ and $e^{(k-1)}$ is at least 3.

Then we obtain:

$$\begin{aligned} [2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} &= [I - 2C_2 e^{(k)} + 2C_3 e^{(k-1)} e^{(k)} - C_3 (e^{(k-1)})^2 + 4(C_2^2 - C_3)(e^{(k)})^2][F'(\alpha)]^{-1} \\ &\quad + \mathcal{O}_3(e^{(k-1)}, e^{(k)}). \end{aligned} \tag{7}$$

Being $b = B^{(k)}$ defined in (6), from (7) we have

$$I + B^{(k)} F'(\alpha) = 2C_2 e^{(k)} - 2C_3 e^{(k-1)} e^{(k)} + C_3 (e^{(k-1)})^2 - 4(C_2^2 - C_3)(e^{(k)})^2 + \mathcal{O}_3(e^{(k-1)}, e^{(k)}) \sim e^{(k)}. \tag{8}$$

On the other hand, from the error equation (3) we have $e^{(k+1)} \sim (I + B^{(k)} F'(\alpha))(e^{(k)})^2$, and by using (8), we obtain:

$$e^{(k+1)} \sim e^{(k)} (e^{(k)})^2 = (e^{(k)})^3.$$

The previous relation shows that method FM3 has order of convergence 3. □

2.1. Extension to higher-order methods with memory

Based on the iterative scheme (2), in this section we develop a new method with memory with order of convergence five. Previously, we analyze the resulting two-step family obtained from the composition of structure (2). This composition gives the following derivative-free family of iterative methods with order of convergence 3:

$$\begin{array}{rcl} y^{(k)} &=& x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\ x^{(k+1)} &=& y^{(k)} - [w^{(k)}, y^{(k)}; F]^{-1} F(y^{(k)}) \end{array}, \qquad k = 0, 1, \ldots \tag{9}$$

Theorem 3. *Let us consider a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ sufficiently differentiable in an open convex set D such that $F'(x)$ is continuous and nonsingular in $\alpha \in D$. If the initial estimation $x^{(0)} \in \mathbb{R}^n$ is close enough to α, family (9) has order of convergence 3 for any value of b, with the following error equation:*

$$e^{(k+1)} = C_2(I + bF'(\alpha))C_2(I + bF'(\alpha))(e^{(k)})^3 + \mathcal{O}((e^{(k)})^4), \tag{10}$$

where $e^{(k)} = x^{(k)} - \alpha$, $k = 1, 2, \ldots$, $C_j = \frac{1}{j!}[F'(\alpha)]^{-1} F^{(j)}(\alpha)$, $j \geq 2$.

Proof. In a similar way as in previous results,

$$\begin{aligned} [w^{(k)}, x^{(k)}; F] &= F'(\alpha)[F'(x^{(k)}) + \frac{1}{2}F''(x^{(k)})(w^{(k)} - x^{(k)}) + \frac{1}{6}F'''(x^{(k)})(w^{(k)} - x^{(k)})^2] + \mathcal{O}((e^{(k)})^4) \\ &= F'(\alpha)[I + (2C_2 + bC_2F'(\alpha))e^{(k)} + (3C_3 + bC_2F'(\alpha)C_2 + 3bC_3F'(\alpha) + b^2C_3F'(\alpha)^2)(e^{(k)})^2] \\ &\quad + (bC_2F'(\alpha)C_3 + 3bC_3F'(\alpha)C_2 + 6bC_4F'(\alpha) + b^2C_3F'(\alpha)^2C_2 \\ &\quad + b^2C_3F'(\alpha)C_2F'(\alpha) + 4b^2C_4F'(\alpha)^2 \\ &\quad + 4C_4)(e^{(k)})^3 + \mathcal{O}((e^{(k)})^4). \end{aligned}$$

Then, the inverse of the divided difference operator is given by an expression of the form:

$$[w^{(k)}, x^{(k)}; F]^{-1} = [X_1 + X_2e^{(k)} + X_3(e^{(k)})^2] + \mathcal{O}((e^{(k)})^3).$$

The coefficients X_i are obtained from the equality $[w^{(k)}, x^{(k)}; F]^{-1}[w^{(k)}, x^{(k)}; F] = I$, so we have $X_1 = I$ and

$$\begin{aligned} X_2 &= -2C_2 - bC_2F'(\alpha), \\ X_3 &= -3C_3 + bC_2F'(\alpha)C_2 + 2bC_2^2F'(\alpha) - 3bC_3F'(\alpha) - b^2C_3F'(\alpha)^2 + b^2C_2F'(\alpha)C_2F'(\alpha) + 4C_2^2. \end{aligned}$$

Then, we have for the first step of family (9):

$$\begin{aligned} y^{(k)} - \alpha &= e^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}) \\ &= C_2(I + bF'(\alpha))(e^{(k)})^2 + (2C_3 - 2C_2^2 - 2bC_2^2F'(\alpha) + 3bC_3F'(\alpha) + b^2C_3F'(\alpha)^2 \\ &\quad - b^2C_2F'(\alpha)C_2F'(\alpha))(e^{(k)})^3 + \mathcal{O}((e^{(k)})^4). \end{aligned}$$

By Developing $F(y^{(k)})$ and its derivatives around α, we have

$$\begin{aligned} [w^{(k)}, y^{(k)}; F] &= F'(y^{(k)}) + \frac{1}{2}F''(y^{(k)})(w^{(k)} - y^{(k)}) + \frac{1}{6}F'''(y^{(k)})(w^{(k)} - y^{(k)})^2 + \mathcal{O}((e^{(k)})^3) \\ &= F'(\alpha)[I + C_2(I + bF'(\alpha))e^{(k)} + (2C_2^2(I + bF'(\alpha)) + C_2(bF'(\alpha)C_2 - C_2 - bC_2F'(\alpha)) \\ &\quad + C_3(I + bF'(\alpha))^2)(e^{(k)})^2] + \mathcal{O}((e^{(k)})^3). \end{aligned}$$

By solving $[w^{(k)}, y^{(k)}; F]^{-1}[w^{(k)}, y^{(k)}; F] = I$, we obtain:

$$\begin{aligned}[w^{(k)}, y^{(k)}; F]^{-1} = {} & [I - C_2(I + bF'(\alpha))e^{(k)} + (-2C_2^2(I + bF'(\alpha)) - C_2(bF'(\alpha)C_2 - C_2 \\ & - bC_2F'(\alpha)) - C_3(I + bF'(\alpha))^2 \\ & + C_2(I + bF'(\alpha))C_2(I + bF'(\alpha)))(e^{(k)})^2][F'(\alpha)]^{-1} + \mathcal{O}((e^{(k)})^3).\end{aligned}$$

Then, the error equation of family (9) is:

$$e^{(k+1)} = y^{(k)} - \alpha - [w^{(k)}, y^{(k)}; F]^{-1}F(y^{(k)}) = C_2(I + bF'(\alpha))C_2(I + bF'(\alpha))(e^{(k)})^3 + \mathcal{O}((e^{(k)})^4),$$

and the proof is finished. □

Let us observe that factor $I+bF'(\alpha)$ appears again in this error equation. As in Section 2, the Kurchatov's divided difference operator is used to approximate the parameter b. In this way, a new method with memory, denoted by FM5 is obtained, whose iterative expression is:

$$\begin{aligned} w^{(k)} &= x^{(k)} - \left[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F\right]^{-1} F(x^{(k)}), \\ y^{(k)} &= x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - [w^{(k)}, y^{(k)}; F]^{-1}F(y^{(k)}), \qquad k = 1, 2, \ldots \end{aligned}$$

In the following result, we establish the order of convergence of FM5.

Theorem 4. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and satisfying the same conditions than Theorem 2. If the initial estimations $x^{(0)}$ and $x^{(1)}$ are close enough to the solution α, then the order of convergence of method FM5 is 5.*

Proof. From the proof of Theorem 2 we have $I + B^{(k)}F'(\alpha) \sim e^{(k)}$ and by replacing it in the error equation (10), it is immediate that

$$e^{(k+1)} \sim e^{(k)}e^{(k)}(e^{(k)})^3 = (e^{(k)})^5. \tag{11}$$

As a consequence, method FM5 has order of convergence 5. □

3. Numerical results

In this section, we test numerically our proposed methods with memory in order to confirm the theoretical results described in Theorems 2 and 4. We also compare our methods with other known ones as the bi-parametric iterative family presented in [4], with order of convergence $p = \dfrac{3+\sqrt{13}}{2} \approx 3.30$ when the parameters satisfy the relation $\delta - \gamma = 1$. In our tests, we choose $\gamma = 1$ and $\delta = 2$ and the resulting method is denoted by $M1_{3.30}$. On the other hand, the two-step bi-parametric family with memory presented in [3], with order of convergence $p = 2 + \sqrt{6} \approx 4.45$ when $a = 3$, and order $p = 2 + \sqrt{5} \approx 4.24$ in other case. In our examples, we use $a = 3$ and $c = -0.01$, and we call the method $M2_{4.45}$.

Let us note that all the considered methods are derivative-free so the divided difference operators used instead of the Jacobian matrices have been computed for the numerical performance of the iterative methods using the first-order divided difference operator defined for all $1 \le i, j \le n$ by

$$[x, y; F]_{i,j} = \frac{F_i(y_1, \ldots, y_{j-1}, y_j, x_{j+1}, \ldots, x_n) - F_i(y_1, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_n)}{y_j - x_j}.$$

Table 1
Numerical results for Example 1.

$x^{(0)}$	Method	$\|x^{(1)}-x^{(0)}\|$	$\|x^{(2)}-x^{(1)}\|$	$\|x^{(3)}-x^{(2)}\|$	$\|F(x^{(1)})\|$	$\|F(x^{(2)})\|$	$\|F(x^{(3)})\|$	ACOC
$\begin{pmatrix}0.9\\0.9\\\vdots\\0.9\end{pmatrix}$	FM3	0.03749	6.92e−6	7.427e−16	2.076e−5	2.228e−15	5.232e−31	2.67
	FM5	0.0375	1.287e−9	1.89e−46	3.861e−9	5.245e−46	1.079e−184	4.9344
	$M1_{3.30}$	0.3076	0.008655	1.151e−8	0.0259	3.454e−8	9.633e−28	3.7895
	$M2_{4.45}$	0.316	0.0002238	4.096e−18	0.0006712	1.229e−17	2.669e−71	4.3612
$\begin{pmatrix}1.1\\1.1\\\vdots\\1.1\end{pmatrix}$	FM3	0.02874	3.026e−6	8.25e−17	9.078e−6	2.475e−16	6.457e−33	2.656
	FM5	0.02874	3.205e−10	2.555e−49	9.616e−10	7.093e−49	3.604e−196	4.9164
	$M1_{3.30}$	0.3118	0.004429	1.73e−9	0.0133	5.189e−9	2.546e−30	3.4684
	$M2_{4.45}$	0.3159	0.000325	1.001e−17	0.000975	3.003e−17	9.52e−70	4.5223
$\begin{pmatrix}-0.5\\-0.5\\\vdots\\-0.5\end{pmatrix}$	FM3	0.1066	0.001496	6.02e−10	0.004489	1.806e−9	1.284e−25	3.4512
	FM5	0.1051	2.08e−5	1.177e−22	6.239e−5	3.53e−22	1.818e−89	4.6568
	$M1_{3.30}$	4.399	0.316	0.02848	0.925	0.08467	2.089e−6	0.91391
	$M2_{4.45}$	0.7246	4.587	0.5377	3.225	2.03	0.09263	–

All the proposed methods require two initial guesses, $x^{(0)}$ and $x^{(1)}$, to start the iteration process. However, following the guidelines in [3,4] an initial value is given for the parameter $B^{(0)} = -0.01I$ in methods FM3 and FM5, $[u^{(0)}, v^{(0)}; F] = -0.01I$ for $M1_{3.30}$ and $[u^{(0)}, x^{(0)}; F] = -0.01I$ for $M2_{4.45}$, where I denotes the identity matrix. Then, only an initial estimation $x^{(0)}$ is required.

The numerical results have been carried out using the software program Matlab R2019a with variable precision arithmetics of 2000 digits of mantissa. The iterative process stops after three iterations, so we can compare the error of each method in all the iterations analyzing the values of $\|x^{(k+1)}-x^{(k)}\|$ and $\|F(x^{(k+1)})\|$ for $k = 0, 1, 2$. Tables 1–3 collect the main results obtained for the selected nonlinear test systems. In addition, the order of convergence of each method is compared with the approximated computational order of convergence, ACOC, presented in [8].

For the numerical performance we have taken different initial estimations more and less near the already known solution of each nonlinear system for showing the behavior of the methods after three iterations. In general, we can observe in Tables 1–3 that all the methods achieve the expected order of convergence, being FM5 the method with higher order after the last iteration. Moreover, method FM5 provides better approximations to the solution than methods FM3, $M1_{3.30}$ and $M2_{4.45}$. This fact can be analyzed in the columns corresponding to $\|x^{(3)}-x^{(2)}\|$ and $\|F(x^{(3)})\|$, where method FM5 has always lower values than the others. We write — in the ACOC when the method requires more than three iterations to converge to the solution and vector ACOC is not stable.

Example 1. Exact solution $\alpha = (1, 1, \ldots, 1)^T$,

$$\begin{aligned} x_i^2 x_{i+1} - 1 &= 0, \qquad i = 1, 2, \ldots, 9, \\ x_{10}^2 x_1 - 1 &= 0. \end{aligned}$$

Example 2. Approximated solution is $\alpha \approx (0.57735, 0.57735, 0.57735, -0.28867)^T$.

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) &= 0, \\ x_1x_3 + x_4(x_1 + x_3) &= 0, \\ x_1x_2 + x_4(x_1 + x_2) &= 0, \\ x_1x_2 + x_1x_3 + x_2x_3 - 1 &= 0. \end{aligned}$$

Example 3. Estimated solution $\alpha \approx (0.1757, 0.1757, \ldots, 0.1757)^T$.

$$\arctan x_i - 2\left(\sum_{j=1}^{20} x_j^2\right) + 2x_i^2 = -1, \qquad i = 1, 2, \ldots, 20.$$

Table 2
Numerical results for Example 2.

$x^{(0)}$	Method	$\|x^{(1)} - x^{(0)}\|$	$\|x^{(2)} - x^{(1)}\|$	$\|x^{(3)} - x^{(2)}\|$	$\|F(x^{(1)})\|$	$\|F(x^{(2)})\|$	$\|F(x^{(3)})\|$	ACOC
$\begin{pmatrix} 0.8 \\ 0.8 \\ \vdots \\ 0.8 \end{pmatrix}$	FM3	0.297	0.002474	7.219e−9	0.004924	1.444e−8	3.354e−25	2.6619
	FM5	0.2946	1.993e−6	3.443e−31	3.966e−6	6.887e−31	1.336e−173	4.7899
	$M1_{3.30}$	1.074	0.08273	1.631e−6	0.1747	3.354e−6	1.619e−23	4.2266
	$M2_{4.45}$	1.135	0.02034	4.263e−12	0.04171	8.543e−12	9.242e−42	5.5409
$\begin{pmatrix} 0.75 \\ 0.75 \\ \vdots \\ 0.75 \end{pmatrix}$	FM3	0.2396	0.00108	5.543e−10	0.002154	1.109e−9	5.202e−25	2.6812
	FM5	0.2385	3.7e−7	2.12e−33	7.376e−7	4.24e−33	1.82e−183	4.5171
	$M1_{3.30}$	1.028	0.05387	2.245e−7	0.1122	4.598e−7	2.757e−26	4.2017
	$M2_{4.45}$	1.071	0.01053	8.809e−14	0.02144	1.763e−13	1.216e−44	5.5185
$\begin{pmatrix} 0.3 \\ 0.3 \\ \vdots \\ 0.3 \end{pmatrix}$	FM3	0.6649	0.031	1.331e−5	0.06098	2.662e−5	9.098e−17	2.5291
	FM5	0.6945	0.0003301	1.25e−19	0.0006494	2.5e−19	3.628e−94	4.6408
	$M1_{3.30}$	0.8105	1.376	0.01725	1.969	0.03494	1.069e−8	–
	$M2_{4.45}$	4.513	5.309	0.1816	4.854	0.3626	6.352e−10	–

Table 3
Numerical results for Example 3.

$x^{(0)}$	Method	$\|x^{(1)} - x^{(0)}\|$	$\|x^{(2)} - x^{(1)}\|$	$\|x^{(3)} - x^{(2)}\|$	$\|F(x^{(1)})\|$	$\|F(x^{(2)})\|$	$\|F(x^{(3)})\|$	ACOC
$\begin{pmatrix} 0.5 \\ 0.5 \\ \vdots \\ 0.5 \end{pmatrix}$	FM3	0.3821	0.01455	1.443e−6	0.1821	1.787e−5	2.185e−17	2.821
	FM5	0.396	0.0006666	5.622e−17	0.008262	6.965e−16	1.757e−48	4.7132
	$M1_{3.30}$	1.13	0.3122	0.008046	4.843	0.1002	9.878e−7	2.8447
	$M2_{4.45}$	1.278	0.1716	0.0001324	2.38	0.00164	3.751e−17	3.5698
$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$	FM3	0.9402	0.1176	0.0006277	1.585	0.00778	1.504e−9	2.5179
	FM5	1.039	0.0198	7.519e−10	0.2486	9.315e−9	3.341e−40	4.3144
	$M1_{3.30}$	2.409	1.108	0.1682	29.77	2.347	0.01729	2.4261
	$M2_{4.45}$	2.799	0.853	0.03435	17.72	0.4356	1.727e−6	2.7034
$\begin{pmatrix} 0.3 \\ 0.3 \\ \vdots \\ 0.3 \end{pmatrix}$	FM3	0.1033	0.0004325	3.989e−11	0.005359	4.941e−10	3.283e−26	2.9581
	FM5	0.1038	1.93e−6	6.056e−28	2.391e−5	7.502e−27	4.019e−59	4.5458
	$M1_{3.30}$	0.5007	0.05483	4.34e−5	0.7055	0.0005377	2.596e−14	3.229
	$M2_{4.45}$	0.5384	0.01718	6.614e−9	0.2153	8.193e−8	1.884e−33	4.2872

4. Conclusions

From iterative methods for solving nonlinear systems of order two and three and by using the Kurchatov's divided difference operator, we design iterative schemes with memory of order three and five, respectively. With these methods we extend the set of iterative schemes with memory to solve nonlinear systems, an area that is still underdeveloped. The proposed methods are compared with other known ones of similar characteristics with satisfactory results.

CRediT authorship contribution statement

Francisco I. Chicharro: Conceptualization, Methodology, Software. **Alicia Cordero:** Writing - original draft. **Neus Garrido:** Validation, Visualization, Investigation. **Juan R. Torregrosa:** Supervision, Writing - review & editing.

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