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A novel bi-parametric sixth order iterative scheme for solving nonlinear systems and its dynamics[☆]

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ABSTRACT

In this paper, we propose a general bi-parametric family of sixth order iterative methods to solve systems of nonlinear equations. The presented scheme contains some well known existing methods as special cases. The stability of the proposed class, presented as an appendix, is used for selecting the most stable members of the family with optimum numerical performance. From the comparison with some existing methods of similar nature, it is observed that the presented methods show robust and efficient character.

1. Introduction

Constructing fixed point methods for solving nonlinear equations and systems of nonlinear equations is one of the most attractive topics in the theory of numerical analysis, with wide applications in science and engineering. A great importance of this topic has led to the development of many numerical methods, most frequently of iterative nature (see [1–5]). With the advancement of computer hardware and software, the problem of solving nonlinear equations by numerical methods has gained an additional importance.

In this paper, we consider the problem of finding solution of the system of nonlinear equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0},$$

by iterative methods of a high order of convergence. This problem can be precisely stated as to find a vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)^T$ such that $\mathbf{F}(\mathbf{r}) = \mathbf{0}$, where $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the given nonlinear vector function $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. The solution vector $\mathbf{r}$ of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ can be obtained as a fixed point of some function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by means of the fixed point iteration

$$\mathbf{x}^{(k+1)} = \Phi(\mathbf{x}^{(k)}), \quad k = 0, 1, 2, \ldots$$

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One of the basic procedures for solving systems of nonlinear equations is the quadratically convergent Newton method (see [1–3]), which is given as,

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}), \tag{1}$$

where $\mathbf{F}'(\mathbf{x})^{-1}$ is the inverse of first Fréchet derivative $\mathbf{F}'(\mathbf{x})$ of the function $\mathbf{F}(\mathbf{x})$. In terms of computational cost the Newton method requires the evaluations of one $\mathbf{F}$, one $\mathbf{F}'$ and the solution of a linear system per iteration.

To improve the order of convergence of Newton method, a number of third and fourth order methods have been proposed in literature (see [6–17]).

In quest of more fast algorithms, researchers have also proposed fifth and sixth order methods in recent years (see [10,13,15,18–23]). Very recently, Narang et. al. [17] extended Babajee's fourth order scheme [24] to multidimensional case and further developed a two-parameter family of Chebyshev–Halley-like methods of sixth order convergence. The resulting scheme requires evaluations of two $\mathbf{F}$, two $\mathbf{F}'$ and the solution of two linear systems per iteration. However, with the same number of evaluations, we can develop a simpler yet efficient bi-parametric scheme of sixth order methods for solving nonlinear systems. This is the motivation behind the present work.

The rest of the paper is organized as follows. Section 2 includes the development of the fourth order method with its analysis of convergence. In Section 3, the sixth order scheme is introduced and its analysis of convergence is presented along with some special cases of the family. In Section 4, we select some elements of the family from the dynamical analysis appearing in the appendix as some of the more stable members of the family. Then, we present numerical work to confirm the theoretical results and to compare convergence properties of the proposed method with existing ones. Concluding remarks are given in Section 5.

2. Development of the fourth order scheme

We consider the three parameter iteration scheme proposed by Sharma and Sharma in [27], for solving a nonlinear equation $f(x) = 0$, defined by

$$x_{k+1} = x_k - \left[1 + \frac{\mathrm{M}(x_k)}{2\theta}\left(1 + \frac{\beta \mathrm{M}(x_k)}{2\theta - \alpha \mathrm{M}(x_k)}\right)\right]\frac{f(x_k)}{f'(x_k)}, \quad \theta \neq 0 \tag{2}$$

where

$$\mathrm{M}(x_k) = 1 - \frac{f'(y_k)}{f'(x_k)}, \quad y_k = x_k - \theta\frac{f(x_k)}{f'(x_k)}.$$

It is noteworthy that this scheme is two-point optimal fourth order family of variants of Kou-Li-Wang method for computing simple roots.

In this paper, we extend this scheme for solving a system of nonlinear equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. For this purpose, we write (2) in the generalised form as:

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \theta\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \left[\mathbf{I} + \frac{1}{2\theta}\mathrm{M}(\mathbf{x}^{(k)})\left(\mathbf{I} + \frac{\beta}{2\theta}\left(\mathbf{I} - \frac{\alpha}{2\theta}\mathrm{M}(\mathbf{x}^{(k)})\right)^{-1}\mathrm{M}(\mathbf{x}^{(k)})\right)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),
\end{aligned} \tag{3}$$

where

$$\mathrm{M}(\mathbf{x}^{(k)}) = \mathbf{I} - \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)}), \tag{4}$$

$\alpha, \ \beta, \ \theta \in \mathbb{R}, \ \theta \neq 0$ and $\mathbf{I}$ denotes the $n \times n$ identity matrix.

In order to analyze the convergence property of scheme (3), we need the following results of Taylor's expansion on vector functions (see [2]).

Lemma 1. *Let $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a q-times Fréchet differentiable in a convex set $D \subseteq \mathbb{R}^n$, then for any $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$ the following expression holds:*

$$\mathbf{F}(\mathbf{x}+\mathbf{h}) = \mathbf{F}(\mathbf{x}) + \mathbf{F}'(\mathbf{x})\mathbf{h} + \frac{1}{2!}\mathbf{F}''(\mathbf{x})\mathbf{h}^2 + \frac{1}{3!}\mathbf{F}'''(\mathbf{x})\mathbf{h}^3 + \cdots + \frac{1}{q!}\mathbf{F}^{(q-1)}(\mathbf{x})\mathbf{h}^{q-1} + \mathbf{R}_q, \tag{5}$$

where

$$||\mathbf{R}_q|| \leqslant \frac{1}{q!}\sup_{0<t<1}||\mathbf{F}^{(q)}(\mathbf{x}+t\mathbf{h})||||\mathbf{h}||^q \text{ and } \mathbf{h}^q = (\mathbf{h}, \mathbf{h}, \overset{q-times}{\cdots\cdots}, \mathbf{h}). \tag{6}$$

We now prove the following result.

Theorem 1. *Let $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in a convex set D containing a zero $\mathbf{r}$ of $\mathbf{F}(\mathbf{x})$. Suppose that $\mathbf{F}'(\mathbf{x})$ is continuous and nonsingular in $\mathbf{r}$. If an initial approximation $\mathbf{x}^{(0)}$ is sufficiently close to $\mathbf{r}$, then $\forall\ \alpha \in \mathbb{R}$, the local convergence order of the method* (3) *is at least four, provided $\theta = \frac{2}{3}$ and $\beta = 2$.*

Proof. Developing $\mathbf{F}(\mathbf{x}^{(k)})$ in the neighborhood of $\mathbf{r}$ and assuming that $\Gamma = \mathbf{F}'(\mathbf{r})^{-1}$ exists, we write

$$\mathbf{F}(\mathbf{x}^{(k)}) = \mathbf{F}'(\mathbf{r})\big[\mathbf{e}^{(k)} + \mathbf{A}_2(\mathbf{e}^{(k)})^2 + \mathbf{A}_3(\mathbf{e}^{(k)})^3 + \mathbf{A}_4(\mathbf{e}^{(k)})^4 + \mathbf{A}_5(\mathbf{e}^{(k)})^5 + \mathbf{A}_6(\mathbf{e}^{(k)})^6 + \mathbf{O}((\mathbf{e}^{(k)})^7)\big],$$

where $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{r},\ \mathbf{A}_i = \dfrac{1}{i!}\Gamma\,\mathbf{F}^{(i)}(\mathbf{r}),\ \mathbf{F}^{(i)}(\mathbf{r}) \in \mathcal{L}(\mathbb{R}^n \times \overset{i}{\cdots} \times \mathbb{R}^n, \mathbb{R}^n),\ \Gamma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $(\mathbf{e}^{(k)})^i = (\mathbf{e}^{(k)}, \mathbf{e}^{(k)}, \overset{i-times}{\cdots\cdots}, \mathbf{e}^{(k)})$ with $\mathbf{e}^{(k)} \in \mathbb{R}^n,\ i = 2, 3, \ldots$

Also,

$$\mathbf{F}'(\mathbf{x}^{(k)}) = \mathbf{F}'(\mathbf{r})\big[\mathbf{I} + 2\mathbf{A}_2(\mathbf{e}^{(k)}) + 3\mathbf{A}_3(\mathbf{e}^{(k)})^2 + 4\mathbf{A}_4(\mathbf{e}^{(k)})^3 + 5\mathbf{A}_5(\mathbf{e}^{(k)})^4 + 6\mathbf{A}_6(\mathbf{e}^{(k)})^5 + \mathbf{O}((\mathbf{e}^{(k)})^6)\big]. \tag{7}$$

Inversion of $\mathbf{F}'(\mathbf{x}^{(k)})$ yields,

$$\mathbf{F}'(\mathbf{x}^{(k)})^{-1} = \big[\mathbf{I} + \mathbf{B}_1(\mathbf{e}^{(k)}) + \mathbf{B}_2(\mathbf{e}^{(k)})^2 + \mathbf{B}_3(\mathbf{e}^{(k)})^3 + \mathbf{B}_4(\mathbf{e}^{(k)})^4 + \mathbf{B}_5(\mathbf{e}^{(k)})^5 + \mathbf{O}((\mathbf{e}^{(k)})^6)\big]\Gamma, \tag{8}$$

where

$$\begin{aligned}
\mathbf{B}_1 &= -2\mathbf{A}_2,\ \mathbf{B}_2 = 4\mathbf{A}_2^2 - 3\mathbf{A}_3,\ \mathbf{B}_3 = -(8\mathbf{A}_2^3 - 6\mathbf{A}_2\mathbf{A}_3 - 6\mathbf{A}_3\mathbf{A}_2 + 4\mathbf{A}_4),\\
\mathbf{B}_4 &= 8\mathbf{A}_2\mathbf{A}_4 + 9\mathbf{A}_3^2 + 8\mathbf{A}_4\mathbf{A}_2 - 12\mathbf{A}_2^2\mathbf{A}_3 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_3\mathbf{A}_2^2 + 16\mathbf{A}_2^4 - 5\mathbf{A}_5,\\
\mathbf{B}_5 &= 10\mathbf{A}_2\mathbf{A}_5 + 12\mathbf{A}_3\mathbf{A}_4 + 12\mathbf{A}_4\mathbf{A}_3 + 10\mathbf{A}_5\mathbf{A}_2 - 16\mathbf{A}_2^2\mathbf{A}_4 - 18\mathbf{A}_2\mathbf{A}_3^2 - 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\\
&\quad - 18\mathbf{A}_3^2\mathbf{A}_2 - 16\mathbf{A}_4\mathbf{A}_2^2 + 24\mathbf{A}_2^3\mathbf{A}_3 + 24\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 24\mathbf{A}_3\mathbf{A}_2^3 - 32\mathbf{A}_2^5 - 6\mathbf{A}_6.
\end{aligned}$$

Post multiplication of the above equation by $\mathbf{F}(\mathbf{x}^{(k)})$ yields,

$$\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}) = \mathbf{e}^{(k)} + \mathbf{C}_1(\mathbf{e}^{(k)})^2 + \mathbf{C}_2(\mathbf{e}^{(k)})^3 + \mathbf{C}_3(\mathbf{e}^{(k)})^4 + \mathbf{C}_4(\mathbf{e}^{(k)})^5 + \mathbf{C}_5(\mathbf{e}^{(k)})^6 + \mathbf{O}((\mathbf{e}^{(k)})^7), \tag{9}$$

where

$$\begin{aligned}
\mathbf{C}_1 &= -\mathbf{A}_2,\ \mathbf{C}_2 = 2(\mathbf{A}_2^2 - \mathbf{A}_3),\ \mathbf{C}_3 = -4\mathbf{A}_2^3 + 4\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_2 - 3\mathbf{A}_4,\\
\mathbf{C}_4 &= 6\mathbf{A}_2\mathbf{A}_4 + 6\mathbf{A}_3^2 + 4\mathbf{A}_4\mathbf{A}_2 - 8\mathbf{A}_2^2\mathbf{A}_3 - 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 6\mathbf{A}_3\mathbf{A}_2^2 + 8\mathbf{A}_2^4 - 4\mathbf{A}_5,\\
\mathbf{C}_5 &= 8\mathbf{A}_2\mathbf{A}_5 + 9\mathbf{A}_3\mathbf{A}_4 + 8\mathbf{A}_4\mathbf{A}_3 + 5\mathbf{A}_5\mathbf{A}_2 - 12\mathbf{A}_2^2\mathbf{A}_4 - 12\mathbf{A}_2\mathbf{A}_3^2 - 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 9\mathbf{A}_3^2\mathbf{A}_2\\
&\quad -8\mathbf{A}_4\mathbf{A}_2^2 + 16\mathbf{A}_2^3\mathbf{A}_3 + 12\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 12\mathbf{A}_3\mathbf{A}_2^3 - 16\mathbf{A}_2^5 - 5\mathbf{A}_6.
\end{aligned}$$

Taking $\tilde{e}^{(k)} = \mathbf{y}^{(k)} - \mathbf{r}$, where $\mathbf{y}^{(k)} = \mathbf{x}^{(k)} - \theta\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)})$, and using (9), we obtain

$$\tilde{e}^{(k)} = (1-\theta)\mathbf{e}^{(k)} + \theta\mathbf{A}_2(\mathbf{e}^{(k)})^2 - 2\theta(\mathbf{A}_2^2 - \mathbf{A}_3)(\mathbf{e}^{(k)})^3 + \theta(4\mathbf{A}_2^3 - 4\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2 + 3\mathbf{A}_4)(\mathbf{e}^{(k)})^4 + \mathbf{O}((\mathbf{e}^{(k)})^5). \tag{10}$$

Expanding $\mathbf{F}'(\mathbf{y}^{(k)})$ about $\mathbf{r}$ and using above result, we have

$$\begin{aligned}
\mathbf{F}'(\mathbf{y}^{(k)}) &= \mathbf{F}'(\mathbf{r})\big[\mathbf{I} + 2\mathbf{A}_2(\tilde{e}^{(k)}) + 3\mathbf{A}_3(\tilde{e}^{(k)})^2 + 4\mathbf{A}_4(\tilde{e}^{(k)})^3 + \mathbf{O}((\tilde{e}^{(k)})^4)\big]\\
&= \mathbf{F}'(\mathbf{r})\big[\mathbf{I} + 2(1-\theta)\mathbf{A}_2\mathbf{e}^{(k)} + (2\theta\mathbf{A}_2^2 + 3(1-\theta)^2\mathbf{A}_3)(\mathbf{e}^{(k)})^2 + (-4\theta(\mathbf{A}_2^3 - \mathbf{A}_2\mathbf{A}_3)\\
&\qquad + 6\theta(1-\theta)\mathbf{A}_3\mathbf{A}_2 + 4(1-\theta)^3\mathbf{A}_4)(\mathbf{e}^{(k)})^3 + \mathbf{O}((\mathbf{e}^{(k)})^4)\big].
\end{aligned} \tag{11}$$

Then, from (8) and (11), it follows that

$$\begin{aligned}
\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)}) &= \mathbf{I} - 2\theta\mathbf{A}_2\mathbf{e}^{(k)} + \theta(6\mathbf{A}_2^2 - 3(2-\theta)\mathbf{A}_3)(\mathbf{e}^{(k)})^2 + \theta((16-6\theta)\mathbf{A}_2\mathbf{A}_3\\
&\qquad + 6(2-\theta)\mathbf{A}_3\mathbf{A}_2 - 16\mathbf{A}_2^3 - 4(3 - 3\theta + \theta^2)\mathbf{A}_4)(\mathbf{e}^{(k)})^3 + \mathbf{O}((\mathbf{e}^{(k)})^4).
\end{aligned} \tag{12}$$

Substituting (9) and (12) in second step of (3), we can get

$$\begin{aligned}
\mathbf{e}^{(k+1)} &= \frac{1}{2}\big[(4-2\beta)\mathbf{A}_2^2 + (3\theta-2)\mathbf{A}_3\big](\mathbf{e}^{(k)})^3 + \frac{1}{2}\big[(-2\alpha\beta + 14\beta - 18)\mathbf{A}_2^3 + (3\beta\theta - 6\theta - 6\beta + 12)\mathbf{A}_2\mathbf{A}_3\\
&\quad + (3\beta\theta - 9\theta - 6\beta + 12)\mathbf{A}_3\mathbf{A}_2 + (-4\theta^2 + 12\theta - 6)\mathbf{A}_4\big](\mathbf{e}^{(k)})^4 + \mathbf{O}((\mathbf{e}^{(k)})^5).
\end{aligned} \tag{13}$$

For obtaining order four, we must have

$$4 - 2\beta = 0,\ 3\theta - 2 = 0,$$

which implies

$$\theta = \frac{2}{3} \text{ and } \beta = 2. \tag{14}$$

By using (14) in (13), we get the error equation as

$$\mathbf{e}^{(k+1)} = \left((5-2\alpha)\mathbf{A}_2^3 - \mathbf{A}_3\mathbf{A}_2 + \frac{\mathbf{A}_4}{9}\right)(\mathbf{e}^{(k)})^4 + \mathbf{O}((\mathbf{e}^{(k)})^5). \tag{15}$$

This shows the fourth order convergence of the proposed family. □

Thus the presented scheme (3) is finally given as

$$\begin{aligned}\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\ \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \left[\mathbf{I} + \frac{3\mathrm{M}(\mathbf{x}^{(k)})}{4}\left(\mathbf{I} + 6\left(4\mathbf{I} - 3\alpha\mathrm{M}(\mathbf{x}^{(k)})\right)^{-1}\mathrm{M}(\mathbf{x}^{(k)})\right)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\end{aligned} \tag{16}$$

where $\mathrm{M}(\mathbf{x}^{(k)})$ is given in (4).

Clearly, this iterative scheme defines a one parameter family of fourth order methods.

3. The sixth order scheme

Based on the two-step fourth order scheme (16), we propose the following three-step scheme:

$$\begin{aligned}\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\ \mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left[\mathbf{I} + \frac{3\mathrm{M}(\mathbf{x}^{(k)})}{4}\left(\mathbf{I} + 6(4\mathbf{I} - 3\alpha\mathrm{M}(\mathbf{x}^{(k)}))^{-1}\mathrm{M}(\mathbf{x}^{(k)})\right)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\ \mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left[\left(\gamma\mathbf{F}'(\mathbf{x}^{(k)}) + \lambda\mathbf{F}'(\mathbf{y}^{(k)})\right)^{-1}\left(\mathbf{F}'(\mathbf{x}^{(k)}) + \delta\mathbf{F}'(\mathbf{y}^{(k)})\right)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}).\end{aligned} \tag{17}$$

where $\alpha \in \mathbb{R}$, $\mathbf{I}$ denotes the $n \times n$ identity matrix, γ, δ and λ are newly introduced real parameters. Note that this scheme requires one additional function evaluation to that of the evaluations of the fourth-order scheme (16). The method has order of convergence six, as we shall prove in the following result.

Theorem 2. *Let $\mathbf{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a function sufficiently differentiable in a convex set D containing the zero $\mathbf{r}$ of $\mathbf{F}(\mathbf{x})$. Suppose that $\mathbf{F}'(\mathbf{x})$ is continuous and nonsingular in $\mathbf{r}$. If an initial approximation $\mathbf{x}^{(0)}$ is sufficiently close to $\mathbf{r}$, then $\forall\ \alpha \in \mathbb{R}$ and $\forall\ \lambda \in \mathbb{R} \setminus \{-1\}$, the local convergence order of the method* (17) *is at least six, provided $\gamma = \dfrac{2-3\lambda}{5}$ and $\delta = \dfrac{-3+2\lambda}{5}$.*

Proof. Let $\mathbf{z}^{(k)} = \hat{e}^{(k)} + \mathbf{r}$, then from the error Eq. (15) of the fourth order scheme proved above, we can write

$$\hat{e}^{(k)} = \mathbf{z}^{(k)} - \mathbf{r} = \left((5-2\alpha)\mathbf{A}_2^3 - \mathbf{A}_3\mathbf{A}_2 + \frac{\mathbf{A}_4}{9}\right)(\mathbf{e}^{(k)})^4 + \mathbf{O}((\mathbf{e}^{(k)})^5). \tag{18}$$

By using (7), (11) and simple algebraic manipulations, yield

$$\left(\gamma\mathbf{F}'(\mathbf{x}^{(k)}) + \lambda\mathbf{F}'(\mathbf{y}^{(k)})\right)^{-1}\left(\mathbf{F}'(\mathbf{x}^{(k)}) + \delta\mathbf{F}'(\mathbf{y}^{(k)})\right) = \mathbf{D}_0\mathbf{I} + \mathbf{D}_1\mathbf{e}^{(k)} + \mathbf{D}_2(\mathbf{e}^{(k)})^2 + \mathbf{D}_3(\mathbf{e}^{(k)})^3 + \mathbf{O}((\mathbf{e}^{(k)})^4), \tag{19}$$

where

$$\mathbf{D}_0 = \frac{\delta+1}{\gamma+\lambda},\ \mathbf{D}_1 = \frac{4(\lambda-\gamma\delta)}{3(\gamma+\lambda)^2}\mathbf{A}_2,\ \mathbf{D}_2 = \frac{4\left((9\gamma+5\lambda)\mathbf{A}_2^2 - 6(\gamma+\lambda)\mathbf{A}_3\right)(\gamma\delta-\lambda)}{9(\gamma+\lambda)^3},$$

$$\begin{aligned}\text{and}\quad \mathbf{D}_3 = \frac{8(\gamma\delta-\lambda)}{27(\gamma+\lambda)^4}&\Big[3\left(9\gamma^2+14\gamma\lambda+5\lambda^2\right)\mathbf{A}_2\mathbf{A}_3 + 6\left(3\gamma^2+4\gamma\lambda+\lambda^2\right)\mathbf{A}_3\mathbf{A}_2\\ &-(36\gamma^2+36\gamma\lambda+8\lambda^2)\mathbf{A}_2^3 - (13\lambda^2+13\gamma^2+26\gamma\lambda)\mathbf{A}_4\Big].\end{aligned}$$

By using Taylor's series of $\mathbf{F}(\mathbf{z}^{(k)})$ about $\mathbf{r}$, we have

$$\mathbf{F}(\mathbf{z}^{(k)}) = \mathbf{F}'(\mathbf{r})\left[\hat{e}^{(k)} + \mathbf{A}_2(\hat{e}^{(k)})^2 + O((\hat{e}^{(k)})^3)\right]. \tag{20}$$

By employing (8), (18)–(20) in the third step of (17) and simplifying, we get

$$\begin{aligned}\mathbf{e}^{(k+1)} = &-\frac{(1-\gamma+\delta-\lambda)}{\gamma+\lambda}\hat{e}^{(k)} + \frac{2((5\delta+3)\gamma+(3\delta+1)\lambda)}{3(\gamma+\lambda)^2}\mathbf{A}_2\mathbf{e}^{(k)}\hat{e}^{(k)}\\ &-\frac{1}{9(\gamma+\lambda)^3}\Big[\left((-51\delta-27)\gamma^2+(-78\delta\lambda-30\lambda)\gamma-3(9\delta+1)\lambda^2\right)\mathbf{A}_3\\ &+\left((96\delta+36)\gamma^2+(116\delta\lambda+12\lambda)\gamma+(36\delta-8)\lambda^2\right)\mathbf{A}_2^2\Big](\mathbf{e}^{(k)})^2\hat{e}^{(k)} + O((\mathbf{e}^{(k)})^7).\end{aligned} \tag{21}$$

Our aim is to find the values of parameters $\alpha\,\gamma$ and δ in such a way that the proposed iterative scheme may produce order of convergence as high as possible. Thus, if we take $\gamma = \dfrac{2-3\lambda}{5}$ and $\delta = \dfrac{-3+2\lambda}{5}$, the above equation yields

$$\mathbf{e}^{(k+1)} = -\frac{\left(3(\lambda+1)\mathbf{A}_3 + 2(\lambda-9)\mathbf{A}_2^2\right)}{3(\lambda+1)}(\mathbf{e}^{(k)})^2\hat{e}^{(k)} + O((\mathbf{e}^{(k)})^7). \tag{22}$$

Combining (18) and (22),

$$\mathbf{e}^{(k+1)} = \frac{1}{729(1+\lambda)}\Big(1458\alpha(1+\lambda)\mathbf{A}_3\mathbf{A}_2^3 + (972\alpha\lambda - 8748\alpha - 2430\lambda + 21870)\mathbf{A}_2^5 - 81(1+\lambda)\mathbf{A}_3\mathbf{A}_4 + (486-54\lambda)\mathbf{A}_2^2\mathbf{A}_4 + 729(1+\lambda)\mathbf{A}_3^2\mathbf{A}_2 + (486\lambda - 4374)\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 3645(1+\lambda)\mathbf{A}_3\mathbf{A}_2^3\Big)(\mathbf{e}^{(k)})^6 + O((\mathbf{e}^{(k)})^7).$$

This shows that $\forall\ \alpha \in \mathbb{R}$ and $\forall\ \lambda \in \mathbb{R}, \lambda \neq -1$, the presented scheme (17) has sixth order of convergence. Hence, the proof is complete. □

Thus, the proposed bi-parametric sixth order scheme is finally expressed as

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left[\mathbf{I} + \frac{3\mathrm{M}(\mathbf{x}^{(k)})}{4}\Big(\mathbf{I} + 6\big(4\mathbf{I} - 3\alpha\mathrm{M}(\mathbf{x}^{(k)})\big)^{-1}\mathrm{M}(\mathbf{x}^{(k)})\Big)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left[\left(\frac{2-3\lambda}{5}\ \mathbf{F}'(\mathbf{x}^{(k)}) + \lambda\mathbf{F}'(\mathbf{y}^{(k)})\right)^{-1}\left(\mathbf{F}'(\mathbf{x}^{(k)}) + \frac{2\lambda-3}{5}\ \mathbf{F}'(\mathbf{y}^{(k)})\right)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}),
\end{aligned} \tag{23}$$

where $\alpha, \lambda \in \mathbb{R}$ and $\mathrm{M}(\mathbf{x}^{(k)})$ is given in (4).

3.1. Some special cases of scheme (23)

1. If we take $\lambda = 0,\ \alpha \neq 0,\ \alpha \in \mathbb{R}$, in (23), we get a sixth-order uniparametric scheme requiring the evaluations of two functions, two first derivatives and only one linear system per iteration. In this case, the scheme reads as

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left[\mathbf{I} + \frac{3\mathrm{M}(\mathbf{x}^{(k)})}{4}\Big(\mathbf{I} + 6\big(4\mathbf{I} - 3\alpha\mathrm{M}(\mathbf{x}^{(k)})\big)^{-1}\mathrm{M}(\mathbf{x}^{(k)})\Big)\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left[\frac{5}{2}\mathbf{I} - \frac{3}{2}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}).
\end{aligned} \tag{24}$$

2. Further taking $\alpha = 2$ and $\lambda \in \mathbb{R} \setminus \{0\}$, in the scheme (23), we get another sixth order scheme requiring the evaluations of two functions, two first derivatives and two matrix inversions per iteration. In particular, for $\lambda = \frac{3}{2}$, this family reduces to already existing modified Newton–Jarratt composition given by Cordero et al. [15].
3. If we consider $\alpha = 0$ and $\lambda = 0$, the scheme corresponds to existing Jarratt like sixth order method given by Sharma and Arora [20].

Remark 1. Recently, a three-step sixth order scheme has also been presented by Narang et al. [17]. The scheme is composed of three steps of which the first two steps are generalisation of fourth order Babajee's method [24] where as the third is same as in (23). The present scheme is, therefore, different and hence new.

4. Numerical results

In this section, we solve some numerical examples by using some elements of the proposed family $M_i,\ i = 1, 2$ (for $\alpha = 2,\ \lambda = 3/2$ and $\alpha = 0,\ \lambda = 3/2$, respectively). These values have been selected by using the information provided by the stability analysis presented in the appendix: let us observe that, when $\lambda = 3/2$, the number of free critical points is reduced for $\alpha = 0$ and $\alpha = 2$ and also there are no pathological behavior as attracting strange fixed points nor attracting periodic orbits. Here, we compare the performance with some existing sixth order methods viz. sixth order method by Cordero et al. [29], sixth order generalised Jarratt's method by Sharma and Arora [20], sixth order methods by Lotfi et al. [21], Soleymani et al. [33] and Narang et al. [17] to solve this system numerically. The above mentioned methods are given as follows:

Method by Cordero et al. (M_3):

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{1}{2}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \frac{1}{3}\big(4\mathbf{y}^{(k)} - \mathbf{x}^{(k)}\big),\\
\mathbf{u}^{(k)} &= \mathbf{y}^{(k)} + \big(\mathbf{F}'(\mathbf{x}^{(k)}) - 3\mathbf{F}'(\mathbf{z}^{(k)})\big)^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{u}^{(k)} + 2\big(\mathbf{F}'(\mathbf{x}^{(k)}) - 3\mathbf{F}'(\mathbf{z}^{(k)})\big)^{-1}\mathbf{F}(\mathbf{u}^{(k)}).
\end{aligned}$$

Table 1
Abscissas and weights of Gauss–Legendre quadrature formula for $n = 8$.

j	t_j	ϖ_j
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

Sharma–Arora method (M_4):

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left[\frac{23}{8}\mathbf{I} - \left(3\mathbf{I} - \frac{9}{8}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})\right)\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \frac{1}{2}\left(5\mathbf{I} - 3\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})\right)\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}).
\end{aligned}$$

Method by Lotfi et al. (M_5):

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - 2\left(\mathbf{F}'(\mathbf{x}^{(k)}) + \mathbf{F}'(\mathbf{y}^{(k)})\right)^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left[\frac{7}{2}\mathbf{I} - 4\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)}) + \frac{3}{2}\left(\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})\right)^2\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}).
\end{aligned}$$

Method by Soleymani et al. (M_6):

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}^{(k)}) - \mathbf{F}'(\mathbf{x}^{(k)})\right)^{-1}\left(3\mathbf{F}'(\mathbf{y}^{(k)}) + \mathbf{F}'(\mathbf{x}^{(k)})\right)\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left[\left(\frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}^{(k)}) - \mathbf{F}'(\mathbf{x}^{(k)})\right)^{-1}\left(3\mathbf{F}'(\mathbf{y}^{(k)}) + \mathbf{F}'(\mathbf{x}^{(k)})\right)\right)^2\right]\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}).
\end{aligned}$$

Method by Narang et al. (M_7):

$$\begin{aligned}
\mathbf{y}^{(k)} &= \mathbf{x}^{(k)} - \frac{2}{3}\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{z}^{(k)} &= \mathbf{x}^{(k)} - \left(\mathbf{I} + \frac{1}{2a}\mathbf{G}(\mathbf{x}^{(k)})\right)\mathbf{H}(\mathbf{G}(\mathbf{x}^{(k)}))\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{x}^{(k)}),\\
\mathbf{x}^{(k+1)} &= \mathbf{z}^{(k)} - \left(\mathbf{I} + \frac{3}{2}\mathbf{G}(\mathbf{x}^{(k)})\right)\mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}),
\end{aligned}$$

where

$\mathbf{G}(\mathbf{x}^{(k)}) = \mathbf{I} - \mathbf{F}'(\mathbf{x}^{(k)})^{-1}\mathbf{F}'(\mathbf{y}^{(k)})$, $\mathbf{H}(\mathbf{G}(\mathbf{x}^{(k)})) = \mathbf{I} + \frac{1}{4}\frac{-2+3a}{a}(\mathbf{G}(\mathbf{x}^{(k)})) + \frac{1}{8}\frac{-3a+9a^2+2}{a^2}(\mathbf{G}(\mathbf{x}^{(k)}))^2$ and value of parameter a is taken as 2/5.

In Tables 2–9, we exhibit numerical results obtained by implementing the methods M_i, $(i = 1, 2, \ldots, 7)$. All computations are performed in the programming package Mathematica [30] using multiple-precision arithmetic with 4096 digits. For every method, we analyze the number of iterations (k) needed to converge to the solution such that the stopping criterion $||\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}|| + ||\mathbf{F}(\mathbf{x}^{(k)})|| < 10^{-100}$ is satisfied. To confirm the theoretical order of convergence, we calculate the computational order of convergence (ρ_k) using the formula (see [31,36])

$$\rho_k = \frac{\log(||\mathbf{F}(\mathbf{x}^{(k)})||/||\mathbf{F}(\mathbf{x}^{(k-1)})||)}{\log(||\mathbf{F}(\mathbf{x}^{(k-1)})||/||\mathbf{F}(\mathbf{x}^{(k-2)})||)},$$

where $\mathbf{x}^{(k-1)}$, $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$ are the three consecutive iterations close to the zero $\mathbf{r}$ of $\mathbf{F}(\mathbf{x})$. In numerical results, we also include CPU time (measured in milliseconds) used in the execution of program which is computed by the Mathematica command "TimeUsed[ ]". Displayed in the table are the necessary iterations (k), the computational order of convergence ρ_k, the residual of the function at the last iteration, $||\mathbf{F}(\mathbf{x}^{(k)})||$ and the CPU time (CPU-Time). For numerical tests we consider the following problems:

Table 2
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 1				
M_1	5	$6 \pm 8.59 \times 10^{-7}$	2.87×10^{-2448}	0.288
M_2	5	$6 \pm 1.85 \times 10^{-6}$	1.22×10^{-1883}	0.312
M_3	5	$6 \pm 8.59 \times 10^{-7}$	2.87×10^{-2448}	0.92
M_4	5	$6 \pm 4.87 \times 10^{-8}$	1.14×10^{-1572}	1.248
M_5	5	$6 \pm 2.97 \times 10^{-8}$	3.79×10^{-1589}	1.013
M_6	5	$6 \pm 7.19 \times 10^{-6}$	3.28×10^{-2634}	0.91
M_7	5	$6 \pm 1.42 \times 10^{-7}$	3.44×10^{-2192}	0.389

Table 3
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 2				
M_1	4	$6 \pm 1.82 \times 10^{-4}$	1.54×10^{-708}	0.124
M_2	4	$6 \pm 1.83 \times 10^{-4}$	3.34×10^{-580}	0.132
M_3	4	$6 \pm 1.82 \times 10^{-4}$	1.54×10^{-708}	0.467
M_4	5	$6 \pm 2.57 \times 10^{-5}$	2.40×10^{-2971}	0.736
M_5	5	$6 \pm 9.96 \times 10^{-1}$	1.02×10^{-1718}	0.784
M_6	5	$6 \pm 0.53 \times 10^{-4}$	2.79×10^{-761}	1.203
M_7	4	$6 \pm 0.01 \times 10^{-3}$	1.55×10^{-626}	0.12

Table 4
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 3				
M_1	2	$6 \pm 2.13 \times 10^{-5}$	5.72×10^{-6}	0.008
M_2	2	$6 \pm 2.67 \times 10^{-2}$	5.72×10^{-6}	0.016
M_3	2	$6 \pm 2.13 \times 10^{-5}$	0.01×10^{-30}	0.047
M_4	2	$6 \pm 1.71 \times 10^{-5}$	5.72×10^{-6}	1.073
M_5	2	$6 \pm 8.12 \times 10^{-1}$	0.15×10^{-4}	1.142
M_6	5	$6 \pm 7.13 \times 10^{-15}$	5.72×10^{-6}	0.091
M_7	2	$6 \pm 1.42 \times 10^{-3}$	5.74×10^{-6}	0.0159

Table 5
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 4				
M_1	2	$6 \pm 7.81 \times 10^{-4}$	7.21×10^{-152}	0.172
M_2	3	$6 \pm 0.12 \times 10^{-1}$	4.90×10^{-25}	0.187
M_3	4	$6 \pm 1.81 \times 10^{-4}$	1.30×10^{-951}	0.302
M_4	4	$6 \pm 1.79 \times 10^{-4}$	1.79×10^{-868}	0.574
M_5	3	$6 \pm 9.38 \times 10^{-1}$	7.56×10^{-27}	0.742
M_6	5	$6 \pm 6.51 \times 10^{-3}$	3.57×10^{-161}	0.806
M_7	3	$6 \pm 1.48 \times 10^{-2}$	6.00×10^{-158}	0.187

Problem 1. We begin with the system of two equations (selected from [19]):

$$\begin{cases} x_1 + e^{x_2} - \cos x_2 = 0, \\ 3x_1 - \sin x_1 - x_2 = 0, \end{cases}$$

with initial value $\mathbf{x}^{(0)} = \{-1, 1\}^T$ towards the solution: $\mathbf{r} = \{0, 0\}^T$.

Problem 2. Here we consider the system of three nonlinear equations (selected from [32]):

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 - 9 = 0, \\ x_1 x_2 x_3 - 1 = 0, \\ x_1 + x_2 - x_3^2 = 0. \end{cases}$$

with initial approximation $\mathbf{x}^{(0)} = \{3, 1, 2\}^T$ towards the solution:

$$\mathbf{r} = \{2.4913756968306888..., 0.24274587875713651..., 1.6535179393002742...\}^T.$$

Table 6
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 5				
M_1	3	$6 \pm 3.42 \times 10^{-10}$	2.09×10^{-306}	4.985
M_2	3	$6 \pm 1.89 \times 10^{-9}$	2.20×10^{-280}	5.985
M_3	3	$6 \pm 3.42 \times 10^{-10}$	2.09×10^{-306}	4.942
M_4	3	$6 \pm 6.18 \times 10^{-9}$	2.65×10^{-262}	4.71
M_5	3	$6 \pm 3.21 \times 10^{-9}$	7.95×10^{-273}	5.02
M_6	5	$6 \pm 1.59 \times 10^{-10}$	7.06×10^{-317}	6.24
M_7	5	$6 \pm 2.32 \times 10^{-8}$	8.83×10^{-318}	12.33

Table 7
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 6				
M_1	3	$6 \pm 1.98 \times 10^{-49}$	1.92×10^{-1717}	0.83
M_2	2	$6 \pm 0.30 \times 10^{-1}$	1.03×10^{-36}	1.023
M_3	2	$6 \pm 7.17 \times 10^{-3}$	1.86×10^{-46}	1.13
M_4	2	$6 \pm 5.96 \times 10^{-2}$	1.57×10^{-32}	1.079
M_5	2	$6 \pm 7.67 \times 10^{-2}$	3.01×10^{-32}	1.32
M_6	2	$6 \pm 2.38 \times 10^{-1}$	4.90×10^{-53}	1.271
M_7	2	$6 \pm 1.50 \times 10^{-1}$	2.50×10^{-39}	1.83

Table 8
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 7				
M_1	3	$6 \pm 1.92 \times 10^{-2}$	4.33×10^{-15}	0.236
M_2	3	$6 \pm 0.07 \times 10^{-1}$	4.68×10^{-15}	0.187
M_3	3	$6 \pm 0.09 \times 10^{-1}$	4.39×10^{-15}	0.350
M_4	3	$6 \pm 0.01 \times 10^{-1}$	4.70×10^{-15}	0.273
M_5	3	$6 \pm 0.02 \times 10^{-1}$	5.03×10^{-15}	0.312
M_6	3	$6 \pm 0.08 \times 10^{-3}$	4.39×10^{-15}	0.234
M_7	3	$6 \pm 9.61 \times 10^{-4}$	5.03×10^{-15}	0.2395

Table 9
Comparison of the performances of methods.

Methods	k	$(\rho_k \pm \triangle\rho_k)$	$\|\|\mathbf{F}(\mathbf{x}^{(k)})\|\|$	CPU-Time
Problem 8				
M_1	4	$6 \pm 3.04 \times 10^{-52}$	1.62×10^{-1812}	665.06
M_2	4	$6 \pm 3.22 \times 10^{-48}$	1.62×10^{-717}	414.44
M_3	4	$6 \pm 3.04 \times 10^{-52}$	1.62×10^{-1812}	714.286
M_4	4	$6 \pm 4.50 \times 10^{-52}$	1.49×10^{-1807}	707.925
M_5	4	$6 \pm 1.91 \times 10^{-51}$	4.31×10^{-1785}	768.624
M_6	4	$6 \pm 2.99 \times 10^{-49}$	4.68×10^{-7}	780.81
M_7	4	$6 \pm 1.84 \times 10^{-50}$	4.76×10^{-7}	665.67

Problem 3. The next example we consider is the following mixed Hammerstein integral equation (see [19]):

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 dt,$$

where $x \in C[0, 1]$, $s, t \in [0, 1]$ and the kernel G is

$$G(s,t) = \begin{cases} (1-s)t, & \text{if } t \le s\ , \\ s(1-t), & \text{if } s < t. \end{cases}$$

We transform the above equation into a finite-dimensional problem by using Gauss–Legendre quadrature formula given as

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{n} \varpi_j f(t_j),$$

where the abscissas t_j and the weights ϖ_j are determined for $n=8$. Denoting the approximation of $x(t_i)$ by x_i $(i=1,2,\ldots,8)$, we obtain the system of nonlinear equations

$$5x_i-\sum_{j=1}^{8}a_{ij}x_j^3-5=0,$$

where for $i=1,2,\ldots,8$,

$$a_{ij}=\begin{cases}\varpi_j t_j(1-t_i), & \text{if } j\le i\ ,\\ \varpi_j t_i(1-t_j), & \text{if } i<j,\end{cases}$$

wherein the abscissas t_j and the weights ϖ_j are known and given in Table 1 for $n=8$. The initial approximation assumed is $\mathbf{x}^{(0)}=\{-0.5,-0.5,\ldots,-0.5\}^T$ for obtaining the solution of this problem,

$$\mathbf{r}=\{1.0020962450311568..., 1.0099003161874888..., 1.0197269609931769..., 1.0264357430306205..., 1.0264357430306205..., 1.0197269609931769..., 1.0099003161874888..., 1.0020962450311568...\}^T.$$

Problem 4. Next, consider the boundary value problem (see [34]):

$$y''+y^3=0,\quad y(0)=0,\quad y(1)=1,$$

Consider the following partitioning of the interval [0,1]:
$u_0=0<u_1<u_2<\cdots<u_{m-1}<u_m=1,\quad u_{j+1}=u_j+h,\quad h=1/m.$
Let us define $y_0=y(u_0)=0, y_1=y(u_1),\ldots,y_{m-1}=y(u_{m-1}), y_m=y(u_m)=1$. If we discretize the problem by using the numerical formula for second derivative

$$y_k''=\frac{y_{k-1}-2y_k+y_{k+1}}{h^2},\quad k=1,2,3,\ldots,m-1,$$

we obtain a system of $m-1$ nonlinear equations in $m-1$ variables:

$$y_{k-1}-2y_k+y_{k+1}+h^2y_k^3=0,\quad k=1,2,3,\ldots,m-1.$$

In particular, we solve this problem for $m=11$ so that $n=10$ by selecting $\mathbf{y}^{(0)}=\{1,0,1,0,\ldots,1,0\}^T$ as the initial value. The solution of this problem is,

$$\mathbf{r}=\{0.09596073069865263 4..., 0.19191415849873986..., 0.28780916977931909..., 0.38350715280542599..., 0.47873897579953763..., 0.57306399988514091..., 0.66583368873854480..., 0.75616381522678097..., 0.84292070075827740..., 0.92472793289206890...\}^T.$$

Problem 5. In this example, we present a system of twenty nonlinear equations (selected from [34]):

$$x_i-\cos\left(2x_i-\sum_{j=1}^{20}x_j\right)=0,\quad 1\le i\le 20.$$

In this problem a closer choice of initial approximation to the required solution is very much needed since the problem has many solution vectors with the same value of each component of magnitude less than one in every solution vector. That means each solution vector satisfies $\|\mathbf{r}\|=\sqrt{\sum_{i=1}^{20}|r_i|^2}<\sqrt{20}$. We choose the initial approximation $\mathbf{x}^{(0)}=\{-0.9,-0.9,\ldots,-0.9\}^T$ for obtaining the solution,

$$\mathbf{r}=\{-0.89797814194212824..., -0.89797814194212824..., \cdots, -0.89797814194212824...\}^T.$$

Problem 6. Next, we consider the system of nonlinear equations (selected from [10]):

$$\begin{cases}x_ix_{i+1}-e^{-x_i}-e^{-x_{i+1}}=0, & 1\le i\le n-1,\\ x_nx_1-e^{-x_n}-e^{-x_1}=0.\end{cases}$$

In particular, we solve this problem for $n=35$ by selecting $\mathbf{x}^{(0)}=\{1.2,1.2,\ldots,1.2\}^T$ as initial value. The solution of this problem is,

$$\mathbf{r}=\{0.901201031729666145..., 0.901201031729666145..., \cdots, 0.901201031729666145...\}^T.$$

Problem 7. Further, we solve a system of nonlinear equations which arise while solving the following nonlinear partial differential equation, (see [35])

$$u_{xx}+u_{yy}=u^2,\quad (x,y)\in[0,1]\times[0,1]$$

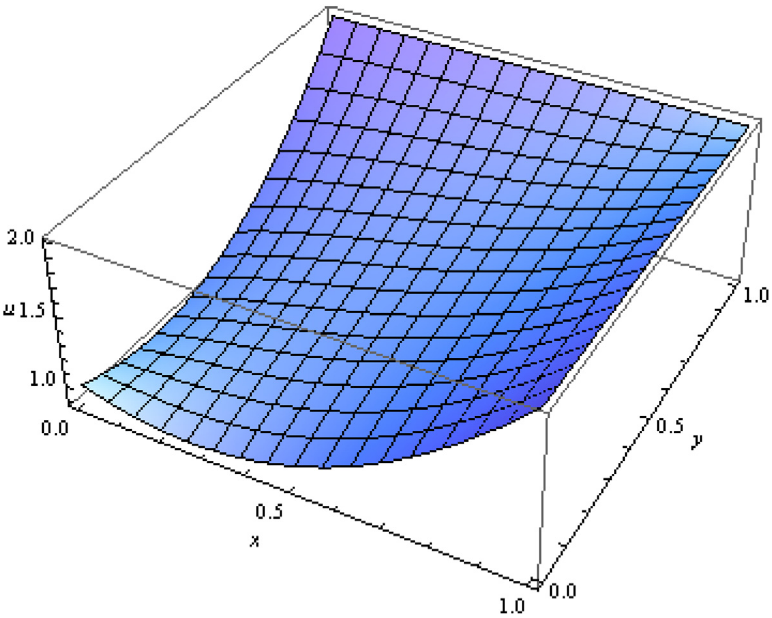

Fig. 1. Approximate solution of Poisson's equation.

with boundary conditions

$$u(x,0) = 2x^2 - x + 1, \quad u(x,1) = 2,$$
$$u(0,y) = 2y^2 - y + 1, \quad u(1,y) = 2.$$

The solution of a nonlinear partial differential equation can be found using finite difference discretization. Let $u = u(x,y)$ be the exact solution of this poisson equation.

Let $w_{i,j} = u(x_i, y_j)$ be its approximate solution at the grid points of the mesh. Let M and N be the number of steps in x and y directions and h and k be the respective step size.

If we discretize the problem by using the central divided differences i.e.

$u_{xx}(x_i, y_j) = (w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h^2$ and $u_{yy}(x_i, y_j) = (w_{i,j+1} - 2w_{i,j} + w_{i,j-1})/k^2$, we get the following system of nonlinear equations:

$$w_{i+1,j} - 4w_{i,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - h^2 w_{i,j}^2 = 0,$$
$$i = 1, 2, \ldots, \ M, \ \ j = 1, 2, \ldots, \ N.$$

We here consider $M = 11$ and $N = 11$ and transform the problem of solving a PDE to a nonlinear system of 100 equations in 100 unknowns using boundary conditions. For the sake of brevity, we have renamed the unknowns as:

$$x_1 = w_{1,1}, \ x_2 = w_{1,2}, \ \ldots x_{10} = w_{1,10},$$
$$x_{11} = w_{2,1}, \ x_{12} = w_{2,2}, \ \ldots x_{20} = w_{2,10},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{91} = w_{10,1}, \ x_{92} = w_{10,2}, \ \ldots x_{100} = w_{10,10}.$$

We have taken the initial guess as $\mathbf{x}^{(0)} = \{1, 1, \cdots, 1\}^T$ towards the solution of the problem given by $\mathbf{r} = \{0.925418, 0.928755, \ldots, 1.9493\}^T$. The approximate solution found has also been plotted in Fig. 1.

Problem 8. Lastly, we consider the following system of equations for $n = 999$ (see [17]):

$$\begin{cases} x_i \sin x_{i+1} - 1 = 0, & 1 \le i \le n-1, \\ x_n \sin x_1 - 1 = 0. \end{cases}$$

with initial approximation $\mathbf{x}^{(0)} = \{-1, -1, \ldots, -1\}^T$ towards the solution:

$$\mathbf{r} = \{-1.114157140871930087\ldots, -1.114157140871930087\ldots, \cdots, -1.114157140871930087\ldots\}^T.$$

5. Concluding remarks

In the foregoing study, we have obtained a novel bi-parametric family of sixth order iterative methods for solving systems of nonlinear equations. Direct computation by Taylor's expansion is used to prove the local convergence order of the methods. By applying some members of this family selected on the basis of dynamical analysis presented in the appendix, we obtain competitive and robust numerical behavior of the corresponding schemes. These particular members have less computational cost and hence more efficient. It is observed that for large-scale systems, the present method has an edge over similar existing methods. In addition, the dynamical analysis (presented in the appendix) shows that the stability of a member of the family depends clearly on the value of the parameter that defines it.

Acknowledgements

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Appendix A. Dynamical analysis

Although the major consideration behind the construction of iterative schemes is to get improve order of convergence retaining high computational efficiency, their stability is also of great concern to be taken into account. The wideness of basins of attraction and the nature of convergence to attracting elements other than the root are issues of great significance in this regard. The analysis of complex dynamics of the iterative methods gives us interesting information about these topics.

Here, we study the dynamical behavior of operator associated to the bi-parametric scheme (23) on quadratic polynomials using some tools from complex dynamics. In order to study the dynamical behavior of an iterative method when it is applied on a polynomial $p(z)$, the basic dynamical concepts of rational functions can be found in, for example, [25].

The following Scaling Theorem shows that the complex dynamics of two rational functions related by means of an affine transformation is conjugated. It allows to affirm that the analysis made on an specific second degree polynomial is conjugate to the obtained by any other polynomial of the same degree.

Theorem 3 (Scaling Theorem)**.** *Let $A(z) = az + b$ be an affine transformation. Let also $f(z)$ be an analytic function and $g(z) = (f o A)(z)$. Then, the associated operators to family (23) H_f and H_g are affinely conjugated by A, that is, $(A \circ H_g \circ A^{-1})(z) = H_f(z)$, for all z.*

So, the behavior of this family of methods on quadratic polynomials can be reduced to the study of this scheme applied on $p(z) = (z - a)(z - b)$. Then, the resulting operator is a rational map, $H_p(z, \alpha, \lambda, a, b)$, given by the quotient of a 14th-degree polynomial and a 13th-degree polynomial and depending on parameters a, b, α and λ.

Further on, to simplify the expression and eliminate the parameters a and b, we here consider the Möbius transformation (see [25])

$$h(z) = \frac{z-a}{z-b},$$

having the following properties:

$$h(\infty) = 1, \quad h(a) = 0, \quad h(b) = \infty.$$

Then, operator $H_p(z, \alpha, \lambda, a, b)$ is conjugated to

$$O_p(z,\alpha,\lambda) = (h \circ H_p \circ h^{-1})(z) = z^6 \frac{\left(z^2 - (\alpha-4)z + 5 - 2\alpha\right) g_p(z,\alpha,\lambda)}{\left((2\alpha-5)z^2 + (\alpha-4)z - 1\right) q_p(z,\alpha,\lambda)}, \tag{25}$$

where

$$\begin{aligned} g_p(z,\alpha,\lambda) = {} & 3(\lambda+1)z^6 + z^5(-3\alpha(\lambda+1) + 14\lambda + 24) + z^4(-2\alpha(4\lambda+9) + 24\lambda + 84) \\ & + z^3(6(3\lambda+28) - 5\alpha(\lambda+9)) + z^2(6\alpha(\lambda-9) - 5(\lambda-39)) \\ & + 2z(\alpha(\lambda-9) - 2\lambda + 48) - 2(\lambda-9), \\ q_p(z,\alpha,\lambda) = {} & 2(\lambda-9)z^6 - 2z^5(\alpha(\lambda-9) - 2\lambda + 48) + z^4(5(\lambda-39) - 6\alpha(\lambda-9)) \\ & + z^3(5\alpha(\lambda+9) - 6(3\lambda+28)) + 2z^2(\alpha(4\lambda+9) - 6(2\lambda+7)) \\ & + z(3\alpha(\lambda+1) - 2(7\lambda+12)) - 3(\lambda+1). \end{aligned}$$

Nevertheless, there are several values of the parameters that simplify the rational function, decreasing the degrees of the involved polynomials, as

- If $\alpha = 2$, the operator is reduced to

$$O_p(z,2,\lambda) = z^6 \frac{-2(\lambda-9) + 4z(\lambda+6) + z^2(\lambda+21) + 2z^3(\lambda+6) + 3z^4(\lambda+1)}{3(\lambda+1) + 2z(\lambda+6) + z^2(\lambda+21) + 4z^3(\lambda+6) - 2z^4(\lambda-9)}.$$

- If $\alpha = \dfrac{10}{3}$, the resulting rational function is
$$O_p\left(z, \frac{10}{3}, \lambda\right) = z^6 \frac{(5+3z)q_1(z,\lambda)}{(3+5z)q_2(z,\lambda)},$$
where $q_1(z,\lambda) = (-6-8z^4)(-9+\lambda) + (45z^2+9z^6)(1+\lambda) + 6z^5(7+2\lambda) + 4z(27+2\lambda) + z^3(54+4\lambda)$ and $q_2(z,\lambda) = (8z^2+6z^6)(-9+\lambda) - (9+45z^4)(1+\lambda) - 6z(7+2\lambda) - (2z^3+4z^5)(27+2\lambda)$.
- If $\lambda = \dfrac{3}{2}$, then the fixed point operator is reduced to
$$O_p\left(z, \alpha, \frac{3}{2}\right) = -z^6 \frac{\left(2+z^4+z^2(7-2\alpha) - (2z+z^3)(-4+\alpha)\right)\left(5+z^2-z(-4+\alpha)-2\alpha\right)}{\left(1+2z^4+z^2(7-2\alpha) - (z+2z^3)(-4+\alpha)\right)\left(-1+z(-4+\alpha)+z^2(-5+2\alpha)\right)}.$$

Taking into account the singularities of the bi-parametric rational function, different subfamilies can be analyzed, depending only on one parameter. The special cases $\alpha = 2$, $\alpha = \frac{10}{3}$ and $\lambda = \frac{3}{2}$ have been analyzed, but the second one is discarded, as $z = 1$ is not a fixed point of $O_p\left(z, \frac{10}{3}, \lambda\right)$ but forms a superattracting periodic orbit joint with $z = -1$. So, for the family corresponding to $\alpha = \frac{10}{3}$ there is no possibility of having only convergence to the roots and it is rejected. In what follows, we discuss in detail, the dynamical analysis of the subclasses of methods corresponding to $\lambda = \frac{3}{2}$ and $\alpha = 2$.

A1. The uniparametric family $O_p\left(z, \alpha, \frac{3}{2}\right)$

When we consider $\lambda = \dfrac{3}{2}$ in (25), the rational operator is simplified and takes the form,
$$O_p\left(z, \alpha, \frac{3}{2}\right) = -z^6 \frac{\left(z^4 - (\alpha-4)(z^3+2z) + (7-2\alpha)z^2 + 2\right)\left(-2\alpha + z^2 - (\alpha-4)z + 5\right)}{\left(2z^4 - (\alpha-4)(2z^3+z) + (7-2\alpha)z^2 + 1\right)\left((2\alpha-5)z^2 + (\alpha-4)z - 1\right)}.$$
Further, there exist some values of parameter α that simplify the expression of operator $O_p\left(z, \alpha, \dfrac{3}{2}\right)$:

1. If $\alpha = 2$, then
$$O_p\left(z, 2, \frac{3}{2}\right) = z^6 \frac{z^2+2}{2z^2+1}.$$
2. If $\alpha = \dfrac{5}{2}$, then
$$O_p\left(z, \frac{5}{2}, \frac{3}{2}\right) = z^7 \frac{(2z+3)\left(2z^4+3z^3+4z^2+6z+4\right)}{(3z+2)\left(4z^4+6z^3+4z^2+3z+2\right)}.$$
3. When $\alpha = \dfrac{22}{5}$, then the rational function results
$$O_p\left(z, \frac{22}{5}, \frac{3}{2}\right) = -z^6 \frac{(5z^2-2z-19)(5z^3+3z^2-6z-10)}{(19z^2+2z-5)(10z^3+6z^2-3z-5)}.$$

It is useful to now that there exist different special cases in the presented scheme with lower degree polynomials in their associated rational function, as they lead to lower number of strange fixed points and free critical points and therefore reducing the unstable behavior. In general, $z = 0$ and $z = \infty$ are fixed points of the operator $O_p\left(z, \alpha, \frac{3}{2}\right)$ (related to the roots a and b, respectively, of the polynomial $p(z)$). Also, $z = 1$ is a strange fixed point, which is associated with the original convergence to infinity. The following results summarize the number of strange fixed points and their stability.

Proposition 1. *The point $z = 1$ is, for $\alpha \neq \dfrac{10}{3}$ and $\alpha \neq \dfrac{22}{5}$, a fixed point of the operator $O_p\left(z, \alpha, \frac{3}{2}\right)$. Its stability depends on the value of $\alpha \in \mathbb{C}$, as it is attracting if $64|(\alpha-4)^2| < |(3\alpha-10)(5\alpha-22)|$, superattracting for $\alpha = 4$, parabolic if $64|(\alpha-4)^2| = |(3\alpha-10)(5\alpha-22)|$ and repulsive if $64|(\alpha-4)^2| > |(3\alpha-10)(5\alpha-22)|$.*

Proposition 2. *The strange fixed points of the operator $O_p\left(z, \alpha, \frac{3}{2}\right)$ different from $z = 1$ are the ten roots of the symmetric polynomial $q(z) = -z^{10} + (2\alpha-9)(z^9+z) - \left(\alpha^2 - 14\alpha + 37\right)(z^8+z^2) + \left(-5\alpha^2 + 44\alpha - 93\right)(z^7+z^3) + \left(-11\alpha^2 + 84\alpha - 162\right)(z^6+z^4) + \left(-15\alpha^2 + 108\alpha - 200\right)z^5 - 1$, which are called $ex_i(\alpha)$, $i = 1, \ldots, 10$. Two of them ($ex_i(\alpha)$, $i = 1, 2$) are always repulsive and the rest can be attracting in specific areas of the complex plane. However, the number of fixed points can be lower for specific values of parameter α,*

- *If $\alpha = \dfrac{2}{49}\left(99 \pm i4\sqrt{3}\right)$, there exist eight strange fixed points of the operator $O_p\left(z, \alpha, \frac{3}{2}\right)$ different from $z = 1$, all of them repulsive.*

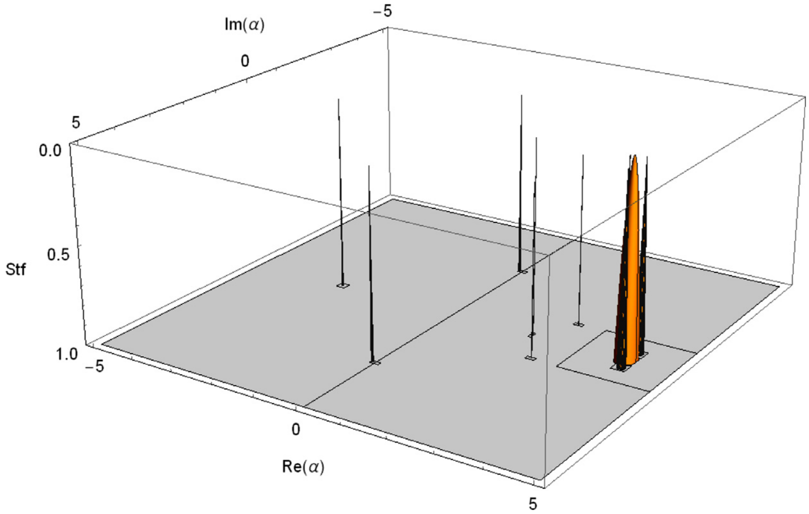


Fig. 2. Stability functions of all strange fixed points of $O_p\left(z, \alpha, \frac{3}{2}\right)$.

- *If* $\alpha = 2$, *the rational function has six strange fixed points different from* $z = 1$, *all of them repulsive.*
- *If* $\alpha \approx -2.79198$, *there are eight strange fixed points different from* $z = 1$, *two of them parabolic and the rest of them, repulsive.*

In Fig. 2 the combined stability functions of all the strange fixed points is presented in the complex plane. It can be observed that the regions of unstable behavior are very small and there exist a wide region of stable values of parameter α. However, it is possible that the convergence of the iterative scheme leads us to other attracting elements, such as periodic orbits. In order to detect this kind of behavior, the analysis of the critical points is necessary.

To determine the critical points, we calculate the first derivative of $O_p(z, \alpha, 3/2)$,

$$O'_p\left(z, \alpha, \frac{3}{2}\right) = -4z^5(z+1)^4\Bigg[\frac{3(2\alpha-5)(1+z^8)+(-17\alpha^2+90\alpha-124)(z+z^7)}{\left(2z^4-2(\alpha-4)z^3+(7-2\alpha)z^2-(\alpha-4)z+1\right)^2\left((2\alpha-5)z^2+(\alpha-4)z-1\right)^2}$$
$$+\frac{+2(8\alpha^3-67\alpha^2+198\alpha-204)(z^6+z^2)(-5\alpha^4+64\alpha^3-335\alpha^2+814\alpha-756)(z^5+z^3)}{\left(2z^4-2(\alpha-4)z^3+(7-2\alpha)z^2-(\alpha-4)z+1\right)^2\left((2\alpha-5)z^2+(\alpha-4)z-1\right)^2}$$
$$+\frac{\left(-5\alpha^4+76\alpha^3-416\alpha^2+1004\alpha-914\right)z^4}{\left(2z^4-2(\alpha-4)z^3+(7-2\alpha)z^2-(\alpha-4)z+1\right)^2\left((2\alpha-5)z^2+(\alpha-4)z-1\right)^2}\Bigg],$$

where

It is a well known fact that there is at least one critical point associated with each invariant Fatou component. Clearly, $z = 0$ and $z = \infty$ are critical points (as they are superattracting fixed points for being the order of the iterative method higher than two), which are related to the roots of polynomial $p(z)$ and give rise to their respective Fatou components. There can also exist critical points, not related to the roots, called free critical points.

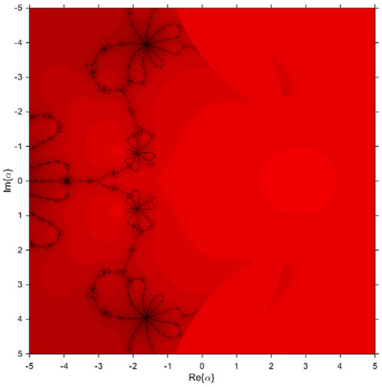
By solving the equation $O'_p(z, \alpha, 3/2) = 0$, eight free critical points are obtained:

$$cr_i(\alpha) = \frac{1}{2}(\alpha - 2 \pm \sqrt{\alpha-4}\sqrt{\alpha}), \quad i = 1, 2,$$

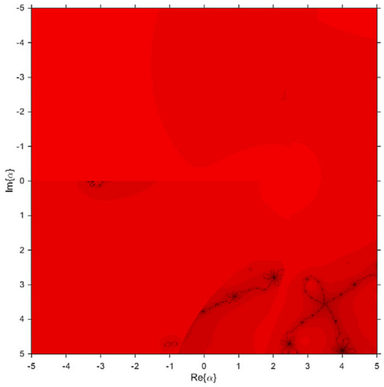
that are preimages of strange fixed point $z = 1$ and the roots of the symmetric six-degree polynomial $(6\alpha-15)(1+t^6)+(-94+63\alpha-11\alpha^2)(t+t^5)+(-205+170\alpha-49\alpha^2+5\alpha^3)(t^2+t^4)+(-252+206\alpha-56\alpha^2+5\alpha^3)t^3$.

Following [28], we solve this six-degree polynomial by dividing it by t^3 and then applying the change of variable $x = t + \frac{1}{t}$. This transforms the sixth-degree polynomial into the cubic one $(6\alpha-15)x^3+(-94+63\alpha-11\alpha^2)x^2+(-160+152\alpha-49\alpha^2+5\alpha^3)x+(-430+80\alpha-34\alpha^2+5\alpha^3)$, whose roots $x_i(\alpha)$, $i = 1, 2, 3$ can be found analytically and then, the critical points that are the roots of the sixth-degree polynomial are

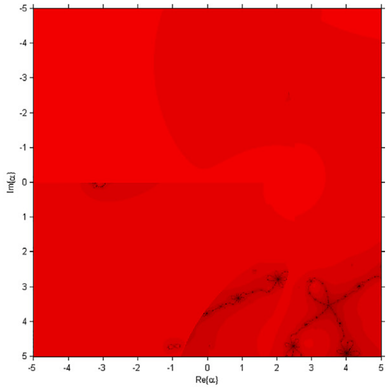
- $cr_3(\alpha) = \frac{x_1(\alpha)+\sqrt{x_1(\alpha)^2-4}}{2} = \frac{1}{cr_4(\alpha)}$
- $cr_5(\alpha) = \frac{x_2(\alpha)+\sqrt{x_2(\alpha)^2-4}}{2} = \frac{1}{cr_6(\alpha)}$



(a) $cr_3(\alpha)$

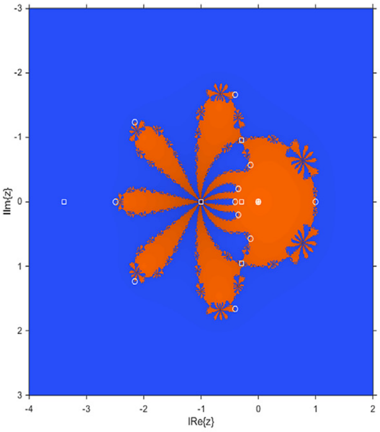


(b) $cr_5(\alpha)$

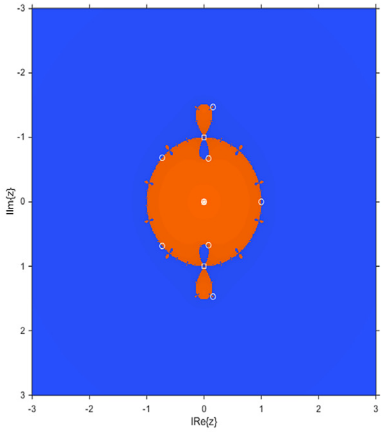


(c) $cr_7(\alpha)$

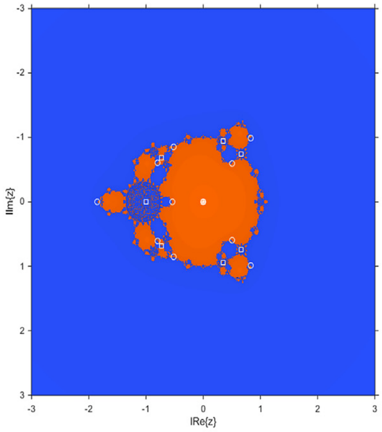
Fig. 3. Parameter planes of free independent critical points of $O_p\left(z, \alpha, \frac{3}{2}\right)$.



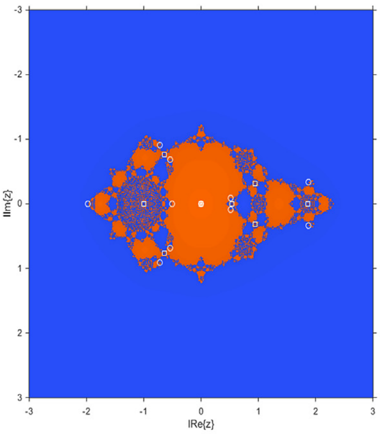
(a) $\alpha = 0$



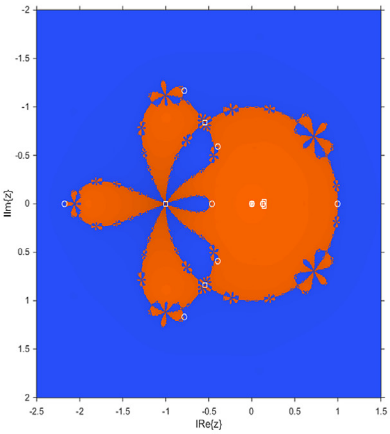
(b) $\alpha = 2$



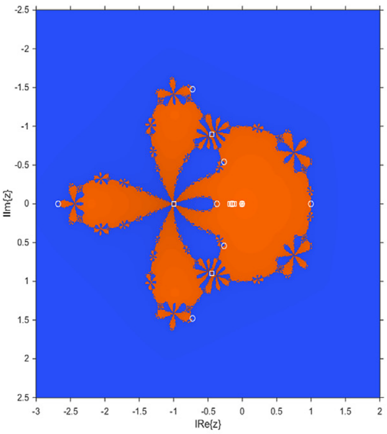
(c) $\alpha = \frac{10}{3}$



(d) $\alpha = \frac{22}{5}$

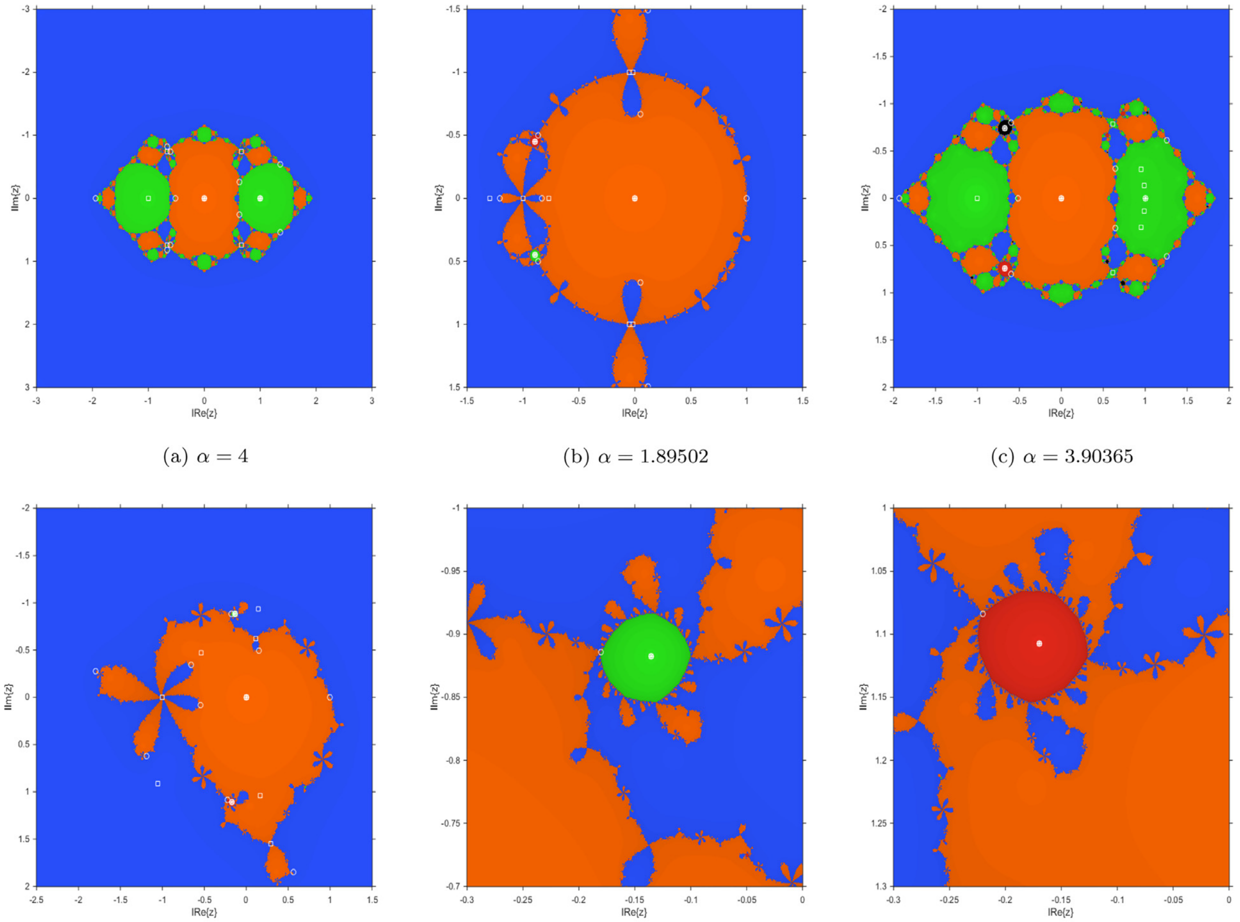


(e) $\alpha = 10$



(f) $\alpha = -5$

Fig. 4. Dynamical planes with stable behavior. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.).



(a) $\alpha = 4$ (b) $\alpha = 1.89502$ (c) $\alpha = 3.90365$

(d) $\alpha = 2.4087 + i0.931271$ (e) $\alpha = 2.4087 + i0.931271$, a detail (f) $\alpha = 2.4087 + i0.931271$, a detail

Fig. 5. Dynamical planes with unstable behavior.

- $cr_7(\alpha) = \frac{x_3(\alpha)+\sqrt{x_3(\alpha)^2-4}}{2} = \frac{1}{cr_8(\alpha)}$

So, the following results can be stated.

- If $\alpha = 2$, then there exist three free critical points: $z = -1$, and $z = \pm i$ that are preimages of $z = 1$.
- If $\alpha = 4$, then there are six free critical points $z = \pm 1$, $z = \frac{1}{3}(-2 \pm i\sqrt{5})$ and $z = \frac{1}{3}(2 \pm i\sqrt{5})$.
- If $\alpha = 0$, then there are five free critical points, $z = 1$, $z = \frac{1}{30}(-32 - 4\sqrt{34} \pm 15\sqrt{\frac{668}{225} + \frac{256\sqrt{34}}{225}})$ and $z = \frac{1}{15}(-16 + 2\sqrt{34} \pm i\sqrt{-167 + 64\sqrt{34}})$
- If $\alpha = \frac{5}{2}$, then there exist seven free critical points: $z = -1$, $z = \frac{1}{4}(1 \pm i\sqrt{(5)})$ and the roots of $21 + \frac{65}{2}z + \frac{71}{2}z^2 + \frac{65}{2}z^3 + 21z^4$, that is, $z = \frac{1}{168}(-65 - \sqrt{6409} \pm 84i\sqrt{\frac{8795}{3528} - \frac{65\sqrt{6409}}{3528}})$ and $z = \frac{1}{168}(-65 + \sqrt{6409} \pm i\sqrt{10(1759 + 13\sqrt{6409})})$.
- Finally, if $\alpha = \frac{22}{5}$, there are also seven critical points, $z = -1$, $z = \frac{1}{5}(6 \pm \sqrt{11})$, $z = \frac{1}{190}(29 + \sqrt{22881} \pm 95i\sqrt{\frac{12378}{9025} - \frac{58\sqrt{22881}}{9025}})$ and $z = \frac{1}{190}(29 - \sqrt{22881} \pm i\sqrt{2(6189 + 29\sqrt{22881})})$.

To observe the dynamical behavior of operator $O_p(z, \alpha, \frac{3}{2})$, we analyze the parameter space associated to it. It is obtained by applying the operator on an independent free critical point as initial estimation and coloring the point of the plane corresponding to the value of α (we have used a mesh of 800×800 points), that is, with an element of the family of iterative methods. A point is painted in red if the iteration of the method starting with the critical point goes to the basin of attraction of 0 or ∞ and in black if it converges to attracting strange fixed point or attracting periodic orbits or even to $z=1$, or if the maximum number of 200 iterations is reached.

As previously stated, free critical points $cr_1(\alpha)$ and $cr_2(\alpha)$ are preimages of the strange fixed point $z=1$, so there is no sense in analyzing their associated parameter space. On the other hand, we know that $cr_3(\alpha)=\frac{1}{cr_4(\alpha)}$, $cr_5(\alpha)=\frac{1}{cr_6(\alpha)}$, and $cr_7(\alpha)=\frac{1}{cr_8(\alpha)}$. Then, there exist only three independent free critical points, whose associated parameter space can be seen in Fig. 3.

It can be observed that the stability regions of the strange fixed points appear in some of the parameter planes as black areas and even some smaller black areas appear, that correspond to values of the parameter α whose dynamical planes contain a periodic orbit of any period. Those regions that appear in red simultaneously in all the parameter planes correspond to the set of values of α where only convergence to the roots ($z=0$ and $z=\infty$ after the Möbius map) is allowed.

In order to fully interpret the obtained parameter space, we draw some dynamical planes corresponding to selected values of parameter α. We have used the software presented in [26] implemented in Matlab, as well as for plotting the parameter planes. To obtain these pictures, we have used a mesh of 800×800, a maximum number of iterations of 80 and 10^{-3} as a tolerance. The colors used in different pictures also give us important information: the convergence to fixed point 0 appears in orange; blue regions correspond to the basin of the infinity and in black, the zones with no convergence to the roots are shown in different colors if they are fixed points or in black color if it is the basin of a periodic orbit. The strange fixed points are represented by white circles, with white stars if they are superattracting, while critical points are marked as white squares.

In Fig. 4, the dynamical planes associated with different stable values of α are drawn. Some of the values of α used correspond to those that simplify the rational function (as $\alpha=0$, $\alpha=2$, $\alpha=\frac{10}{3}$ and $\alpha=\frac{22}{5}$ in Fig. 4a–d), and others, that have been selected in the common red region of the parameter planes ($\alpha=10$ and $\alpha=-5$ in Fig. 4e and f). Their common feature in the existence of only two basins of attraction, corresponding to $z=0$ and $z=\infty$.

With respect to unstable behavior, in Fig. 5, different values of the parameter have been selected where any of the strange fixed point is superattracting. It is the case of Fig. 5a, where $\alpha=4$ and $z=1$ is superattracting, or Fig. 5b and c, where $ex_3(\alpha)$ and $ex_4(\alpha)$ are simultaneously superattracting and also the case of $\alpha=2.4087+i0.931271$ in Fig. 5c; for this value of the parameter $ex_7(\alpha)$ and $ex_8(\alpha)$ are simultaneously superattracting, but their respective basins of attraction are so small that they have been detailed in Fig. 5e and f.

A2. The uniparametric family $O_p(z, 2, \lambda)$

If we replace $\alpha=2$ in (25), the rational function is simplified,

$$O_p(z,2,\lambda)=x^6\frac{\left(-2(-9+\lambda)+3z^4(1+\lambda)+4z(6+\lambda)+2z^3(6+\lambda)+z^2(21+\lambda)\right)}{-2z^4(-9+\lambda)+3(1+\lambda)+2z(6+\lambda)+4z^3(6+\lambda)+z^2(21+\lambda)}.$$

As in the previous case, there are values of parameter λ that simplify the expression of operator $O_p(z, 2, \lambda)$:

1. If $\lambda=-1$, then
$$O_p(z,2,-1)=z^6\frac{2+2z+2z^2+z^3}{1+2z+2z^2+2z^3}.$$
2. If $\lambda=\dfrac{3}{2}$, then
$$O_p\left(z,2,\frac{3}{2}\right)=z^6\frac{2+z^2}{1+2z^2}.$$

Regarding the number and stability of the strange fixed points, specially $z=1$ that comes from the divergence of the original iterative class of methods, they are summarized in the following results.

Proposition 3. *The point $z=1$ is, for $\lambda\neq\dfrac{39}{4}$, a fixed point of the operator $O_p(z, 2, \lambda)$. Its stability depends on the value of $\alpha\in\mathbb{C}$, as it is attracting if $\left|\lambda+\frac{499}{84}\right|<\frac{10}{21}$, superattracting for $\alpha=-6$, parabolic if $\left|\lambda+\frac{499}{84}\right|=\frac{10}{21}$ and repulsive if $\left|\lambda+\frac{499}{84}\right|>\frac{10}{21}$.*

In Fig. 6, the stability function of $z=1$ is presented.

The number and stability of the rest of strange fixed points are described in the following result, that can be proved in a similar way as in Proposition 2.

Proposition 4. *The strange fixed points of the operator $O_p(z, 2, \lambda)$ different from $z=1$ are the eight roots of the symmetric polynomial $r(z)=(3+3\lambda)(1+z^8)+(15+5\lambda)(z+z^7)+(36+6\lambda)(z^2+z^6)+(60+10\lambda)(z^3+z^5)+(78+8\lambda)z^4$, which are called $s_i(\lambda)$, $i=1,\dots,8$. All of them are attracting in specific areas of the complex plane. However, the number of fixed points can be lower for $\lambda=-\frac{153}{28}$; for this value $z=1$ is parabolic and six more strange fixed points, all of them repulsive.*

Now, we present in Fig. 7 the combined stability functions of all the strange fixed points. We observe that the regions of unstable behavior are quite small, located mainly around real values of $\lambda\approx 20$ and $\lambda\approx -7$ and there also exist a wide region of stable values of the parameter.

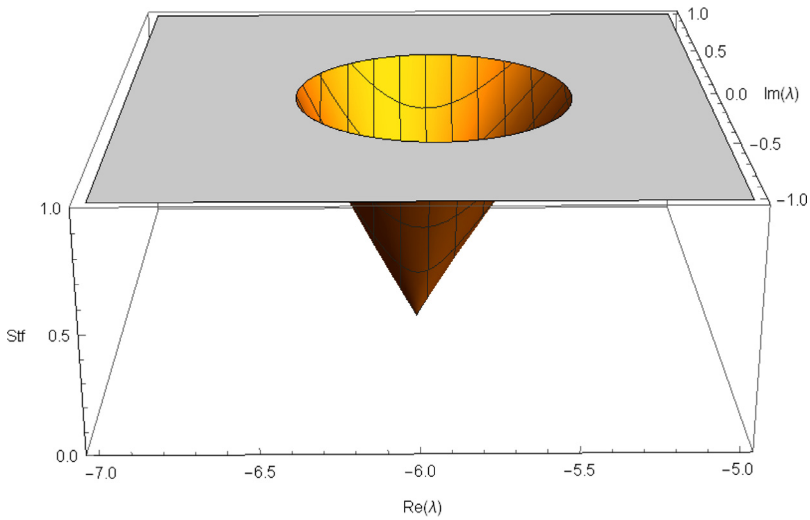


Fig. 6. Stability functions of fixed point $z = 1$.

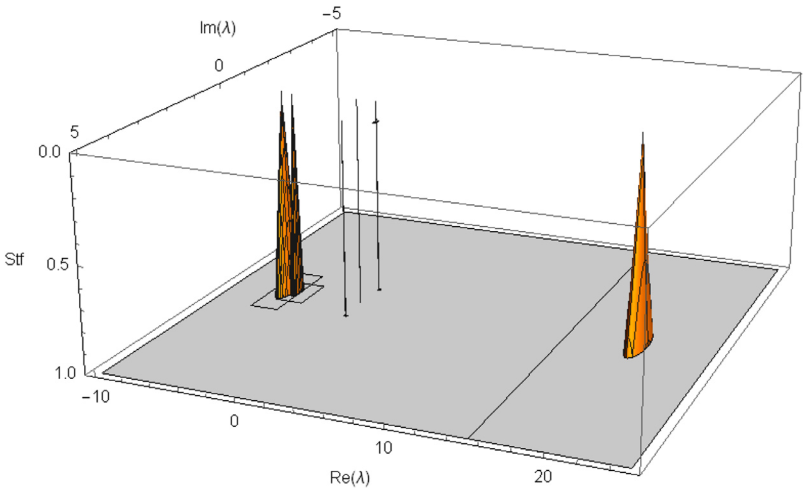


Fig. 7. Stability functions of all strange fixed points of $O_p(z, 2, \lambda)$.

In order to complete the analysis, we determine the critical points, calculating the first derivative of $O_p(z, 2, \lambda)$,

$$O'_p(z, 2, \lambda) = -4z^5(z+1)^1(1+z^2)\frac{9R(\lambda) + 9z^4R(\lambda) - 2zS(\lambda) - 2z^3S(\lambda) + z^2\left(-306 - 72\lambda + 34\lambda^2\right)}{\left(-2z^4(-9+\lambda) + 3(1+\lambda) + 2z(6+\lambda) + 4z^3(6+\lambda) + z^2(21+\lambda)\right)^2},$$

where $R(\lambda) = -9 - 8\lambda + \lambda^2$ and $S(\lambda) = 117 + 9\lambda + 17\lambda^2$.

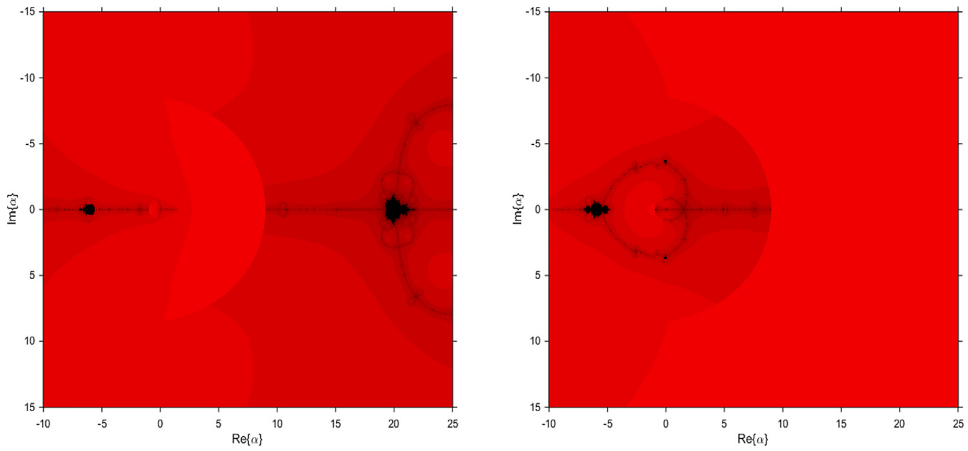
Again, $z = 0$ and $z = \infty$ are critical points and give rise to their respective Fatou components. There exist also three free critical points $cr_1(\lambda) = -1,\ cr_j(\lambda) = \pm i,\ j = 1, 2$ that are preimages of the fixed point $z = 1$ and also the roots of the symmetric fourth-degree polynomial $(-81 - 72\lambda + 9\lambda^2)(1 + z^4) + (-234\lambda - 18\lambda - 34\lambda^2)(z + z^3) + (-306 - 72\lambda + 34\lambda^2)z^2$.

Following the same procedure as in the previous section, we divide this polynomial by z^2 and then applying the change of variable $x = z + \dfrac{1}{z}$, we get a second degree-polynomial whose roots are

$$x_1(\lambda) = \frac{126x^2\lambda + 17x^2\lambda^2 - \frac{1}{2}\sqrt{M(x, \lambda)}}{9(1+x^4)(-9 - 8\lambda + \lambda^2)}$$

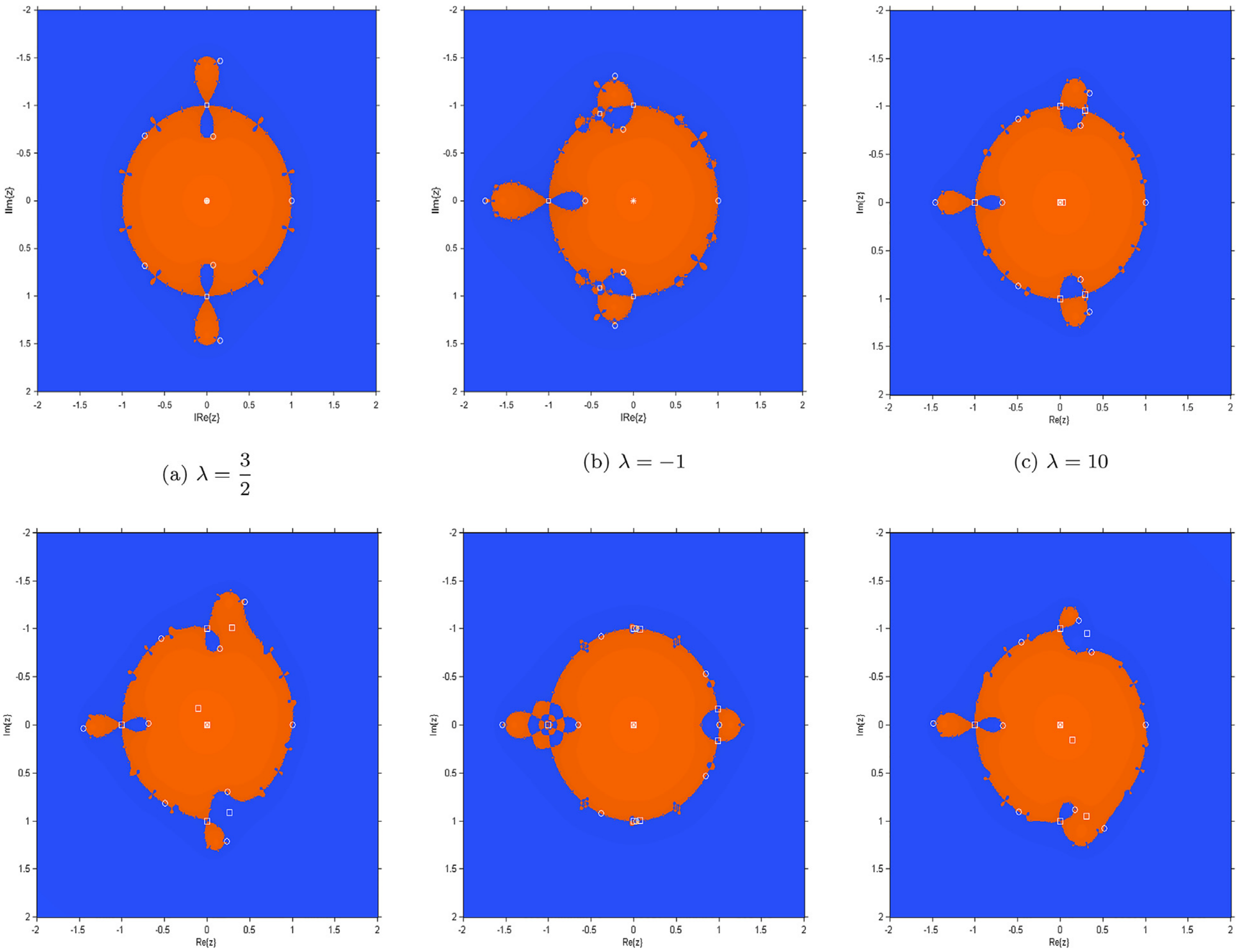
and

$$x_2(\lambda) = \frac{126x^2\lambda + 17x^2\lambda^2 + \frac{1}{2}\sqrt{M(x, \lambda)}}{9(1+x^4)(-9 - 8\lambda + \lambda^2)},$$



(a) $cr_4(\lambda)$ (b) $cr_6(\lambda)$

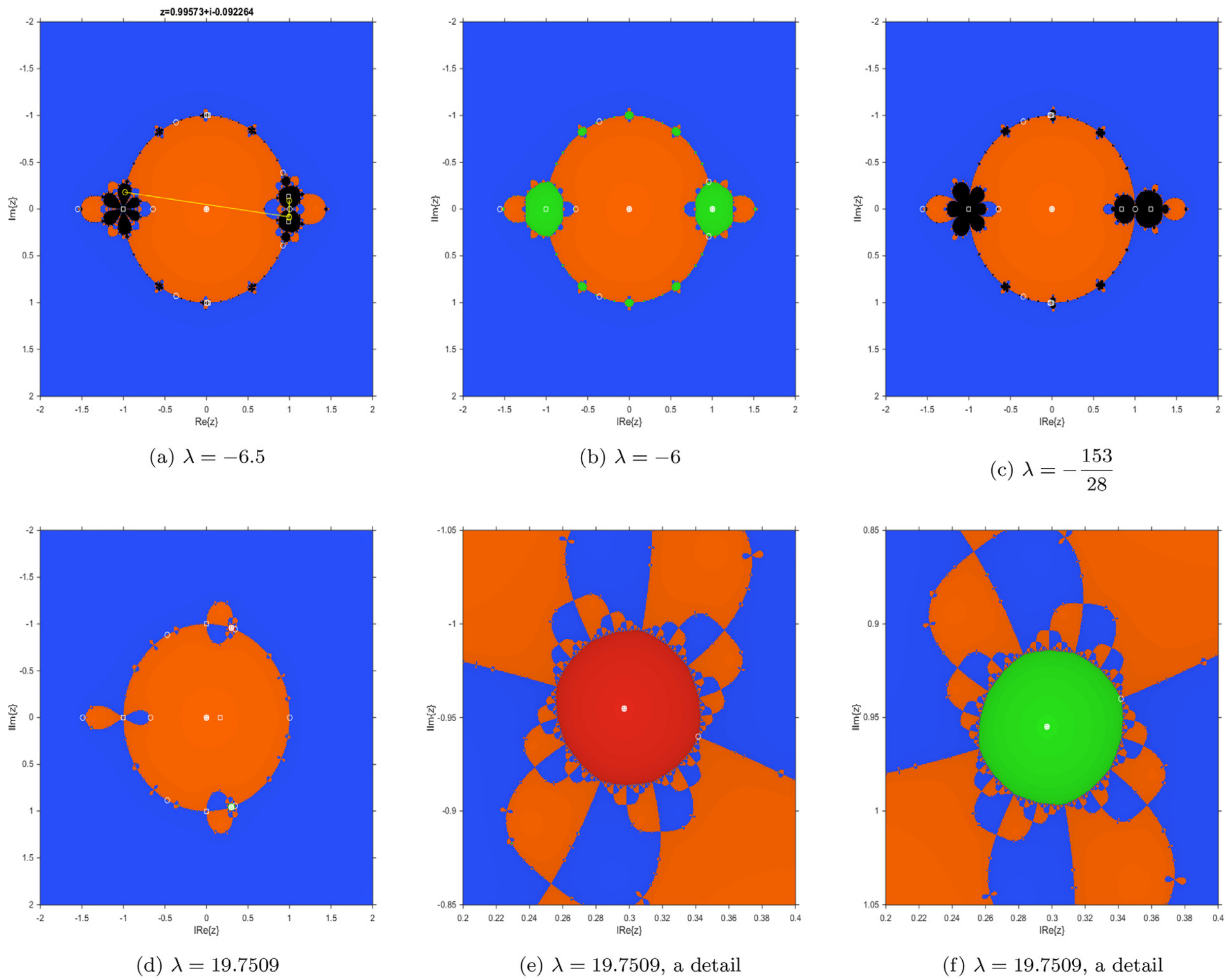
Fig. 8. Parameter planes of free independent critical points of $O_p(z, 2, \lambda)$ (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.).



(a) $\lambda = \frac{3}{2}$ (b) $\lambda = -1$ (c) $\lambda = 10$

(d) $\lambda = 5 - 3i$ (e) $\lambda = -8$ (f) $\lambda = 10 + 10i$

Fig. 9. Dynamical planes with stable behavior. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.).



(a) $\lambda = -6.5$ (b) $\lambda = -6$ (c) $\lambda = -\frac{153}{28}$

(d) $\lambda = 19.7509$ (e) $\lambda = 19.7509$, a detail (f) $\lambda = 19.7509$, a detail

Fig. 10. Dynamical planes with unstable behavior. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.).

where $M(x,\lambda) = 4x^4\lambda^2(126+17\lambda)^2 + 72(1+x^4)(-9-8\lambda+\lambda^2)N(x,\lambda)$ and $N(x,\lambda) = x^2(153+36\lambda-17\lambda^2) + 9(-9-8\lambda+\lambda^2) + 9x^4(-9-8\lambda+\lambda^2)$. Then, the rest of critical points are

- $cr_4(\lambda) = \dfrac{x_1(\lambda) + \sqrt{x_1(\lambda)^2 - 4}}{2} = \dfrac{1}{cr_5(\lambda)}$
- $cr_6(\lambda) = \dfrac{x_2(\lambda) + \sqrt{x_2(\lambda)^2 - 4}}{2} = \dfrac{1}{cr_7(\lambda)}$

Moreover, the number of critical points can be reduced for specific values of the parameter.

- If $\lambda = 0$, then there are five free critical points: $z = -1$, and $z = \pm i$ and $z = \frac{1}{9}\left(-4 \pm i\sqrt{65}\right)$.
- If $\lambda = \frac{3}{2}$, then exist three free critical points: $z = -1$, and $z = \pm i$ that are preimages of $z = 1$.
- Finally, if $\lambda = -6$, there are only two free critical points, $z = \pm 1$.

Regarding the parameter planes, we plot with the same conditions as in the previous uniparametric family the corresponding to independent free critical points $cr_4(\lambda)$ and $cr_6(\lambda)$, that can be seen in Fig. 8. It them, the stability regions of the strange fixed points appear as black region and other black areas appear, corresponding to periodic orbits of different periods.

We also plot some dynamical planes in order to show the different behaviors of the members of the family, that, as has been observed in the mainly red parameter planes, is basically stable.

In Fig. 9, some cases are shown: some of them, as Fig. 9a and b, correspond to values of parameter λ that simplify the operator $O_p(z, 2, \lambda)$ meanwhile the rest of values of λ (Fig. 9c–f) have been selected in the red area of both parameter planes.

In Fig. 10, some kinds of unstable behavior are shown: in Fig. 10a, the basin of attraction of a periodic orbit of period 2 appears in black and the orbit is marked in yellow lines. In Fig. 10b, the dynamical plane shows the appearance of the basins of attraction when $z = 1$ is superattracting; so, three basins appear, being those in green color corresponding to the convergence to $z = 1$. Nevertheless, in Fig. 10c, $z = 1$ is parabolic, being located in the Julia set and the rest of strange fixed points are repulsive. $ex_5(\lambda)$ and $ex_6(\lambda)$ are superattracting for $\lambda = 19.7509$ (Fig. 10d), but even in this case their respective basins of attraction are so small that they have been amplified in Fig. 10e and f.

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