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Increasing the order of convergence of iterative schemes for solving nonlinear systems[☆]

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ABSTRACT

A set of multistep iterative methods with increasing order of convergence is presented, for solving systems of nonlinear equations. One of the main advantages of these schemes is to achieve high order of convergence with few Jacobian and functional evaluations, joint with the use of the same matrix of coefficients in the most of the linear systems involved in the process. Indeed, the application of the pseudocomposition technique on these proposed schemes allows us to increase their order of convergence, obtaining new high-order, efficient methods. Finally, some numerical tests are performed in order to check their practical behavior.

1. Introduction

Many applied problems in Science and Engineering are reduced to solve nonlinear systems numerically. The numerical methods commonly used for solving these problems are iterative. A detailed study of the techniques in construction of iterative methods can be found in the text of Traub [1].

Recently, for $n = 1$, many robust and efficient methods have been proposed in order to obtain high orders of convergence, but in most cases the methods cannot be extended to several variables. However, Babajee et al. in [2] design Chebyshev-like schemes for solving nonlinear systems. In general, few papers for the multidimensional case introduce methods with high order of convergence. The authors design in [3] a modified Newton–Jarratt scheme of sixth-order; in [4] a third-order method is presented for computing real and complex roots of nonlinear systems; Darvishi et al. in [5] improve the order of convergence of known methods from quadrature formulas; Shin et al. compare in [6] Newton–Krylov methods and Newton-like schemes for solving big-sized nonlinear systems; in [7] a general procedure to design high-order methods for problems in several variables is presented; moreover, the Adomian Decomposition has shown to be a useful tool to design new high-order methods (see [8,9]).

The pseudocomposition technique (see [10]) consists of the following: we consider a method of order of convergence p as a predictor, whose penultimate step is of order q, and then we use a corrector step based on the Gaussian quadrature. So, we obtain a family of iterative schemes whose order of convergence is $\min\{q + p, 3q\}$. This is a general procedure to improve the order of convergence of known methods.

To analyze and compare the efficiency of the proposed methods we use the classic efficiency index $I = p^{1/d}$ due to Ostrowski [11], where p is the order of convergence and d is the number of functional evaluations, per iteration.

In this paper, we present three new Newton-like schemes, of order of convergence four, six and eight, respectively. After the analysis of convergence of the new methods, we apply the pseudocomposition technique in order to get higher order procedures.

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The convergence theorem in Section 2 is demonstrated by means of the n-dimensional Taylor expansion of the functions involved. Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Frechet differentiable in D. By using the notation introduced in [3], the qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \ldots, v_q) \in \mathbb{R}^n$. In the following, we will denote as

(a) $F^{(q)}(u)(v_1, \ldots, v_q) = F^{(q)}(u)v_1 \ldots v_q$,
(b) $F^{(q)}(u)v^{q-1}F^{(p)}v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

Indeed, it is well known that, for $\xi + h \in \mathbb{R}^n$ lying in a neighborhood of a solution ξ of the nonlinear system $F(x) = 0$, Taylor's expansion can be applied (assuming that the Jacobian matrix $F'(\xi)$ is nonsingular), and

$$F(\xi + h) = F'(\xi)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O[h^p], \tag{1}$$

where $C_q = (1/q!)[F'(\xi)]^{-1}F^{(q)}(\xi)$, $q \geq 2$. We observe that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\xi) \in \mathcal{L}(R^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

In addition, we can express the Jacobian matrix of F, F', as

$$F'(\xi + h) = F'(\xi)\left[I + \sum_{q=2}^{p-1} qC_q h^{q-1}\right] + O[h^p], \tag{2}$$

where I is the identity matrix.

We denote $e_k = x^{(k)} - \xi$ the error in the kth iteration. The equation $e_{(k+1)} = L{e_k}^p + O[{e_k}^{p+1}]$, where L is a p-linear function $L \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$, is called the *error equation* and p is the *order of convergence*.

We have organized the rest of the paper as follows: in Section 2, we present the new methods of order four, six and eight, respectively. Then, the pseudocomposition technique is applied on them and some new higher-order schemes are obtained, which have also more interesting properties. Section 3 is devoted to the comparison of the different methods by means of several numerical tests.

2. Proposed high-order methods

In the following, we will present a new multistep Newton-type scheme which reaches eighth-order of convergence with five steps, and we will denote it as M8. In the analysis of convergence, we will proof that its first three steps are a fourth-order scheme, denoted by M4, and its four first steps become a sixth-order method that will be denoted by M6. The coefficients involved have been obtained optimizing the order the convergence and the whole scheme requires three functional evaluations of F and two of F' to attain eighth-order of convergence. Let us also note that no linear system must be solved at the second step and the linear systems to be solved in the last three steps have the same matrix. So, the number of operations involved is not as high as it can seem.

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in a neighborhood of $\xi \in D$ which is a solution of the nonlinear system $F(x) = 0$. We suppose that $F'(x)$ is continuous and nonsingular at ξ and $x^{(0)}$ close enough to the solution. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by*

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{1}{2}\left[F'\left(x^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right), \\
z^{(k)} &= \frac{1}{3}\left(4y^{(k)} - x^{(k)}\right), \\
u^{(k)} &= y^{(k)} + \left[F'\left(x^{(k)}\right) - 3F'\left(z^{(k)}\right)\right]^{-1} F\left(x^{(k)}\right), \\
v^{(k)} &= u^{(k)} + 2\left[F'\left(x^{(k)}\right) - 3F'\left(z^{(k)}\right)\right]^{-1} F\left(u^{(k)}\right), \\
x^{(k+1)} &= v^{(k)} + 2\left[F'\left(x^{(k)}\right) - 3F'\left(z^{(k)}\right)\right]^{-1} F\left(v^{(k)}\right),
\end{aligned} \tag{3}$$

converges to ξ with order of convergence eight. The error equation is:

$$e_{k+1} = \frac{1}{9}\left(C_3 - C_2^2\right)\left(C_4 - 9C_3C_2 + 9C_2^3\right)e_k^8 + O[e_k^9].$$

Proof. From (1) and (2) we obtain

$$\begin{aligned}
F(x^{(k)}) &= F'(\xi)\left[e_k + C_2e_k^2 + C_3e_k^3 + C_4e_k^4 + C_5e_k^5 + C_6e_k^6 + C_7e_k^7 + C_8e_k^8\right] + O[e_k^9], \\
F'(x^{(k)}) &= F'(\xi)\left[I + 2C_2e_k + 3C_3e_k^2 + 4C_4e_k^3 + 5C_5e_k^4 + 6C_6e_k^5 + 7C_7e_k^6 + 8C_8e_k^7\right] + O[e_k^8].
\end{aligned} \tag{4}$$

As $\left[F'(x^{(k)})\right]^{-1} F'(x^{(k)}) = I$, we calculate

$$[F'(x^{(k)})]^{-1} = \left[I + X_2 e_k + X_3 e_k^2 + X_4 e_k^3 + X_5 e_k^4 + X_6 e_k^5 + X_7 e_k^6 + X_8 e_k^7\right] [F'(\xi)]^{-1} + O[e_k^8],$$

where $X_1 = I$ and $X_s = -\sum_{j=2}^{s} jX_{s-j+1}C_j$, for $s = 1, 2, \ldots$.

Then,

$$y^{(k)} = \xi + \frac{1}{2}\left(e_k + C_2 e_k^2 - M\right)$$

and

$$z^{(k)} = \xi + \frac{1}{3}e_k + \frac{1}{2}\left(C_2 e_k^2 - M\right),$$

where $M = +M_1 e_k^3 + M_2 e_k^4 + M_3 e_k^5 + M_4 e_k^6 + M_5 e_k^7 + M_6 e_k^8 + O[e_k^9]$ and $M_s = C_{s+2} + \sum_{j=1}^{s} X_{j+1}C_{s-j+2} + X_{s+2},\ s = 1, 2, \ldots$.

The Taylor expansion of $F'(z^{(k)})$ is

$$F'\left(z^{(k)}\right) = F'(\xi)\left[I + \frac{2}{3}C_2 e_k + Q_2 e_k^2 + Q_3 e_k^3 + Q_4 e_k^4 + Q_5 e_k^5 + Q_6 e_k^6 + Q_7 e_k^7 + Q_8 e_k^8\right] + O[e_k^9],$$

where

$$\begin{aligned}
Q_2 &= \frac{3}{4}C_2^2 + \frac{1}{3}C_3,\\
Q_3 &= -\frac{4}{3}C_2M_1 + \frac{4}{3}C_3C_2 + \frac{4}{27}C_4,\\
Q_4 &= -\frac{4}{3}C_2M_2 + \frac{4}{3}C_3\alpha_1 + \frac{8}{9}C_4C_2 + \frac{5}{81}C_5,\\
Q_5 &= -\frac{4}{3}C_2M_3 - \frac{4}{3}C_3\alpha_2 - 4C_4\beta_1 + \frac{40}{81}C_5C_2 + \frac{2}{81}C_6,\\
Q_6 &= -\frac{4}{3}C_2M_4 + \frac{4}{3}C_3\alpha_3 - 4C_4\beta_2 - 5C_5\gamma_1 + \frac{20}{81}C_6C_2 + \frac{7}{243}C_7,\\
Q_7 &= -\frac{4}{3}C_2M_5 + \frac{4}{3}C_3\alpha_4 - 4C_4\beta_3 - 5C_5\gamma_2 - 6C_6\delta_1 + \frac{28}{81}C_7C_2 + \frac{8}{729}C_8,
\end{aligned}$$

being

$$\begin{aligned}
\alpha_1 &= -\frac{4}{9}(M_1 + C_2^2),\\
\alpha_2 &= -\frac{4}{9}(M_2 - C_2M_1 - M_1C_2),\\
\alpha_3 &= -\frac{4}{9}(M_3 - C_2M_2 + M_1^2 - M_2C_2),\\
\alpha_4 &= -\frac{4}{9}(M_4 - C_2M_3 + M_1M_2 + M_2M_1 - M_3C_2),\\
\beta_1 &= -\frac{8}{27}(2\alpha_1 + C_2^2 - 4M_1),\\
\beta_2 &= -\frac{8}{27}(2\alpha_2 + C_2\alpha_1 - M_1C_2 - 4M_2),\\
\beta_3 &= -\frac{8}{27}(2\alpha_3 + C_2\alpha_2 - M_1\alpha_1 - M_2C_2 - 4M_3),\\
\gamma_1 &= -\frac{1}{3}\beta_1 + \frac{4}{27}C_2^2 - \frac{2}{81}M_1,\\
\gamma_2 &= -\frac{1}{3}\beta_2 - \frac{2}{3}C_2\beta_1 - \frac{4}{27}3M_1C_2 - \frac{2}{81}M_2,
\end{aligned}$$

and

$$\delta_1 = -\frac{1}{3}\gamma_1 + \frac{16}{243}C_2^2 - \frac{2}{243}M_1.$$

Then, the following Taylor expansion can be obtained,

$$F'(x^{(k)}) - 3F'(z^{(k)}) = F'(\xi)[-2I + T_2 e_k^2 + T_3 e_k^3 + T_4 e_k^4 + T_5 e_k^5 + T_6 e_k^6 + T_7 e_k^7] + O[e_k^8],$$

where $T_s = (s+1)C_{s+1} - 3Q_s$, $s = 2, 3, \ldots$.

So,

$$[F'(x^{(k)}) - 3F'(z^{(k)})]^{-1} = \left[-\frac{1}{2}I + Y_3e_k^2 + Y_4e_k^3 + Y_5e_k^4 + Y_6e_k^5 + Y_7e_k^6 + Y_8e_k^7\right] + O[e_k^8],$$

where

$$\begin{aligned}
Y_3 &= -\frac{1}{4}T_2,\\
Y_4 &= -\frac{1}{4}T_3,\\
Y_5 &= -\frac{1}{8}(2T_4 + T_2^2),\\
Y_6 &= -\frac{1}{16}(4T_5 + 2T_2T_3 + 2T_4T_2 + T_2^3),\\
Y_7 &= -\frac{1}{16}(4T_6 + 2T_2T_4 + 2T_3^2 + 2T_4T_2 + T_2^3),\\
Y_8 &= -\frac{1}{32}(8T_7 + 4T_2T_5 + 4T_3T_4 + 4T_4T_3 + 2T_2^2T_3 + 4T_5T_2 + 2T_2T_3T_2 + 2T_4T_2^2 + T_2^4).
\end{aligned}$$

The following calculation is necessary to obtain the error equation of the third step of the iterative process:

$$[F'(x^{(k)}) - 3F'(z^{(k)})]^{-1}F(x^{(k)}) = -\frac{1}{2}e_k - \frac{1}{2}C_2e_k^2 + N_3e_k^3 + N_4e_k^4 + N_5e_k^5 + N_6e_k^6 + N_7e_k^7 + N_8e_k^8 + O[e_k^9],$$

where $N_3 = -\frac{1}{2}C_3 + Y_3$ and $N_s = -\frac{1}{2}C_s + \sum_{j=4}^{s} Y_{j-1}C_{s-j+2} + Y_s$, $s = 4, 5, \ldots$. Then,

$$\begin{aligned}
u^{(k)} - \xi &= y^{(k)} - \xi + [F'(x^{(k)}) - 3F'(z^{(k)})]^{-1}F(x^{(k)})\\
&= L_4e_k^4 + L_5e_k^5 + L_6e_k^6 + L_7e_k^7 + L_8e_k^8 + O[e_k^9],
\end{aligned}$$

being $L_s = N_s - \frac{1}{2}M_{s-2}$, $s = 4, 5, \ldots$.

In order to obtain the error of the fourth step, $v^{(k)} - \xi$, we calculate

$$[F'(x^{(k)}) - 3F'(z^{(k)})]^{-1}F(u^{(k)}) = -\frac{1}{2}L_4e_k^4 - \frac{1}{2}L_5e_k^5 + R_6e_k^6 + R_7e_k^7 + R_8e_k^8 + O[e_k^9],$$

where

$$\begin{aligned}
R_6 &= -\frac{1}{2}L_6 + Y_3L_4,\\
R_7 &= -\frac{1}{2}L_7 + Y_3L_5 + Y_4L_4,\\
R_8 &= -\frac{1}{2}L_8 - C_2L_4 + Y_3L_6 + Y_4L_5 + Y_5L_4
\end{aligned}$$

and

$$v^{(k)} = \xi + 2Y_3L_4e_k^6 + 2(Y_3L_5 + Y_4L_4)e_k^7 + 2(-C_2L_4 + Y_3L_6 + Y_4L_5 + Y_5L_4)e_k^8 + O[e_k^9]$$

Finally,

$$[F'(x^{(k)}) - 3F'(z^{(k)})]^{-1}F(v^{(k)}) = -Y_3L_4e_k^6 - (Y_3L_5 + Y_4L_4)e_k^7 + [C_2L_4 - Y_4L_5 - Y_5L_4 + Y_3R_6]e_k^8 + O[e_k^9]$$

and the error equation of the method is:

$$\begin{aligned}
x^{(k+1)} &= v^{(k)} + 2[F'(x^{(k)}) - 3F'(z^{(k)})]^{-1}F(v^{(k)})\\
&= \xi - \frac{1}{2}T_2(L_6 + 2R_6)e_k^8 + O[e_k^9].
\end{aligned}$$

This error can be expressed, in terms of C_i, $s = 2, 3, \ldots$ as

$$e_{k+1} = \frac{1}{9}(C_3 - C_2^2)(C_4 - 9C_3C_2 + 9C_2^3)e_k^8 + O[e_k^9]. \quad \square$$

Table 1
Quadratures used.

Number of nodes	Quadratures							
	Chebyshev		Legendre		Lobatto		Radau	
	σ	σ_1	σ	σ_1	σ	σ_1	σ	σ_1
1	π	0	2	0	2	0	2	-1
2	π	0	2	0	2	0	2	0
3	π	0	2	0	2	0	2	0

Let us note that the number of operations (per iteration) needed to execute this procedure is not as high as it can seem, as the linear system to be solved in steps third to fifth have the same matrix of coefficients. So, this multistep procedure is very competitive as it will be showed in the numerical section.

It is known (see [10]) that, by applying the pseudocomposition technique, it is possible to design methods with higher order of convergence. We will see in the following how this technique modify the properties of the proposed schemes.

Theorem 2 *([10]). Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be differentiable enough D and $\xi \in D$ a solution of the nonlinear system $F(x) = 0$. We suppose that $F'(x)$ is continuous and nonsingular at ξ and $x^{(0)}$ close enough to the solution. Let $y^{(k)}$ and $z^{(k)}$ be the penultimate and final steps of orders q and p, respectively, of a certain iterative method. Taking this scheme as a predictor we get a new approximation $x^{(k+1)}$ of ξ given by*

$$x^{(k+1)} = y^{(k)} - 2\left[\sum_{i=1}^{m} \omega_i F'(\eta_i^{(k)})\right]^{-1} F(y^{(k)}),$$

where $\eta_i^{(k)} = \frac{1}{2}\left[(1+\tau_i)z^{(k)} + (1-\tau_i)y^{(k)}\right]$ and $\tau_i, \omega_i, i = 1, \ldots, m$ are the nodes and weights of the orthogonal polynomial corresponding to the Gaussian quadrature used. Then,

(1) *the obtained set of families will have an order of convergence at least q;*
(2) *if $\sigma = 2$ is satisfied, then the order of convergence will be at least $2q$;*
(3) *if, also, $\sigma_1 = 0$ the order of convergence will be $\min\{p+q, 3q\}$;*

where $\sum_{i=1}^{n} \omega_i = \sigma$ and $\sum_{i=1}^{n} \frac{\omega_i \tau_i^j}{\sigma} = \sigma_j$ with $j = 1, 2$.

Each of the families obtained will consist of subfamilies that are determined by the orthogonal polynomial corresponding to the Gaussian quadrature used. Furthermore, in these subfamilies it can be obtained methods using different number of nodes corresponding to the orthogonal polynomial used (see Table 1). According to the proof of Theorem 2 the order of convergence of the obtained methods does not depend on the number of nodes used; so, the method will be more efficient as lower is the number of nodes employed.

Let us note that these methods, obtained by means of Gaussian quadratures, seem to be known interpolation quadrature schemes such as midpoint, trapezoidal or Simpson's method (see [12]). It is only a similitude, as they are not applied on the last iteration $x^{(k)}$, and the last step of the predictor, but on the two last steps of the predictor. In the following, we will use a midpoint-like as a corrector step, which corresponds to a Gauss–Legendre quadrature with one node; for this scheme the order of convergence will be at least $\min\{q+p, 3q\}$, by applying Theorem 2. As this corrector on any of the new methods does only need a new functional evaluation of the Jacobian matrix, the efficiency of the resulting procedure will be maximum. So, by pseudocomposing on M6 and M8 there can be obtained two procedures of order of convergence 10 and 14 (denoted by PsM10 and PsM14), respectively. It is also possible to pseudocompose on M4, but the resulting scheme would be of third order of convergence, which is worst than the original M4, so it will not be considered.

Following the notation used in (3), the last step of PsM10 is

$$x^{(k+1)} = u^{(k)} - \left[F'\left(\frac{v^{(k)} + u^{(k)}}{2}\right)\right]^{-1} F(u^{(k)}), \tag{5}$$

and the last three steps of PsM14 can be expressed as

$$\begin{aligned} v^{(k)} &= u^{(k)} + 2\left[F'\left(x^{(k)}\right) - 3F'\left(z^{(k)}\right)\right]^{-1} F\left(u^{(k)}\right), \\ w^{(k)} &= v^{(k)} + 2\left[F'\left(x^{(k)}\right) - 3F'\left(z^{(k)}\right)\right]^{-1} F\left(v^{(k)}\right), \\ x^{(k+1)} &= v^{(k)} - \left[F'\left(\frac{w^{(k)} + v^{(k)}}{2}\right)\right]^{-1} F(v^{(k)}). \end{aligned} \tag{6}$$

If we analyze the efficiency indices (see Fig. 1), we deduce the following conclusions: the new methods M4, M6 and M8 (and also the pseudocomposed PsM10 and PsM14) improve Newton and Jarratt's schemes (in fact, the indices of M4 and

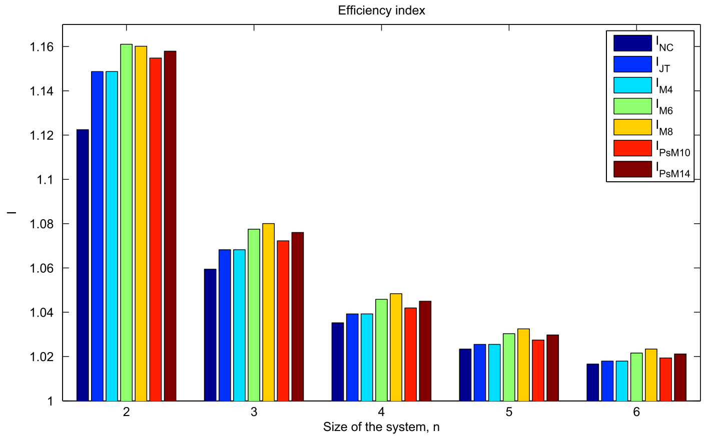


Fig. 1. Efficiency index of the different methods for different sizes of the system.

Jarratt's are equal). Indeed, for $n \geq 3$ the best index is that of M8. Nevertheless, none of the pseudocomposed methods improve the efficiency index of their original partners. However, in the next section the application of PsM10 and PsM14 on certain cases will show some interesting properties.

3. Numerical results

Now, we test the presented schemes in order to check their effectiveness. Numerical computations have been performed in MATLAB R2011a by using variable-precision arithmetic, which uses floating-point representation of 2000 decimal digits of mantissa. The computer specifications are: Intel(R) Core(TM) i5-2500 CPU @ 3.30 GHz with 16.00 GB of RAM. Each iteration is obtained from the former by means of an iterative expression $x^{(k+1)} = x^{(k)} - A^{-1}b$, where $x^{(k)} \in \mathbb{R}^n$, A is a nonsingular real matrix $n \times n$ and $b \in \mathbb{R}^n$. The matrix A and vector b are different according to the method proposed, but in any case, we use to calculate inverse $-A^{-1}b$ the solution of the linear system $Ay = b$, with Gaussian elimination with partial pivoting. The stopping criterion used is $\|x^{(k+1)} - x^{(k)}\| < 10^{-200}$ or $\|F(x^{(k)})\| < 10^{-200}$.

Firstly, let us consider the following nonlinear systems of different sizes:

(1) $F_1 = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$f_i(x) = x_i x_{i+1} - 1, \quad i = 1, 2, \ldots, n-1,$$
$$f_n(x) = x_n x_1 - 1.$$

When n is odd, the exact zeros of $F_1(x)$ are: $\xi_1 = (1, 1, \ldots, 1)^T$ and $\xi_2 = (-1, -1, \ldots, -1)^T$.

(2) $F_2(x_1, x_2) = (x_1^2 - x_1 - x_2^2 - 1, -\sin(x_1) + x_2)$ and the solutions are $\xi_1 \approx (-0.845257, -0.748141)^T$ and $\xi_2 \approx (1.952913, 0.927877)^T$.

(3) $F_3(x_1, x_2) = (x_1^2 + x_2^2 - 4, -\exp(x_1) + x_2 - 1)$, being the solutions $\xi_1 \approx (1.004168, -1.729637)^T$ and $\xi_2 \approx (-1.816264, 0.837368)^T$.

(4) $F_4(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 - 9, x_1 x_2 x_3 - 1, x_1 + x_2 - x_3^2)$ with three roots $\xi_1 \approx (2.14025, -2.09029, -0.223525)^T$, $\xi_2 \approx (2.491376, 0.242746, 1.653518)^T$ and $\xi_3 \approx (0.242746, 2.491376, 1.653518)^T$.

Table 2 presents results showing the following information: the different iterative methods employed (Newton (NC), Jarratt (JT), the new methods M4, M6 and M8 and the pseudocomposed PsM10 and PsM14), the number of iterations *Iter* needed to converge to the solution Sol, the value of the stopping factors at the last step and the computational order of convergence ρ (see [13]) approximated by the formula:

$$\rho \approx \frac{\ln(\|x^{(k+1)} - x^{(k)}\|)/(\|x^{(k)} - x^{(k-1)}\|)}{\ln(\|x^{(k)} - x^{(k-1)}\|)/(\|x^{(k-1)} - x^{(k-2)}\|)}. \tag{7}$$

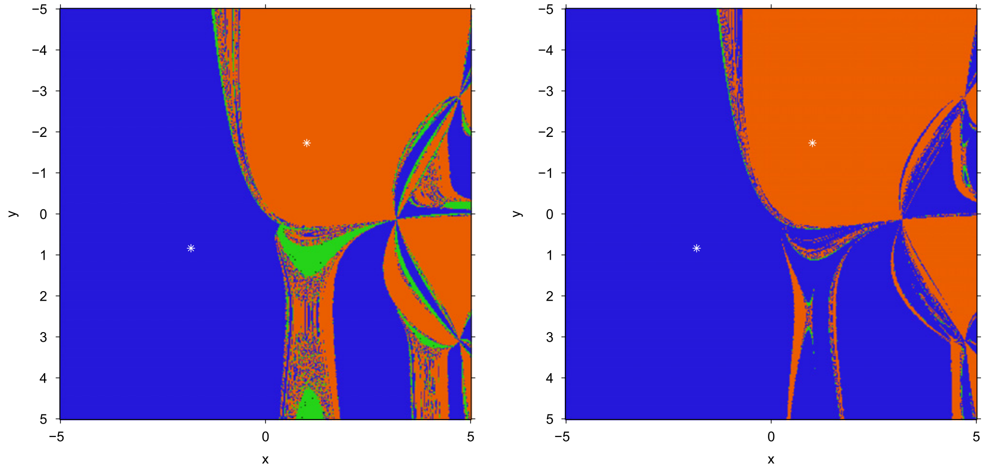
The value of ρ which appears in Table 2 is the last coordinate of the vector ρ when the variation between their coordinates is small. Also the elapsed time, in seconds, appears in Table 2, being the mean execution time for 100 performances of the method (the command cputime of Matlab has been used).

We observe from Table 2 that, not only the order of convergence and the number of new functional evaluations and operations is important in order to obtain new efficient iterative methods to solve nonlinear systems of equations. A key factor is the range of applicability of the methods. Although they are slower than the original methods when the initial estimation is quite good, when we are far from the solution or inside a region of instability, the original schemes do not converge or do it more slowly, the corresponding pseudocomposed procedures usually still converge or do it faster.

Table 2
Numerical results for functions F_1 to F_4.

Function	Method	Iter	Sol	$\|x^{(k)} - x^{(k-1)}\|$	$\|F(x^{(k)})\|$	ρ	e-time (s)
F_1, with $n = 99$	NC	9	ξ_1	1.43e−121	2.06e−243	2.0000	9.1371
$x^{(0)} = (0.5, \dots, 0.5)$	JT	5	ξ_1	1.43e−121	1.07e−487	4.0000	8.8336
	M4	5	ξ_1	1.43e−121	1.07e−487	4.0000	8.2902
	M6	4	ξ_1	7.81e−092	2.92e−553	5.9995	9.1314
	M8	3	ξ_1	1.90e−025	1.12e−206	8.3236	8.7721
	PsM10	3	ξ_1	1.83e−044	3.36e−449	10.3015	8.9779
	PsM14	3	ξ_1	7.24e−082	2.26e−1152	14.2939	10.6858
F_1, with $n = 99$	NC	18	ξ_1	2.83e−113	8.02e−227	2.0000	18.5877
$x^{(0)} = (0.001, \dots, 0.001)$	JT	9	ξ_1	2.37e−056	8.02e−227	4.0000	15.8859
	M4	9	ξ_1	2.37e−056	8.02e−227	4.0000	14.7889
	M6	8	ξ_1	1.14e−139	2.76e−840	6.0000	17.8668
	M8	7	ξ_1	1.49e−099	1.58e−799	7.9928	19.8457
	PsM10	6	ξ_1	5.07e−067	9.22e−675	9.8423	17.0695
	PsM14	5	ξ_1	4.22e−019	1.20e−273	–	17.2963
F_2	NC	9	ξ_1	2.45e−181	5.92e−362	2.0148	0.4069
$x^{(0)} = (-0.5, -0.5)$	JT	5	ξ_1	9.48e−189	8.13e−754	4.0279	0.4338
	M4	5	ξ_1	9.48e−189	8.13e−754	4.0279	0.4401
	M6	4	ξ_1	1.34e−146	2.14e−878	5.9048	0.4209
	M8	3	ξ_1	3.38e−042	9.08e−335	7.7943	0.4590
	PsM10	3	ξ_1	1.09e−068	1.88e−685	10.2609	0.4660
	PsM14	3	ξ_1	1.65e−130	3.07e−1822	13.8766	0.5256
F_2	NC	13	ξ_1	2.20e−182	2.73e−364	1.9917	0.5524
$x^{(0)} = (-5, -3)$	JT	7	ξ_1	2.10e−179	4.51e−716	3.9925	0.5111
	M4	7	ξ_1	2.10e−179	4.51e−716	3.9925	0.5390
	M6	8	ξ_1	2.55e−036	5.81e−216	–	0.8304
	M8	nc					
	PsM10	5	ξ_1	5.05e−131	3.95e−1306	10.3772	0.7004
	PsM14	5	ξ_1	6.67e−102	6.21e−1422	–	0.8777
F_3	NC	11	ξ_2	1.82e−164	3.33e−328	2.0000	0.3668
$x^{(0)} = (1, 4)$	JT	6	ξ_2	4.88e−059	3.59e−235	3.9998	0.3749
	M4	6	ξ_2	4.88e−059	3.59e−235	3.9998	0.3844
	M6	18	ξ_2	1.33e−106	4.33e−638	–	1.5248
	M8	23	ξ_2	3.73e−097	3.65e−775	–	2.5126
	PsM10	6	ξ_2	6.26e−130	2.93e−1297	9.9820	0.7165
	PsM14	nc					
F_3	NC	14	ξ_2	3.95e−173	1.56e−345	2.0000	0.5614
$x^{(0)} = (0.8, 0.5)$	JT	7	ξ_2	1.22e−073	1.42e−293	3.9999	0.5494
	M4	7	ξ_2	1.22e−073	1.42e−293	3.9999	0.4967
	M6	8	ξ_1	6.09e−051	3.72e−303	–	0.8240
	M8	nc					
	PsM10	5	ξ_2	7.36e−164	1.48e−1636	9.9935	0.7192
	PsM14	6	ξ_1	1.14e−167	0	13.8332	1.0149
F_4	NC	10	ξ_1	1.09e−135	1.55e−270	1.9995	0.6975
$x^{(0)} = (1, -1.5, -0.5)$	JT	5	ξ_1	9.94e−073	2.09e−289	4.0066	0.6639
	M4	5	ξ_1	9.94e−073	2.09e−289	4.0066	0.6831
	M6	4	ξ_1	9.36e−057	4.86e−338	5.9750	0.7474
	M8	4	ξ_1	2.18e−124	1.26e−991	8.0041	0.9560
	PsM10	3	ξ_1	5.52e−028	5.38e−276	9.7714	0.7516
	PsM14	3	ξ_1	1.36e−050	1.27e−702	13.7136	0.9029
F_4	NC	9	ξ_3	8.90e−149	1.34e−296	2.0001	0.5830
$x^{(0)} = (1, 3, 2)$	JT	5	ξ_3	3.64e−156	3.99e−623	3.9999	0.6203
	M4	5	ξ_3	3.64e−156	3.99e−623	3.9999	0.6218
	M6	4	ξ_3	1.79e−118	1.54e−708	5.9943	0.6959
	M8	3	ξ_3	7.20e−034	8.89e−268	7.7015	0.7022
	PsM10	3	ξ_3	2.16e−057	1.29e−570	9.7953	0.6933
	PsM14	3	ξ_3	1.02e−105	4.62e−1475	13.7602	0.8504

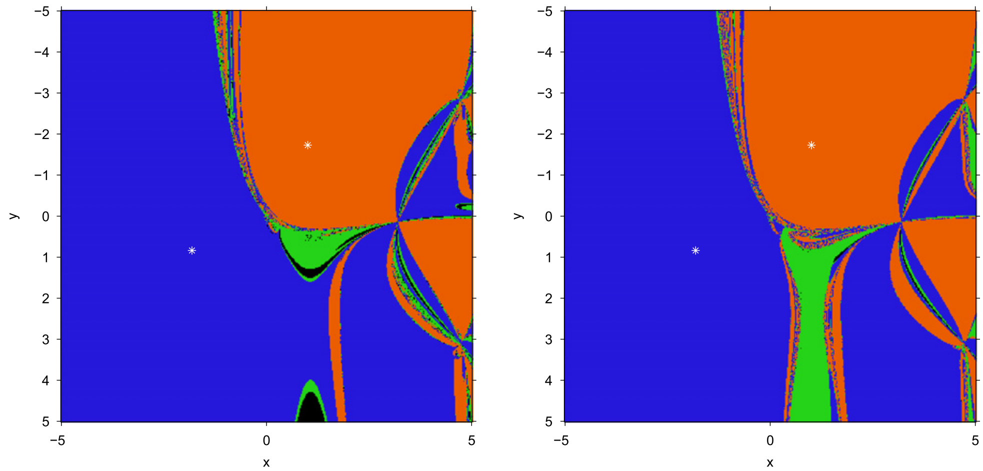
The advantage of pseudocomposition can be observed in Figs. 2(a) and (b) (methods M6 and PsM10) and 3(a) and (b) (methods M8 and PsM14) where the dynamical plane on $\mathbb{R}^2$ is represented: let us consider a system of two equations and two unknowns (the case $F_3(x) = 0$ is showed), for any initial estimation in $\mathbb{R}^2$ represented by its position in the plane, a different color (blue or orange, as there exist only two solutions) is used for the different solutions found (marked by a white point in the figure). Black color represents an initial point in which the method converges to infinity, and the green one means that no convergence is found (usually because any linear system cannot be solved). It is clear that when many initial estimations tend to infinity (see Fig. 3(a)), the pseudocomposition "cleans" the dynamical plane, making the method more



(a) M6.

(b) PsM10.

Fig. 2. Real dynamical planes for system $F_3(x) = 0$ and methods M6 and PsM10.



(a) M8.

(b) PsM14.

Fig. 3. Real dynamical planes for system $F_3(x) = 0$ and methods M8 and PsM14.

stable as it can find one of the solutions by using starting points that do not allow convergence with the original scheme (see Fig. 2(b)).

We conclude that the presented schemes M4, M6 and M8 show to be excellent, in terms of order of convergence and efficiency, but also that the pseudocomposition technique achieves to transform them in competent and more robust new schemes.

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