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Third and fourth order iterative methods free from second derivative for nonlinear systems ☆

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ABSTRACT

In this work, we develop a family of predictor–corrector methods free from second derivative for solving systems of nonlinear equations. In general, the obtained methods have order of convergence three but, in some particular cases the order is four. We also perform different numerical tests that confirm the theoretical results and allow us to compare these methods with Newton's classical method and with other recently published methods.

1. Introduction

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, that is, a real solution α, of the nonlinear system $F(x) = 0$, of n equations with n unknowns. The most known iterative method is the classical Newton's method that converges quadratically under certain conditions. Recently, for $n = 1$, many robust and efficient methods have been proposed with higher convergence order. In this paper, the predictor–corrector technique of third order for single equations (see [1]) is generalized to nonlinear systems in a simple way, without using the decomposition of the nonlinear operator $N(x)$ proposed in the mentioned paper.

The method obtained uses the second derivative, what is a serious drawback, specially for functions of several variables. To remove the second derivative, we extend the technique due to Kou et al. [2], to the multi-dimensional case. As in [2,3] for single equations, we obtain a family of predictor–corrector methods for nonlinear systems. Analysis of convergence shows that, in general, this family is of third order convergence and of fourth order convergence in some particular cases. We present some examples to illustrate the efficiency of the studied methods and compare them with Newton's method and with a recent method with third order convergence and free from second derivative introduced by Frontini in [4]:

$$x^{(k+1)} = x^{(k)} - \left[F'\left(x^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F(x^{(k)})\right)\right]^{-1} F(x^{(k)}). \tag{1}$$

In the next sections, we use the same notation for the n-dimensional case as for the one-dimensional case, interpreting the symbols appropriately.

2. Description of the methods

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function and α a zero of the system of nonlinear equations

$$F(x) = 0, \tag{2}$$

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with nonsingular jacobian in a neighborhood of α. The considered methods are obtained from Taylor's approximation. If we substitute $F(x)$ by its second order Taylor's approximation about an initial estimation $x^{(0)}$ of α, system (2) becomes:

$$F(x^{(0)}) + F'(x^{(0)})(x - x^{(0)}) + \frac{1}{2}F''(x^{(0)})(x - x^{(0)})^2 = 0. \tag{3}$$

Newton's method ignores the terms of second order, so that its approximation to the solution is:

$$x_N = x^{(0)} - F'(x^{(0)})^{-1}F(x^{(0)}).$$

If we take this estimation as a predicted value, substitute x_N in the second order term of (3) and solve again the system we obtain a new estimation:

$$x_{NN} = x^{(0)} - F'(x^{(0)})^{-1}F(x^{(0)}) - \frac{1}{2}F'(x^{(0)})^{-1}F''(x^{(0)})(x_N - x^{(0)})^2. \tag{4}$$

This results in the well known Chebyshev's method:

Predictor-step:

$$x_N^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}),$$

Corrector-step:

$$x^{(k+1)} = x_N^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(x_N^{(k)} - x^{(k)})^2. \tag{5}$$

Let us now define a three-step iterative method by using the result of (4), x_{NN}, as a new predicted value in (3). By solving this system we obtain:

$$x = x^{(0)} - F'(x^{(0)})^{-1}F(x^{(0)}) - \frac{1}{2}F'(x^{(0)})^{-1}F''(x^{(0)})(x_{NN} - x^{(0)})^2.$$

This process can be formulated as three steps predictor–corrector method for solving nonlinear systems that generalizes the one presented by Noor in [1] for the uni-dimensional case.

Algorithm 1. Let $x^{(0)}$ be an initial guess of the solution of system $F(x) = 0$. Compute approximations $x^{(k+1)},\quad k = 0, 1, \ldots$ by:

Predictor-step 1:

$$x_N^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \tag{6}$$

Predictor-step 2:

$$y_N^{(k)} = -\frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(x_N^{(k)} - x^{(k)})^2, \tag{7}$$

$$x_{NN}^{(k)} = x_N^{(k)} + y_N^{(k)}, \tag{8}$$

Corrector-step:

$$x^{(k+1)} = x_N^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(x_{NN}^{(k)} - x^{(k)})^2. \tag{9}$$

This method uses the second derivative, what is a serious drawback, specially for functions of several variables. We use the technique due to Kou and Li [3] in order to remove the second derivative.

Define $x_p^{(k)} = x^{(k)} - \theta F'(x^{(k)})^{-1}F(x^{(k)})$, where θ is a real parameter. Then we can express

$$y_p^{(k)} = -\frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(x_p^{(k)} - x^{(k)})^2, \tag{10}$$

$$x_{pp}^{(k)} = x_N^{(k)} + y_p^{(k)}, \tag{11}$$

$$x^{(k+1)} = x_N^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(x_{pp}^{(k)} - x^{(k)})^2. \tag{12}$$

We consider Taylor expression of $F(x_p^{(k)})$ about $x^{(k)}$

$$F(x_p^{(k)}) = F(x^{(k)}) + F'(x^{(k)})(x_p^{(k)} - x^{(k)}) + \frac{1}{2}F''(x^{(k)})(x_p^{(k)} - x^{(k)})^2 + O(\|x_p^{(k)} - x^{(k)}\|^3),$$

so we can obtain the following approximation:

$$\frac{1}{2}F''(x^{(k)})(x_p^{(k)} - x^{(k)})^2 \approx F(x_p^{(k)}) - F(x^{(k)}) - F'(x^{(k)})(x_p^{(k)} - x^{(k)}) = F(x_p^{(k)}) + (\theta - 1)F(x^{(k)}). \tag{13}$$

Using (6) and (13) in (11) we have

$$x_{pp}^{(k)} - x^{(k)} = -F'(x^{(k)})^{-1}\left[F(x_p^{(k)}) + \theta F(x^{(k)})\right] \tag{14}$$

and by Taylor approximation of $F(x_{pp}^{(k)})$ about $x^{(k)}$ we can get:

$$\frac{1}{2}F''(x^{(k)})(x_{pp}^{(k)} - x^{(k)})^2 = F(x_{pp}^{(k)}) + F(x_p^{(k)}) + (\theta - 1)F(x^{(k)}),$$

what substituted in (12) gives us, finally, a free from second derivative method that can be expressed as:

Algorithm 2. Let $x^{(0)}$ be an initial guess of the solution of system $F(x) = 0$. Compute approximations $x^{(k+1)}, \quad k = 0, 1, \ldots$, by:

Predictor-step 1:

$$x_N^{(k)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \tag{15}$$

$$x_p^{(k)} = x^{(k)} - \theta F'(x^{(k)})^{-1}F(x^{(k)}), \tag{16}$$

Predictor-step 2:

$$y_p^{(k)} = -F'(x^{(k)})^{-1}[F(x_p^{(k)}) + (\theta - 1)F(x^{(k)})], \tag{17}$$

$$x_{pp}^{(k)} = x_N^{(k)} + y_p^{(k)}, \tag{18}$$

Corrector-step:

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}[F(x_{pp}^{(k)}) + F(x_p^{(k)}) + \theta F(x^{(k)})]. \tag{19}$$

In the following section, we will be prove that, under certain conditions, the method described by Algorithm 1 has convergence order three and that the method described by Algorithm 2 reaches convergence order four for $\theta = \pm 1$.

3. Convergence of the methods

Theorem 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in a neighborhood D of α, that is a solution of the system $F(x) = 0$, which has jacobian matrix continuous and nonsingular in D. Let $x^{(0)}$ be an initial guess sufficiently close to α. Then,*

(a) *the iterative method defined by Algorithm 1 has order of convergence three;*
(b) *the iterative method defined by Algorithm 2 has third order convergence for every $\theta \neq \pm 1$, and fourth order convergence for $\theta = \pm 1$.*

Proof

(a) The Taylor's series of $F(x)$ about $x^{(k)}$ is:

$$F(x) = F(x^{(k)}) + F'(x^{(k)})(x - x^{(k)}) + \frac{1}{2}F''(x^{(k)})(x - x^{(k)})^2 + O(\|x - x^{(k)}\|^3).$$

Setting $x = \alpha$ we obtain:

$$0 = F(\alpha) = F(x^{(k)}) + F'(x^{(k)})(\alpha - x^{(k)}) + \frac{1}{2}F''(x^{(k)})(\alpha - x^{(k)})^2 + O(\|\alpha - x^{(k)}\|^3)$$

and replacing $e^{(k)} = x^{(k)} - \alpha$ it results

$$F(x^{(k)}) = F'(x^{(k)})e^{(k)} - \frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + O(\|e^{(k)}\|^3). \tag{20}$$

Substituting (6)–(8) in (9), the iterates of Algorithm 1 can be expressed as:

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}\left[F(x^{(k)}) + \frac{1}{2}F''(x^{(k)})(x_N^{(k)} + y_N^{(k)} - x^{(k)})^2\right].$$

Adding the term $-\alpha$ in both sides and using (20), we have

$$e^{(k+1)} = e^{(k)} - F'(x^{(k)})^{-1}\left[F'(x^k)e^{(k)} - \frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + \frac{1}{2}F''(x^{(k)})(x_N^{(k)} + y_N^{(k)} - x^{(k)})^2 + O(\|e^{(k)}\|^3)\right]. \tag{21}$$

Simple algebraic manipulations using (6) and (20) give us:

$$x_N^{(k)} - x^{(k)} = -F'(x^{(k)})^{-1}F(x^{(k)}) = -e^{(k)} + \frac{1}{2}F'(x^{(k)})^{-1}[F''(x^{(k)})(e^{(k)})^2 + O(\|e^{(k)}\|^3)]$$

and

$$F''(x^{(k)})(x_N^{(k)} - x^{(k)})^2 = F''(x^{(k)})(e^{(k)})^2 + O(\|e^{(k)}\|^3),$$

so we get:

$$y_N^{(k)} = -\frac{1}{2}F'(x^{(k)})^{-1}[F''(x^{(k)})(e^{(k)})^2 + O(\|e^{(k)}\|^3)].$$

Now, the other term involving the quadratic form in (21) can be expressed as:

$$\begin{aligned} F''(x^{(k)})(x_N^{(k)} + y_N^{(k)} - x^{(k)})^2 &= F''(x^{(k)})\left[-e^{(k)} + \frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^2 - \frac{1}{2}F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^2 + O(\|e^{(k)}\|^3)\right]^2 \\ &= F''(x^{(k)})[(e^{(k)})^2 + O(\|e^{(k)}\|^3)]. \end{aligned}$$

Then, by substituting this expression in (21), we obtain:

$$e^{(k+1)} = -F'(x^{(k)})^{-1}[O(\|e^{(k)}\|^3)].$$

Therefore, by the nonsingularity of the Jacobian in a neighborhood of α we have

$$e^{(k+1)} = O(\|e^{(k)}\|^3),$$

what allows to conclude that the sequence obtained by Algorithm 1 converges to α with convergence order three.

(b) In order to obtain fourth order convergence, we have to include terms of order three in the Taylor's series of $F(x^k)$ about x^k. Using the notation introduced in (a), we obtain:

$$F(x^{(k)}) = F'(x^{(k)})e^{(k)} - \frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + \frac{1}{3!}F'''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4). \tag{22}$$

The term

$$\begin{aligned} x_p^{(k)} - x^{(k)} &= -\theta F'(x^k)^{-1}\left[F'(x^{(k)})e^{(k)} - \frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + \frac{1}{3!}F'''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4)\right] \\ &= -\theta e^{(k)} + \frac{1}{2}\theta F'(x^k)^{-1}F''(x^{(k)})(e^{(k)})^2 - \frac{1}{6}\theta F'(x^k)^{-1}F'''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4), \end{aligned} \tag{23}$$

gives us that

$$(x_p^{(k)} - x^{(k)})^2 = \theta^2(e^{(k)})^2 - \theta^2 F'(x^k)^{-1}F''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4) \tag{24}$$

and

$$(x_p^{(k)} - x^{(k)})^3 = -\theta^3(e^{(k)})^3 + O(\|e^{(k)}\|^4), \tag{25}$$

$$(x_p^{(k)} - x^{(k)})^4 = O(\|e^{(k)}\|^4). \tag{26}$$

Taylor's expansion of $F(x_p^{(k)})$ about $x^{(k)}$ is:

$$F(x_p^{(k)}) = F(x^{(k)}) + F'(x^{(k)})(x_p^{(k)} - x^{(k)}) + \frac{1}{2}F''(x^{(k)})(x_p^{(k)} - x^{(k)})^2 + \frac{1}{3!}F'''(x^{(k)})(x_p^{(k)} - x^{(k)})^3 + O(\|x_p^{(k)} - x^{(k)}\|^4). \tag{27}$$

By substituting (22)–(26) in (27), we have

$$\begin{aligned} F(x_p^{(k)}) &= F'(x^{(k)})e^{(k)} - \frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + \frac{1}{3!}F'''(x^{(k)})(e^{(k)})^3 \\ &\quad + F'(x^{(k)})\left[-\theta e^{(k)} + \frac{1}{2}\theta F'(x^k)^{-1}F''(x^{(k)})(e^{(k)})^2 - \frac{1}{6}\theta F'(x^k)^{-1}F'''(x^{(k)})(e^{(k)})^3\right] + \frac{1}{2}\theta^2 F''(x^{(k)})(e^{(k)})^2 \\ &\quad - \frac{1}{2}\theta^2 F''(x^{(k)})F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^3 - \frac{1}{6}F'''(x^{(k)})\theta^3(e^{(k)})^3 + O(\|e^{(k)}\|^4). \end{aligned}$$

Grouping similar terms we obtain

$$\begin{aligned} F(x_p^{(k)}) &= (1-\theta)F'(x^{(k)})e^{(k)} + (\theta^2 + \theta - 1)\frac{1}{2}F''(x^{(k)})(e^{(k)})^2 + (1 - \theta - \theta^3)\frac{1}{3!}F'''(x^{(k)})(e^{(k)})^3 \\ &\quad - \frac{1}{2}\theta^2 F''(x^{(k)})F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4). \end{aligned} \tag{28}$$

Now, we consider the Taylor's expansion of $F(x_{pp}^{(k)})$ about $x^{(k)}$

$$F(x_{pp}^{(k)}) = F(x^{(k)}) + F'(x^{(k)})(x_{pp}^{(k)} - x^{(k)}) + \frac{1}{2}F''(x^{(k)})(x_{pp}^{(k)} - x^{(k)})^2 + + \frac{1}{3!}F'''(x^{(k)})(x_{p}^{(k)} - x^{(k)})^3 + O(\|x_{pp}^{(k)} - x^{(k)}\|^4). \tag{29}$$

For establishing the order of the approximation, we use in

$$x_{pp}^{(k)} - x^{(k)} = -F'(x^k)^{-1}[F(x_p^{(k)}) + \theta F(x^{(k)})]$$

expressions (22) and (28), getting

$$\begin{aligned} x_{pp}^{(k)} - x^{(k)} = &-(e^{(k)}) - \frac{1}{2}(\theta^2 - 1)F'(x^k)^{-1}F''(x^{(k)})(e^{(k)})^2 + \frac{1}{2}\theta^2 F'(x^{(k)})^{-1}F''(x^{(k)})F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^3 \\ &- \frac{1}{3!}(1-\theta^3)F'(x^{(k)})^{-1}F'''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4). \end{aligned} \tag{30}$$

Hence, considering only the terms of inferior order, we can write:

$$(x_{pp}^{(k)} - x^{(k)})^2 = (e^{(k)})^2 + (\theta^2 - 1)F'(x^k)^{-1}F''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4), \tag{31}$$

$$(x_{pp}^{(k)} - x^{(k)})^3 = -(e^{(k)})^3 + O(\|e^{(k)}\|^4) \tag{32}$$

and

$$(x_{pp}^{(k)} - x^{(k)})^4 = O(\|e^{(k)}\|^4). \tag{33}$$

By substituting (22), (30)–(33) in (29) and grouping terms with the same order, we have:

$$F(x_{pp}^{(k)}) = -\frac{1}{2}(1-\theta^2)F''(x^{(k)})(e^{(k)})^2 + \frac{1}{3!}(\theta^3 - 1)F'''(x^{(k)})(e^{(k)})^3 + \frac{1}{2}(2\theta^2 - 1)F''(x^{(k)})F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4). \tag{34}$$

Finally, from (19) we get:

$$e^{(k+1)} = e^{(k)} - F'(x^{(k)})^{-1}[F(x_{pp}^{(k)}) + F(x_p^{(k)}) + \theta F(x^{(k)})]. \tag{35}$$

Substituting the three Taylor's expansion obtained in (22), (28) and (34), we have:

$$e^{(k+1)} = \frac{1}{2}(1-\theta^2)F'(x^{(k)})^{-1}F''(x^{(k)})F'(x^{(k)})^{-1}F''(x^{(k)})(e^{(k)})^3 + O(\|e^{(k)}\|^4).$$

Then, by the nonsingularity of the jacobian in a neighborhood of α, we conclude that for $\theta = \pm 1$ the methods obtained are of order four and for other values of θ, of order three. □

Notice that Algorithm 2 generates new third and fourth order methods free from second derivative. For $\theta = 0$, the third order method that we obtain is the extension for nonlinear systems of the so called Potra–Pták method [5].

4. Numerical results

In this section, we compare the performance of the numerical methods introduced in our work with that of Newton's method and Frontini's method (1), in order to check their effectiveness. We denote by $A1$ the method defined in Algorithm 1 and by $A2_\theta$ the method defined in Algorithm 2 for a particular value of θ. We consider the following values of θ: $\theta = \pm 1$, which produce methods of fourth order convergence and $\theta = \phi = \frac{\sqrt{5}-1}{2}$ which results in a third order method.

In Table 1, we compare the number of functional evaluations and linear systems to be solved per iteration in each method. Notice that $F(x^k)$, $F'(x^k)$ and $F''(x^k)$ require n, n^2 and $\frac{n^3}{2} + \frac{n^2}{2}$ functional evaluations, respectively, so that, the total cost per iteration of method A1 is $\frac{1}{2}n^3 + \frac{3}{2}n^2 + n$, Frontini's method costs $2n^2 + n$ and method $A2_\theta$ needs $n^2 + 3n$ functional evaluations. We remark that, for $n > 2$, method $A2_\theta$ has the smallest cost and, in particular, the methods $A2_1$ and $A2_{-1}$ are of fourth order of convergence, greater than that of the other methods. Furthermore, Frontini's method needs to solve two linear systems per iteration, whereas the other methods need just one.

Table 1
Functional evaluations and linear systems solved.

	Newton	$A1$	Frontini	$A2_\theta$
$F(x^k)$	1	1	1	3
$F'(x^k)$	1	1	2	1
$F''(x^k)$	0	1	0	0
Linear system	1	1	2	1

Some of the examples used in our test appear in [4]. For each example we have performed 100 tests, choosing the starting point on a 10 by 10 square grid T centered on the root. Numerical computations have been carried out in MATLAB 2006b.

The stopping criterion used is $\|x^{(k+1)} - x^{(k)}\| + \|F(x^{(k)})\| < 10^{-15}$. Therefore, we check that the iterates converge to a limit and that this limit is a solution of the nonlinear system. For every method and every initial guess, we calculate the number of iterations and the order of convergence p, approximated by (see [6]):

$$\frac{\ln(\|x^{(k+1)} - x^{(k)}\|/\|x^{(k)} - x^{(k-1)}\|)}{\ln(\|x^{(k)} - x^{(k-1)}\|/\|x^{(k-1)} - x^{(k-2)}\|)}. \tag{36}$$

The methods have been compared using mean values of the 100 obtained results. The average value of iterations and convergence order have been calculated discarding the eventual divergent tests or tests converging to other roots.

Example 1

$$\left.\begin{aligned} x^2 - y - 0.2 = 0,\\ -x + y^2 - 0.3 = 0. \end{aligned}\right\} \tag{37}$$

This system has two solutions:

$$\alpha_1 = (-0.28603216362886, -0.11818560136979),$$
$$\alpha_2 = (1.1923091251488, 1.2216010499131).$$

We take initial guesses in the following grids:

$$T_1: \ x_i = -0.786 + ih, \quad y_k = -0.6186 + kh,$$
$$T_2: \ x_i = 0.692 + ih, \quad y_k = 0.722 + kh$$

with $i, k = 0, 1, \ldots, 9$ and $h = 0.111$ (see Tables 2 and 3).

Example 2

$$\left.\begin{aligned} e^x - 2 - y = 0,\\ \cos x + y - 1 = 0. \end{aligned}\right\}$$

This system has a solution at $\alpha = (0.85023291641695, 0.34019185755559)$. We take initial guesses in the grid:

$$T: \ x_i = 0.35 + ih, \quad y_k = -0.16 + kh$$

with $i, k = 0, 1, \ldots, 9$ and $h = 0.111$ (see Table 4).

Example 3

$$\left.\begin{aligned} x - y^2 + 3\log(x) = 0,\\ 2x^2 - xy - 5x + 1 = 0. \end{aligned}\right\}$$

Table 2
Numerical results for Example 1, convergence to α_1.

α_1	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	7.0	5.3	4.8	5.3	5.3	5.3
Average order	1.8	3.3	2.9	3.6	3.7	2.8

Table 3
Numerical results for Example 1, convergence to α_2.

α_2	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	6.1	4.2	4.2	4.2	4.2	4.2
Average order	2.0	3.8	3.0	3.8	3.8	2.8

Table 4
Numerical results for Example 2, convergence to α.

α	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	5.6	4.1	4.0	3.8	3.6	4.1
Average order	1.5	2.1	1.6	2.0	2.6	2.3

Table 5
Numerical results for Example 3, convergence to α_1.

α_1	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	5.5	3.8	4.0	3.7	3.7	4.0
Average order	2.0	3.5	3.0	3.9	3.8	2.3

Table 6
Numerical results for Example 3, convergence to α_2.

α_2	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	5.9	4.2	4.2	4.0	4.0	4.2
Average order	1.6	3.0	2.7	3.6	3.6	2.5

This system has two solutions:

$$\alpha_1 = (3.75683400801277, 2.77984959281790),$$
$$\alpha_2 = (1.37347835340981, -1.52496483637952).$$

We take initial guesses in the grids:

$$T_1 : \; x_i = 3.257 + ih, \quad y_k = 2.279 + kh,$$
$$T_2 : \; x_i = 0.873 + ih, \quad y_k = -2.025 + kh$$

with $i, k = 0, 1, \ldots, 9$ and $h = 0.111$ (see Tables 5 and 6).

Example 4

$$\left.\begin{array}{l} x^2 + y^2 - 1 = 0, \\ x^2 - y^2 + 0.5 = 0. \end{array}\right\}$$

This system has a solution at $\alpha = (0.5, 0.8660254037844)$. We take initial guesses in the grid:

$$T : \; x_i = ih, \quad y_k = 0.366 + kh$$

with $i, k = 0, 1, \ldots, 9$ and $h = 0.111$ (see Table 7).

Table 7
Numerical results for Example 4, convergence to α.

α	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	6.3	4.1	4.4	4.1	4.1	4.3
Average order	2.0	3.8	3.0	3.8	3.7	2.8

Table 8
Numerical results for Example 5, convergence to α_1.

α_1	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	6.6	5.3	4.6	4.2	4.6	4.5
Average order	2.0	2.7	2.8	3.0	3.1	2.4

Table 9
Numerical results for Example 5, convergence to α_2.

α_2	Newton	$A1$	Frontini	$A2_1$	$A2_{-1}$	$A2_\phi$
Average iterations	6.7	4.3	4.4	4.7	4.0	5.1
Average order	2.0	2.7	2.8	3.0	3.1	2.4

Example 5

$$\left.\begin{array}{l} -e^{x^2}+8x\,sin(y)=0, \\ x+y-1=0. \end{array}\right\}$$

This system has two solutions:

$$\alpha_1=(0.7042469666489, 0.2957530333511),$$
$$\alpha_2=(0.1755989241777, 0.8244010758223).$$

We take initial guesses in the grids:

$$T_1:\ x_i=0.204+ih,\quad y_k=-0.204+kh,$$
$$T_2:\ x_i=-0.324+ih,\quad y_k=0.324+kh$$

with $i,k=0,1,\ldots,9$ and $h=0.111$ (see Tables 8 and 9).

We can observe that the number of iterations of methods A1, Frontini and $A2_\theta$ is similar and lower than that of Newton's method. In addition, the computational order of convergence of Newton's method is 2.0, for methods $A1$, Frontini, and $A2_\phi$ is about 3, whereas for $A2_1$ and $A2_{-1}$ is generally greater than 3 and always higher than that of the other methods.

5. Conclusions

We have obtained an iterative method for nonlinear systems (Algorithm 1) and proved that its convergence order is three. Then we have defined a family of numerical methods of the same order, but that do not use the second derivative. We have also proved that some members of this family have convergence order four. The theoretical results have been checked with some numerical examples, comparing our algorithms with Newton's method and Frontini's method.

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