Instituto tecnológico de Costa Rica

Sede central Cartago

Escuela de Ingeniería Electrónica
EL4703 - Señales y sistemas

Formulario general

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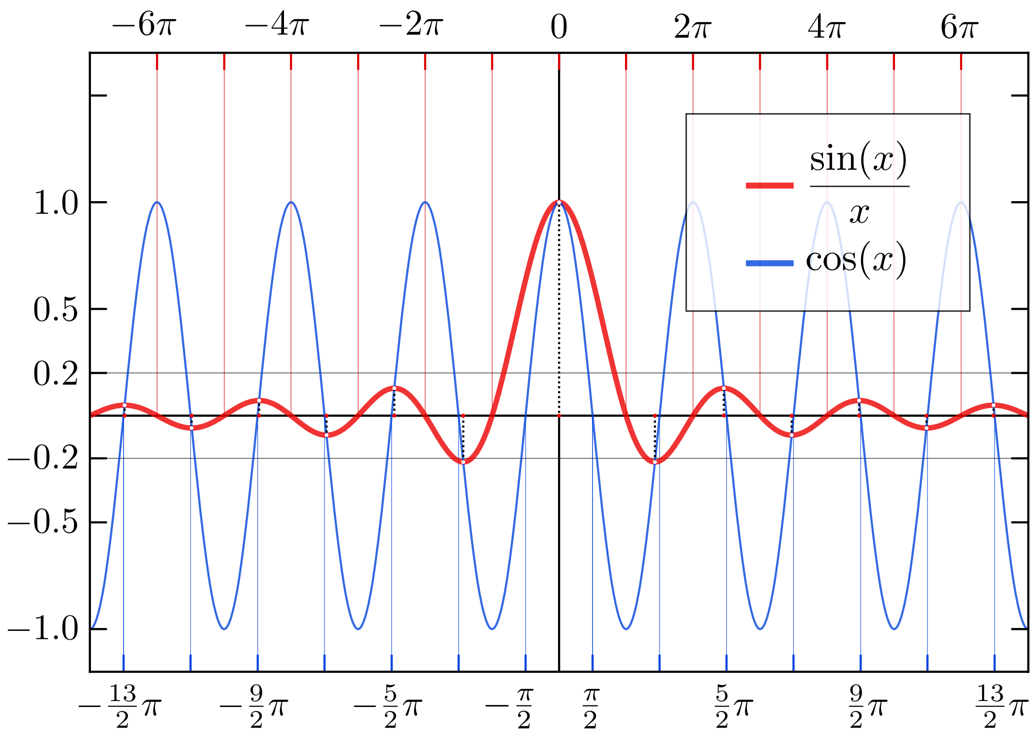


Cartago, Costa Rica.

Noviembre del 2019

Mapeos e Integración

Mapeo de inversión

Mapeo de círculos

$$|w - w_0| = r_w$$

$$r_w = \left|\frac{r}{\alpha}\right|$$

$$w_0 = -\frac{z_0^*}{\alpha}$$

$$\alpha = r^2 - |z_0|^2$$

$$v = \frac{x_0}{y_0}u - \frac{1}{2y_0}$$

Mapeo de rectas

$$v = \frac{\operatorname{Re}\{a-b\}}{\operatorname{Im}\{a-b\}}u$$

$$w_0 = \frac{(a-b)^*}{\beta}$$

$$r_w = \left|\frac{a-b}{\beta}\right|$$

$$\beta = |a|^2 - |b|^2$$

Integración trigonometrica

$$\operatorname{sen}\theta = \frac{1}{2j}\left(z - \frac{1}{z}\right)$$

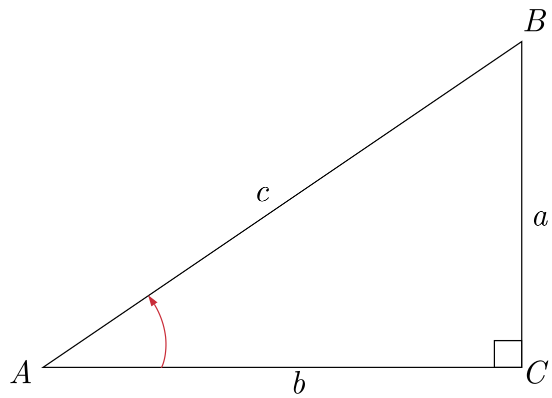
$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$d\theta = \frac{dz}{jz}$$

$$z = e^{j\theta}$$

Funciones trigonométricas

Definición en un triangulo rectángulo



$$\operatorname{sen} A = \frac{a}{c} = \frac{\text{Cateto opuesto}}{\text{Hipotenusa}}$$

$$\cos A = \frac{b}{c} = \frac{\text{Cateto adyacente}}{\text{Hipotenusa}}$$

$$\tan A = \frac{a}{b} = \frac{\text{Cateto opuesto}}{\text{Cateto adyacente}}$$

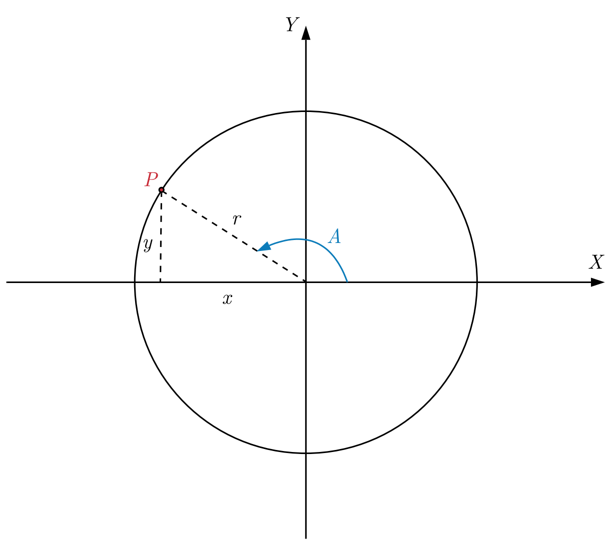
$$\cot A = \frac{b}{a} = \frac{\text{Cateto adyacente}}{\text{Cateto opuesto}}$$

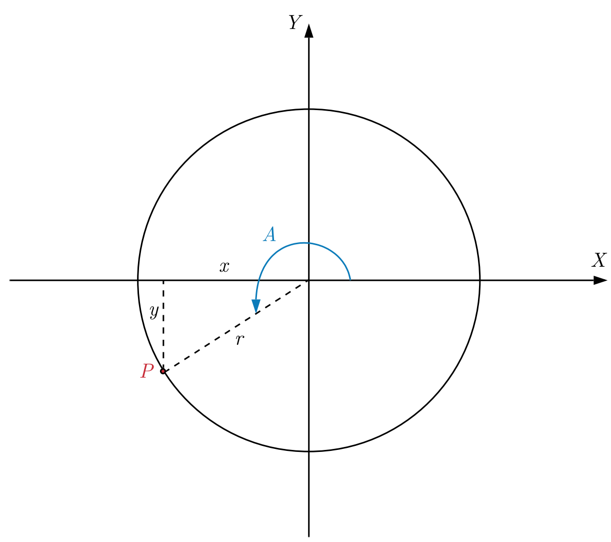
$$\sec A = \frac{c}{b} = \frac{\text{Hipotenusa}}{\text{Cateto adyacente}}$$

$$\csc A = \frac{c}{a} = \frac{\text{Hipotenusa}}{\text{Cateto opuesto}}$$

Funciones para cualquier angulo A en el plano

Considere un sistema de coordenadas XY, donde las coordenadas de un punto $P = (x, y)$, la distancia del punto P al origen $r = \sqrt{x^2 + y^2}$ y el ángulo A positivo como se muestra en la figura.





Funciones trigonométricas para un ángulo A en cualquier cuadrante:

$$\operatorname{sen} A = \frac{y}{r} \qquad \csc A = \frac{r}{y}$$

$$\cos A = \frac{x}{r} \qquad \sec A = \frac{r}{x}$$

$$\tan A = \frac{y}{x} \qquad \cot A = \frac{x}{y}$$

Relaciones entre grados y radianes

1 radián $= \frac{180}{\pi}^{\circ}$

$1^{\circ} = \frac{\pi}{180}$ radianes

$\theta \text{ rad} = \frac{\theta^{\circ} \cdot \pi \text{ rad}}{180^{\circ}}$

$\theta^{\circ} = \frac{\theta \text{ rad} \cdot 180^{\circ}}{\pi \text{ rad}}$

Relaciones entre las funciones trigonométricas

$$\tan A = \frac{\operatorname{sen} A}{\cos A} \qquad \operatorname{sen}^2 A + \cos^2 A = 1$$

$$\cot A = \frac{\cos A}{\operatorname{sen} A} \qquad \sec^2 A - \tan^2 A = 1$$

$$\sec A = \frac{1}{\cos A} \qquad \csc^2 A - \cot^2 A = 1$$

$$\csc A = \frac{1}{\operatorname{sen} A}$$

Signos e intervalos de variación

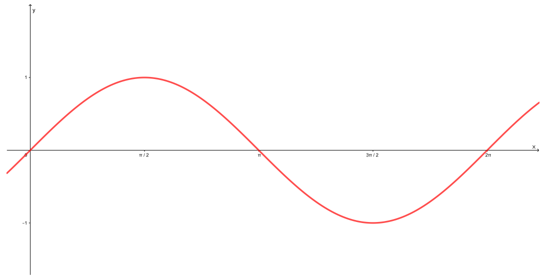
Cuadrante	sen A	cos A	tan A	cot A	sec A	csc A
I	$+$ 0 a 1	$+$ 1 a 0	$+$ 0 a ∞	$+$ ∞ a 0	$+$ 1 a ∞	$+$ ∞ a 1
II	$+$ 1 a 0	$-$ 0 a -1	$-$ $-\infty$ a 0	$-$ 0 a $-\infty$	$-$ $-\infty$ a -1	$+$ 1 a ∞
III	$-$ 0 a -1	$-$ -1 a 0	$+$ 0 a ∞	$+$ ∞ a 0	$-$ -1 a $-\infty$	$-$ $-\infty$ a -1
IV	$-$ -1 a 0	$+$ 0 a 1	$-$ $-\infty$ a 0	$-$ 0 a $-\infty$	$+$ ∞ a 1	$-$ -1 a $-\infty$

Valores exactos de las funciones para ciertos ángulos

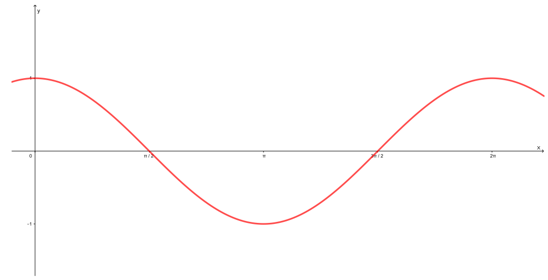
$A°$	A rad	$\operatorname{sen} A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
$0°$	0	0	1	0	∞	1	∞
$15°$	$\frac{\pi}{12}$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$45°$	$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
$75°$	$\frac{5\pi}{12}$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
$90°$	$\frac{\pi}{2}$	1	0	$\pm\infty$	0	$\pm\infty$	1
$105°$	$\frac{7\pi}{12}$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-2-\sqrt{3}$	$-2+\sqrt{3}$	$-\sqrt{6}-\sqrt{2}$	$-\sqrt{6}+\sqrt{2}$
$120°$	$\frac{2\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
$135°$	$\frac{3\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$150°$	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
$165°$	$\frac{11\pi}{12}$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-2+\sqrt{3}$	$-2-\sqrt{3}$	$-\sqrt{6}+\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
$180°$	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
$195°$	$\frac{13\pi}{12}$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-\sqrt{6}+\sqrt{2}$	$-\sqrt{6}-\sqrt{2}$
$210°$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
$225°$	$\frac{5\pi}{4}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$240°$	$\frac{4\pi}{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
$255°$	$\frac{17\pi}{12}$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-\sqrt{6}-\sqrt{2}$	$-\sqrt{6}+\sqrt{2}$
$270°$	$\frac{3\pi}{2}$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
$285°$	$\frac{19\pi}{12}$	$-\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$-2-\sqrt{3}$	$-2+\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$-\sqrt{6}+\sqrt{2}$
$300°$	$\frac{5\pi}{3}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
$315°$	$\frac{7\pi}{4}$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$330°$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
$345°$	$\frac{23\pi}{12}$	$-\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$	$\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$	$-2+\sqrt{3}$	$-2-\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$-\sqrt{6}-\sqrt{2}$
$360°$	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

Representación gráfica de las funciones

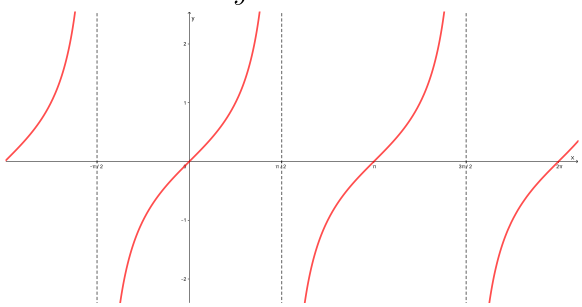
$y = \operatorname{sen} x$



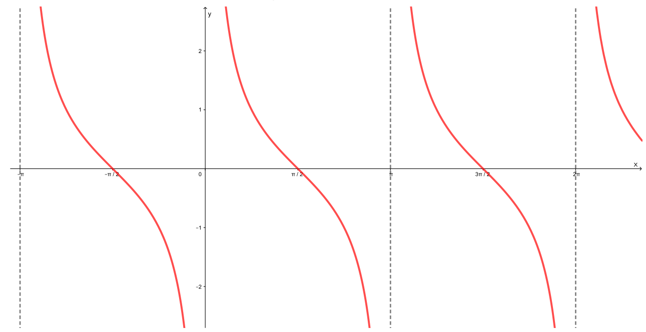
$y = \cos x$



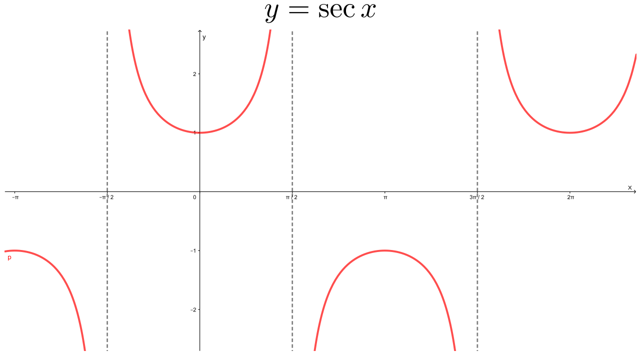
$y = \tan x$



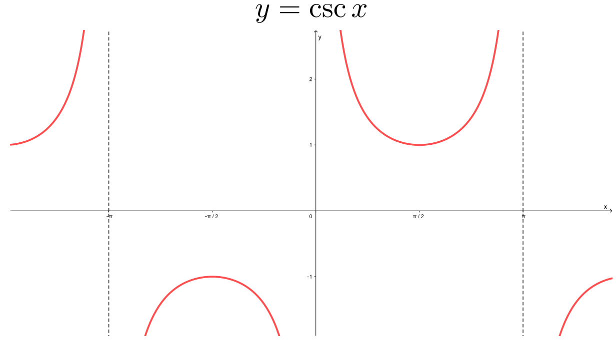
$y = \cot x$



$y = \sec x$



$y = \csc x$



Funciones de ángulos negativos

$\operatorname{sen}(-A) = -\operatorname{sen} A \qquad \cos(-A) = \cos A \qquad \tan(-A) = -\tan A$

$\csc(-A) = -\csc A \qquad \sec(-A) = \sec A \qquad \cot(-A) = -\cot A$

Formulas de adicion

$$\operatorname{sen}(A \pm B) = \operatorname{sen} A \cos B \pm \cos A \operatorname{sen} B \qquad \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cos(A \pm B) = \cos A \cos B \mp \operatorname{sen} A \operatorname{sen} B \qquad \cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$$

Funciones de ángulos reducidos al primer cuadrante

(con $k \in \mathbb{Z}$)

Función	$-A$	$\frac{\pi}{2} \pm A$	$\pi \pm A$	$\frac{3\pi}{2} \pm A$	$2k\pi + A$
sen	$-\operatorname{sen} A$	$\cos A$	$\mp \operatorname{sen} A$	$-\cos A$	$\pm \operatorname{sen} A$
cos	$\cos A$	$\mp \operatorname{sen} A$	$-\cos A$	$\pm \operatorname{sen} A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

Relaciones entre funciones de los ángulos en el primer cuadrante

Función	$\operatorname{sen} A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
$\operatorname{sen} A$	u	$\sqrt{1-u^2}$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{1}{u}$
$\cos A$	$\sqrt{1-u^2}$	u	$\frac{1}{\sqrt{1+u^2}}$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\frac{\sqrt{1-u^2}}{u}$
$\tan A$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{\sqrt{1-u^2}}{u}$	u	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{1}{\sqrt{1-u^2}}$
$\cot A$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{1}{u}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\sqrt{1-u^2}$
$\sec A$	$\frac{1}{\sqrt{1-u^2}}$	$\frac{1}{u}$	$\sqrt{1+u^2}$	$\frac{\sqrt{1-u^2}}{u}$	u	$\frac{u}{\sqrt{1-u^2}}$
$\csc A$	$\frac{1}{u}$	$\frac{1}{\sqrt{1-u^2}}$	$\frac{\sqrt{1+u^2}}{u}$	$\sqrt{1+u^2}$	$\frac{u}{\sqrt{1-u^2}}$	u

Formulas del ángulo doble

$$\operatorname{sen} 2A = 2 \operatorname{sen} A \cos A$$

$$\cos 2A = \cos^2 A - \operatorname{sen}^2 A = 1 - 2\operatorname{sen}^2 A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulas de medio ángulo

$$\operatorname{sen}\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad \begin{bmatrix} + & \text{si} & A/2 & \text{esta} & \text{I o II cuadrante} \\ - & \text{si} & A/2 & \text{esta} & \text{III o IV cuadrante} \end{bmatrix}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \begin{bmatrix} + & \text{si} & A/2 & \text{esta} & \text{I o IV cuadrante} \\ - & \text{si} & A/2 & \text{esta} & \text{II o III cuadrante} \end{bmatrix}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1-\cos A}} \quad \begin{bmatrix} + & \text{si} & A/2 & \text{esta} & \text{I o III cuadrante} \\ - & \text{si} & A/2 & \text{esta} & \text{II o IV cuadrante} \end{bmatrix}$$

$$= \frac{\operatorname{sen} A}{1+\cos A} = \frac{1-\cos A}{\operatorname{sen} A} = \csc A - \cot A$$

Formulas de ángulos múltiplos

$$\operatorname{sen} 3A = 3\operatorname{sen} A - 4\operatorname{sen}^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\operatorname{sen} 4A = 4\operatorname{sen} A\cos A - 8\operatorname{sen}^3 A\cos A$$

$$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

$$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

$$\operatorname{sen} 5A = 5\operatorname{sen} A - 20\operatorname{sen}^3 A + 16\operatorname{sen}^5 A$$

$$\cos 5A = 16\cos^3 A - 20\cos^3 A + 5\cos A$$

$$\tan 5A = \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}$$

Potencias de funciones trigonométricas

$$\operatorname{sen}^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$$

$$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$$

$$\operatorname{sen}^3 A = \frac{3}{4}\operatorname{sen} A - \frac{1}{4}\operatorname{sen} 3A$$

$$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$$

$$\operatorname{sen}^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$$

$$\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$$

$$\operatorname{sen}^5 A = \frac{5}{8}\operatorname{sen} A - \frac{5}{16}\operatorname{sen} 3A + \frac{1}{16}\operatorname{sen} 5A$$

$$\cos^5 A = \frac{5}{8}\cos A + \frac{5}{16}\cos 3A + \frac{1}{16}\cos 5A$$

Suma, diferencia y producto de las funciones

$$\operatorname{sen} A + \operatorname{sen} B = 2\operatorname{sen}\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\operatorname{sen} A - \operatorname{sen} B = 2\cos\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = 2\operatorname{sen}\left(\frac{A+B}{2}\right)\operatorname{sen}\left(\frac{A-B}{2}\right)$$
$$\operatorname{sen} A \operatorname{sen} B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$
$$\operatorname{sen} A \cos B = \frac{1}{2}\left[\operatorname{sen}(A-B) + \operatorname{sen}(A+B)\right]$$
$$\cos A \operatorname{sen} B = \frac{1}{2}\left[\operatorname{sen}(A+B) - \operatorname{sen}(A-B)\right]$$

Formulas generales

$$\operatorname{sen} nA = \operatorname{sen} A\left[(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{1}(2\cos A)^{n-5} - \cdots\right]$$

$$\cos nA = \frac{1}{2}\left[(2\cos A)^{n} - n(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\right]$$

$$\operatorname{sen}^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}}\left[\operatorname{sen}(2n-1)A - \binom{2n-2}{1}\operatorname{sen}(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\operatorname{sen} A\right]$$

$$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\left[\cos(2n-1)A + \binom{2n-2}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right]$$

$$\operatorname{sen}^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^{n}}{2^{2n-1}}\left[\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right]$$

$$\cos^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left[\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right]$$

Funciones trigonométricas inversas

Valores principales para $x \geq 0$	Valores principales para $x < 0$
$0 \leq \operatorname{sen}^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \operatorname{sen}^{-1} x < 0$
$0 \leq \cos^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \leq \pi$
$0 \leq \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 \leq \cot^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \leq \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \leq \pi$
$0 \leq \csc^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \csc^{-1} x < 0$

Relaciones entre trigonométricas inversas

$$\operatorname{sen}^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

$$\csc^{-1} x = \operatorname{sen}^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{sen}^{-1}(-x) = -\operatorname{sen}^{-1} x$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x$$

$$\csc^{-1}(-x) = -\csc^{-1} x$$

Funciones inversas en su forma compleja

$$\operatorname{sen}^{-1} x = -j \ln\left(jx + \sqrt{1 - x^2}\right)$$

$$\cos^{-1} x = -j \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tan^{-1} x = \frac{j}{2} \ln\left(\frac{j + x}{j - x}\right)$$

$$\csc^{-1} x = -j \ln\left(\frac{j}{x} + \sqrt{1 - \frac{1}{x^2}}\right)$$

$$\sec^{-1} x = -j \ln\left(\frac{1}{x} + \sqrt{1 - \frac{i}{x^2}}\right)$$

$$\cot^{-1} x = \frac{j}{2} \ln\left(\frac{j - x}{j + x}\right)$$

Relaciones entre lados y ángulos en un triangulo plano

Ley de senos

$$\frac{a}{\operatorname{sen} A} = \frac{b}{\operatorname{sen} B} = \frac{c}{\operatorname{sen} C}$$

Ley de cosenos

$$c^2 = a^2 + b^2 - 2ab\cos C$$

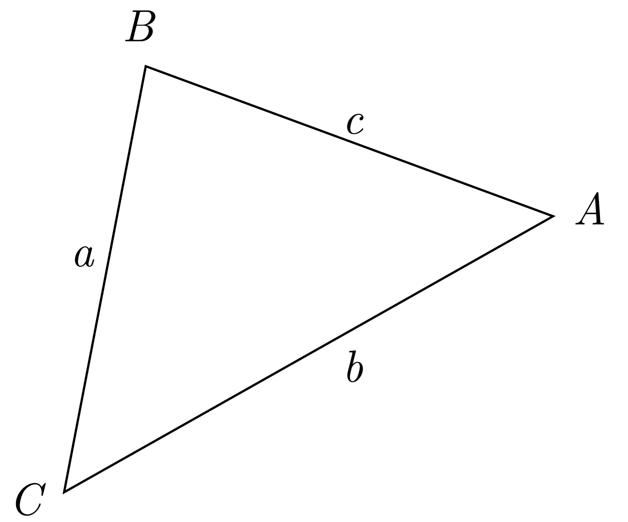
Ley de tangentes

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

Relacion de Heron

$$\operatorname{sen} A = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

con $s = \frac{a+b+c}{2}$



Relaciones entre lados y ángulos en un triangulo esférico

Ley de senos

$$\frac{\operatorname{sen} a}{\operatorname{sen} A} = \frac{\operatorname{sen} b}{\operatorname{sen} B} = \frac{\operatorname{sen} c}{\operatorname{sen} C}$$

Ley de cosenos

$$\cos a = \cos b\cos c + \operatorname{sen} b\operatorname{sen} c\cos A$$

$$\cos A = -\cos B\cos C + \operatorname{sen} B\operatorname{sen} C\cos a$$

Ley de tangentes

$$\frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} = \frac{\tan\left(\frac{a+b}{2}\right)}{\tan\left(\frac{a-b}{2}\right)}$$

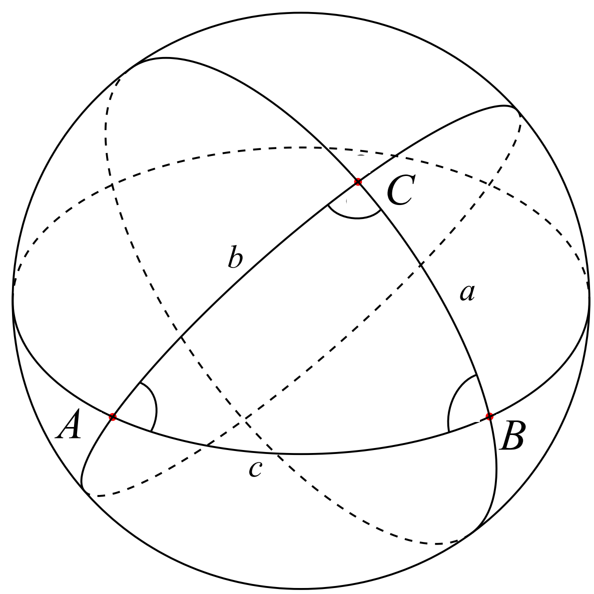
Relacion de Heron

$$\cos\frac{A}{2} = \sqrt{\frac{\operatorname{sen} s\operatorname{sen}(s-c)}{\operatorname{sen} b\operatorname{sen} c}}$$

con $s = \frac{a+b+c}{2}$

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\operatorname{sen} B\operatorname{sen} C}}$$

con $S = \frac{A+B+C}{2}$



Funciones exponenciales y logarítmicas

Forma polar de los números complejos

$$z = x + jy = r(\cos\theta + j\,\mathrm{sen}\,\theta) = re^{j\theta}$$

$$\text{Donde } r = \sqrt{x^2 + y^2} \text{ y } \theta = \mathrm{Arg}\{z\}$$

Operaciones con números complejos en forma polar

$$\left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1+\theta_2)}$$

$$\left(re^{j\theta}\right)^p = r^p e^{jp\theta}$$

$$\left(re^{j\theta}\right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\theta+2k\pi}{n}}$$

Logaritmo de un numero complejo

$$\ln\left(re^{j\theta}\right) = \ln r + i(\theta + 2k\pi)$$

$$\text{con } \; k \in \mathbb{Z}$$

Funciones hiperbólicas

Definición de las funciones

$$\operatorname{senh} x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$
$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$
$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

Relaciones entre funciones

$$\tanh x = \frac{\operatorname{senh} x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\operatorname{senh} x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\operatorname{senh} x}$$
$$\cosh^2 x - \operatorname{senh}^2 x = 1$$
$$\operatorname{sech}^2 x + \tanh^2 x = 1$$
$$\coth^2 x - \operatorname{csch}^2 x = 1$$

Funciones de ángulos negativos

$$\operatorname{senh}(-x) = -\operatorname{senh} x \qquad \cosh(-x) = \cosh x \qquad \tanh(-x) = -\tanh x$$

$$\operatorname{csch}(-x) = -\operatorname{csch} x \qquad \operatorname{sech}(-x) = \operatorname{sech} x \qquad \coth(-x) = -\coth x$$

Formulas de adicion

$$\operatorname{senh}(x \pm y) = \operatorname{senh} x \cosh y \pm \cosh x \operatorname{senh} y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tan y}$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \operatorname{senh} x \operatorname{senh} y$$

$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth x \pm \coth y}$$

Formulas del ángulo doble

$$\begin{aligned}
\operatorname{senh} 2x &= 2\operatorname{senh} x\cosh x\\
\cosh 2x &= \cosh^2 x + \operatorname{senh}^2 x = 1 + 2\operatorname{senh}^2 x = 2\cosh^2 x - 1\\
\tanh 2x &= \frac{2\tanh x}{1+\tanh^2 x}
\end{aligned}$$

Formulas de medio ángulo

$$\begin{aligned}
\operatorname{senh}\frac{A}{2} &= \pm\sqrt{\frac{\cosh x - 1}{2}}\\
\cosh\frac{A}{2} &= \sqrt{\frac{\cosh x + 1}{2}}\\
\tanh\frac{A}{2} &= \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}}\\
&= \frac{\operatorname{senh} x}{\cosh x + 1} = \frac{\cosh x - 1}{\operatorname{senh} x}
\end{aligned}$$

$$(+ \text{ si } x > 0,\ - \text{ si } x < 0)$$

Formulas de ángulos múltiplos

$$\begin{aligned}
\operatorname{senh} 3x &= 3\operatorname{senh} x + 4\operatorname{senh}^3 x\\
\cosh 3x &= 4\cosh^3 x - 3\cosh x\\
\tanh 3x &= \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}\\
\operatorname{senh} 4x &= 4\operatorname{senh} x\cosh^3 x + 4\operatorname{senh}^3 x\cosh x\\
\cosh 4x &= 8\cosh^4 x - 8\cosh^2 x + 1\\
\tanh 4x &= \frac{4\tanh x + 4\tanh^3 x}{1 + 6\tanh^2 x + \tanh^4 x}
\end{aligned}$$

Potencias de funciones trigonométricas

$$\operatorname{senh}^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2} \qquad \operatorname{senh}^3 x = \frac{1}{4}\operatorname{senh} 3x - \frac{3}{4}\operatorname{senh} x \qquad \operatorname{senh}^4 x = \frac{3}{8} - \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$$

$$\cosh^2 x = \frac{1}{2} + \frac{1}{2}\cosh 2x \qquad \cosh^3 x = \frac{3}{4}\cosh x + \frac{1}{4}\cosh 3x \qquad \cosh^4 x = \frac{3}{8} + \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$$

Suma, diferencia y producto de las funciones

$$\operatorname{senh} x + \operatorname{senh} y = 2 \operatorname{senh}\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-b}{2}\right)$$
$$\operatorname{senh} x - \operatorname{senh} y = 2 \cosh\left(\frac{x+y}{2}\right) \operatorname{senh}\left(\frac{x-y}{2}\right)$$
$$\cosh x + \cosh y = 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$$
$$\cosh x - \cosh y = 2 \operatorname{senh}\left(\frac{x+y}{2}\right) \operatorname{senh}\left(\frac{x-y}{2}\right)$$
$$\operatorname{senh} x \operatorname{senh} y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$$
$$\cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$$
$$\operatorname{senh} x \cosh y = \frac{1}{2}\left[\operatorname{senh}(x+y) + \operatorname{senh}(x-y)\right]$$

Relaciones entre funciones hiperbólicas

Función	$\operatorname{senh} A = u$	$\cosh A = u$	$\tanh A = u$	$\coth A = u$	$\operatorname{sech} A = u$	$\operatorname{csch} A = u$
$\operatorname{senh} A$	u	$\sqrt{u^2-1}$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{1}{\sqrt{u^2-1}}$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{1}{u}$
$\cosh A$	$\sqrt{1+u^2}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\frac{u}{\sqrt{u^2-1}}$	$\frac{1}{u}$	$\frac{\sqrt{1+u^2}}{u}$
$\tanh A$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{\sqrt{u^2-}}{u}$	u	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{1}{\sqrt{1+u^2}}$
$\coth A$	$\frac{\sqrt{u^2+1}}{u}$	$\frac{u}{\sqrt{u^2-1}}$	$\frac{1}{u}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\sqrt{1+u^2}$
$\operatorname{sech} A$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{\sqrt{u^2-1}}{u}$	u	$\frac{u}{\sqrt{1+u^2}}$
$\operatorname{csch} A$	$\frac{1}{u}$	$\frac{1}{\sqrt{u^2-1}}$	$\frac{\sqrt{1-u^2}}{u}$	$\sqrt{u^2-1}$	$\frac{u}{\sqrt{1-u^2}}$	u

Funciones hiperbólicas inversas

$$\operatorname{senh}^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$$

Relaciones entre hiperbólicas inversas

$$\operatorname{csch}^{-1} x = \operatorname{senh}^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{senh}^{-1}(-x) = -\operatorname{senh}^{-1} x$$

$$\tanh^{-1}(-x) = -\tanh^{-1} x$$

$$\operatorname{sech}^{-1}(-x) = -\operatorname{sech}^{-1} x$$

$$\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$$

Relaciones entre funciones trigonométricas con hiperbólicas

$$\operatorname{sen}(jx) = j \operatorname{senh} x \qquad \cos(jx) = \cosh x \qquad \tan(jx) = j \tan x$$

$$\csc(jx) = -j \operatorname{csch} x \qquad \sec(jx) = \operatorname{sech} x \qquad \cot(jx) = -j \coth x$$

$$\operatorname{senh}(jx) = j \operatorname{sen} x \qquad \cosh(jx) = \cos x \qquad \tanh(jx) = j \tan x$$

$$\operatorname{csch}(jx) = -j \csc x \qquad \operatorname{sech}(jx) = \sec x \qquad \coth(jx) = -j \cot x$$

Periodicidad de funciones hiperbólicas

$$\operatorname{senh}(x + 2k\pi j) = \operatorname{senh} x \qquad \cosh(x + 2k\pi j) = \cosh x \qquad \tanh(x + k\pi j) = \tanh x$$

$$\operatorname{csch}(x + 2k\pi j) = \operatorname{csch} x \qquad \operatorname{sech}(x + 2k\pi j) = \operatorname{sech} x \qquad \coth(x + k\pi j) = \coth x$$

Con $k \in \mathbb{Z}$.

Relaciones entre hiperbólicas inversas y trigonométricas inversas

$\operatorname{senh}^{-1}(jx) = j \operatorname{sen}^{-1} x$

$\cosh^{-1} x = \pm j \cos^{-1} x$

$\tanh^{-1}(jx) = j \tan^{-1} x$

$\coth^{-1}(jx) = -j \cot^{-1} x$

$\operatorname{sech}^{-1} x = \pm j \sec^{-1} x$

$\operatorname{csch}^{-1}(jx) = -j \csc^{-1} x$

$\operatorname{sen}^{-1}(jx) = j \operatorname{senh}^{-1} x$

$\cos^{-1} x = \pm j \cosh^{-1} x$

$\tan^{-1}(jx) = j \tanh^{-1} x$

$\cot^{-1}(jx) = -j \coth^{-1} x$

$\sec^{-1} x = \pm j \operatorname{sech}^{-1} x$

$\csc^{-1}(jx) = -j \operatorname{csch}^{-1} x$

Soluciones de ecuaciones algebraicas

Ecuación cuadrática $ax^2 + bx + c = 0$

Soluciones:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Si a, b , c son reales, y $\Delta = b^2 - 4ac$ es el *discriminante*, entonces las raíces son

i reales y desiguales si $\Delta > 0$

ii reales e iguales si $\Delta = 0$

iii conjugadas complejas si $\Delta < 0$

Si x_1, x_2 son las raíces entonces $x_1 + x_2 = -\frac{b}{a}$ y $x_1 x_2 = \frac{c}{a}$

Ecuacion cubica $x^3 + a_1 x^2 + a_2 x + a_3 = 0$

Sea

$$Q = \frac{3a_2 - a_1^2}{9} \qquad R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}$$

$$S = \sqrt{R + \sqrt{Q^3 + R^2}} \qquad T = \sqrt{R - \sqrt{Q^3 + R^2}}$$

Soluciones:

$$\begin{cases} x_1 = & S + T - \frac{1}{3}a_1 \\ x_2 = & -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{1}{2}j\sqrt{3}(S-T) \\ x_3 = & -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{1}{2}j\sqrt{3}(S-T) \end{cases}$$

Si a_1, a_2 , a_3 son reales, y $\Delta = Q^3 + R^2$ es el *discriminante*, entonces las raíces son

i una real y dos complejas conjugadas si $\Delta > 0$

ii todas reales y por lo menos dos de ellas iguales si $\Delta = 0$

iii todas reales y distintas si $\Delta < 0$

Simplificación si $\Delta < 0$.
Soluciones:

$$\begin{cases} x_1 = & 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) \\ x_2 = & 2\sqrt{-Q}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right) \\ x_3 = & 2\sqrt{-Q}\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right) \end{cases}$$

donde $\cos\theta = -\dfrac{R}{\sqrt{-Q^3}}$
Y ademas:
$x_1 + x_2 + x_3 = -a_1$, $x_1x_2 + x_2x_3 + x_3x_1 = a_2$ y $x_1x_2x_3 = -a_3$ donde x_1, x_2 y x_3 son las tres raíces.

Integrales

$$\int x^n\ dx = \frac{x^{n+1}}{n+1},\ \ n \neq -1$$
$$\int \frac{dx}{x} = \ln|x|$$
$$\int \frac{dx}{a+bx} = \frac{\ln(a+bx)}{b}$$
$$\int \frac{x}{a+bx} = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$$
$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\ln\left(\frac{x+a}{x}\right)$$
$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$
$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$$
$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$$
$$\int \frac{x}{x^2 \pm a^2}\ dx = \pm\frac{1}{2}\ln(a^2 \pm x^2)$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{sen}^{-1}\left(\frac{x}{a}\right),\quad a^2-x^2>0$$
$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x+\sqrt{x^2 \pm a^2}\right)$$
$$\int \frac{x}{\sqrt{a^2-x^2}}\ dx = -\sqrt{a^2-x^2}$$
$$\int \frac{x}{\sqrt{x^2 \pm a^2}}\ dx = \sqrt{x^2 \pm a^2}$$
$$\int \sqrt{a^2-x^2}\ dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\operatorname{sen}^{-1}\left(\frac{x}{a}\right)\right]$$
$$\int x\sqrt{a^2-x^2}\ dx = -\frac{1}{3}(a^2-x^2)^{3/2}$$
$$\int \sqrt{x^2 \pm a^2}\ dx = \frac{1}{3}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x+\sqrt{x^2 \pm a^2}\right)\right]$$
$$\int x\sqrt{x^2 \pm a^2}\ dx = \frac{1}{3}(a^2 \pm a^2)^{3/2}$$
$$\int e^{ax}\ dx = \frac{1}{a}e^{ax}$$
$$\int \ln(ax)\ dx = x\ln(ax) - x$$
$$\int xe^{ax}\ dx = \frac{e^{ax}}{a^2}(ax-1)$$
$$\int \frac{dx}{a+bx^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln\left(a+bx^{cx}\right)$$
$$\int \operatorname{sen}(ax)\ dx = -\frac{1}{a}\cos(ax)$$
$$\int \cos(ax)\ dx = \frac{1}{a}\operatorname{sen}(ax)$$
$$\int \tan(ax)\ dx = \frac{1}{a}\ln[\sec(ax)]$$
$$\int \cot(ax)\ dx = \frac{1}{a}\ln[\operatorname{sen}(ax)]$$
$$\int \sec(ax)\ dx = \frac{1}{a}\ln[\sec(ax)+\tan(ax)]$$
$$\int \csc(ax)\ dx = \frac{1}{a}\ln[\csc(ax)-\cot(ax)]$$
$$\int \operatorname{sen}^2(ax)\ dx = \frac{x}{2} - \frac{\operatorname{sen}(2ax)}{4a}$$
$$\int \cos^2(ax)\ dx = \frac{x}{2} + \frac{\operatorname{sen}(2ax)}{4a}$$
$$\int \frac{dx}{\operatorname{sen}^2(ax)} = -\frac{1}{a}\cot(ax)$$
$$\int \frac{dx}{\cos^2(ax)} = \frac{1}{a}\tan(ax)$$
$$\int \tan^2(ax)\ dx = \frac{1}{a}\tan(ax) - x$$
$$\int \cot^2(ax)\ dx = -\frac{1}{a}\cot(ax) - x$$
$$\int \operatorname{sen}^{-1}(ax)\ dx = x\operatorname{sen}^{-1}(ax) + \frac{\sqrt{1-a^2x^2}}{a}$$
$$\int \cos^{-1}(ax)\ dx = x\cos^{-1}(ax) - \frac{\sqrt{1-a^2x^2}}{a}$$
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2+a^2}}$$
$$\int \frac{x}{(x^2+a^2)^{3/2}}\ dx = -\frac{1}{\sqrt{x^2+a^2}}$$

$$\int x \operatorname{sen}(ax)\ dx = \frac{1}{a^2}\left(\sin(ax) - ax\cos(ax)\right)$$

$$\int x\cos(ax)\ dx = \frac{1}{a^2}\left(\cos(ax) + ax\operatorname{sen}(ax)\right)$$

$$\int x^2\operatorname{sen}(ax)dx = \frac{1}{a^3}\left(-a^2x^2\cos(ax) + 2\cos(ax) + 2ax\operatorname{sen}(ax)\right)$$

$$\int x^2\cos(ax)dx = \frac{1}{a^3}\left(-a^2x^2\operatorname{sen}(ax) - 2\operatorname{sen}(ax) + 2ax\cos(ax)\right)$$

$$\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}\ dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}}\ dx = -\sqrt{a^2 - x^2}$$

$$\int x\sqrt{a^2 - x^2}\ dx = -\frac{x}{3}\left(a^2 - x^2\right)^{3/2}$$

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} - a^2\operatorname{sen}^{-1}\left(\frac{x}{a}\right)\right]$$

$$\int x\sqrt{x^2 \pm a^2}dx = \frac{x}{3}\left(x^2 - a^2\right)^{3/2}$$

Integral de probabilidad de Gauss

$$I_0 = \int_0^\infty e^{-ax^2}\ dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$I_1 = \int_0^\infty xe^{-ax^2}\ dx = \frac{1}{2a}$$

$$I_2 = \int_0^\infty x^2e^{-ax^2}\ dx = -\frac{dI_0}{da} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^\infty x^3e^{-ax^2}\ dx = -\frac{dI_1}{da} = \frac{1}{2a}$$

$$I_4 = \int_0^\infty x^4e^{-ax^2}\ dx = \frac{d^2I_0}{da^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

$$I_5 = \int_0^\infty x^5e^{-ax^2}\ dx = \frac{d^2I_1}{da^2} = \frac{1}{a^3}$$

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$$I_{2n} = (-1)^n\frac{d^n}{da^n}I_0$$

$$I_{2n+1} = (-1)^n\frac{d^n}{da^n}I_1$$