

Instituto tecnológico de Costa Rica

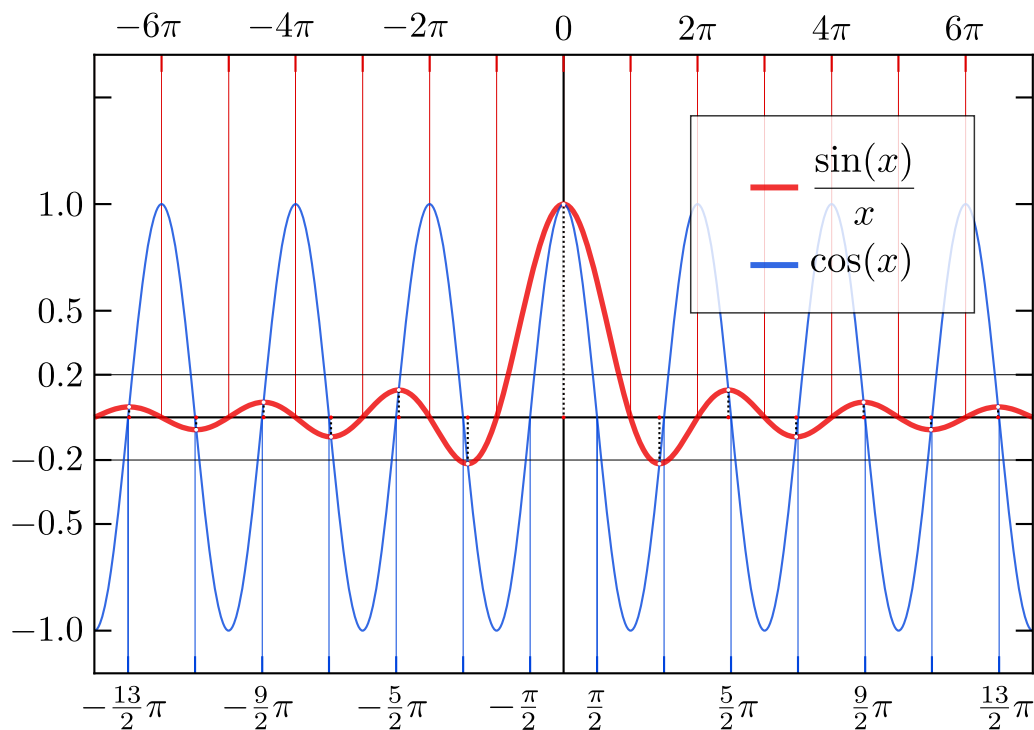
Sede central Cartago

Escuela de Ingeniería Electrónica
EL4703 - Señales y sistemas



Formulario general

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Mapeos e Integración

Mapeo de inversión

Mapeo de círculos

$$|w - w_0| = r_w$$

$$r_w = \left| \frac{r}{\alpha} \right|$$

$$w_0 = -\frac{z_0^*}{\alpha}$$

$$\alpha = r^2 - |z_0|^2$$

$$v = \frac{x_0}{y_0}u - \frac{1}{2y_0}$$

Mapeo de rectas

$$v = \frac{\operatorname{Re}\{a - b\}}{\operatorname{Im}\{a - b\}}u$$

$$w_0 = \frac{(a - b)^*}{\beta}$$

$$r_w = \left| \frac{a - b}{\beta} \right|$$

$$\beta = |a|^2 - |b|^2$$

Integración trigonométrica

$$\operatorname{sen} \theta = \frac{1}{2j} \left(z - \frac{1}{z} \right)$$

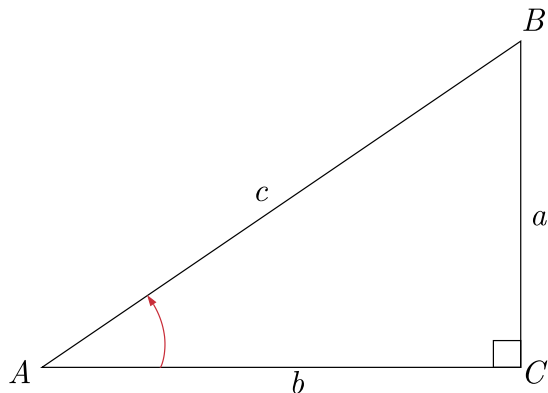
$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

$$d\theta = \frac{dz}{jz}$$

$$z = e^{j\theta}$$

Funciones trigonométricas

Definición en un triángulo rectángulo



$$\operatorname{sen} A = \frac{a}{c} = \frac{\text{Cateto opuesto}}{\text{Hipotenusa}}$$

$$\cos A = \frac{b}{c} = \frac{\text{Cateto adyacente}}{\text{Hipotenusa}}$$

$$\tan A = \frac{a}{b} = \frac{\text{Cateto opuesto}}{\text{Cateto adyacente}}$$

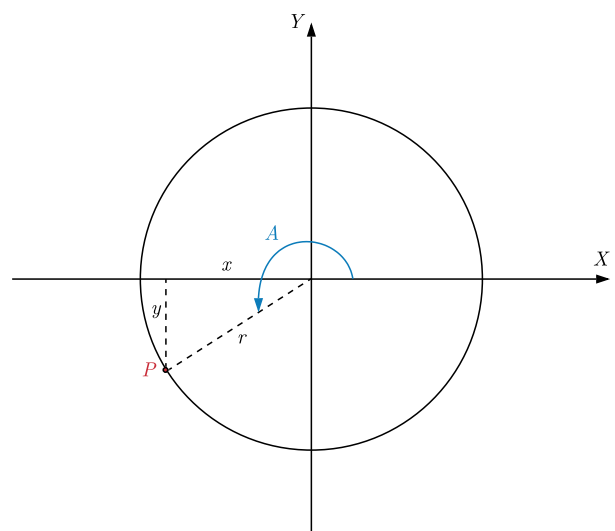
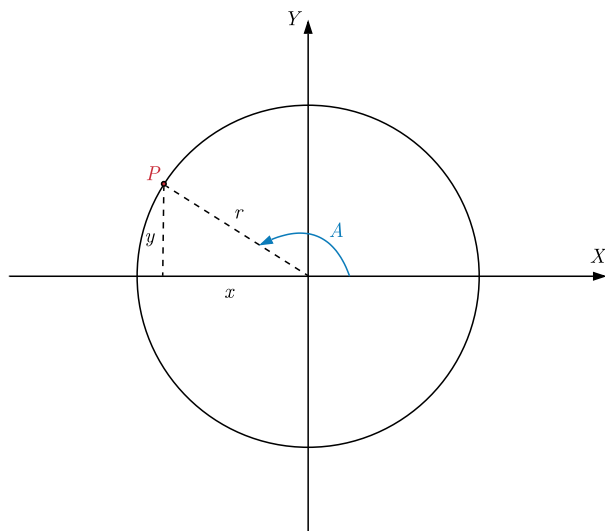
$$\cot A = \frac{b}{a} = \frac{\text{Cateto adyacente}}{\text{Cateto opuesto}}$$

$$\sec A = \frac{c}{b} = \frac{\text{Hipotenusa}}{\text{Cateto adyacente}}$$

$$\csc A = \frac{c}{a} = \frac{\text{Hipotenusa}}{\text{Cateto opuesto}}$$

Funciones para cualquier ángulo A en el plano

Considere un sistema de coordenadas XY , donde las coordenadas de un punto $P = (x, y)$, la distancia del punto P al origen $r = \sqrt{x^2 + y^2}$ y el ángulo A positivo como se muestra en la figura.



Funciones trigonométricas para un ángulo A en cualquier cuadrante:

$$\operatorname{sen} A = \frac{y}{r}$$

$$\operatorname{csc} A = \frac{r}{y}$$

$$\cos A = \frac{x}{r}$$

$$\sec A = \frac{r}{x}$$

$$\tan A = \frac{y}{x}$$

$$\cot A = \frac{x}{y}$$

Relaciones entre grados y radianes

$$1 \text{ radián} = \frac{180^\circ}{\pi}$$

$$\theta \text{ rad} = \frac{\theta^\circ \cdot \pi \text{ rad}}{180^\circ}$$

$$1^\circ = \frac{\pi}{180} \text{ radianes}$$

$$\theta^\circ = \frac{\theta \text{ rad} \cdot 180^\circ}{\pi \text{ rad}}$$

Relaciones entre las funciones trigonométricas

$$\tan A = \frac{\operatorname{sen} A}{\cos A}$$

$$\operatorname{sen}^2 A + \cos^2 A = 1$$

$$\cot A = \frac{\cos A}{\operatorname{sen} A}$$

$$\sec^2 A - \tan^2 A = 1$$

$$\sec A = \frac{1}{\cos A}$$

$$\csc^2 A - \cot^2 A = 1$$

$$\csc A = \frac{1}{\operatorname{sen} A}$$

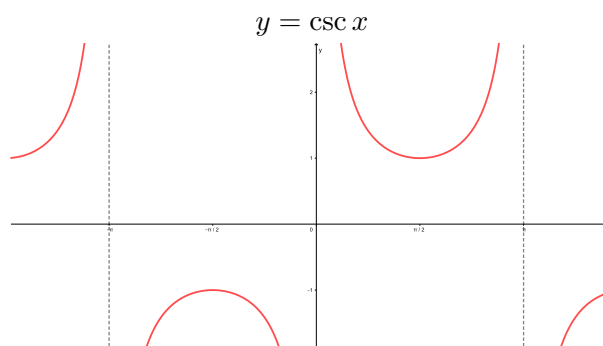
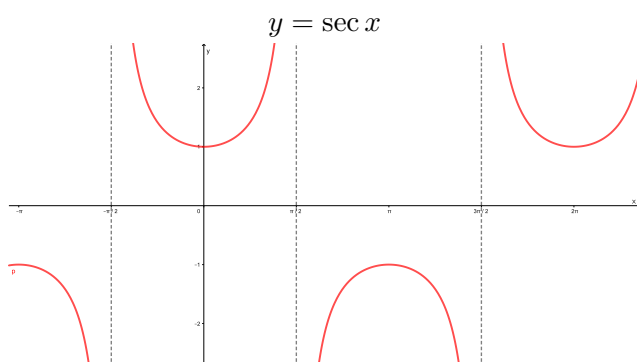
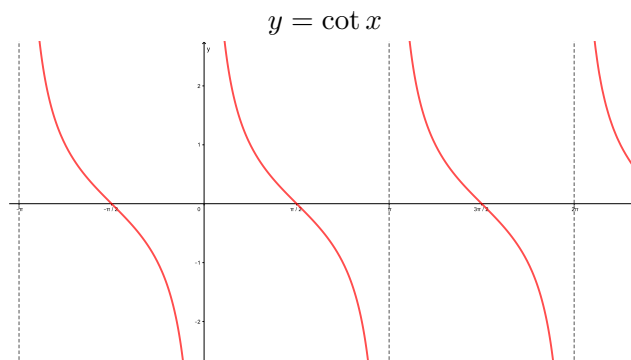
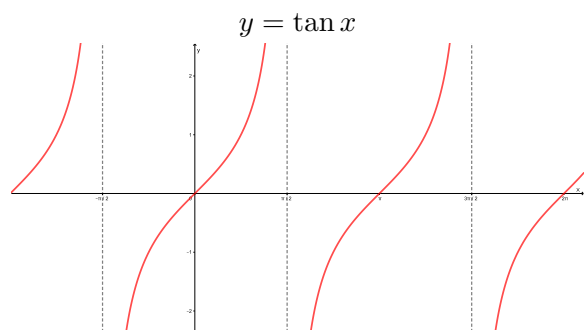
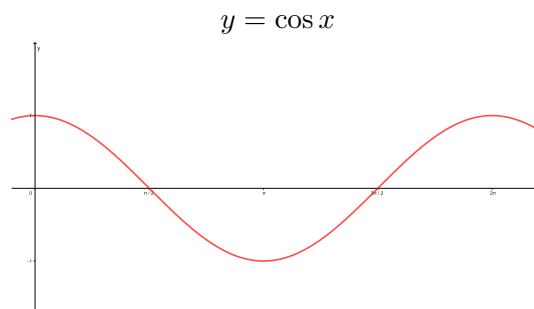
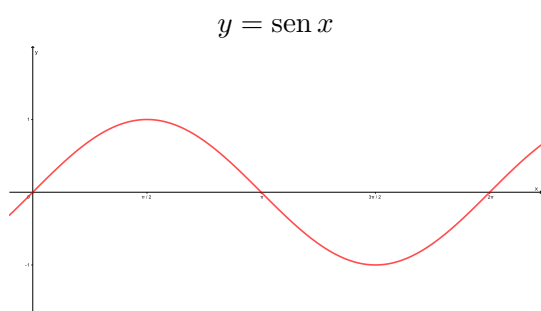
Signos e intervalos de variación

Cuadrante	$\operatorname{sen} A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
I	+	+	+	+	+	+
	0 a 1	1 a 0	0 a ∞	∞ a 0	1 a ∞	∞ a 1
II	+	-	-	-	-	+
	1 a 0	0 a -1	$-\infty$ a 0	0 a $-\infty$	$-\infty$ a -1	1 a ∞
III	-	-	+	+	-	-
	0 a -1	-1 a 0	0 a ∞	∞ a 0	-1 a $-\infty$	$-\infty$ a -1
IV	-	+	-	-	+	-
	-1 a 0	0 a 1	$-\infty$ a 0	0 a $-\infty$	∞ a 1	-1 a $-\infty$

Valores exactos de las funciones para ciertos ángulos

A°	A rad	$\operatorname{sen} A$	$\operatorname{cos} A$	$\tan A$	$\cot A$	$\sec A$	$\operatorname{csc} A$
0°	0	0	1	0	∞	1	∞
15°	$\frac{\pi}{12}$	$\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$2 - \sqrt{3}$	$2 + \sqrt{3}$	$\sqrt{6} - \sqrt{2}$	$\sqrt{6} + \sqrt{2}$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
75°	$\frac{5\pi}{12}$	$\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$2 + \sqrt{3}$	$2 - \sqrt{3}$	$\sqrt{6} + \sqrt{2}$	$\sqrt{6} - \sqrt{2}$
90°	$\frac{\pi}{2}$	1	0	$\pm\infty$	0	$\pm\infty$	1
105°	$\frac{7\pi}{12}$	$\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$-\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$-2 - \sqrt{3}$	$-2 + \sqrt{3}$	$-\sqrt{6} - \sqrt{2}$	$-\sqrt{6} + \sqrt{2}$
120°	$\frac{2\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
165°	$\frac{11\pi}{12}$	$\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$-\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$-2 + \sqrt{3}$	$-2 - \sqrt{3}$	$-\sqrt{6} + \sqrt{2}$	$\sqrt{6} + \sqrt{2}$
180°	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
195°	$\frac{13\pi}{12}$	$-\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$-\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$2 - \sqrt{3}$	$2 + \sqrt{3}$	$-\sqrt{6} + \sqrt{2}$	$-\sqrt{6} - \sqrt{2}$
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$\frac{5\pi}{4}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
255°	$\frac{17\pi}{12}$	$-\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$-\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$2 + \sqrt{3}$	$2 - \sqrt{3}$	$-\sqrt{6} - \sqrt{2}$	$-\sqrt{6} + \sqrt{2}$
270°	$\frac{3\pi}{2}$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
285°	$\frac{19\pi}{12}$	$-\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$-2 - \sqrt{3}$	$-2 + \sqrt{3}$	$\sqrt{6} + \sqrt{2}$	$-\sqrt{6} + \sqrt{2}$
300°	$\frac{5\pi}{3}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
345°	$\frac{23\pi}{12}$	$-\frac{1}{4}(\sqrt{6} - \sqrt{2})$	$\frac{1}{4}(\sqrt{6} + \sqrt{2})$	$-2 + \sqrt{3}$	$-2 - \sqrt{3}$	$\sqrt{6} - \sqrt{2}$	$-\sqrt{6} - \sqrt{2}$
360°	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

Representación gráfica de las funciones



Funciones de ángulos negativos

$$\operatorname{sen}(-A) = -\operatorname{sen} A$$

$$\cos(-A) = \cos A$$

$$\tan(-A) = -\tan A$$

$$\csc(-A) = -\csc A$$

$$\sec(-A) = \sec A$$

$$\cot(-A) = -\cot A$$

Formulas de adición

$$\operatorname{sen}(A \pm B) = \operatorname{sen} A \cos B \pm \cos A \operatorname{sen} B \qquad \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cos(A \pm B) = \cos A \cos B \mp \operatorname{sen} A \operatorname{sen} B \qquad \cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$$

Funciones de ángulos reducidos al primer cuadrante

(con $k \in \mathbb{Z}$)

Función	$-A$	$\frac{\pi}{2} \pm A$	$\pi \pm A$	$\frac{3\pi}{2} \pm A$	$2k\pi + A$
sen	$-\operatorname{sen} A$	$\cos A$	$\mp \operatorname{sen} A$	$-\cos A$	$\pm \operatorname{sen} A$
cos	$\cos A$	$\mp \operatorname{sen} A$	$-\cos A$	$\pm \operatorname{sen} A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

Relaciones entre funciones de los ángulos en el primer cuadrante

Función	$\operatorname{sen} A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
sen A	u	$\sqrt{1-u^2}$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{1}{u}$
cos A	$\sqrt{1-u^2}$	u	$\frac{1}{\sqrt{1+u^2}}$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\frac{\sqrt{1-u^2}}{u}$
tan A	$\frac{u}{\sqrt{1-u^2}}$	$\frac{\sqrt{1-u^2}}{u}$	u	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{1}{\sqrt{1-u^2}}$
cot A	$\frac{\sqrt{1-u^2}}{u}$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{1}{u}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\sqrt{1-u^2}$
sec A	$\frac{1}{\sqrt{1-u^2}}$	$\frac{1}{u}$	$\sqrt{1+u^2}$	$\frac{\sqrt{1-u^2}}{u}$	u	$\frac{u}{\sqrt{1-u^2}}$
csc A	$\frac{1}{u}$	$\frac{1}{\sqrt{1-u^2}}$	$\frac{\sqrt{1+u^2}}{u}$	$\sqrt{1+u^2}$	$\frac{u}{\sqrt{1-u^2}}$	u

Formulas del ángulo doble

$$\operatorname{sen} 2A = 2 \operatorname{sen} A \cos A$$

$$\cos 2A = \cos^2 A - \operatorname{sen}^2 A = 1 - 2 \operatorname{sen}^2 A = 2 \cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulas de medio ángulo

$$\begin{aligned}\operatorname{sen} \frac{A}{2} &= \pm \sqrt{\frac{1 - \cos A}{2}} \left[\begin{array}{llll} + & \text{si } A/2 \text{ esta} & \text{I o II cuadrante} \\ - & \text{si } A/2 \text{ esta} & \text{III o IV cuadrante} \end{array} \right] \\ \cos \frac{A}{2} &= \pm \sqrt{\frac{1 + \cos A}{2}} \left[\begin{array}{llll} + & \text{si } A/2 \text{ esta} & \text{I o IV cuadrante} \\ - & \text{si } A/2 \text{ esta} & \text{II o III cuadrante} \end{array} \right] \\ \tan \frac{A}{2} &= \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \left[\begin{array}{llll} + & \text{si } A/2 \text{ esta} & \text{I o III cuadrante} \\ - & \text{si } A/2 \text{ esta} & \text{II o IV cuadrante} \end{array} \right] \\ &= \frac{\operatorname{sen} A}{1 + \cos A} = \frac{1 - \cos A}{\operatorname{sen} A} = \csc A - \cot A\end{aligned}$$

Formulas de ángulos múltiples

$$\begin{aligned}\operatorname{sen} 3A &= 3 \operatorname{sen} A - 4 \operatorname{sen}^3 A \\ \cos 3A &= 4 \cos^3 A - 3 \cos A \\ \tan 3A &= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \\ \operatorname{sen} 4A &= 4 \operatorname{sen} A \cos A - 8 \operatorname{sen}^3 A \cos A \\ \cos 4A &= 8 \cos^4 A - 8 \cos^2 A + 1 \\ \tan 4A &= \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} \\ \operatorname{sen} 5A &= 5 \operatorname{sen} A - 20 \operatorname{sen}^3 A + 16 \operatorname{sen}^5 A \\ \cos 5A &= 16 \cos^3 A - 20 \cos^5 A + 5 \cos A \\ \tan 5A &= \frac{\tan^5 A - 10 \tan^3 A + 5 \tan A}{1 - 10 \tan^2 A + 5 \tan^4 A}\end{aligned}$$

Potencias de funciones trigonométricas

$$\begin{aligned}\operatorname{sen}^2 A &= \frac{1}{2} - \frac{1}{2} \cos 2A & \operatorname{sen}^4 A &= \frac{3}{8} - \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A \\ \cos^2 A &= \frac{1}{2} + \frac{1}{2} \cos 2A & \cos^4 A &= \frac{3}{8} + \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A \\ \operatorname{sen}^3 A &= \frac{3}{4} \operatorname{sen} A - \frac{1}{4} \operatorname{sen} 3A & \operatorname{sen}^5 A &= \frac{5}{8} \operatorname{sen} A - \frac{5}{16} \operatorname{sen} 3A + \frac{1}{16} \operatorname{sen} 5A \\ \cos^3 A &= \frac{3}{4} \cos A + \frac{1}{4} \cos 3A & \cos^5 A &= \frac{5}{8} \cos A + \frac{5}{16} \cos 3A + \frac{1}{16} \cos 5A\end{aligned}$$

Suma, diferencia y producto de las funciones

$$\operatorname{sen} A + \operatorname{sen} B = 2 \operatorname{sen} \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\operatorname{sen} A - \operatorname{sen} B = 2 \cos \left(\frac{A+B}{2} \right) \operatorname{sen} \left(\frac{A-B}{2} \right)$$

$$\cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\cos A - \cos B = 2 \operatorname{sen} \left(\frac{A+B}{2} \right) \operatorname{sen} \left(\frac{A-B}{2} \right)$$

$$\operatorname{sen} A \operatorname{sen} B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\operatorname{sen} A \cos B = \frac{1}{2} [\operatorname{sen}(A-B) + \operatorname{sen}(A+B)]$$

$$\cos A \operatorname{sen} B = \frac{1}{2} [\operatorname{sen}(A+B) - \operatorname{sen}(A-B)]$$

Formulas generales

$$\operatorname{sen} nA = \operatorname{sen} A \left[(2 \cos A)^{n-1} - \binom{n-2}{1} (2 \cos A)^{n-3} + \binom{n-3}{1} (2 \cos A)^{n-5} - \dots \right]$$

$$\cos nA = \frac{1}{2} \left[(2 \cos A)^n - n(2 \cos A)^{n-2} + \frac{n}{2} \binom{n-3}{1} (2 \cos A)^{n-4} - \frac{n}{3} \binom{n-4}{2} (2 \cos A)^{n-6} + \dots \right]$$

$$\operatorname{sen}^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}} \left[\operatorname{sen}(2n-1)A - \binom{2n-2}{1} \operatorname{sen}(2n-3)A + \dots + (-1)^{n-1} \binom{2n-1}{n-1} \operatorname{sen} A \right]$$

$$\cos^{2n-1} A = \frac{1}{2^{2n-2}} \left[\cos(2n-1)A + \binom{2n-2}{1} \cos(2n-3)A + \dots + \binom{2n-1}{n-1} \cos A \right]$$

$$\operatorname{sen}^{2n} A = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \left[\cos 2nA - \binom{2n}{1} \cos(2n-2)A + \dots + (-1)^{n-1} \binom{2n}{n-1} \cos 2A \right]$$

$$\cos^{2n} A = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{1}{2^{2n-1}} \left[\cos 2nA + \binom{2n}{1} \cos(2n-2)A + \dots + \binom{2n}{n-1} \cos 2A \right]$$

Funciones trigonométricas inversas

Valores principales para $x \geq 0$	Valores principales para $x < 0$
$0 \leq \operatorname{sen}^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \operatorname{sen}^{-1} x < 0$
$0 \leq \operatorname{cos}^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \operatorname{cos}^{-1} x \leq \pi$
$0 \leq \operatorname{tan}^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \operatorname{tan}^{-1} x < 0$
$0 \leq \operatorname{cot}^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \operatorname{cot}^{-1} x < \pi$
$0 \leq \operatorname{sec}^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \operatorname{sec}^{-1} x \leq \pi$
$0 \leq \operatorname{csc}^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \operatorname{csc}^{-1} x < 0$

Relaciones entre trigonométricas inversas

$$\operatorname{sen}^{-1} x + \operatorname{cos}^{-1} x = \frac{\pi}{2}$$

$$\operatorname{tan}^{-1} x + \operatorname{cot}^{-1} x = \frac{\pi}{2}$$

$$\operatorname{sec}^{-1} x + \operatorname{csc}^{-1} x = \frac{\pi}{2}$$

$$\operatorname{csc}^{-1} x = \operatorname{sen}^{-1} \left(\frac{1}{x} \right)$$

$$\operatorname{sec}^{-1} x = \operatorname{cos}^{-1} \left(\frac{1}{x} \right)$$

$$\operatorname{tan}^{-1} x = \operatorname{cot}^{-1} \left(\frac{1}{x} \right)$$

$$\operatorname{sen}^{-1}(-x) = -\operatorname{sen}^{-1} x$$

$$\operatorname{cos}^{-1}(-x) = \pi - \operatorname{cos}^{-1} x$$

$$\operatorname{tan}^{-1}(-x) = -\operatorname{tan}^{-1} x$$

$$\operatorname{cot}^{-1}(-x) = \pi - \operatorname{cot}^{-1} x$$

$$\operatorname{sec}^{-1}(-x) = \pi - \operatorname{sec}^{-1} x$$

$$\operatorname{csc}^{-1}(-x) = -\operatorname{csc}^{-1} x$$

Funciones inversas en su forma compleja

$$\operatorname{sen}^{-1} x = -j \ln \left(jx + \sqrt{1 - x^2} \right)$$

$$\operatorname{cos}^{-1} x = -j \ln \left(x - \sqrt{x^2 - 1} \right)$$

$$\operatorname{tan}^{-1} x = \frac{j}{2} \ln \left(\frac{j+x}{j-x} \right)$$

$$\operatorname{csc}^{-1} x = -j \ln \left(\frac{j}{x} + \sqrt{1 - \frac{1}{x^2}} \right)$$

$$\operatorname{sec}^{-1} x = -j \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right)$$

$$\operatorname{cot}^{-1} x = \frac{j}{2} \ln \left(\frac{x-j}{x+j} \right)$$

Relaciones entre lados y ángulos en un triángulo plano

Ley de senos

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Ley de cosenos

$$c^2 = a^2 + b^2 - 2ab \cos C$$

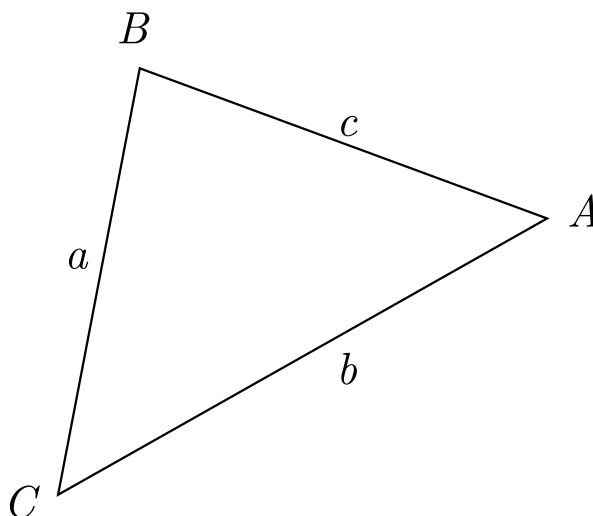
Ley de tangentes

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

Relacion de Heron

$$\sin A = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{con } s = \frac{a+b+c}{2}$$



Relaciones entre lados y ángulos en un triángulo esférico

Ley de senos

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Ley de cosenos

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

Ley de tangentes

$$\frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} = \frac{\tan\left(\frac{a+b}{2}\right)}{\tan\left(\frac{a-b}{2}\right)}$$

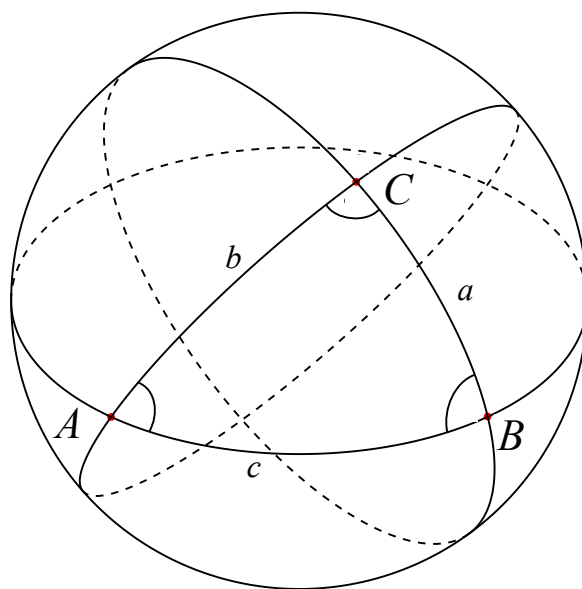
Relacion de Heron

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin(s-c)}{\sin b \sin c}}$$

$$\text{con } s = \frac{a+b+c}{2}$$

$$\cos \frac{a}{2} = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}}$$

$$\text{con } S = \frac{A+B+C}{2}$$



Funciones exponenciales y logarítmicas

Forma polar de los números complejos

$$z = x + jy = r(\cos \theta + j \operatorname{sen} \theta) = re^{j\theta}$$

$$\text{Donde } r = \sqrt{x^2 + y^2} \text{ y } \theta = \operatorname{Arg}\{z\}$$

Operaciones con números complejos en forma polar

$$\left(r_1 e^{j\theta_1}\right) \left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

$$\left(r e^{j\theta}\right)^p = r^p e^{jp\theta}$$

$$\left(r e^{j\theta}\right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\theta + 2k\pi}{n}}$$

Logaritmo de un numero complejo

$$\ln \left(r e^{j\theta}\right) = \ln r + j(\theta + 2k\pi)$$

$$\text{con } k \in \mathbb{Z}$$

Funciones hiperbólicas

Definición de las funciones

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

Relaciones entre funciones

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

Funciones de ángulos negativos

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\tanh(-x) = -\tanh x$$

$$\operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\operatorname{sech}(-x) = \operatorname{sech} x$$

$$\coth(-x) = -\coth x$$

Formulas de adición

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth x \pm \coth y}$$

Formulas del ángulo doble

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

Formulas de medio ángulo

$$\sinh \frac{A}{2} = \pm \sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh \frac{A}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\begin{aligned} \tanh \frac{A}{2} &= \pm \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \\ &= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x} \end{aligned}$$

$$(+ \text{ si } x > 0, - \text{ si } x < 0)$$

Formulas de ángulos múltiples

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$\sinh 4x = 4 \sinh x \cosh^3 x + 4 \sinh^3 x \cosh x$$

$$\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$$

$$\tanh 4x = \frac{4 \tanh x + 4 \tanh^3 x}{1 + 6 \tanh^2 x + \tanh^4 x}$$

Potencias de funciones trigonométricas

$$\sinh^2 x = \frac{1}{2} \cosh 2x - \frac{1}{2} \quad \sinh^3 x = \frac{1}{4} \sinh 3x - \frac{3}{4} \sinh x \quad \sinh^4 x = \frac{3}{8} - \frac{1}{2} \cosh 2x + \frac{1}{8} \cosh 4x$$

$$\cosh^2 x = \frac{1}{2} + \frac{1}{2} \cosh 2x \quad \cosh^3 x = \frac{3}{4} \cosh x + \frac{1}{4} \cosh 3x \quad \cosh^4 x = \frac{3}{8} + \frac{1}{2} \cosh 2x + \frac{1}{8} \cosh 4x$$

Suma, diferencia y producto de las funciones

$$\sinh x + \sinh y = 2 \sinh \left(\frac{x+y}{2} \right) \cosh \left(\frac{x-y}{2} \right)$$

$$\sinh x - \sinh y = 2 \cosh \left(\frac{x+y}{2} \right) \sinh \left(\frac{x-y}{2} \right)$$

$$\cosh x + \cosh y = 2 \cosh \left(\frac{x+y}{2} \right) \cosh \left(\frac{x-y}{2} \right)$$

$$\cosh x - \cosh y = 2 \sinh \left(\frac{x+y}{2} \right) \sinh \left(\frac{x-y}{2} \right)$$

$$\sinh x \sinh y = \frac{1}{2} [\cosh(x+y) - \cosh(x-y)]$$

$$\cosh x \cosh y = \frac{1}{2} [\cosh(x+y) + \cosh(x-y)]$$

$$\sinh x \cosh y = \frac{1}{2} [\sinh(x+y) + \sinh(x-y)]$$

Relaciones entre funciones hiperbólicas

Función	$\sinh A = u$	$\cosh A = u$	$\tanh A = u$	$\coth A = u$	$\operatorname{sech} A = u$	$\operatorname{csch} A = u$
$\sinh A$	u	$\sqrt{u^2 - 1}$	$\frac{u}{\sqrt{1-u^2}}$	$\frac{1}{\sqrt{u^2-1}}$	$\frac{\sqrt{1-u^2}}{u}$	$\frac{1}{u}$
$\cosh A$	$\sqrt{1+u^2}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\frac{u}{\sqrt{u^2-1}}$	$\frac{1}{u}$	$\frac{\sqrt{1+u^2}}{u}$
$\tanh A$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{\sqrt{u^2-1}}{u}$	u	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{1}{\sqrt{1+u^2}}$
$\coth A$	$\frac{\sqrt{u^2+1}}{u}$	$\frac{u}{\sqrt{u^2-1}}$	$\frac{1}{u}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\sqrt{1+u^2}$
$\operatorname{sech} A$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{\sqrt{u^2-1}}{u}$	u	$\frac{u}{\sqrt{1+u^2}}$
$\operatorname{csch} A$	$\frac{1}{u}$	$\frac{1}{\sqrt{u^2-1}}$	$\frac{\sqrt{1-u^2}}{u}$	$\sqrt{u^2-1}$	$\frac{u}{\sqrt{1-u^2}}$	u

Funciones hiperbólicas inversas

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right)$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$\operatorname{sech}^{-1} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right)$$

$$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)$$

$$\coth^{-1} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right)$$

$$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right)$$

Relaciones entre hiperbólicas inversas

$$\operatorname{csch}^{-1} x = \sinh^{-1} \left(\frac{1}{x} \right)$$

$$\operatorname{sech}^{-1} x = \cosh^{-1} \left(\frac{1}{x} \right)$$

$$\coth^{-1} x = \tanh^{-1} \left(\frac{1}{x} \right)$$

$$\sinh^{-1}(-x) = -\sinh^{-1} x$$

$$\tanh^{-1}(-x) = -\tanh^{-1} x$$

$$\operatorname{sech}^{-1}(-x) = -\operatorname{sech}^{-1} x$$

$$\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$$

Relaciones entre funciones trigonométricas con hiperbólicas

$$\sin(jx) = j \sinh x$$

$$\csc(jx) = -j \operatorname{csch} x$$

$$\sinh(jx) = j \sin x$$

$$\operatorname{csch}(jx) = -j \csc x$$

$$\cos(jx) = \cosh x$$

$$\sec(jx) = \operatorname{sech} x$$

$$\cosh(jx) = \cos x$$

$$\operatorname{sech}(jx) = \sec x$$

$$\tan(jx) = j \tan x$$

$$\cot(jx) = -j \coth x$$

$$\tanh(jx) = j \tan x$$

$$\coth(jx) = -j \cot x$$

Periodicidad de funciones hiperbólicas

$$\sinh(x + 2k\pi j) = \sinh x$$

$$\operatorname{csch}(x + 2k\pi j) = \operatorname{csch} x$$

$$\cosh(x + 2k\pi j) = \cosh x$$

$$\operatorname{sech}(x + 2k\pi j) = \operatorname{sech} x$$

$$\tanh(x + k\pi j) = \tanh x$$

$$\coth(x + k\pi j) = \coth x$$

Con $k \in \mathbb{Z}$.

Relaciones entre hiperbólicas inversas y trigonométricas inversas

$$\sinh^{-1}(jx) = j \operatorname{sen}^{-1} x$$

$$\cosh^{-1} x = \pm j \cos^{-1} x$$

$$\tanh^{-1}(jx) = j \tan^{-1} x$$

$$\coth^{-1}(jx) = -j \cot^{-1} x$$

$$\operatorname{sech}^{-1} x = \pm j \sec^{-1} x$$

$$\operatorname{csch}^{-1}(jx) = -j \csc^{-1} x$$

$$\operatorname{sen}^{-1}(jx) = j \sinh^{-1} x$$

$$\cos^{-1} x = \pm j \cosh^{-1} x$$

$$\tan^{-1}(jx) = j \tanh^{-1} x$$

$$\cot^{-1}(jx) = -j \coth^{-1} x$$

$$\sec^{-1} x = \pm j \operatorname{sech}^{-1} x$$

$$\csc^{-1}(jx) = -j \operatorname{csch}^{-1} x$$

Soluciones de ecuaciones algebraicas

Ecuación cuadrática $ax^2 + bx + c = 0$

Soluciones:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Si a, b, c son reales, y $\Delta = b^2 - 4ac$ es el *discriminante*, entonces las raíces son

i reales y desiguales si $\Delta > 0$

ii reales e iguales si $\Delta = 0$

iii conjugadas complejas si $\Delta < 0$

Si x_1, x_2 son las raíces entonces $x_1 + x_2 = -\frac{b}{a}$ y $x_1 x_2 = \frac{c}{a}$

Ecuacion cubica $x^3 + a_1 x^2 + a_2 x + a_3 = 0$

Sea

$$Q = \frac{3a_2 - a_1^2}{9}$$

$$R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}$$

$$S = \sqrt{R + \sqrt{Q^3 + R^2}}$$

$$T = \sqrt{R - \sqrt{Q^3 + R^2}}$$

Soluciones:

$$\begin{cases} x_1 = S + T - \frac{1}{3}a_1 \\ x_2 = -\frac{1}{2}(S + T) - \frac{1}{3}a_1 + \frac{1}{2}j\sqrt{3}(S - T) \\ x_3 = -\frac{1}{2}(S + T) - \frac{1}{3}a_1 - \frac{1}{2}j\sqrt{3}(S - T) \end{cases}$$

Si a_1, a_2, a_3 son reales, y $\Delta = Q^3 + R^2$ es el *discriminante*, entonces las raíces son

i una real y dos complejas conjugadas si $\Delta > 0$

ii todas reales y por lo menos dos de ellas iguales si $\Delta = 0$

iii todas reales y distintas si $\Delta < 0$

Simplificación si $\Delta < 0$.

Soluciones:

$$\begin{cases} x_1 = 2\sqrt{-Q} \cos\left(\frac{\theta}{3}\right) \\ x_2 = 2\sqrt{-Q} \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \\ x_3 = 2\sqrt{-Q} \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) \end{cases}$$

donde $\cos \theta = -\frac{R}{\sqrt{-Q^3}}$

Y además:

$x_1 + x_2 + x_3 = -a_1$, $x_1x_2 + x_2x_3 + x_3x_1 = a_2$ y $x_1x_2x_3 = -a_3$ donde x_1 , x_2 y x_3 son las tres raíces.

Integrales

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int \frac{dx}{a+bx} = \frac{\ln(a+bx)}{b}$$

$$\int \frac{x}{a+bx} = \frac{x}{b} - \frac{a}{b^2} \ln(a+bx)$$

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \ln\left(\frac{x+a}{x}\right)$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x}{x^2 \pm a^2} dx = \pm \frac{1}{2} \ln(a^2 \pm x^2)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a^2-x^2 > 0$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{x}{\sqrt{a^2-x^2}} dx = -\sqrt{a^2-x^2}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$$

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2} \left[x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) \right]$$

$$\int x\sqrt{a^2-x^2} dx = -\frac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{3} \left[x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left(x + \sqrt{x^2 \pm a^2}\right) \right]$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}(a^2 \pm x^2)^{3/2}$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \ln(ax) dx = x \ln(ax) - x$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$\int \frac{dx}{a+bx^{cx}} = \frac{x}{a} - \frac{1}{ac} \ln(a+bx^{cx})$$

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax)$$

$$\int \cos(ax) dx = \frac{1}{a} \sin(ax)$$

$$\int \tan(ax) dx = \frac{1}{a} \ln[\sec(ax)]$$

$$\int \cot(ax) dx = \frac{1}{a} \ln[\sin(ax)]$$

$$\int \sec(ax) dx = \frac{1}{a} \ln[\sec(ax) + \tan(ax)]$$

$$\int \csc(ax) dx = \frac{1}{a} \ln[\csc(ax) - \cot(ax)]$$

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax) dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int \frac{dx}{\sin^2(ax)} = -\frac{1}{a} \cot(ax)$$

$$\int \frac{dx}{\cos^2(ax)} = \frac{1}{a} \tan(ax)$$

$$\int \tan^2(ax) dx = \frac{1}{a} \tan(ax) - x$$

$$\int \cot^2(ax) dx = -\frac{1}{a} \cot(ax) - x$$

$$\int \sin^{-1}(ax) dx = x \sin^{-1}(ax) + \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \cos^{-1}(ax) dx = x \cos^{-1}(ax) - \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x}{(x^2+a^2)^{3/2}} dx = -\frac{1}{\sqrt{x^2+a^2}}$$

$$\begin{aligned}
\int x \sin(ax) dx &= \frac{1}{a^2} (\sin(ax) - ax \cos(ax)) & \int \frac{x}{\sqrt{x^2 \pm a^2}} dx &= \sqrt{x^2 \pm a^2} \\
\int x \cos(ax) dx &= \frac{1}{a^2} (\cos(ax) + ax \sin(ax)) & \int \frac{x}{\sqrt{a^2 - x^2}} dx &= -\sqrt{a^2 - x^2} \\
\int x^2 \sin(ax) dx &= \frac{1}{a^3} (-a^2 x^2 \cos(ax) + 2 \cos(ax) + 2ax \sin(ax)) & \int x \sqrt{a^2 - x^2} dx &= -\frac{x}{3} (a^2 - x^2)^{3/2} \\
\int x^2 \cos(ax) dx &= \frac{1}{a^3} (-a^2 x^2 \sin(ax) - 2 \sin(ax) + 2ax \cos(ax)) & \int \sqrt{a^2 - x^2} dx &= \frac{1}{2} \left[x \sqrt{a^2 - x^2} - a^2 \sin^{-1} \left(\frac{x}{a} \right) \right] \\
\int \sqrt{x^2 \pm a^2} dx &= \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \left(x + \sqrt{x^2 \pm a^2} \right) & \int x \sqrt{x^2 \pm a^2} dx &= \frac{x}{3} (x^2 - a^2)^{3/2}
\end{aligned}$$

Integral de probabilidad de Gauss

$$\begin{aligned}
I_0 &= \int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \\
I_1 &= \int_0^\infty x e^{-ax^2} dx = \frac{1}{2a} \\
I_2 &= \int_0^\infty x^2 e^{-ax^2} dx = -\frac{dI_0}{da} = \frac{1}{4} \sqrt{\frac{\pi}{a^3}} \\
I_3 &= \int_0^\infty x^3 e^{-ax^2} dx = -\frac{dI_1}{da} = \frac{1}{2a} \\
I_4 &= \int_0^\infty x^4 e^{-ax^2} dx = \frac{d^2 I_0}{da^2} = \frac{3}{8} \sqrt{\frac{\pi}{a^5}} \\
I_5 &= \int_0^\infty x^5 e^{-ax^2} dx = \frac{d^2 I_1}{da^2} = \frac{1}{a^3} \\
&\cdot \\
&\cdot \\
I_{2n} &= (-1)^n \frac{d^n}{da^n} I_0 \\
I_{2n+1} &= (-1)^n \frac{d^n}{da^n} I_1
\end{aligned}$$