I. Literature Outline

Two classifications for elasticities of substitution

- **Gross/net elasticities**:
  - Net elasticities are computed holding output constant
  - Gross elasticities are computed by letting output vary optimally (gross)
- **Scalar, asymmetric ratio, or symmetric ratio elasticities**
  - Scalar elasticities (Allen-Uzawa) measure the effect of a change in the price of another factor scaled by its cost share on the quantity of a factor demanded
  - Asymmetric ratio elasticities (Morishima) measure the effect on the factor input ratio of a change in a ratio of prices
  - Symmetric elasticities can be found by putting restrictions on asymmetric elasticities; holding cost constant on the Morishima elasticities produces the shadow elasticities of substitution

A. Formulas

Translog

Own price elasticities are given by:

$$\eta_{ii} = \frac{\partial \ln X_i(y,p)}{\partial \ln p_i} = \frac{\beta_{ii} + S_i^2 - S_i}{S_i} \qquad \Longrightarrow \qquad n_{ii}(1,\mathbb{1}) = \frac{\beta_{i,i} - \beta_i^2 - \beta_i}{\beta_i}$$

where X_i is demand for input i, p_i is its price, and S_i is its predicted cost share. And, B_{ij} is the second order parameter from the translog cost function, y is the output, and p is a vector of factor prices.

Cross price elasticities are given by:

$$\eta_{ij} = \frac{\partial \ln X_i(y,p)}{\partial \ln p_j} = \frac{B_{ij} + S_i S_j}{S_i} \qquad \Longrightarrow \qquad n_{ii}(1,\mathbb{1}) = \frac{\beta_{i,i} - \beta_j \beta_i}{\beta_i}$$

where B_i is the first-order parameter of the translog cost function.

Morishima

The elasticity of substitution here is given by:

$$\mu_{ij} = \left. \frac{\partial \ln(X_j(y,p)/X_i(y,p))}{\partial \ln(p_i/p_j)} \right|_{p_j} = \eta_{ji} - \eta_{ii}$$

Shadow

The shadow elasticity of substitution is given by:

$$\sigma_{ij} = \left. \frac{\partial \ln(X_j(y,p)/X_i(y,p))}{\partial \ln(p_i/p_j)} \right|_{C} = \frac{S_i}{S_i + S_j}\mu_{ij} + \frac{S_j}{S_i + S_j}\mu_{ji}$$

II. Model of Electricity Production/Consumption

A. Formulation

The model involves a consumer and producers reaching equilibrium in a two-period setting. The consumer maximizes a Cobb-Douglas utility function, and the producers allocate capital into two energy sources: one which provides a constant output and the other which generates an intermittent output. The producers' energy output, say kWh, at time t is

$$Y_t = \xi_{1,t}X_1 + \xi_{2,t}X_2$$

where X_1 is investment into the constant output source, X_2 is the investment into the intermittent output source, and $\xi_{i,t}$ is the conversion factor from input X_i into kWh at time t. Referring to the next period as period s, we also have:

$$Y_s = \xi_{1,s}X_1 + \xi_{2,s}X_2$$

Since X_1 is a constant output source, $\xi_{1,t} = \xi_{1,s}$. And, since X_2 is an intermittent source, we may have $\xi_{2,t} > \xi_{2,s}$ without loss of generality. For example, X_1 may be coal and X_2 may be solar. Letting $\xi_{2,t} > \xi_{2,s}$ reflects the fact that the conversion rate from tons of coal to kWh is independent of the time of day, while the conversion rate for solar panels into kWh is higher during the day. Next, the cost of production for inputs X_i is given by:

$$C(X_1, X_2) = p_1(X_1 - c_1)^{\eta_1} + p_2(X_2 - c_1)^{\eta_2}$$

where $p_i > 0$, $c_i > 0$, $\eta_i > 2$. When $\eta_i = 3$, this particular function has:

$$\begin{aligned} &\partial C/\partial X_i > 0 \\ &\partial^2 C/\partial X_i^2 < 0 \text{ when } X_i < c_i \\ &\partial^2 C/\partial X_i^2 > 0 \text{ when } X_i > c_i \end{aligned}$$

which shows concave costs followed by convex costs. That is, average cost initially decreases with production and then increases. This might be more realistic, but the following results can also be obtained with $c_i = 0$ and $\eta = 2$ which is a simpler model with linearly increasing average costs. Lastly, the producer has the following profit function:

$$\Pi = p_tY_t + p_sY_2 - C(X_1, X_2)$$

And, the consumer maximizes utility while constrained by their budget:

$$U = Y_t^\alpha \cdot Y_s^\beta$$

$$\text{s.t. } p_tY_t + p_sY_2 \leq B$$

B. Numerical Solution

First, we assume that the social planner maximizes utility while keeping profit non-negative. Additionally, we keep all prices and quantities non-negative.

Optimizing a Cobb-Douglas function implies that the demand for energy is given by:

$$\begin{aligned} Q_{D,t} &= \alpha * B/p_t \\ Q_{D,s} &= \beta * B/p_s \end{aligned}$$

These demand functions maximize utility while ensuring the consumer budget constraint is satisfied.

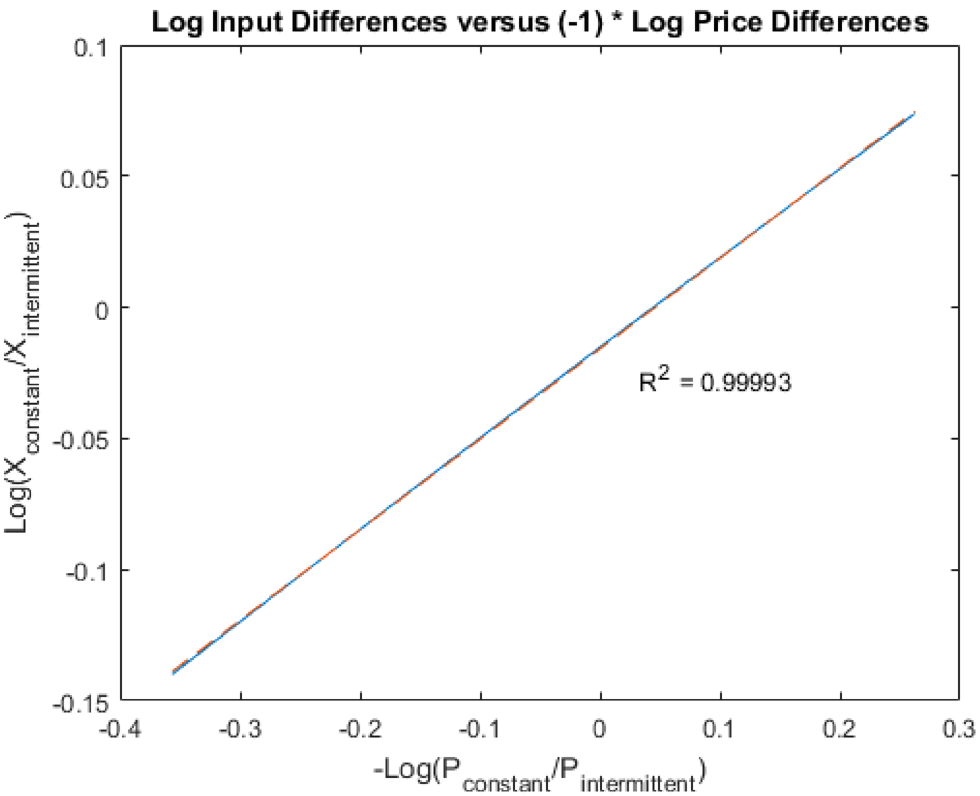
The energy production functions remain the same:

$$Q_{S,t} = \xi_{1,t} X_1 + \xi_{2,t} X_2$$
$$Q_{S,s} = \xi_{1,s} X_1 + \xi_{2,s} X_2$$

But, the producer profit is now:

$$\Pi = p_t \min(Q_{D,t}, Q_{S,t}) + p_s \min(Q_{D,s}, Q_{S,s}) - C(X_1, X_2)$$

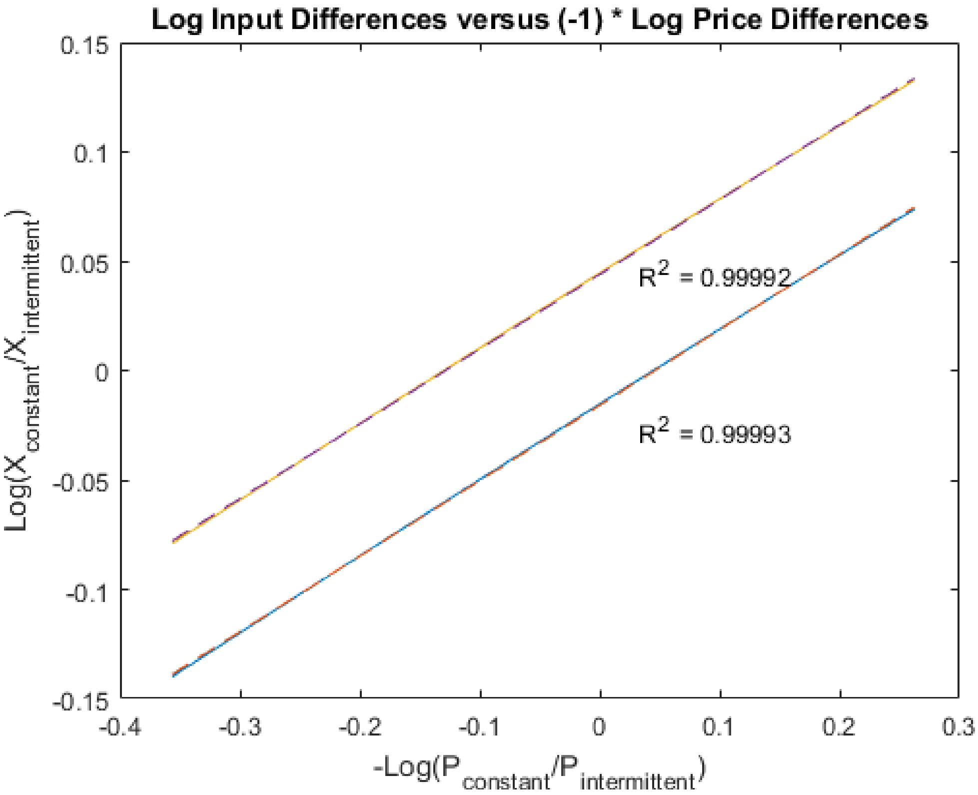
Running an optimization routine provides the following results:



Graphed above is $\ln(X_1/X_2)$ versus $\ln(P_2/P_1)$. The blue line shows the results while the dashed orange line is an OLS fit. Similar to a CES function, this relationship is practically linear; error may be due to the optimization routine. The results were generated with $p_1 = 10, c_1 = 0.5, c_2 = 0.5, , \eta_1 = 3, \eta_2 = 3, \xi_{1,t} = 5, \xi_{1,s} = 5, \xi_{2,t} = 10, \xi_{2,s} = 4, \alpha = 0.3, \beta = 0.7, B = 50$, and with p_2 allowed to vary over iterations from 7 to 13. That is, with:

$$U = Y_t^{0.3} \cdot Y_s^{0.7}$$
$$p_t Y_t + p_s Y_2 \leq 50$$
$$Y_t = 5X_1 + 10X_2$$
$$Y_s = 5X_1 + 4X_2$$
$$C = 10(X_1 - 0.5)^3 + p_2(X_2 - 0.5)^3$$

Running the routine again but with $\xi_{1,t} = \xi_{1,s} = 6$, we get the following results:



In the graph above, the yellow line is a second set of results obtained by modifiying ξ. We can see that the intercept has shifted without changing the slope very much ($< 1\%$ change in the slope), which is similar to what would be seen with a CES model. That is, the optimization of a standard CES model would result in

$$\ln(X_i/X_j) = \sigma \ln(P_j/P_i) + \sigma \ln(\xi_i/\xi_j)$$

where changes in ξ would affect the intercept but not the slope (elasticity of substitution) σ as seen in this two-period model. The estimated coefficient of the slope $\hat{\sigma}$ with these parameters is 0.3416. This implies that the constant and intermittent energy sources are complements in energy production, which reflects reality since an intermittent source could not substitute well for a constant output source.

Part of the reason the model implies that they are complements may be due to the fact that the cost functions show increasing average cost, so it would be more expensive to completely substitute one for the other. However setting $\eta_1 = \eta_2 = 2$, results in $\hat{\sigma} = 0.6695$. The two are much weaker complements but still remain complements even if average cost is linearly increasing. Generally, as η_i decreases, the elasticity of substitution rises and the two become increasingly better substitutes. On the other hand, when the output of the second source becomes more intermittent, when ξ_{2_d}/ξ_{2_n} increases, the elasticity of substitution declines as expected.

Overall, this result - the near linear relationship between the log of inputs and prices - essentially implies that we could approximate a two-period model of electricity into a one-period model by using a CES production function. This is useful, because we do not have electricity output by source, hour, and state for 2016. Since modeling intermittency directly with a multi-period SAM would be impossible without data to estimate the parameters, a single period SAM with this theoretical approach would be more persuasive. Additionally, using a CES function without $\sigma \to \infty \iff \phi \to 1$ would always imply that

1 kWh from one source does not substitute for 1 kWh from another. When estimating the CES function:

$$Y = \theta \cdot \left(\Pi_i \alpha_i X_i^{\phi}\right)^{1/\phi}$$

the estimates will always have $\hat{\theta} \to 1$, $\hat{\alpha_i}$ approaching the average conversion factors, and $\hat{\phi} \to 1$. When the electricity data comes from a single period, such as 2016 in our data, this must be the result; even theoretically this is true. For instance, this is true with the MATLAB simulation when aggregating the two-period model data into one period and fitting a CES function. We see $\theta \to 1$, $\phi \to 1$, and $\alpha_i \to (\xi_{i,t} + \xi_{i,s})/2$. To get $\hat{\phi} \neq 1$ would be wrong in any single period model, but would make sense in a multi-period model. Implementing this in a CGE would still imply that 1 kWh of solar stops translating into 1 kWh of electricity when a shock occurs. But, this can make sense if we suppose reality follows a multi-period model and such a kWh of solar was just over-generated in one period as a result of optimal investment targeting all periods. So, starting from a multi-period theoretical foundation would better justify using $\phi \neq 1$ in the CGE.

Cobb-Douglas Utility

Firstly, the demand function for each commodity is derived from the Cobb-Douglas utility function. Hence, we have:

$$Y_t = (\alpha \cdot B)/p_t \qquad Y_s = (\beta \cdot B)/p_s$$

Therefore, the consumer surplus is given by:

$$\text{CS} = \int_{p_t}^{\infty} (\alpha \cdot B)/p \, dp \; + \; \int_{p_s}^{\infty} (\beta \cdot B)/p \, dp$$

Equivalently, using inverse demand, we have:

$$\text{CS} = \int_{0}^{Y_t} (\alpha \cdot B)/y \, dy \; + \; \int_{0}^{Y_s} (\beta \cdot B)/y \, dy$$

Producer surplus remains equivalent to profit; here I use a simple cost function.

$$\text{PS} = \Pi = p_t Y_t + p_s Y_2 - (p_1 X_1^2 + p_2 X_2^2)$$

Then, total welfare is $W = \text{PS} + \text{CS}$. So, we have:

$$\frac{\partial W}{\partial X_1} = \frac{\partial CS}{\partial X_1} + \frac{\partial PS}{\partial X_1}$$

Working with consumer surplus, we get:

$$\begin{aligned} CS &= (\alpha \cdot B) \cdot (\ln(Y_t) - \ln(0)) + (\beta \cdot B) \cdot (\ln(Y_s) - \ln(0)) \\ \implies \frac{\partial CS}{\partial X_1} &= (\alpha \cdot B) \cdot (\xi_{1,t}/Y_t) + (\beta \cdot B) \cdot (\xi_{1,s}/Y_s) \\ \implies \frac{\partial CS}{\partial X_2} &= (\alpha \cdot B) \cdot (\xi_{2,t}/Y_t) + (\beta \cdot B) \cdot (\xi_{2,s}/Y_s) \end{aligned}$$

And, for profit, we have:

$$\frac{\partial PS}{\partial X_1} = p_t \cdot \xi_{1,t} + p_2 \cdot \xi_{1,s} - 2\, p_1 X_1$$

$$\frac{\partial PS}{\partial X_2} = p_t \cdot \xi_{2,t} + p_2 \cdot \xi_{2,s} - 2\, p_2 X_2$$

Hence, the first-order conditions are:

$$\begin{aligned} \frac{\partial W}{\partial X_1} = 0 &\implies (\alpha \cdot B) \cdot (\xi_{1,t}/Y_t) + (\beta \cdot B) \cdot (\xi_{1,s}/Y_s) + p_t \cdot \xi_{1,t} + p_2 \cdot \xi_{1,s} - 2\, p_1 X_1 = 0 \\ \frac{\partial W}{\partial X_2} = 0 &\implies (\alpha \cdot B) \cdot (\xi_{2,t}/Y_t) + (\beta \cdot B) \cdot (\xi_{2,s}/Y_s) + p_t \cdot \xi_{2,t} + p_2 \cdot \xi_{2,s} - 2\, p_2 X_2 = 0 \end{aligned}$$

To restate the problem, we have production determined by:

$$Y_t = \xi_{1,t} X_1 + \xi_{2,t} X_2$$

$$Y_s = \xi_{1,s} X_1 + \xi_{2,s} X_2$$

with the cost function $C(X_1, X_2) = p_1 X_1^2 + p_2 X_2^2$. And, the consumer attempts to maximize their utility:

$$U = Y_t^\alpha \cdot Y_s^\beta$$

subject to $p_t Y_t + p_s Y_s = B$.

Cobb-Douglas Utility

First, assume producers maximize profit. Then, we have:

$$\begin{aligned} \frac{\partial PS}{\partial X_1} &= p_t \cdot \xi_{1,t} + p_s \cdot \xi_{1,s} - 2\,p_1 X_1 = 0 \\ \frac{\partial PS}{\partial X_2} &= p_t \cdot \xi_{2,t} + p_s \cdot \xi_{2,s} - 2\,p_2 X_2 = 0 \\ \implies X_1 &= (p_t \cdot \xi_{1,t} + p_s \cdot \xi_{1,s})\,/(2p_1) \\ \implies X_2 &= (p_t \cdot \xi_{2,t} + p_s \cdot \xi_{2,s})\,/(2p_2) \end{aligned}$$

Thus, given the demand curves,

$$Y_t = (\alpha \cdot B)/p_t \qquad Y_s = (\beta \cdot B)/p_s$$

we substitute in X_1 and X_2 to get:

$$p_t = \frac{\alpha \cdot B}{\frac{\xi_{1,t}(p_s \cdot \xi_{1,s} + p_t \cdot \xi_{1,t})}{2p_1} + \frac{\xi_{2,t}(p_s \cdot \xi_{2,s} + p_t \cdot \xi_{2,t})}{2p_2}}$$

$$p_t = \frac{\beta \cdot B}{\frac{\xi_{1,s}(p_s \cdot \xi_{1,s} + p_t \cdot \xi_{1,t})}{2p_1} + \frac{\xi_{2,s}(p_s \cdot \xi_{2,s} + p_t \cdot \xi_{2,t})}{2p_2}}$$

Returning to the utility function, $U = Y_t^\alpha \cdot Y_s^\beta$, we expand it to get:

$$U = (\xi_{1,t} X_1 + \xi_{2,t} X_2)^\alpha (\cdot \xi_{1,s} X_1 + \xi_{2,s} X_2)^\beta$$

Furthermore, assuming profit-maximizing firms, we have:

$$\begin{aligned} U = &\left(\xi_{1,t}\,(p_t \cdot \xi_{1,t} + p_s \cdot \xi_{1,s})\,/(2p_1) + \xi_{2,t}\,(p_t \cdot \xi_{2,t} + p_s \cdot \xi_{2,s})\,/(2p_2)\right)^\alpha \\ &* \left(\xi_{1,s}\,(p_t \cdot \xi_{1,t} + p_s \cdot \xi_{1,s})\,/(2p_1) + \xi_{2,s}\,(p_t \cdot \xi_{2,t} + p_s \cdot \xi_{2,s})\,/(2p_2)\right)^\beta \end{aligned}$$

CES Utility

Suppose instead that we have a utility function of the form:

$$U = (\alpha Y_t^{\phi} \; + \; \beta Y_s^{\phi})^{1/\phi}$$

The demand functions are then:

$$Y_t = \left(\frac{\alpha}{p_t}\right)^{\sigma} \cdot \frac{B}{\alpha^{\sigma} p_t^{1-\sigma} \; + \; \beta^{\sigma} p_s^{1-\sigma}}$$
$$Y_s = \left(\frac{\beta}{p_s}\right)^{\sigma} \cdot \frac{B}{\alpha^{\sigma} p_t^{1-\sigma} \; + \; \beta^{\sigma} p_s^{1-\sigma}}$$

where $\sigma = 1/(1-\phi)$ is the elasticity of substitution. And, same as before, we have the following supply curve:

$$X_1^* = (p_t \cdot \xi_{1,t} + p_2 \cdot \xi_{1,s}) / (2p_1)$$
$$X_2^* = (p_t \cdot \xi_{2,t} + p_2 \cdot \xi_{2,s}) / (2p_2)$$

Simpler Model

Suppose that we have the same production function but our first input is equally productive at time t or s, so we have:

$$Y_t = \xi_1 X_1 + \xi_{2,t} X_2$$

$$Y_s = \xi_1 X_1 + \xi_{2,s} X_2$$

Using matrix notation, we have:

$$M = \begin{pmatrix} \xi_1 & \xi_{2,t} \\ \xi_1 & \xi_{2,s} \end{pmatrix} \quad \Longrightarrow \quad Y = M\,X \quad \Longrightarrow \quad X = M^{-1}\,Y$$

Additionally, suppose that we have the cost function $C = (X_1^2 + X_2^2)/2$. Then, we have:

$$\frac{\partial C}{\partial Y_t} = M_{11}^{-1} \cdot X_1 + M_{21}^{-1} \cdot X_2$$
$$\frac{\partial C}{\partial Y_s} = M_{12}^{-1} \cdot X_1 + M_{22}^{-1} \cdot X_2$$

Then, with our original profit function, we get the following first order conditions

$$\Pi = p_t Y_t + p_s Y_s - C(X_1, X_2)$$
$$\Longrightarrow \; p_t = M_{11}^{-1} \cdot X_1 + M_{21}^{-1} \cdot X_2$$
$$\Longrightarrow \; p_y = M_{12}^{-1} \cdot X_1 + M_{22}^{-1} \cdot X_2$$
$$\Longrightarrow \; P = (M^{-1})^T X$$
$$\Longrightarrow \; X = M^T P$$

Therefore, our supply curve is $Y = M\,M^T P$. And, assuming Cobb-Douglas utility, we have:

$$Y_t = (\alpha \cdot B)/p_t \qquad Y_s = (\beta \cdot B)/p_s$$

Hence, setting supply equal to demand, we get:

$$
\begin{aligned}
M\,M^T P &= \begin{pmatrix} \alpha \cdot B/p_t \\ \beta \cdot B/p_s \end{pmatrix} \\
\implies \left(\begin{array}{cc} {\xi_1}^2 + {\xi_{2,t}}^2 & {\xi_1}^2 + \xi_{2,s}\,\xi_{2,t} \\ {\xi_1}^2 + \xi_{2,s}\,\xi_{2,t} & {\xi_1}^2 + {\xi_{2,s}}^2 \end{array} \right) \begin{pmatrix} p_t \\ p_s \end{pmatrix} &= \begin{pmatrix} \alpha \cdot B/p_t \\ \beta \cdot B/p_s \end{pmatrix}
\end{aligned}
$$

Expanding and simplifying, we have:

$$
\begin{aligned}
&({\xi_1}^2 + \xi_{2,t}^2)p_t^2 + ({\xi_1}^2 + \xi_{2,s}\,\xi_{2,t})p_s p_t = \alpha \cdot B \\
&({\xi_1}^2 + \xi_{2,s}\,\xi_{2,t})p_t p_s + ({\xi_1}^2 + \xi_{2,s}^2)p_s^2 = \beta \cdot B \\
&\implies ({\xi_1}^2 + \xi_{2,t}^2)p_t^2 - ({\xi_1}^2 + \xi_{2,s}^2)p_s^2 = (\alpha - \beta) \cdot B
\end{aligned}
$$

References

Stern, D. (2012). "Interfuel Substitution: A Meta-Analysis." Journal of Economic Surveys 26 (2): 307–331. https://doi.org/10.1111/j.1467-6419.2010.00646.x.