I. Introduction

In CGE modeling, energy production often consists of a CES function of energy sources with an empirically estimated elasticity of substitution; this elasticity is meant to capture factors such as intermittency between sources. However, this top down approach may be missing the actual source of the elasticity of substitution. Rather than firms following a general CES function,[1] we may instead have firms simply following a linear production function where energy output is instead the sum of the input of each energy source multiplied by a conversion factor. Although more complex models than this could be conceived, the necessary element of such a model is that increasing the input of one energy source does not decrease the output of another source as it does in a CES function.[2] Without a CES function, we can still find a non-zero elasticity of substitution between energy sources from another source - household preferences. So, instead of a CES of energy inputs on the producer side, consumers might follow a utility function that captures their preferences for particular amounts of energy based on the time of day. Because agents prefer to have energy at different proportions based on the time of day, intermittency then plays a key role in determining the elasticity of substitution. The result of using such preferences in a CGE is that we can capture substitutability between energy sources without having to directly model intermittency on the production side.

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Things I'd like to say:

- Because the marginal product of renewable energy inputs may be time-dependent, renewable energy may complement traditional, constant-output energy sources rather than fully substitute for it. Obvious arugment follows from base load output required at all times.
- How the conversion parameters for energy inputs affect their elasticity of substitution
- How variability in energy output decreases the value of particular energy sources. Obvious argument from concavity of utility implying risk aversion.

[1] In practice, it is difficult to differentiate between firms following a CES production function versus a particular cost function that implies similar factor demand. For example, the cost function $C(X_1, X_2) = X_1^2 + X_2^2$ implies cost minimization when X_1/X_2 is proportional to the marginal rate of technical substitution. Similarly, X_1/X_2 is proportional to the ratio of their prices when using a CES function. Empirically, determining whether we have the above cost function, the latter production function, or both is not possible with only prices and quantities.

[2] Suppose we have the CES function $Y = (X_1^\phi + X_2^\phi)^{1/\phi}$. It's fairly simple to see that $\partial Y/\partial X_1$ is decreasing with X_2. But, realistically, it would not make sense if the energy output of the coal sector decreased with the size of the solar energy sector.

II. Model

In our model of the electricity sector, electricity is treated as a heterogenous good which is solely differentiated by time. Consumers purchase different quantities of electricity at different times in order to maximize utility. They purchase electricity from a set of firms, utility companies, who choose their electricity-generating inputs to maximize profit; these inputs each generate a non-constant amount of energy over time to model intermittency. The firms that produce the inputs are implicitly modeled by the cost functions which capture the supply curve of the inputs.

A. Consumers

First, suppose we define the electricity output at time t as $Y(t)$. Consumers theoretically might prefer that some proportion of their energy arrive at time t_1 while some other fixed proportion arrive at t_2 if all prices at all times are equal. Additionally, suppose they would be willing to shift their consumption between different times of the day, thus substituting between $Y(t)$. Then, we may suppose that consumers follow the CES utility function:

$$U = \left(\int_0^T \alpha(t)Y(t)^{\phi}dt\right)^{1/\phi} \tag{1}$$

where $\phi < 1$ to preserve concavity and $\phi > 0$ to allow for $Y(t) = 0$ at some t. The elasticity of substitution is $\sigma \equiv 1/(1-\phi)$ and the coefficient $\alpha(t)$ capture the desire for particular proportions of energy use at various times of the day. Furthermore, T is a fixed value that represents the length of a day. Consumers purchase the electricity $Y(t)$ at the price $p(t)$ from time 0 to T from a set of firms. This leads to the following budget constraint:

$$\int_0^T Y(t)p(t)dt = I \tag{2}$$

In order to simplify notation, we normalize the time period and budget so $T = I = 1$.

Then, consumers maximize utility with respect to this budget constraint; hence, the Lagrangian L may be set up as $U^{\phi} - \lambda(\int_0^T Y(t)p(t)dt - 1)$ where λ is the multiplier. Solving for the first order condition of $Y(t_i)$, we have:

$$\frac{\partial L}{\partial Y(t_i)} = 0 \implies \phi\, \alpha(t_i)\, Y(t_i)^{\phi-1} = \lambda\, p(t_i) \tag{3}$$

Now, taking the ratio of the FOCs at arbitrary $t_1, t_2 \in [0,1]$ and then simplifying, we get:

$$\begin{aligned}
\implies \frac{Y(t_1)}{Y(t_2)} &= \left(\frac{p(t_1)\,\alpha(t_2)}{p(t_2)\,\alpha(t_1)}\right)^{-\sigma} \\
\implies Y(t_1)p(t_1) &= p(t_1)Y(t_2)\left(\frac{p(t_1)\,\alpha(t_2)}{p(t_2)\,\alpha(t_1)}\right)^{-\sigma}
\end{aligned}$$

Integrating both sides from 0 to 1 with respect to t_1 results in:

$$\begin{aligned}
\implies 1 &= Y(t_2)\,p(t_2)^{\sigma}\,\alpha(t_2)^{-\sigma}\int_0^1 p(t_1)^{1-\sigma}\,\alpha(t_1)^{\sigma}\,dt_1 \\
\implies Y(t_2) &= \frac{(\alpha(t_2)/p(t_2))^{\sigma}}{\int_0^1 p(t_1)^{1-\sigma}\,\alpha(t_1)^{\sigma}\,dt_1}
\end{aligned}$$

Now we use the reciprocal in the previous equation to define the price index:

$$P \equiv \left(\int_0^1 p(t)^{1-\sigma} \, \alpha(t)^{\sigma} \, dt \right)^{1/(1-\sigma)} \tag{4}$$

which allows us to simplify it for the general case to get the demand function:

$$Y(t) = (p(t)/\alpha(t))^{-\sigma} \, P^{1-\sigma} = \left(\frac{p(t)}{\alpha(t)P} \right)^{-\sigma} U \tag{5}$$

where U, our utility, is equivalent to the inverse of P.[3]

B. Firms

Secondly, we have firms under perfect competition maximizing profit by picking an optimal set of energy inputs. For simplicity, suppose we only have two sources of energy, X_1 and X_2, which are converted into energy at rates $\xi_1(t)$ and $\xi_2(t)$. In reality, we might have X_1 be solar panels while X_2 could be a geothermal source; this would imply $\xi_1(t)$ varies with the time of day and $\xi_2(t)$ is be mostly constant. Because the energy output varies with time, this formulation introduces intermittency to the model. The total energy output at any particular time t is determined by the sum of energy outputs of each input at time t. That is, assuming that the $\xi_j(t)$ functions are integrable, we have:

$$Y(t) \equiv \xi_1(t) X_1 + \xi_2(t) X_2 \tag{6}$$

$$Y \equiv \int_0^1 Y(t) \, dt = X_1 \int_0^1 \xi_1(t) dt + X_2 \int_0^1 \xi_1(t) dt \tag{7}$$

Firms pick X_1 and X_2 to maximize profit while facing the cost function $C(X_1, X_2)$; we assume $\partial C / \partial X_i > 0$ to ensure convex costs. To simplify the algebra, suppose that we have the isoelastic cost function $C = (c_1 X_1^{\eta_1} + c_2 X_2^{\eta_2})/2$. Then, we have:

$$\Pi = \int_0^1 Y(t) p(t) dt - (c_1 \, X_1^{\eta_1} + c_2 \, X_2^{\eta_2})/2 \tag{8}$$

Given that we assumed perfect competition, we get the first order conditions for maximizing profit by setting marginal cost to the marginal profit of each input.

$$\frac{\partial \Pi}{\partial X_i} = 0 \implies \int_0^1 \xi_i(t) \, p(t) dt = \eta_i c_i X_i^{\eta_i - 1}$$

Hence, we have the profit-maximizing condition:

$$\frac{X_1}{X_2} = \left(\frac{\eta_2 c_2 \, \int_0^1 \xi_1(t) \, p(t) dt}{\eta_1 c_1 \, \int_0^1 \xi_2(t) \, p(t) dt} \right)^{\frac{\eta_2 - 1}{\eta_1 - 1}} \tag{9}$$

[3] The derivation is:

$$U^{\phi} = \int_0^1 Y(t)^{\phi} dt = \int_0^1 (p(t)/\alpha(t))^{1-\sigma} \, P^{(\sigma-1)\phi} dt$$
$$= P^{(\sigma-1)\phi} \int_0^1 (p(t)/\alpha(t))^{1-\sigma} dt = P^{(\sigma-1)\phi} P^{1-\sigma} = P^{-\phi}$$

In this case, since firms are setting the ratio X_1/X_2 to the ratio of their marginal profit, they are also fixing it to the ratio of their shadow prices faced by consumers. Therefore, with the cost function used here, the market equilibrium is equivalent to a centrally planned equilibrium.

C. Example Two Period Case with CD Utility and Two Technologies

Suppose our utility function is instead given by $Y_t^{\alpha_t} Y_s^{\alpha_s}$ where our two periods are t and s. Optimizing utility given the budget constraint $Y_t p_t + Y_s p_s = I$, our demand curves are $Y_t = \alpha_t/p_t$ and $Y_s = \alpha_s/p_s$.

Next, we introduce the following matrices to simplify the notation:

$$Y = \begin{pmatrix} Y_t \\ Y_s \end{pmatrix} \quad P = \begin{pmatrix} p_t \\ p_s \end{pmatrix} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \xi = \begin{pmatrix} \xi_{1\mathrm{t}} & \xi_{2\mathrm{t}} \\ \xi_{1\mathrm{s}} & \xi_{2\mathrm{s}} \end{pmatrix}$$

where X_i is the quantity of input technologies and ξ_{it} is the output of technology i at time t. This implies that our output $Y = \xi\, X$. Next, we have profit maximizing firms with linear cost. Suppose c_i is the cost of each unit of technology i, and let C be a vector of c_i. Then, our profit function is:

$$\Pi = P^T Y - C^T X = \left(P^T \xi - C^T\right) X$$

Now, assuming that ξ is invertible, we have the first order condition:

$$\begin{aligned} P^T \xi &= C^T \\ P &= (\xi^{-1})^T C \\ P^{opt} &= \begin{pmatrix} -\dfrac{c_1\,\xi_{2\mathrm{s}} - c_2\,\xi_{1\mathrm{s}}}{\xi_{1\mathrm{s}}\,\xi_{2\mathrm{t}} - \xi_{1\mathrm{t}}\,\xi_{2\mathrm{s}}} \\ \dfrac{c_1\,\xi_{2\mathrm{t}} - c_2\,\xi_{1\mathrm{t}}}{\xi_{1\mathrm{s}}\,\xi_{2\mathrm{t}} - \xi_{1\mathrm{t}}\,\xi_{2\mathrm{s}}} \end{pmatrix} \end{aligned}$$

Substituting back into the FOC for consumer demand, we have:

$$Y^{opt} = \begin{pmatrix} \dfrac{\alpha_t\,\left(\xi_{1\mathrm{s}}\,\xi_{2\mathrm{t}} - \xi_{1\mathrm{t}}\,\xi_{2\mathrm{s}}\right)}{c_2\,\xi_{1\mathrm{s}} - c_1\,\xi_{2\mathrm{s}}} \\ \dfrac{\alpha_s\,\left(\xi_{1\mathrm{s}}\,\xi_{2\mathrm{t}} - \xi_{1\mathrm{t}}\,\xi_{2\mathrm{s}}\right)}{c_1\,\xi_{2\mathrm{t}} - c_2\,\xi_{1\mathrm{t}}} \end{pmatrix} \implies X^{opt} = \begin{pmatrix} \dfrac{\alpha_t\,\xi_{2\mathrm{s}}}{c_1\,\xi_{2\mathrm{s}} - c_2\,\xi_{1\mathrm{s}}} + \dfrac{\alpha_s\,\xi_{2\mathrm{t}}}{c_1\,\xi_{2\mathrm{t}} - c_2\,\xi_{1\mathrm{t}}} \\ -\dfrac{\alpha_t\,\xi_{1\mathrm{s}}}{c_1\,\xi_{2\mathrm{s}} - c_2\,\xi_{1\mathrm{s}}} - \dfrac{\alpha_s\,\xi_{1\mathrm{t}}}{c_1\,\xi_{2\mathrm{t}} - c_2\,\xi_{1\mathrm{t}}} \end{pmatrix}$$

If we assume that Y is positive in both periods, then, without loss of generality, we have:

$$\begin{aligned} X_1 \text{ has a comparative advantage in producing in period } s\text{:} &\quad \xi_{1s}/\xi_{1t} > \xi_{2s}/\xi_{2t} \\ X_1 \text{ is more cost effective at producing in period } s\text{:} &\quad \xi_{1s}/c_1 > \xi_{2s}/c_2 \\ X_2 \text{ is more cost effective at producing in period } t\text{:} &\quad \xi_{1t}/c_t < \xi_{2t}/c_2 \end{aligned}$$

Similarly, if we assume X is positive for both technologies, then, without loss of generality, we have:

$$\begin{aligned} \alpha_t \xi_{2s}\left(c_1\xi_{2t} - c_2\xi_{1t}\right) &> \alpha_s \xi_{2t}\left(c_2\xi_1 s - c_1\xi_2 s\right) \\ \alpha_t \xi_{1s}\left(c_1\xi_{2t} - c_2\xi_{1t}\right) &< \alpha_s \xi_{1t}\left(c_2\xi_1 s - c_1\xi_2 s\right) \\ \implies \xi_{2s}/\xi_{1s} &> \xi_{2t}/\xi_{1t} \end{aligned}$$

D. Two Period Two Technology Case with Risk

The intermittency of electricity technologies not only concerns when they produce but how consistently they can produce. Consider the same simplified, two-period model as before with technologies that produce according to a stochastic process. Specifically, suppose that X_1 produces at the rate $N(\xi_{1t}, \sigma_{1t}^2)$ in period t and at the rate $N(\xi_{1s}, \sigma_{1s}^2)$ in period s; also, consider equivalent definitions for X_2. We use the following matrices to simplify notation:

$$\xi = \begin{pmatrix} \xi_{1\mathrm{t}} & \xi_{2\mathrm{t}} \\ \xi_{1\mathrm{s}} & \xi_{2\mathrm{s}} \end{pmatrix} \quad \Sigma_t = \begin{pmatrix} \sigma_{1t}^2 & \sigma_{1t,2t} \\ \sigma_{1t,2t} & \sigma_{2t}^2 \end{pmatrix} \quad \Sigma_s = \begin{pmatrix} \sigma_{1s}^2 & \sigma_{1s,2s} \\ \sigma_{1s,2s} & \sigma_{2s}^2 \end{pmatrix}$$

where Σ_t is the covariance of electricity production in period t and likewise for Σ_s at period s; we the generation processes in both periods are independent for simplicity. This essentially reduces to a portfolio maximization problem in each period; however, there is a significant difference: electricity demand varies by period and the production technologies' output and variance of output vary by period as well. The optimization problem can be defined as:

$$\begin{aligned} \max \quad & E\left[U\right] \\ \text{s.t.} \quad & I \cdot w_i / c_i = X_i \ \forall i \\ & \mathbf{1}^T w_i = 1 \\ & w_i \in [0,1] \ \forall i \end{aligned}$$

where w_i is the percent of our budget that is spent on X_1 and X_2, I is the budget, and c_i is the price of X_i. Since the technologies generate electricity according to a normal process, we have:

$$\mu \equiv \begin{pmatrix} \mu_t \\ \mu_s \end{pmatrix} = \xi\,\mathbf{w} \qquad \sigma_t^2 = \mathbf{w}^T\,\Sigma_t\,\mathbf{w} \qquad \sigma_s^2 = \mathbf{w}^T\,\Sigma_s\,\mathbf{w}$$

$$\begin{aligned} Y_t &\sim N\left(\mu_t, \sigma_t^2\right) \\ Y_s &\sim N\left(\mu_s, \sigma_s^2\right) \end{aligned}$$

where $\mathbf{w}$ is a column vector of weights, μ represents the electricity generation rate in each period, and Σ is the covariance matrix for this rate. To solve, we take the second-order Taylor series around $(\overline{Y}_t, \overline{Y}_s) \equiv E\left[Y_t, Y_s\right]$ which represents our average electricity generation. This is given by:

$$\begin{aligned} U(Y_t, Y_s) \approx\ & \overline{U} + \overline{U}_t(Y_t - \overline{Y_t}) + \overline{U}_s(Y_s - \overline{Y_s}) \\ & + (1/2)\left[\overline{U}_{tt}(Y_t - \overline{Y_t})^2 + 2\overline{U}_{ts}(Y_t - \overline{Y_t})(Y_s - \overline{Y_s}) + \overline{U}_{ss}(Y_s - \overline{Y_s})^2\right] \end{aligned}$$

where $\overline{U} \equiv U(\overline{Y}_t, \overline{Y}_s)$, and $\overline{U}_t$ is the derivative of U with respect to Y_t evaluated at $\overline{Y}_t$. Taking the expectation, the U_t and U_s drop out since $E(Y_s - \overline{Y_s}) = 0$, and the U_{ts} drops out since Y_t and Y_s are independent so $(Y_t - \overline{Y_t})(Y_s - \overline{Y_s}) = 0$. Thus, we are left with:

$$E\left[U(Y_t, Y_s)\right] \approx \overline{U} + (1/2)\left[\overline{U}_{tt}(\sigma_t^2 + \mu_t^2) + \overline{U}_{ss}(\sigma_s^2 + \mu_s^2)\right]$$

III. Empirics

Suppose we continue the assumption of the previous described cost function but generalize it to the an isoelastic form. So, we have:

$$C(X_1, X_2) = c_1 X_1^{\eta_1} + c_2 X_2^{\eta_2}$$

where η_1 and η_2 are the elasticities of cost with respect to the input quantity. Then, profit maximization implies we have:

$$\frac{\eta_1\, X_1^{\eta_1 - 1}}{\eta_2\, X_2^{\eta_2 - 1}} = \frac{c_2\, \int_0^1 \xi_1(t)\, p(t) dt}{c_1\, \int_0^1 \xi_2(t)\, p(t) dt} \tag{10}$$

Since data is limited, we can simplify this function to estimate it empirically. Firstly, we can assume[4] that the solar conversion rate is equal to a fixed factor multiplied by the solar radiation at time t. That is, we have:

$$\xi_1(t) = \theta_1 \cdot r(t)$$

Additionally, we can safely assume that the conversion rate of coal is not time dependent but instead a fixed constant θ_2.

In order to simplify the function, we first rearrange the terms to get:

$$X_1 = \left(\frac{c_2\, \eta_2\, \int_0^1 \xi_1(t)\, p(t) dt}{c_1\, \eta_1\, \int_0^1 \xi_2(t)\, p(t) dt} \cdot X_2^{\eta_2 - 1} \right)^{\frac{1}{\eta_1 - 1}}$$

Next, given that net solar generation is $NG_1 = \theta_1\, r_t\, X_1$, we have:

$$NG_1(t) = \theta_1\, r(t)\, \left(\frac{c_2\, \eta_2\, \int_0^1 \xi_1(t)\, p(t) dt}{c_1\, \eta_1\, \int_0^1 \xi_2(t)\, p(t) dt} \cdot X_2^{\eta_2 - 1} \right)^{\frac{1}{\eta_1 - 1}}$$

Now, with our assumptions regarding the conversion rates, we get:

$$NG_1(t) = \theta_1\, r(t)\, \left(\frac{c_2\, \eta_2\, \theta_1 \int_0^1 r(t)\, p(t) dt}{c_1\, \eta_1\, \theta_2\, \int_0^1 p(t) dt} \cdot X_2^{\eta_2 - 1} \right)^{\frac{1}{\eta_1 - 1}}$$

In order to estimate this given our data, we replace the time terms with monthly indices t and include a new index i to represent each observation at a state level. Additionally, since we cannot integrate in this case, we convert the integrals to summations over the time period of our data which is one year. That is, the equation to estimate becomes:

$$NG_{1,i,t} = \theta_1\, r_{i,t}\, \left(\frac{c_2\, \eta_2\, \theta_1 \sum_{t=1}^{12} r_{i,t}\, p_{i,t}}{c_1\, \eta_1\, \theta_2\, \sum_{t=1}^{12} p_{i,t}} \cdot X_{2,i}^{\eta_2 - 1} \right)^{\frac{1}{\eta_1 - 1}}$$

Taking logs, we get:

$$\log(NG_{1,i,t}) = \log(f(\dots)) + \log(r_{i,t}) + (\tfrac{1}{\eta_1 - 1}) \cdot \log \left(\frac{\sum_{t=1}^{12} r_{i,t}\, p_{i,t}}{\sum_{t=1}^{12} p_{i,t}} \right) + (\tfrac{\eta_2 - 1}{\eta_1 - 1}) \cdot \log \left(X_{2,i} \right)$$

[4] This is reasonable because the actual formula for a solar panel's output is multiplicative with solar radiation.

where $f(\dots)$ is a function of constants that are independent of i and t. We can now estimate this form using the available data.

The results of estimation are:

	$\log(NG_{1,i,t})$
$\log\left(\frac{\sum_{t=1}^{12} r_{i,t}\, p_{i,t}}{\sum_{t=1}^{12} p_{i,t}}\right)$	3.449***
	(0.533)
$\log(X_{2,i})$	-0.0814**
	(0.0277)
$\log(r_{i,t})$	3.386***
	(0.996)
$\log(f(\dots))$	34.52***
	(3.165)
N	576
R^2	0.207
adj. R^2	0.202

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.00 > 1$

This implies that $\eta_2 = 1.28994$ and $\eta_2 = 0.97691$.

References