# I. Literature Outline

Two classifications for elasticities of substitution

- **Gross/net elasticities**:
    - Net elasticities are computed holding output constant
    - Gross elasticities are computed by letting output vary optimally (gross)

- **Scalar, asymmetric ratio, or symmetric ratio elasticities**
    - Scalar elasticities (Allen-Uzawa) measure the effect of a change in the price of another factor scaled by its cost share on the quantity of a factor demanded
    - Asymmetric ratio elasticities (Morishima) measure the effect on the factor input ratio of a change in a ratio of prices
    - Symmetric elasticities can be found by putting restrictions on asymmetric elasticities; holding cost constant on the Morishima elasticities produces the shadow elasticities of substitution

## A. Formulas

**Translog**

Own price elasticities are given by:

$$\eta_{ii} = \frac{\partial \ln X_i(y,p)}{\partial \ln p_i} = \frac{\beta_{ii} + S_i^2 - S_i}{S_i} \quad \implies \quad n_{ii}(1,\mathbb{1}) = \frac{\beta_{i,i} - \beta_i^2 - \beta_i}{\beta_i}$$

where $X_i$ is demand for input $i$, $p_i$ is its price, and $S_i$ is its predicted cost share. And, $B_{ij}$ is the second order parameter from the translog cost function, $y$ is the output, and $p$ is a vector of factor prices.

Cross price elasticities are given by:

$$\eta_{ij} = \frac{\partial \ln X_i(y,p)}{\partial \ln p_j} = \frac{B_{ij} + S_i S_j}{S_i} \quad \implies \quad n_{ii}(1,\mathbb{1}) = \frac{\beta_{i,i} - \beta_j \beta_i}{\beta_i}$$

where $B_i$ is the first-order parameter of the translog cost function.

**Morishima**

The elasticity of substitution here is given by:

$$\mu_{ij} = \left. \frac{\partial \ln(X_j(y,p)/X_i(y,p))}{\partial \ln(p_i/p_j)} \right|_{p_j} = \eta_{ji} - \eta_{ii}$$

**Shadow**

The shadow elasticity of substitution is given by:

$$\sigma_{ij} = \left. \frac{\partial \ln(X_j(y,p)/X_i(y,p))}{\partial \ln(p_i/p_j)} \right|_C = \frac{S_i}{S_i + S_j}\mu_{ij} + \frac{S_j}{S_i + S_j}\mu_{ji}$$

# II. Model of Electricity Production/Consumption

## *A. Formulation*

The model involves a consumer and producers reaching equilibrium in a two-period setting. The consumer maximizes a Cobb-Douglas utility function, and the producers allocate capital into two energy sources: one which provides a constant output and the other which generates an intermittent output. The producers' energy output, say kWh, at time $t$ is

$$Y_t = \xi_{1,t} X_1 + \xi_{2,t} X_2$$

where $X_1$ is investment into the constant output source, $X_2$ is the investment into the intermittent output source, and $\xi_{i,t}$ is the conversion factor from input $X_i$ into kWh at time $t$. Referring to the next period as period $s$, we also have:

$$Y_s = \xi_{1,s} X_1 + \xi_{2,s} X_2$$

Since $X_1$ is a constant output source, $\xi_{1,t} = \xi_{1,s}$. And, since $X_2$ is an intermittent source, we may have $\xi_{2,t} > \xi_{2,s}$ without loss of generality. For example, $X_1$ may be coal and $X_2$ may be solar. Letting $\xi_{2,t} > \xi_{2,s}$ reflects the fact that the conversion rate from tons of coal to kWh is independent of the time of day, while the conversion rate for solar panels into kWh is higher during the day. Next, the cost of production for inputs $X_i$ is given by:

$$C(X_1, X_2) = p_1(X_1 - c_1)^{\eta_1} + p_2(X_2 - c_1)^{\eta_2}$$

where $p_i > 0$, $c_i > 0$, $\eta_i > 2$. When $\eta_i = 3$, this particular function has:

$$\partial C/\partial X_i > 0$$
$$\partial^2 C/\partial X_i^2 < 0 \text{ when } X_i < c_i$$
$$\partial^2 C/\partial X_i^2 > 0 \text{ when } X_i > c_i$$

which shows concave costs followed by convex costs. That is, average cost initially decreases with production and then increases. This might be more realistic, but the following results can also be obtained with $c_i = 0$ and $\eta = 2$ which is a simpler model with linearly increasing average costs. Lastly, the producer has the following profit function:

$$\Pi = p_t Y_t + p_s Y_2 - C(X_1, X_2)$$

And, the consumer maximizes utility while constrained by their budget:

$$U = Y_t^\alpha \cdot Y_s^\beta$$

$$\text{s.t. } p_t Y_t + p_s Y_2 \leq B$$

## *B. Solution*

First, we assume that the social planner maximizes utility while keeping profit non-negative. Additionally, we keep all prices and quantities non-negative.

Optimizing a Cobb-Douglas function implies that the demand for energy is given by:

$$Q_{D,t} = \alpha * B/p_t$$
$$Q_{D,s} = \beta * B/p_s$$

These demand functions maximize utility while ensuring the consumer budget constraint is satisfied.

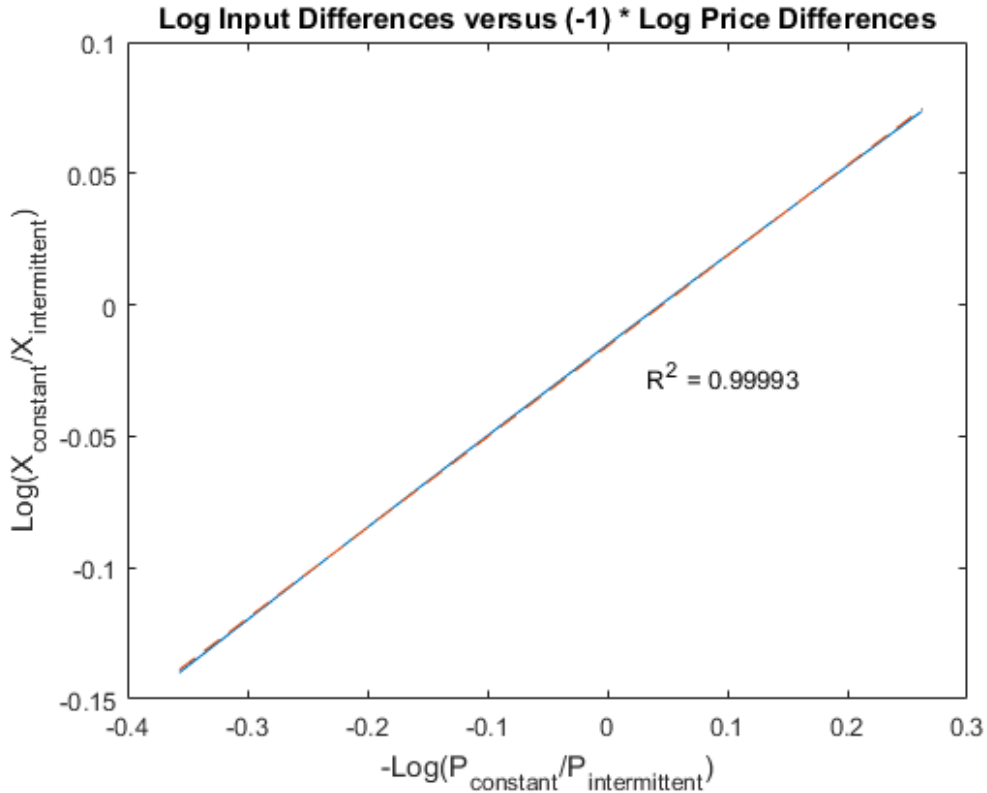The energy production functions remain the same:

$$Q_{S,t} = \xi_{1,t}X_1 + \xi_{2,t}X_2$$
$$Q_{S,s} = \xi_{1,s}X_1 + \xi_{2,s}X_2$$

But, the producer profit is now:

$$\Pi = p_t \min(Q_{D,t}, Q_{S,t}) + p_s \min(Q_{D,s}, Q_{S,s}) - C(X_1, X_2)$$

Running an optimization routine provides the following results:



Graphed above is $\ln(X_1/X_2)$ versus $\ln(P_2/P_1)$. The blue line shows the results while the dashed orange line is an OLS fit. Similar to a CES function, this relationship is practically linear; error may be due to the optimization routine. The results were generated with $p_1 = 10, c_1 = 0.5, c_2 = 0.5,, \eta_1 = 3, \eta_2 = 3, \xi_{1,t} = 5, \xi_{1,s} = 5, \xi_{2,t} = 10, \xi_{2,s} = 4, \alpha = 0.3, \beta = 0.7, B = 50$, and with $p_2$ allowed to vary over iterations from 7 to 13. That is, with:

$$U = Y_t^{0.3} \cdot Y_s^{0.7}$$
$$p_t Y_t + p_s Y_2 \leq 50$$
$$Y_t = 5X_1 + 10X_2$$
$$Y_s = 5X_1 + 4X_2$$
$$C = 10(X_1 - 0.5)^3 + p_2(X_2 - 0.5)^3$$

Running the routine again but with $\xi_{1,t} = \xi_{1,s} = 6$, we get the following results:

**Log Input Differences versus (-1) * Log Price Differences**
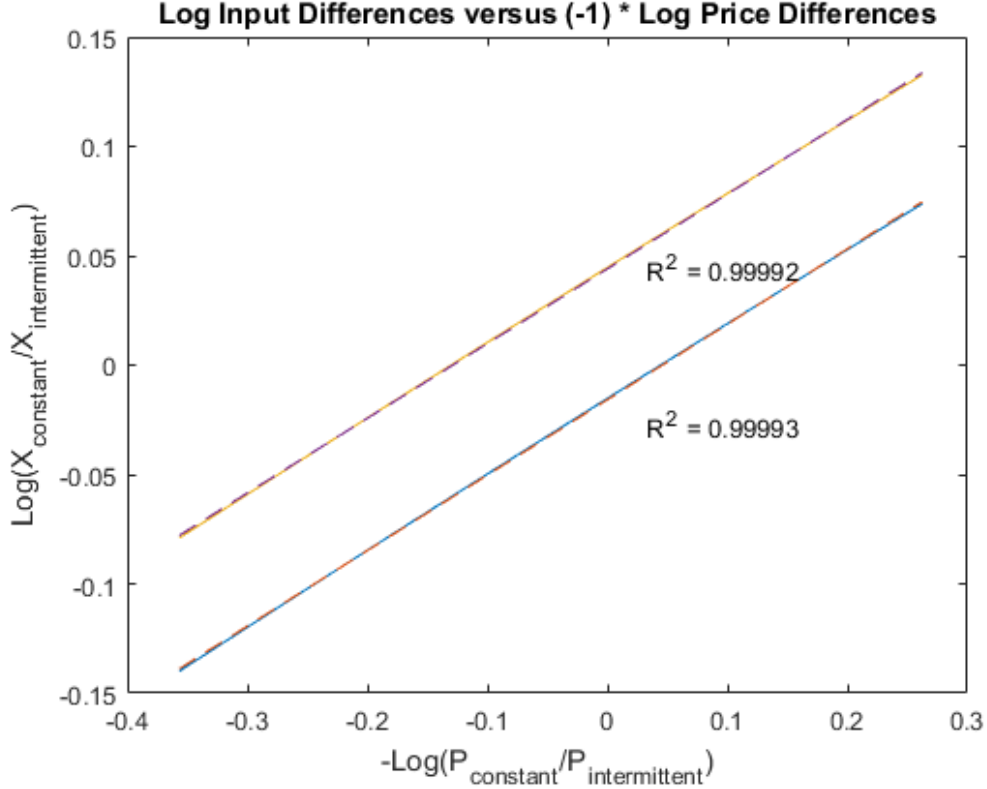
In the graph above, the yellow line is a second set of results obtained by modifiying $\xi$. We can see that the intercept has shifted without changing the slope very much ($< 1\%$ change in the slope), which is similar to what would be seen with a CES model. That is, the optimization of a standard CES model would result in

$$\ln(X_i/X_j) = \sigma \ln(P_j/P_i) + \sigma \ln(\xi_i/\xi_j)$$

where changes in $\xi$ would affect the intercept but not the slope (elasticity of substitution) $\sigma$ as seen in this two-period model. The estimated coefficient of the slope $\hat{\sigma}$ with these parameters is 0.3416. This implies that the constant and intermittent energy sources are complements in energy production, which reflects reality since an intermittent source could not substitute well for a constant output source.

Part of the reason the model implies that they are complements may be due to the fact that the cost functions show increasing average cost, so it would be more expensive to completely substitute one for the other. However setting $\eta_1 = \eta_2 = 2$, results in $\hat{\sigma} = 0.6695$. The two are much weaker complements but still remain complements even if average cost is linearly increasing. Generally, as $\eta_i$ decreases, the elasticity of substitution rises and the two become increasingly better substitutes. On the other hand, when the output of the second source becomes more intermittent, when $\xi_{2_d}/\xi_{2_n}$ increases, the elasticity of substitution declines as expected.

Overall, this result - the near linear relationship between the log of inputs and prices - essentially implies that we could approximate a two-period model of electricity into a one-period model by using a CES production function. This is useful, because we do not have electricity output by source, hour, and state for 2016. Since modeling intermittency directly with a multi-period SAM would be impossible without data to estimate the parameters, a single period SAM with this theoretical approach would be more persuasive. Additionally, using a CES function without $\sigma \to \infty \iff \phi \to 1$ would always imply that

1 kWh from one source does not substitute for 1 kWh from another. When estimating the CES function:

$$Y = \theta \cdot \left( \Pi_i \alpha_i X_i^\phi \right)^{1/\phi}$$

the estimates will always have $\hat{\theta} \to 1$, $\hat{\alpha}_i$ approaching the average conversion factors, and $\hat{\phi} \to 1$. When the electricity data comes from a single period, such as 2016 in our data, this must be the result; even theoretically this is true. For instance, this is true with the MATLAB simulation when aggregating the two-period model data into one period and fitting a CES function. We see $\theta \to 1$, $\phi \to 1$, and $\alpha_i \to (\xi_{i,t} + \xi_{i,s})/2$. To get $\hat{\phi} \neq 1$ would be wrong in any single period model, but would make sense in a multi-period model. Implementing this in a CGE would still imply that 1 kWh of solar stops translating into 1 kWh of electricity when a shock occurs. But, this can make sense if we suppose reality follows a multi-period model and such a kWh of solar was just over-generated in one period as a result of optimal investment targeting all periods. So, starting from a multi-period theoretical foundation would better justify using $\phi \neq 1$ in the CGE.

# References

Stern, D. (2012). "Interfuel Substitution: A Meta-Analysis." Journal of Economic Surveys 26 (2): 307–331. https://doi.org/10.1111/j.1467-6419.2010.00646.x.