AST1420 “Galactic Structure and Dynamics” Final

Due on Dec. 17 at 5pm

The full mark for the final includes the oral defense of the exercises listed here. The breakdown is: 75% written solutions and 25 % oral defense of solutions. The points given for the different problems below only relate to the 75 % written-solutions part of the full mark.

Some of the exercises in this final must be solved on a computer and the best way to hand in the final is as an `ipython notebook`. Rather than sending me the notebook, you can upload it to `GitHub`, which will automatically render the notebook. Rather than starting a repository for a single notebook, you can upload your notebook as a `gist`, which are version-controlled snippets of code that can optionally be made private. If you want to make further changes, you can clone your gist in a separate directory and use it as you would any other git repository. *Please re-run the entire notebook (with `Cell > Run All`) after re-starting the notebook kernel before uploading it*; this will make sure that the input and output are fully consistent.

If you are unfamiliar with notebooks, you can also send in a traditional write-up (in LaTeX), but you also need to send in well-commented code for how you solved the problems. Thus, notebooks are strongly preferred :-)

Problem 1: (25 points, 5 each) Short questions.

(a) In class, we discussed the NFW and Hernquist members of the family of two-power density models. Another commonly used member of this family is the *Jaffe* profile, which can be used to describe the light and mass profile of cuspy elliptical galaxies. The density of the Jaffe profile is

$$\rho(r) = \frac{\rho_0\, a^2}{r^2\,(1+r/a)^2}\,. \tag{1}$$

For this profile, compute the enclosed mass as a function of r/a, determine the total mass M in terms of ρ_0 and a, and compute the gravitational potential and circular velocity as a function of r/a, expressing these in terms of (M, a) rather than (ρ_0, a).

(b) Show that the velocity dispersion for an isotropic spherical system can be calculated as

$$\sigma^2 = \frac{1}{\nu(r)}\int_{\Phi(r)}^{0} \mathrm{d}\Phi'\,\nu(\Phi')\,, \tag{2}$$

where Φ is the gravitational potential and ν is the density.

(c) New observations of the ultra-diffuse galaxy Dragonfly 44 show that it has a line-of-sight velocity dispersion of $33 \pm 3\,\mathrm{km\,s^{-1}}$ at its 3D half-light radius of $4.7 \pm 0.2\,\mathrm{kpc}$ and the velocity dispersion is approximately constant with radius. Using this information, determine the total dynamical mass of Dragonfly 44 within its half-light radius.

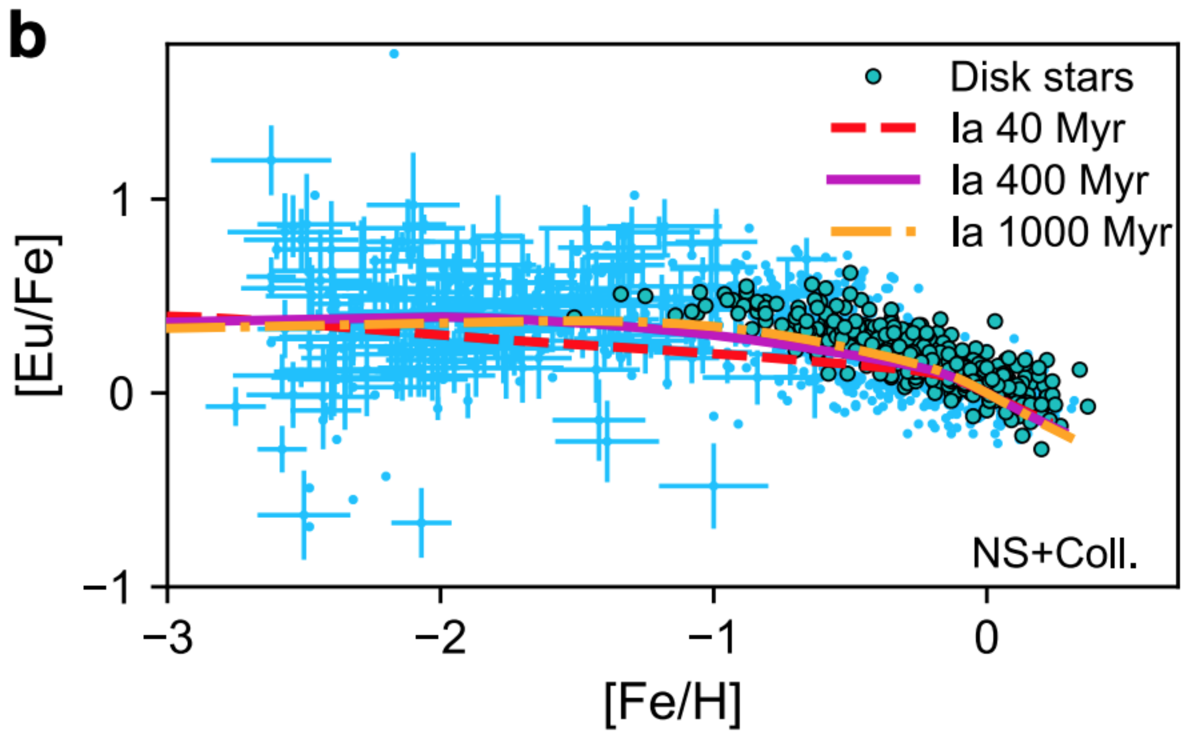

Figure 1: Observed distribution of [Eu/Fe] vs. [Fe/H] for solar neighborhood stars.

(d) The origin of r process elements has long been mysterious, but neutron-star (NS) mergers were long thought to play a potential role and observations of the light curve of the double NS merger GW170817 showed the presence of a significant amount of r-process material in the merger ejecta. The amount of material inferred combined with the rough estimate of the occurence rate obtained from a single NS merger was consistent with *all* r process elements in the Universe being created in NS mergers.

To get to a NS merger, a binary system must first evolve to a binary NS system that then slowly spirals in through gravitational wave radiation unil the NSs merge. This is similar to the favored double-degenerate scenario for type Ia supernovae, where a binary must evolve to a double white-dwarf binary which then slowly spirals in until the white dwarfs merge. The typical time scale for the latter process is a Gyr and we expect the time scale for the double NS merger therefore to be the same.

Given that iron in the solar neighborhood for stars with metallicities like the Sun comes about 50% from type Ia supernovae and about 50% from type II supernovae, while in the double NS merger scenario 100% of r process elements like Eu come from a process with similar time dependence as type Ia supernovae, describe what the expected location of stars looks like in the plane made up of [Eu/Fe] and [Fe/H] (from low metallicity [Fe/H] ≈ -2 to ≈ 0; similar to the [O/Fe] vs. [Fe/H] plane that we discussed in class).

Fig. 1 shows the observed distribution of [Eu/Fe] vs. [Fe/H] in the solar neighborhood. This looks similar to that of [Mg/Fe] vs. [Fe/H]. What can you conclude from this regarding the origin of the r process and the NS merger contribution?

(e) In Section 16.2, we saw that isotropic rotation cannot support the oblate structure of most elliptical galaxies. Thus, anisotropy in the velocity-dispersion tensor is necessary to support the shape of elliptical galaxies. Assuming that the dispersion tensor $\Pi_{zz} = (1-\delta)\,\Pi_{xx}$, with δ the global anisotropy parameter, derive the equivalent of relation (16.68) for $\delta \neq 0$ and plot the resulting relations for $\delta \in \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. Also draw curves for different ratios $\varepsilon_{\mathrm{int}}/\varepsilon_{\mathrm{obs}}$ as we did in the text, making use of the fact that the equivalent to Equation (16.73) for $\delta \neq 0$ modifies each occurrence of W_{xx}/W_{zz} in that equation in the same way. By comparing to the data on v/σ versus ellipticity from van de Sande et al. (2017) shown in the text, what values of δ are required to explain the oblateness of typical elliptical galaxies?

Problem 2: (25 points, 5 each) The epicycle approximation, Bertrand's theorem, and the mass distribution in the center of the Milky Way.

A useful approximation of the orbits in galaxies, especially those of disk stars in disk galaxies, is the *epicycle approximation*. In this approximation, one approximates the gravitational effective potential by Taylor expanding it to second order

$$\Phi_{\mathrm{eff}}(R, z; L_z) \approx \Phi_{\mathrm{eff}}(R_g, 0; L_z) + \frac{1}{2}\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial R^2}\right)\Bigg|_{(R_g,0)} (R-R_g)^2 + \frac{1}{2}\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial z^2}\right)\Bigg|_{(R_g,0)} z^2\,, \quad (3)$$

around the guiding-center radius of a star. This leads to equations of motion that are

$$\ddot{R} = \ddot{R} - \ddot{R}_g = -\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial R^2}\right)\Bigg|_{(R_g,0)} (R-R_g)\,, \quad (4)$$

$$\ddot{z} = -\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial z^2}\right)\Bigg|_{(R_g,0)} z\,, \quad (5)$$

because $\ddot{R}_g = 0$. These are the equations of a decoupled harmonic oscillator in $(R-R_g, z)$, with frequencies. The first of these, κ, is known as the epicycle frequency or as the radial frequency, while the second is known as the vertical frequency. A third important frequency is the circular frequency $\Omega(R_g)$, which is the azimuthal frequency of the circular orbit at R_g and is therefore $\Omega(R_g) = v_c(R_g)/R_g = L_z/R_g^2$. As shown in Section 10.3 in the notes, the equations of motion can be solved fully analytically in terms of these frequencies and the initial conditions and the resulting motion in (R, ϕ) is that of motion along an ellipse whose center is the guiding center, itself orbiting on a circular around the center. The vertical motion is a simple harmonic oscillation decoupled from the planar motion.

(a) For a flat rotation curve $v_c(R) = \text{constant}$, compute κ/Ω.

(b) Betrand's theorem states that the only mass distributions for which all orbits close are the (a) point-mass and (b) homogeneous density sphere. Let's investigate this here and see what it implies about the mass distribution in the Galactic center.

If all orbits in a mass distribution close, then in particular orbits that are close to a circular orbit must close. Close-to-circular orbits are described by the epicycle approximation, so we can use this approximation to see whether orbits close. Using the epicycle approximation for spherical mass distributions—the same as that discussed for disks above, but without the vertical dependence of the potential—demonstrate that the only gravitational potentials for which close-to-circular orbits close have $\phi(r) \propto r^{\beta^2-2}$ where β is a rational number (you can start from the assumption that the potential has a power-law form, because over a small range of radii, all potentials can be approximated as such).

(c) The form $\phi(r) \propto r^{\beta^2-2}$ with β a rational number includes potentials where $\beta = 5/3$ or $\beta = 4/3$. Through explicit orbit integration using `galpy`, investigate whether all orbits close in potentials with $\beta = [1, 16/15, 5/4, 4/3, 3/2, 11/6, 23/12, 2]$. What can you conclude?

(d) Now numerically show that what you claim in (c) is correct by explicitly calculating the radial and azimuthal periods of well-chosen orbits in each of these potentials.

(e) In the Galactic center, we can observe (partial) orbits of the so-called "S stars". In particular, the orbit of the star S2 (or S0-2 depending on who you ask) has been observed through a full azimuthal period and its orbit closes to within the uncertainties. While there are closed orbits in many potentials, the fact that the one orbit that we observe closes (and which is not circular) is good evidence that *all* orbits close in the mass distribution that S2 is orbiting in. From Bertrand's theorem we know that this means that the mass distribution is dominated either by a massive point-like object or that it is homogeneous. These are quite different mass distributions! Discuss what kinds of observations of the S stars could distinguish between these two possibilities.