Retrieving Data and Sorting

Advanced Programming and Algorithmic Design

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Retrieving Data

Retrieving Data

$A =< a_1, \ldots, a_n >$ contains some data, e.g., patient records

Each element is associated to an identifier, $A[i].id$, e.g., SSN

How to find the data associated to the identifier id_1?

A Naïve Solution and Outlook

Scan all the database searching for $A[i].id = \mathrm{id}_1$

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A Naïve Solution and Outlook

Scan all the database searching for $A[i].id = \mathrm{id}_1$

What is the asymptotic complexity in terms of big-O? $O(n)$

Can we do better?

Hint: How do you search a page in a book? Why?

A Better Technique: Dichotomic Search

If $A =< a_1, \ldots, a_n >$ is sorted w.r.t. the id's...

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$$(\text{i.e., if } i < j, \text{ then } A[i].id \leq A[j].id)$$

Look at element in the middle $A[n/2]$

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(i.e., if $i < j$, then $A[i].id \leq A[j].id$)

Look at element in the middle $A[n/2]$

if $A[n/2].id = \mathrm{id}_1$

Done!

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If $A = < a_1, \ldots, a_n >$ is sorted w.r.t. the id's...

$$\text{(i.e., if } i < j \text{, then } A[i].id \leq A[j].id\text{)}$$

Look at element in the middle $A[n/2]$

if $A[n/2].id = \mathrm{id}_1$
    Done!

if $A[n/2].id > \mathrm{id}_1$
    Focus on the 1st half A, i.e, $< a_1, \ldots, a_{n/2-1} >$

A Better Technique: Dichotomic Search

If $A = < a_1, \ldots, a_n >$ is sorted w.r.t. the id's...

(i.e., if $i < j$, then $A[i].id \leq A[j].id$)

Look at element in the middle $A[n/2]$

if $A[n/2].id = \mathrm{id}_1$

Done!

if $A[n/2].id > \mathrm{id}_1$

Focus on the 1st half A, i.e, $< a_1, \ldots, a_{n/2-1} >$

if $A[n/2].id < \mathrm{id}_1$

Focus on the 2nd half A, i.e, $< a_{n/2+1}, \ldots, a_n >$

Repeat until A is not empty

Dichotomic Search: An Example

Search for 2 in $< -4, 0, 1, 2, 5, 6, 7, 11, 12, 13 >$.

1	2	3	4	5	6	7	8	9	10
-4	0	1	2	5	6	7	11	12	13

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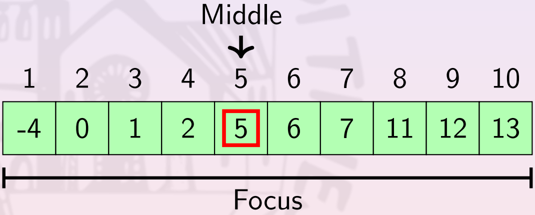

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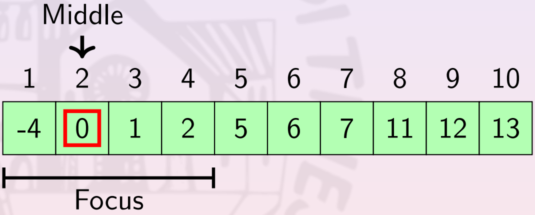

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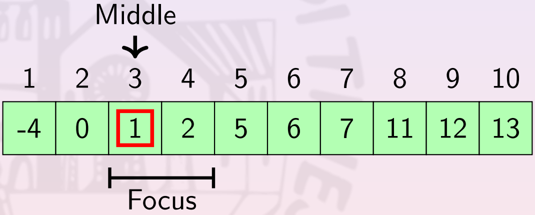

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Focus

Dichotomic Search: An Example

Search for 2 in $< -4, 0, 1, 2, 5, 6, 7, 11, 12, 13 >$.

Middle

1	2	3	4	5	6	7	8	9	10
-4	0	1	$\boxed{2}$	5	6	7	11	12	13

Focus

Found: $A[4] = 2$

Dichotomic Search: Pseudo-Code and Complexity

```
def di_find(A, a):
    (l, r) ← (1, |A|)
    while r ≥ l:
        m ← (l+r)/2
        if A[m]=a:
            return m
        endif
        if A[m]>a:
            r ← m−1
        else
            l ← m+1
        endif
    endwhile

    return 0
enddef
```

At each iteration, $l - r$ is halved.

So, if $|A| = 2^m$, di_find ends after m iterations at most.

The while-block takes time $\Theta(1)$.

The di_find 's complexity is

$$O(\log n)$$

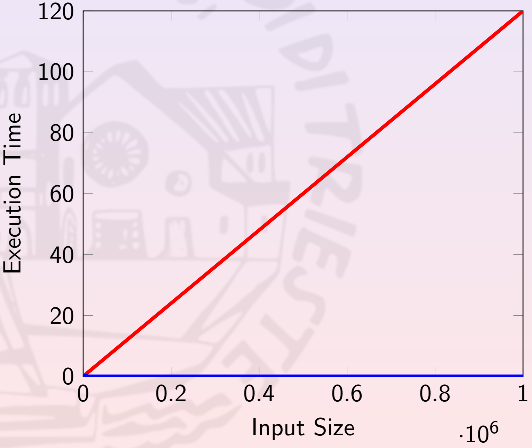
Dichotomic Search vs Linear Search: Experiments

Execution time per 1×10^5 random searches.

Input size	Linear Search	Dichotomic Search
1×10^1	3.3×10^{-3} s	3.2×10^{-3} s
1×10^2	1.4×10^{-2} s	4.3×10^{-3} s
1×10^3	1.2×10^{-1} s	5.9×10^{-3} s
1×10^4	1.2 s	7.8×10^{-3} s
1×10^5	1.2×10^1 s	8.7×10^{-3} s
1×10^6	1.2×10^2 s	1.2×10^{-2} s

Dichotomic Search vs Linear Search: Experiments

Execution time per 1×10^5 random searches.



Sorting

The Sorting Problem

Input: An array A of numbers

Output: The array A *sorted* i.e., if $i < j$, then $A[i] \leq A[j]$

E.g.,

1	2	3	4	5	6	7	8	9	10
13	5	7	2	-4	4	1	11	6	0

$$\Downarrow$$

1	2	3	4	5	6	7	8	9	10
-4	0	1	2	4	5	6	7	11	13

The Sorting Problem

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1	2	3	4	5	6	7	8	9	10
-4	0	1	2	4	5	6	7	11	13

Any idea for a sorting algorithm? What is expected complexity?

Insertion Sort

Insertion Sort: Intuition

If the first fragment of the array is already sorted

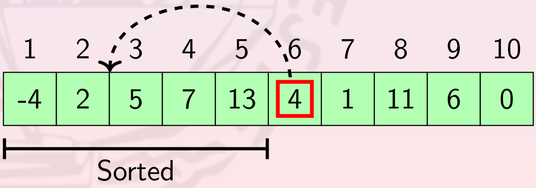
1	2	3	4	5	6	7	8	9	10
-4	2	5	7	13	4	1	11	6	0

Sorted

Insertion Sort: Intuition

If the first fragment of the array is already sorted

we can "*enlarge*" it by inserting **next value** in the right place



Insertion Sort: Intuition

If the first fragment of the array is already sorted

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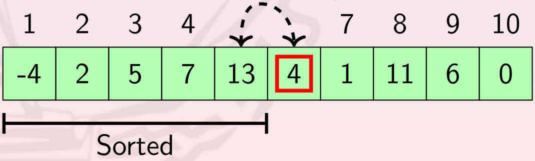
by swapping **it** and the previous value in the array



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1	2	3	4	5	6	7	8	9	10
-4	2	5	7	$\boxed{4}$	13	1	11	6	0

Sorted (positions 1–4)   Sorted (position 6)

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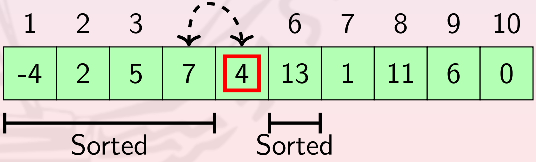

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-4	2	5	4	7	13	1	11	6	0

Sorted (positions 1–3) Sorted (positions 5–6)

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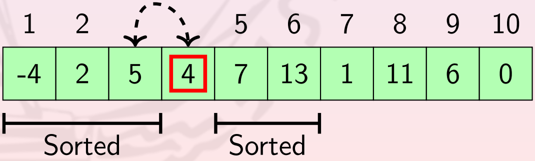

Insertion Sort: Intuition

If the first fragment of the array is already sorted

we can "*enlarge*" it by inserting **next value** in the right place

by swapping **it** and the previous value in the array

until the previous one (if exists) is greater than **it**

1	2	3	4	5	6	7	8	9	10
-4	2	$\boxed{4}$	5	7	13	1	11	6	0

Sorted ≤ 4 Sorted > 4

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1	2	3	4	5	6	7	8	9	10
-4	2	$\boxed{4}$	5	7	13	1	11	6	0

Sorted

Insertion Sort: Code and Complexity

```
def insertion_sort(A):
    for i in 2..|A|:
        j ← i
        while (j>1 and
                  A[j]<A[j−1]):

            swap(A,j−1,j)
            j←j−1

        endwhile
    endfor
enddef
```

The while-loop block costs $\Theta(1)$

It iterates $O(i)$ and $\Omega(1)$ times for all $i \in [2, n]$

$$\sum_{i=2}^{n} O(i) * O(1) = O(\sum_{i=2}^{n} i) = O(n^2)$$

$$\sum_{i=2}^{n} \Omega(1) * \Omega(1) = \Omega(\sum_{i=2}^{n} 1) = \Omega(n)$$

Quick Sort

Quick Sort: Intuition

Select one element of the A: the **pivot**

1	2	3	4	5	6	7	8	9	10
13	$\boxed{5}$	7	2	-4	4	1	11	6	0

Quick Sort: Intuition

Select one element of the A: the **pivot**

partition A in:

- subarray S of the values smaller or equal to the pivot
- the pivot
- subarray G of the values greater than the pivot

1	2	3	4	5	6	7	8	9	10
0	4	1	2	-4	$\boxed{5}$	11	6	7	13

S G

Quick Sort: Intuition

Select one element of the A: the **pivot**

partition A in:

- subarray S of the values smaller or equal to the pivot
- the pivot
- subarray G of the values greater than the pivot

Repeat on the subarrays having more than 1 elements

1	2	3	4	5	6	7	8	9	10
0	4	1	2	-4	$\boxed{5}$	11	6	7	13

$\underbrace{0 \; 4 \; 1 \; 2 \; -4}_{S} \quad 5 \quad \underbrace{11 \; 6 \; 7 \; 13}_{G}$

Quick Sort: Intuition (Cont'd)

At the end of every iteration of above steps:

- the values in S stay in S even after sorting A
- the values in G stay in G even after sorting A
- the pivot is in its "*sorted*" position
- S and G are shorter than A

Quick Sort: Intuition (Cont'd)

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An iteration places at least one element in the correct position

It prepares A for two recursive calls on S and G

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An iteration places at least one element in the correct position

It prepares A for two recursive calls on S and G

Quick Sort: Pseudo-Code

```
def QUICKSORT(A, l=1, r=|A|):
    if l<r:
        p ← PARTITION(A,l,r,l)

        QUICKSORT(A,l,p−1)
        QUICKSORT(A,p+1,r)
    endfi
enddef
```

Quick Sort: Pseudo-Code

The last recursion call is a **tail recursion**

```
def QUICKSORT(A, l=1, r=|A|):
    while l<r:
        p ← PARTITION(A,l,r,l)

        QUICKSORT(A,l,p−1)
        l ← p+1
    endwhile
enddef
```

Quick Sort: Complexity

The time complexity T_Q of quick sort will be

$$T_Q(|A|) = \begin{cases} \Theta(1) & \text{if } |A| = 1 \\ T_Q(|S|) + T_Q(|G|) + T_P(|A|) & \text{otherwise} \end{cases}$$

T_P is the complexity of **partition**

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T_P is the complexity of **partition**

Is the pivot selection relevant?

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T_P is the complexity of **partition**

Is the pivot selection relevant? No, choose whatever you want

Which algorithm is the best for **partition**?

Partition: An In-place Algorithm

Switch the pivot **p** and the first element in A

		3	4	5	6	7	8	9	10
13	$\boxed{5}$	7	2	-4	4	1	11	6	0

Partition: An In-place Algorithm

Switch the pivot **p** and the first element in A

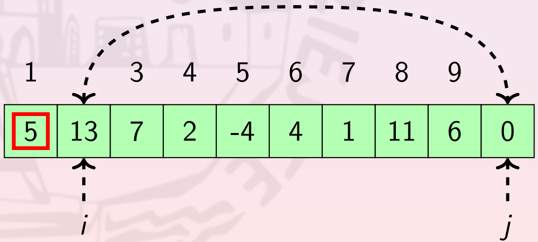
If **A[i]** $>$ **p**,

1	2	3	4	5	6	7	8	9	10
5	13	7	2	-4	4	1	11	6	0
	i								j

Partition: An In-place Algorithm

Switch the pivot **p** and the first element in A

If $\mathbf{A[i]} > \mathbf{p}$, swap $A[i]$ and $A[j]$ and decrease j



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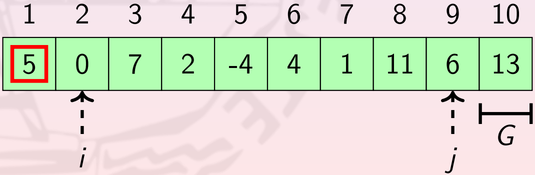
$\boxed{5}$	0	7	2	-4	4	1	11	6	13
	i							j	G

Partition: An In-place Algorithm

Switch the pivot **p** and the first element in A

If $\mathbf{A[i]} > \mathbf{p}$, swap $A[i]$ and $A[j]$ and decrease j

else ($\mathbf{A[i]} \leq \mathbf{p}$),



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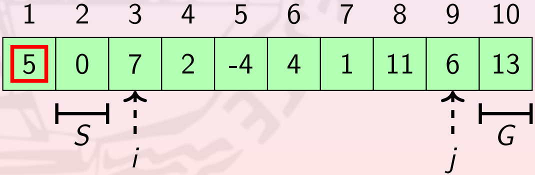

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If $\mathbf{A[i]} > \mathbf{p}$, swap $A[i]$ and $A[j]$ and decrease j

else ($\mathbf{A[i]} \leq \mathbf{p}$), increase i

Repeat until $i \leq j$

1	2	3	4	5	6	7	8	9	10
$\boxed{5}$	0	1	2	-4	4	11	6	7	13

S j i G

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Switch the pivot **p** and the first element in A

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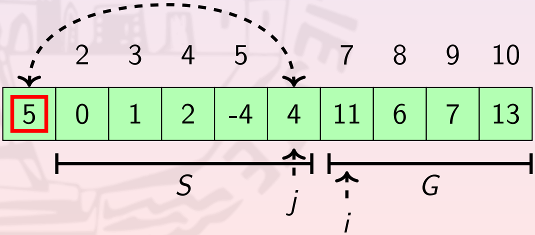

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Switch the pivot **p** and the first element in A

If $\mathbf{A[i]} > \mathbf{p}$, swap $A[i]$ and $A[j]$ and decrease j

else ($\mathbf{A[i]} \leq \mathbf{p}$), increase i

Repeat until $i \leq j$ and swap **p** and $A[j]$

The complexity is $\Theta(|\mathbf{A}|)$

1	2	3	4	5	6	7	8	9	10
4	0	1	2	-4	$\boxed{5}$	11	6	7	13

S $\qquad$ i $\qquad$ G

Partition: Pseudo-Code

```
def PARTITION(A, i, j, p):
    swap(A, i, p)
    (p, i) ← (i, i+1)

    while i≤j:
      if A[i]>A[p]:     # if A[i] is greater than the pivot
        swap(A, i, j)    # place it in G
        j ← j−1         # increase G's size
      else               # otherwise
        i ← i+1         # A[i] is already in S
      endif
    endwhile

    swap(A, p, j)  # place the pivot between S and G
    return j
enddef
```

Quick Sort Complexity: Worst Case

$$T_Q(|A|) = T_Q(|S|) + T_Q(|G|) + \Theta(|A|)$$

Quick Sort Complexity: Worst Case

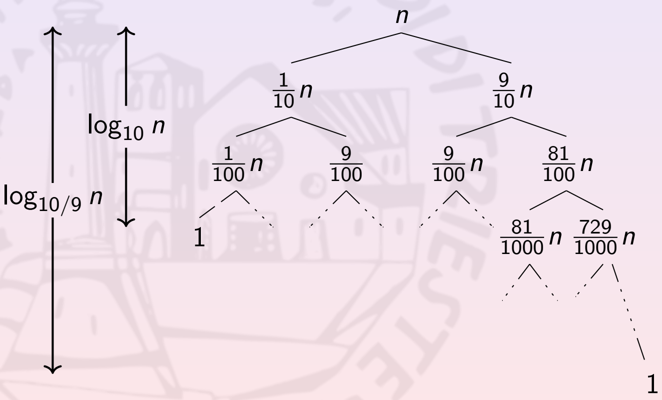
$$T_Q(|A|) = T_Q(|S|) + T_Q(|G|) + \Theta(|A|)$$

Worst Case: $|G| = 0$ or $|S| = 0$ for all recursive call.

$$\begin{aligned} T_Q(n) &= T_Q(n-1) + \Theta(n) \\ &= \sum_{i=0}^{n} \Theta(i) = \Theta\left(\sum_{i=0}^{n} i\right) \\ &= \boldsymbol{\Theta(n^2)} \end{aligned}$$

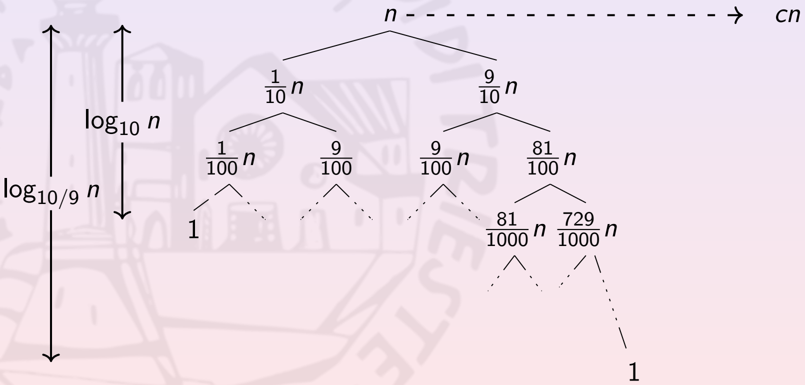
Quick Sort Complexity: Best Case

Best Case: Balanced Partition



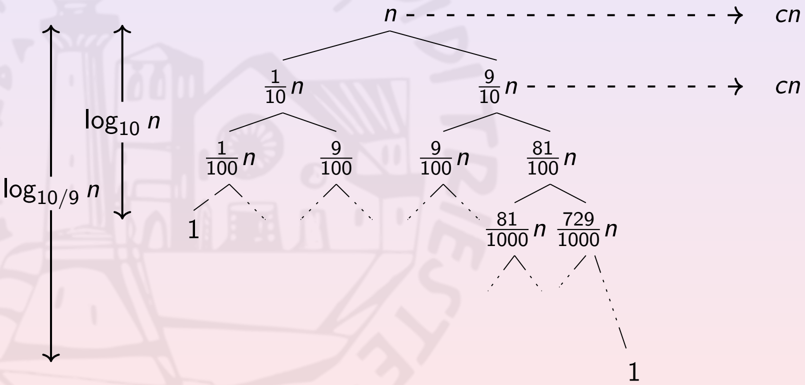
Quick Sort Complexity: Best Case

Best Case: Balanced Partition



Quick Sort Complexity: Best Case

Best Case: Balanced Partition



Quick Sort Complexity: Best Case

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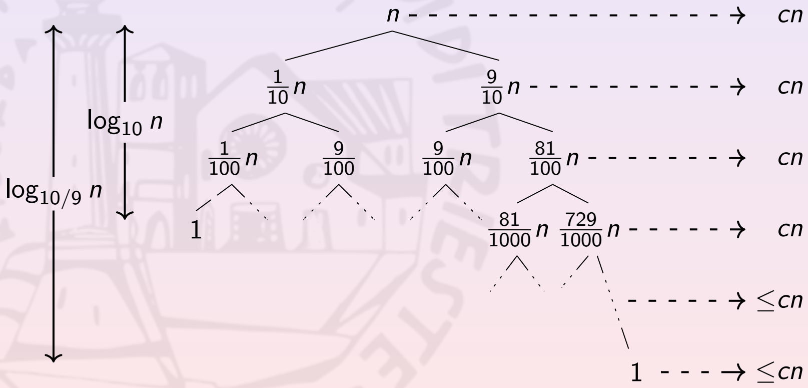

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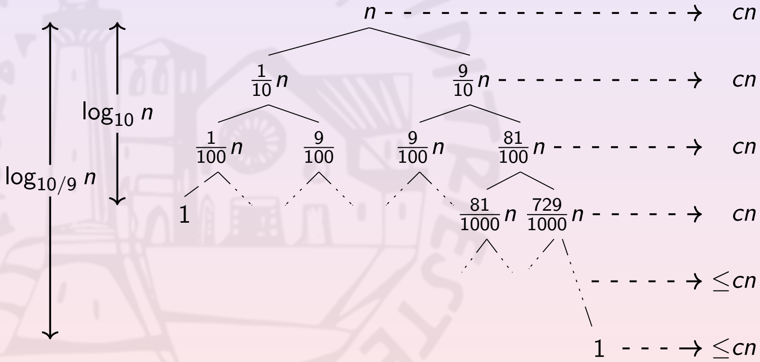


$\Theta(n \log n)$

Quick Sort Complexity: Average Case

"Good" and "bad" cases depend on the ordering of A

If all the permutations of A are equally likely,

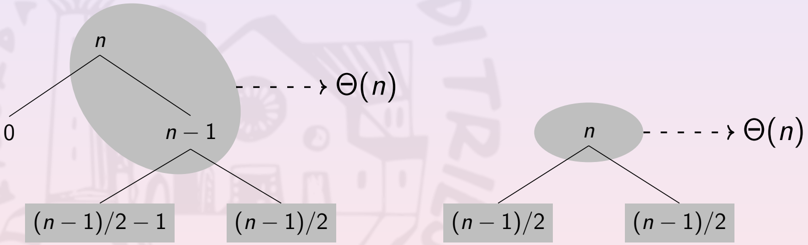
the partition has a ratio more balanced than $1/d$ with probability

$$\frac{d-1}{d+1}$$

e.g., a partition "better" than $1/9$ has probability 0.8

Quick Sort Complexity: Average Case (Cont'd)

Even if "good" and "bad" cases alternate



Quick Sort Complexity: Average Case (Cont'd)

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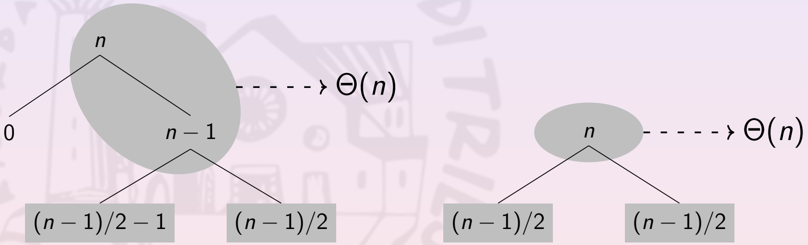


On the average $\Theta(n \log n)$

Finding the Maximum

Sorting by Searching the Maximum

Find the maximum

1	2	3	4	5	6	7	8	9	10
$\boxed{13}$	5	7	2	-4	4	1	11	6	0

Sorting by Searching the Maximum

Find the maximum

Move the maximum at the end of the array

1	2	3	4	5	6	7	8	9	10
0	5	7	2	-4	4	1	11	6	13

Sorting by Searching the Maximum

Find the maximum

Move the maximum at the end of the array

If $|A| > 1$, repeat on the initial fragment of A

1	2	3	4	5	6	7	8	9	10
0	5	7	2	-4	4	1	11	6	13

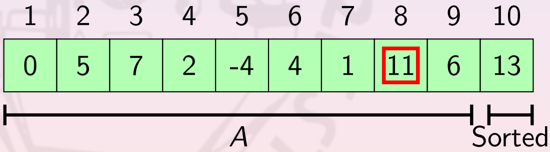
A Sorted

Sorting by Searching the Maximum

Find the maximum

Move the maximum at the end of the array

If $|A| > 1$, repeat on the initial fragment of A



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1	2	3	4	5	6	7	8	9	10
0	5	7	2	-4	4	1	6	11	13

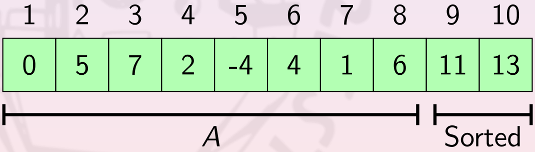
$\underbrace{\text{positions } 1\text{–}8}_{A}$ $\underbrace{\text{positions } 9\text{–}10}_{\text{Sorted}}$

Sorting by Searching the Maximum

Find the maximum

Move the maximum at the end of the array

If $|A| > 1$, repeat on the initial fragment of A



The complexity is $\sum_{i=1}^{|A|} \left(T_{\mathtt{max}}(i) + \Theta(1)\right)$

Finding the Maximum: Solution 1

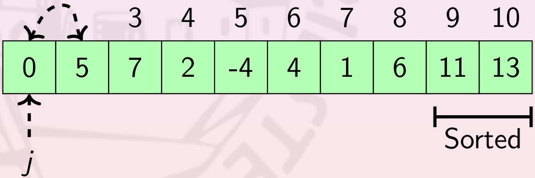
By pair-wise swapping the maximum to the right: **Bubble Sort**

1	2	3	4	5	6	7	8	9	10
0	5	7	2	-4	4	1	6	11	13

Unsorted (positions 1–8) Sorted (positions 9–10)

Finding the Maximum: Solution 1

By pair-wise swapping the maximum to the right: **Bubble Sort**



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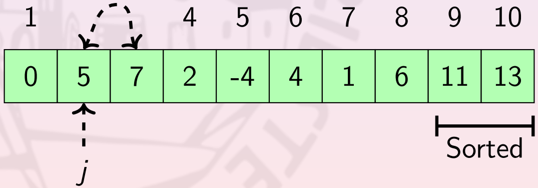

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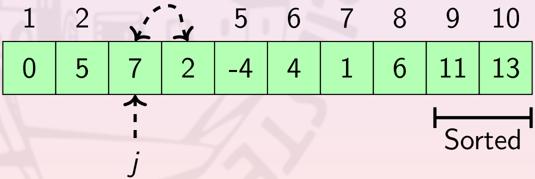

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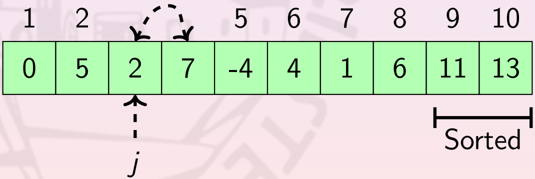

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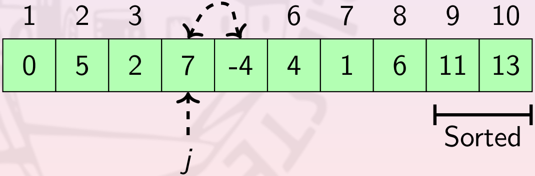

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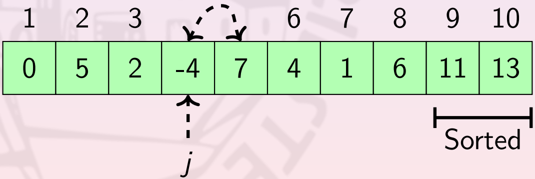
By pair-wise swapping the maximum to the right: **Bubble Sort**

After some swaps. . .



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By pair-wise swapping the maximum to the right: **Bubble Sort**

After some swaps. . .

1	2	3	4	5	6	7	8	9	10
0	5	2	-4	4	1	6	7	11	13

j

Sorted

Bubble Sort: Code and Complexity

```
def BUBBLE_SORT(A):
  for i in |A|..2:
    for j in 1..i-1:
      if A[j]>A[j+1]:
        swap(A,j,j+1)
      endif
    endfor
  endfor
enddef
```

One swap-block costs $\Theta(1)$

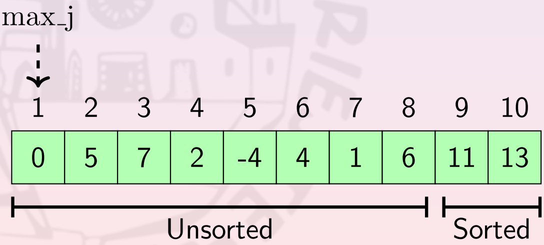
The nested for-loop costs $\Theta(i)$

$$T_B(n) = \sum_{i=2}^{n} \Theta(i) * \Theta(1)$$

$$= \Theta(\sum_{i=2}^{n} i) = \Theta(n^2)$$

Finding the Maximum: Solution 2

By linear scaning the unsorted part: **Selection Sort**



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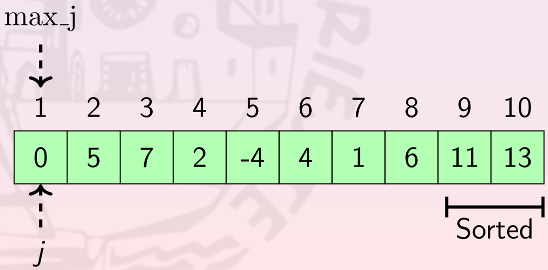

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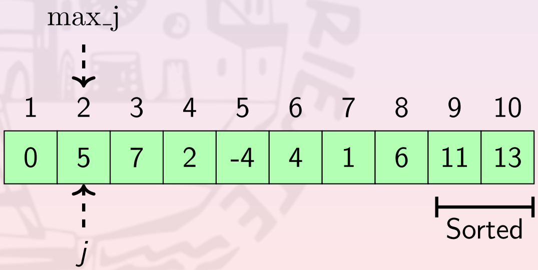
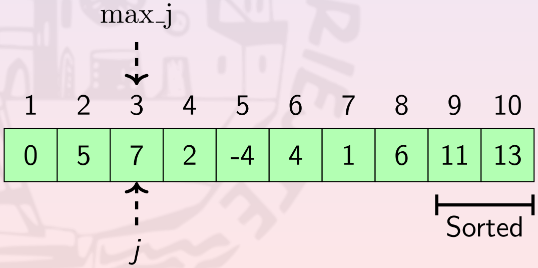


Finding the Maximum: Solution 2

By linear scaning the unsorted part: **Selection Sort**



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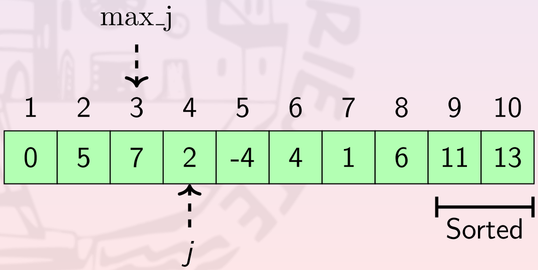
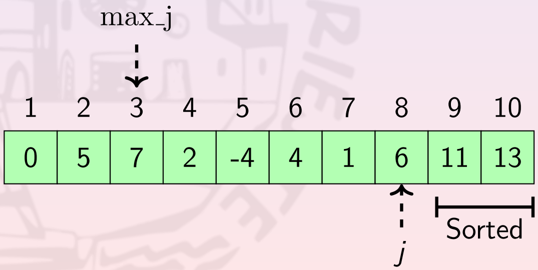


Finding the Maximum: Solution 2

By linear scaning the unsorted part: **Selection Sort**

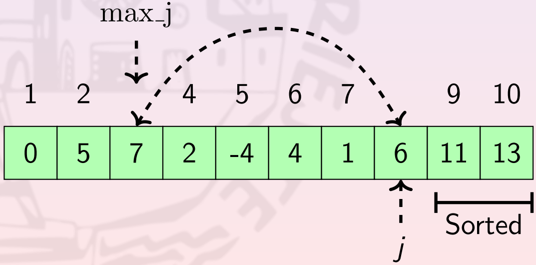
After few steps. . .



Finding the Maximum: Solution 2

By linear scaning the unsorted part: **Selection Sort**

After few steps...



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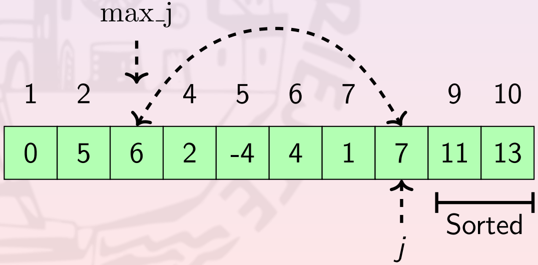

Finding the Maximum: Solution 2

By linear scaning the unsorted part: **Selection Sort**

After few steps. . .

1	2	3	4	5	6	7	8	9	10
0	5	6	2	-4	4	1	7	11	13

Sorted

Selection Sort: Code and Complexity

```
def SELECTION_SORT(A):
  for i in |A|..2:
    max_j ← 1
    for j in 2..i:
      if A[j]>A[max_j]:
        max_j ← j
      endif
    endfor

    swap(A, i, max_j)
  endfor
enddef
```

One if-block costs $\Theta(1)$

The nested for-loop costs $\Theta(i)$

$$T_S(n) = \Theta(1) + \sum_{i=2}^{n} \Theta(i) * \Theta(1)$$

$$= \Theta\left(1 + \sum_{i=2}^{n} i\right) = \Theta(n^2)$$

Finding the Maximum: Solution 3

Any other idea?

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What about using a **max-heap** $H_{\max}$: **Heap Sort**

1. store the elements of A in $H_{\max}$
2. extract the min (i.e., the max) and place it in A
3. repeat from 2 until $H_{\max}$ is not empty

Finding the Maximum: Solution 3

Any other idea?

What about using a **max-heap** $H_{\max}$: **Heap Sort**

1. store the elements of A in $H_{\max}$
2. extract the min (i.e., the max) and place it in A
3. repeat from 2 until $H_{\max}$ is not empty

Array-based representation of heaps $\Rightarrow$ inplace algorithm.

Heap Sort: Pseudo-Code

```
def HEAPSORT(A):
  H ← BUILD_MAX_HEAP(A) # the root is the max

   for i ← |A|..2:
     A[i] ← EXTRACT_MIN(H)
   endfor
 enddef
```

Heap Sort: Complexity

BUILD_MAX_HEAP costs $\Theta(n)$

EXTRACT_MIN costs $O(\log i)$ per iteration and in total

$$
\begin{aligned}
T_H(n) &= \Theta(n) + \sum_{i=2}^{n} O(\log i) \\
&\leq O(n) + O(\sum_{i=2}^{n} \log n) = O(n \log n)
\end{aligned}
$$

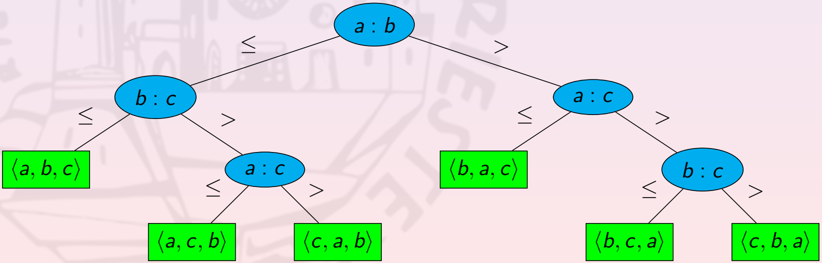
The overall complexity of heap sort is $O(n \log n)$

Sorting in Linear Time

Sorting By Comparison: Lower Bound

The execution of a sorting-by-comparison algorithm can be modeled as a decision-tree model

Any comparison between a and b corresponds to a node which branches the computation according whether $a \leq b$ or $b > a$



Sorting By Comparison: Lower Bound (Cont'd)

The decision tree's leaves are labeled by all the possible permutations of A which are $n!$

The height h is the maximum # of comparisons required by the algorithm

Since a binary tree has no more than 2^h leaves,

$$h \geq \log_2(n!) \in \Omega(n \log n)$$

Sorting By Comparison: Lower Bound (Cont'd)

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Since a binary tree has no more than 2^h leaves,

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The lower bound for comparison-based sorting is $\Omega(n \log n)$

Sorting in Linear Time?

There is no general algorithm to sort in linear time by using comparisons

Sorting in Linear Time?

There is no general algorithm to sort in linear time by using comparisons

This bound does not hold if we introduce minor *ad-hoc* assumptions such as:

- bounded domain for the array values
- uniform distribution of the array values

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	0	0	0	0	0	0	0	0	0

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	$\boxed{13}$	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	0	0	0	0	0	0	0	0	0

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	$\boxed{13}$	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	0	0	0	0	0	0	0	0	$\boxed{0}$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	$\boxed{13}$	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	0	0	0	0	0	0	0	0	$\boxed{1}$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	$\boxed{5}$	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	0	0	0	0	0	0	0	0	1

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	$\boxed{5}$	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	$\boxed{0}$	0	0	0	0	0	0	0	1

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	$\boxed{5}$	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	0	0	0	$\boxed{1}$	0	0	0	0	0	0	0	1

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	1	0	1	1	1	1	0	0	0	1	0	1

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the $\#$ elements $\leq$ to C's indexes

A

1	2	3	4	5	6	7	8	9	10
13	5	7	2	1	4	1	11	6	1

C

1	2	3	4	5	6	7	8	9	10	11	12	13
3	1	0	1	1	1	1	0	0	0	1	0	1

$+$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	4	0	1	1	1	1	0	0	0	1	0	1

$+$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the $\#$ elements $\leq$ to C's indexes

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	4	0	1	1	1	1	0	0	0	1	0	1

$+$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the $\#$ elements $\leq$ to C's indexes

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	4	4	1	1	1	1	0	0	0	1	0	1

$+$

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes

A

1	2	3	4	5	6	7	8	9	10
13	5	7	2	1	4	1	11	6	1

C

1	2	3	4	5	6	7	8	9	10	11	12	13
3	4	4	5	6	7	8	8	8	8	9	9	10

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes
- use C to place the elements of A in the correct positions in B

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	4	4	5	6	7	8	8	8	8	9	9	10

	1	2	3	4	5	6	7	8	9	10
B										

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	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	$\boxed{1}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	3	4	4	5	6	7	8	8	8	8	9	9	10

	1	2	3	4	5	6	7	8	9	10
B										

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
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- use C to place the elements of A in the correct positions in B

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	$\boxed{1}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	$\boxed{3}$	4	4	5	6	7	8	8	8	8	9	9	10

	1	2	3	4	5	6	7	8	9	10
B										

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
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	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	$\boxed{1}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	$\boxed{3}$	4	4	5	6	7	8	8	8	8	9	9	10

	1	2	3	4	5	6	7	8	9	10
B			1							

Values in $[1, k]$: Counting Sort

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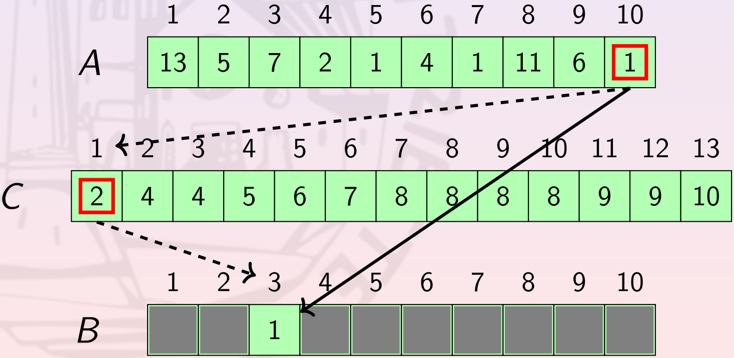

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- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes
- use C to place the elements of A in the correct positions in B

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	$\boxed{6}$	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	2	4	4	5	6	$\boxed{7}$	8	8	8	8	9	9	10

	1	2	3	4	5	6	7	8	9	10
B			1				6			

Values in $[1, k]$: Counting Sort

- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes
- use C to place the elements of A in the correct positions in B

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	$\boxed{11}$	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	2	4	4	5	6	6	8	8	8	8	$\boxed{9}$	9	10

	1	2	3	4	5	6	7	8	9	10
B			1				6		11	

Values in $[1, k]$: Counting Sort

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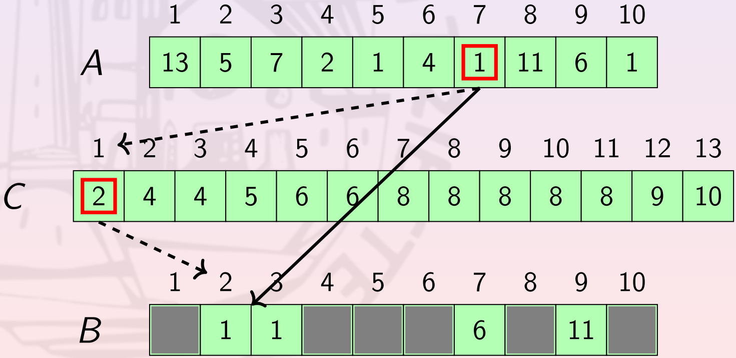

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- count the occurrences of A's values and place them in C
- sums the values in C and get the # elements $\leq$ to C's indexes
- use C to place the elements of A in the correct positions in B

	1	2	3	4	5	6	7	8	9	10
A	13	5	7	2	1	4	1	11	6	1

	1	2	3	4	5	6	7	8	9	10	11	12	13
C	0	3	4	4	5	6	7	8	8	8	8	9	9

	1	2	3	4	5	6	7	8	9	10
B	1	1	1	2	4	5	6	7	11	13

Some Observations about Counting Sort

Why backward placing?

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Why backward placing? For stability, i.e., preserving relative order of equivalent elements

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Generalizing it to deal with any $[k_1, k_2]$ domain is easy

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Generalizing it to deal with any $[k_1, k_2]$ domain is easy

It is not in-place and it requires the array C

Counting Sort: Pseudo-Code

```
def COUNTING_SORT(A,B,k):
  C ← ALLOCATE_ARRAY(k, default_value=0)
  for i ← 1 upto |A|:
    C[A[i]] ← C[A[i]]+1
  endfor  # C[j] is now the # of j in A

  for j ← 2 upto |C|:
    C[j] ← C[j−1] + C[j]
  endfor # C[j] is now the # of A's values ≤j

  for i ← |A| downto 1:
    B[C[A[i]]] ← A[i]
    C[A[i]] ← C[A[i]]−1
  endfor
enddef
```

Counting Sort: Complexity

Allocating C and setting all its elements to 0 $\Theta(k)$

Counting Sort: Complexity

Allocating C and setting all its elements to 0	$\Theta(k)$
Counting the instances of A's values	$\Theta(n)$

Counting Sort: Complexity

Allocating C and setting all its elements to 0	$\Theta(k)$
Counting the instances of A's values	$\Theta(n)$
Setting in $C[j]$ the # of A's values $\leq j$	$\Theta(k)$

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Total complexy	$\Theta(n+k)$

Fixed Number of Digits: Radix Sort

An array A of d-digit values can be sorted digit-by-digit

3	4	7	5
2	4	7	5
3	4	7	1
1	7	3	3

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- for each digit i from the rightmost down to the leftmost
- use a stable algorithm and sort A according the digit i

$$
\begin{array}{cccc}
3 & 4 & 7 & \boxed{5} \\
2 & 4 & 7 & \boxed{5} \\
3 & 4 & 7 & \boxed{1} \\
1 & 7 & 3 & \boxed{3} \\
 & & & \uparrow \\
 & & & i
\end{array}
$$

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3	4	7	1
3	4	7	5

Radix Sort: Complexity

If the digit sorting is in $\Theta(|A| + k)$, radix sort takes time

$$\Theta\left(d\left(|A| + k\right)\right)$$

where d is the number of digits in each of A's values

Uniform Distribution in $[0, 1)$: Bucket Sort

	1	2	3	4	5	6	7	8	9	10
A	0.23	0.60	0.20	0.73	0.97	0.34	0.01	0.50	0.78	0.44

Uniform Distribution in $[0, 1)$: Bucket Sort

- split $[0, 1)$ in n buckets: $\left[\frac{i-1}{n}, \frac{i}{n}\right)$ for $i \in [1, n]$

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B	/	/	/	/	/	/	/	/	/	/

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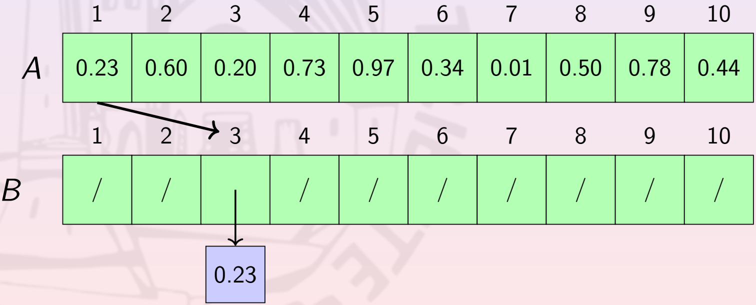

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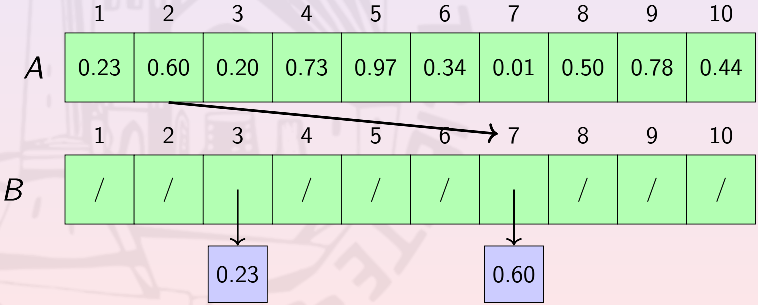

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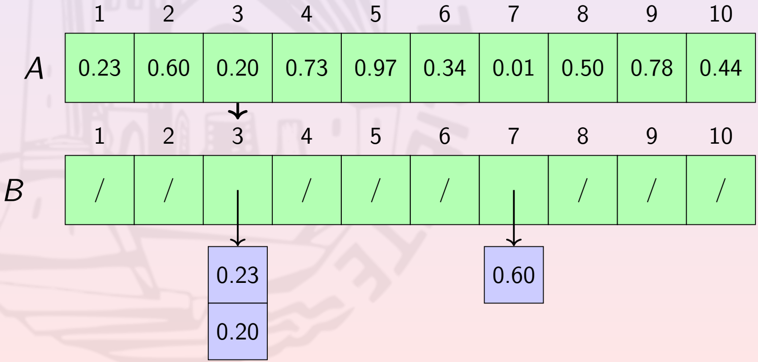

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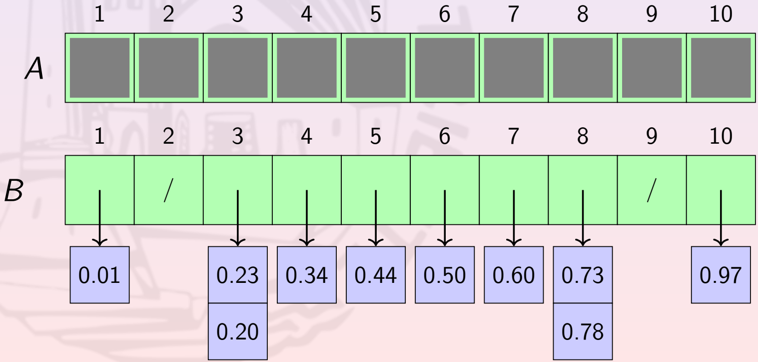

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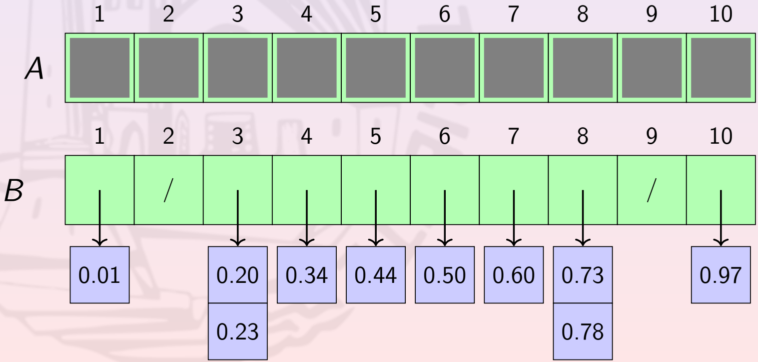

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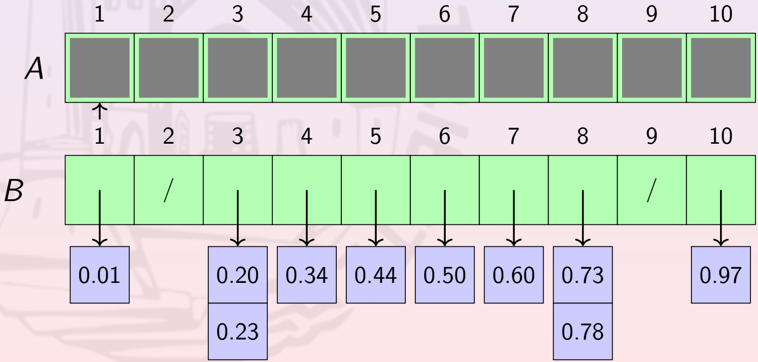

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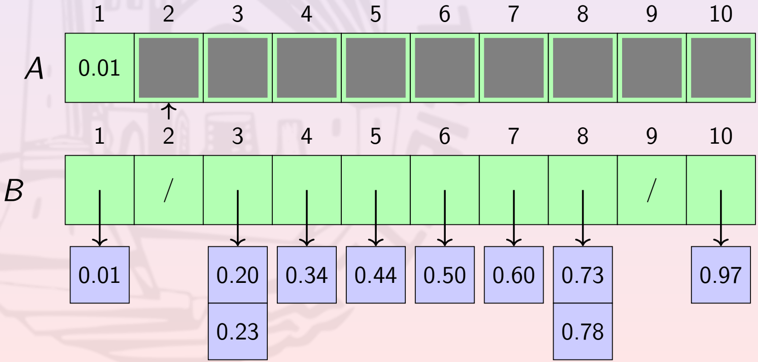

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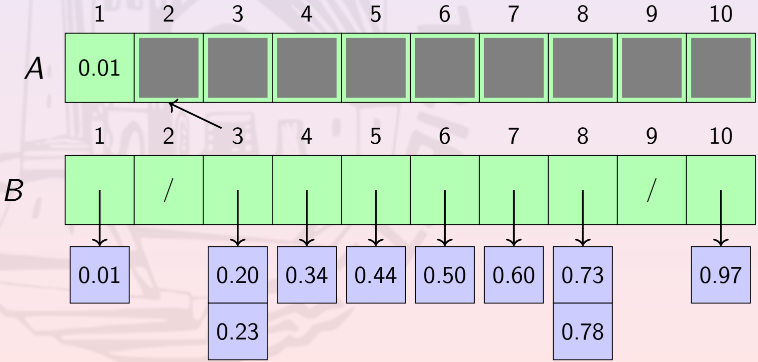

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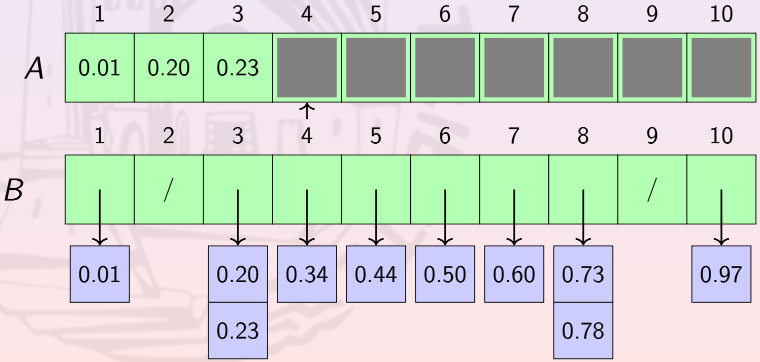

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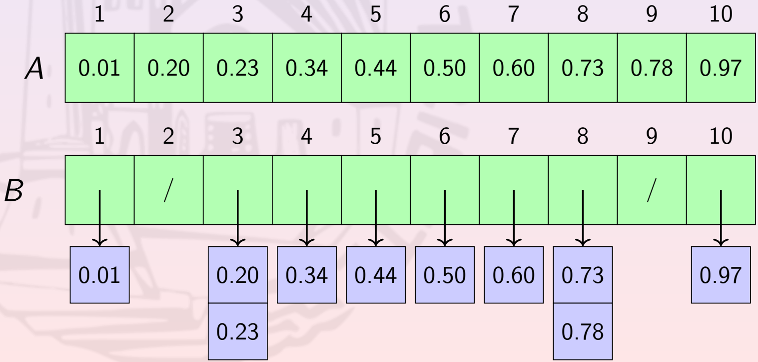

Bucket Sort: Pseudo-Code

```
def BUCKET_SORT(A):
  B ← ALLOCATE_ARRAY_OF_EMPTY_LISTS(|A|)

  for i ← 1 upto |A|:
    B[FLOOR(A[i]*n)+1].append(A[i])
  endfor # now B contains the buckets

  i ← 0
  for j ← 1 upto |B|
    for v in B[j]: # reverse the bucket in A
      A[i] ← v
      i ← i+1
    endfor

    sort(A, i−|B[j]|, |B[j]|) # sort the bucket
  endfor
enddef
```

Bucket Sort: Expected Complexity

Allocating and initializing B $\Theta(n)$

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Allocating and initializing B	$\Theta(n)$
Filling the buckets	$\Theta(n)$
Sorting each bucket (expected)	$O(n)$
Reversing buckets' content into A	$\Theta(n)$
Total expected complexy	$O(n)$

Select

Some Interesting Questions

Let A be unsorted array

How to find the value that, if A was sorted, would be in position:

- 1?

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How to find the value that, if A was sorted, would be in position:

- 1? Complexity? $\Theta(n)$
- n?

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- n? Complexity? $\Theta(n)$
- $i \in [1, n]$? Complexity? $O(n \log n)$

Can we do better?

The Select Problem

Input: a potentially unsorted array A and an index $i \in [1, |A|]$
Output: the value $\bar{A}[i]$ where $\bar{A}$ is the sorted version of A

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We will assume that A does not contains multiple instances of the same value (not necessary, but simplify things)

A Possible Strategy

What about using PARTITION and a “dichotomic approach”?

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
13	5	7	2	-7	4	1	11	6	0	10	15	-1	8	-2	3	14	12	-9	-5	9	-6	16	-4

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What about using PARTITION and a "dichotomic approach"?

- select a pivot $A[j]$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
13	$\boxed{5}$	7	2	-7	4	1	11	6	0	10	15	-1	8	-2	3	14	12	-9	-5	9	-6	16	-4

j

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What about using PARTITION and a "dichotomic approach"?

- select a pivot $A[j]$
- compute k ←PARTITION(A,1,|A|,j) and get S and G

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
-9	3	-1	4	-6	-4	-5	-2	0	1	2	-7	$\boxed{5}$	11	6	7	8	15	10	9	14	12	13	16

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What about using PARTITION and a “dichotomic approach”?

- select a pivot $A[j]$
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- compare i and k

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
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A recursive algorithm can solve the problem!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
-9	3	-1	4	-6	-4	-5	-2	0	1	2	-7	$\boxed{5}$	11	6	7	8	15	10	9	14	12	13	16

$\underbrace{\hspace{10em}}_{S}$ k $\underbrace{\hspace{10em}}_{G}$

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However, constant ratio partitions are QUICKSORT's best case scenarios as well

Is there a smart way to guess an almost-median value for $\bar{A}$?

Choosing the Pivot

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Choosing the Pivot

- split A in $\lceil n/5 \rceil$ chunks $C_1, \ldots, C_{n/5}$ each of size 5

$\mathbf{C_1}$					$\mathbf{C_2}$					$\mathbf{C_3}$					$\mathbf{C_4}$					$\mathbf{C_5}$			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
13	5	7	2	-7	4	1	11	6	0	10	15	-1	8	-2	3	14	12	-9	-5	9	-6	16	-4

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- find the median m_i of C_i

C_1					C_2					C_3					C_4					C_5			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
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- split A in $\lceil n/5 \rceil$ chunks $C_1, \ldots, C_{n/5}$ each of size 5
- find the median m_i of C_i, e.g., by sorting C_i itself
- recursively compute the median m of the m_i's

		$\mathbf{C_1}$					$\mathbf{C_2}$					$\mathbf{C_3}$					$\mathbf{C_4}$					$\mathbf{C_5}$	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
-7	2	5	7	13	0	1	4	6	11	-2	-1	8	10	15	-9	-5	3	12	14	-6	-4	9	16
		m																					

Does the Selected Pivot Partition A Evenly Enough?

Think the chunks as they were the columns of a matrix

		$\mathbf{C_1}$					$\mathbf{C_2}$					$\mathbf{C_3}$					$\mathbf{C_4}$					$\mathbf{C_5}$	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
-7	2	5	7	13	0	1	4	6	11	-2	-1	8	10	15	-9	-5	3	12	14	-6	-4	9	16
		m																					

Does the Selected Pivot Partition *A* Evenly Enough?

Think the chunks as they were the columns of a matrix

C_1	C_2	C_3	C_4	C_5
-7	0	-2	-9	-6
2	1	-1	-5	-4
5	4	8	3	9
7	6	10	12	16
13	11	15	14	

Does the Selected Pivot Partition A Evenly Enough?

Sort the chunks according the medians

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition *A* Evenly Enough?

How many chunks are there?

$$\left\lceil \frac{n}{5} \right\rceil$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition A Evenly Enough?

How many m_i are greater or equal to m?

$$\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition A Evenly Enough?

How many chunks at least have 3 elements greater than m?

$$\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition A Evenly Enough?

How many elements at least are greater than m?

$$3\left(\left\lceil\frac{1}{2}\left\lceil\frac{n}{5}\right\rceil\right\rceil-2\right)$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition A Evenly Enough?

How many elements at least are greater than m?

$$3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5}\right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Does the Selected Pivot Partition A Evenly Enough?

An upper bound for the # of elements smaller or equal to m is

$$n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$$

C_4	C_2	C_1	C_3	C_5
-9	0	-4	-2	-6
-5	1	2	-1	-4
3	4	5	8	9
12	6	7	10	16
14	11	13	15	

Complexity of the Select Algorithm (Substitution Method)

$$T_S(n) = T_S(\lceil n/5 \rceil) + T_S(7n/10 + 6) + \Theta(n)$$

Prove by induction that $T_S(n) \in O(n)$ (Substitution Method)

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$$\begin{aligned} T_S(n) &\leq c\lceil n/5 \rceil + c(7n/10 + 6) + c'n \\ &\leq c(n/5 + 1) + c(7n/10 + 6) + c'n \\ &\leq 9/10cn + c'n + 7c \end{aligned}$$

Hence, $T_S(n) \leq cn$ for $c \geq 20c'$ and $n \geq 140$ and

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Hence, $T_S(n) \leq cn$ for $c \geq 20c'$ and $n \geq 140$ and $T_S(n) \in O(n)$

Select Algorithm: Pseudo-Code

```
def SELECT(A, l=1, r=|A|, i):
  if r−l≤10:          # base case
    SORT(A, l, r)
    return i
  endif

  j ← SELECT_PIVOT(A, l, r)
  k ← PARTITION(A, l, r, j)

  if i=k:              # dichotomic approach
    return k
  endif

  if i<k:              # search in S
    return SELECT(A, l, k−1, i)
  endif

  # search in G
  return SELECT(A, k+1, r, i)
enddef
```

Select Pivot Algorithm: Pseudo-Code

```
def SELECT_PIVOT(A, l=1, r=|A|):
  if r−l≤10:                          # base case
    SORT(A, l, r)
    return (l+r)/2
  endif

  chunks ← (r−l)/5
  for c in 0...chunks−1:              # for each chunk
    (c_l, c_r) ← (1, 5)+c∗5

    SORT(A, c_l, c_r)                 # sort it
    SWAP(A, c_l+2, l+c)               # place the middle elem
                                      # at the beginning of A
  endfor

  # recursive step
  return SELECT(A, l, l+chunks−1, chunks/2)
enddef
```