Sorting (1): Homework Solutions

Emanuele Ballarin

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Exercises 1, 2

Text:
By using the code at:

`https://github.com/albertocasagrande/AD_sorting`

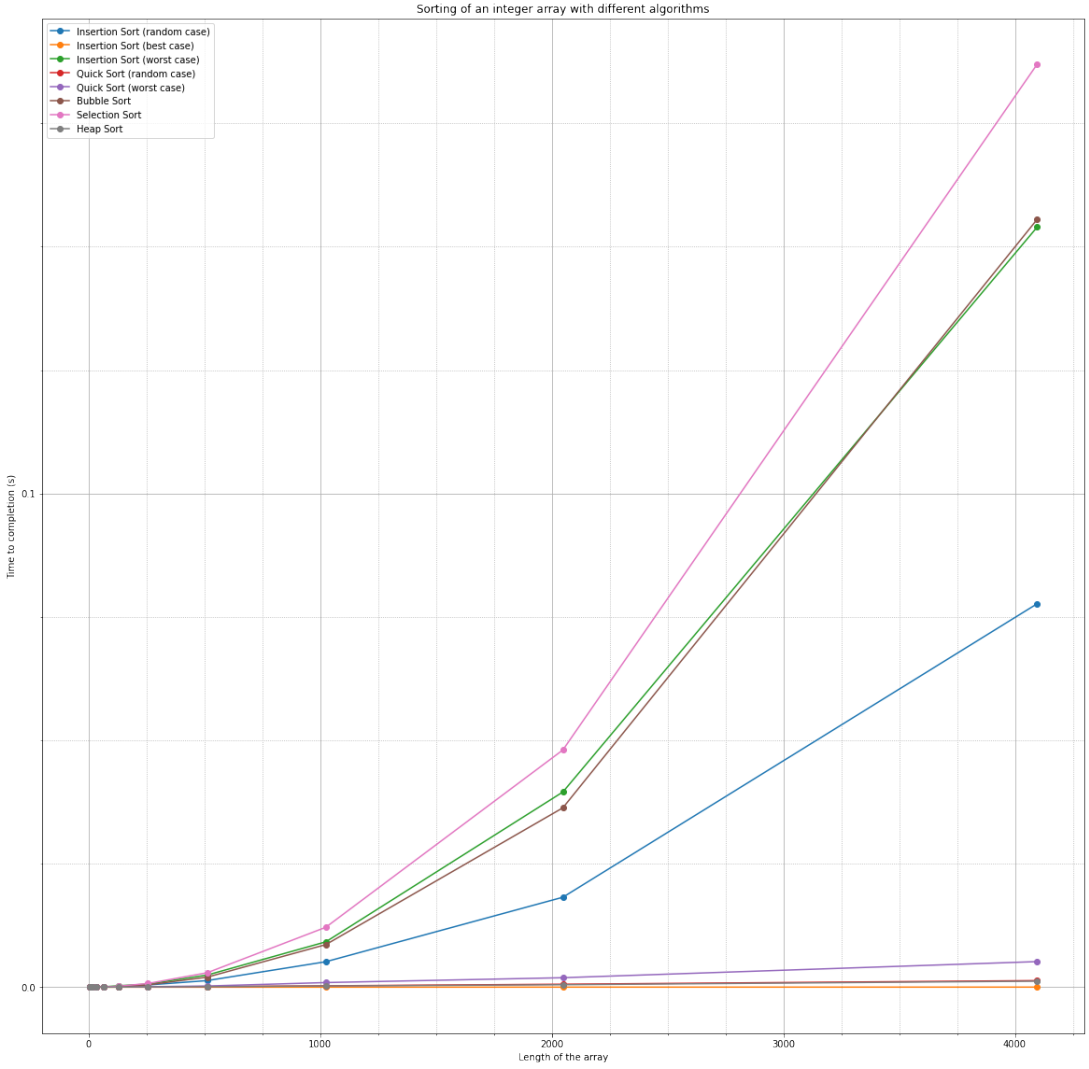
implement `INSERTION SORT`*,* `QUICK SORT`*,* `BUBBLE SORT`*,* `SELECTION SORT`*, and* `HEAP SORT`*.*
For each of the implemented algorithm, draw a curve to represent the relation between the input size and the execution-time.

Solution:
All the requested sorting algorithms have been implemented by faithfully following their description contained in the lecture materials. Most notably, original *Hoare's* partitioning has been used in `QUICK_SORT` instead of other popular variants (namely Lomuto's and Sedgewick's).
Additionally, the *random pivoting* heuristic has been adopted for pivot selection in `QUICK_SORT` as a way to improve non-asymptotic regime execution times.

Benchmarks:

As it is possible to see from the graph, the behavior of the different sorting algorithms is in line with the theoretical predictions obtain from asymptotic complexity analysis, with the only notable exception of *selection sort* being the slowest. Though formally $O(n^2)$ in its best, average and worst cases, in fact *selection sort* involves less swaps w.r.t. other typically $O(n^2)$ approaches like *bubble sort*. This effect may be due to a still non-asymptotic regime for *bubble sort* and *selection sort* or due to some uneven application of compiler optimizations between the two.

Another interesting point is noticing the almost-equivalent behavior of *heap sort* and *quicksort (random case)*. The former, however, offers tighter worst-case bounds.


Sorting of an integer array with different algorithms
Insertion Sort (random case)
Insertion Sort (best case)
Insertion Sort (worst case)
Quick Sort (random case)
Quick Sort (worst case)
Bubble Sort
Selection Sort
Heap Sort
Time to completion (s)
0.1
0.0
0
1000
2000
3000
4000
Length of the array

Exercise 3

Text:
Argue about the following statements and answer the questions:

(a) `HEAP SORT` **on a array** A **whose length is** n **takes time** $O(n)$.

In the general case, the complexity of `HEAP SORT` for an n-element array is determined by the contribution of the `build_heap` function operating once on the unsorted array ($O(n)$) and the repeated calls (n) to the `extract_min` ($O(log(n))$ each), amounting to a total of $O(n) + O(n\ log(n)) \sim O(n\ log(n))$. This holds true for both the (general) *best* and *worst* case scenarios. Thus, the statement is not true in general.

If, however, the statement must be true, it can be the consequence of the (specific) case in which the array is composed of all equal elements, thus reducing the complexity of `extract_min` to $\Theta(1)$ per call. The overall resulting time-complexity in that case amounts to $\Theta(n)$.

(b) `HEAP SORT` **on a array** A **whose length is** n **takes time** $\Omega(n)$.

As better explained in previous answer, $\Theta(n)$ represents the complexity in the specific best case for the `HEAP SORT` algorithm, making this statement true.

(c) **What is the worst case complexity for** `HEAP SORT`**?**

The worst-case time-complexity for `HEAP SORT` is $O(n\ log(n))$. The bound is saturated in the case the maximum number of swaps is performed at each call to the `extract_min` function. This generally tight upper bound (compared to the more common $O(n^2)$) makes `HEAP SORT` a worst-case-robust sorting algorithm for many time-bound scenarios.

(d) `QUICK SORT` **on a array** A **whose length is** n **takes time** $O(n^3)$.

The worst-case time-complexity of `QUICK SORT` is $O(n^2)$. The statement is true, though it provides a very large upper bound. Practical usefulness of this bound is debatable.

(e) **What is the complexity of** `QUICK SORT`**?**

The best-case time-complexity of `QUICK SORT` (balanced partitioning at every step) is $O(n \log(n))$. This is also the average-case time-complexity, assuming that the distribution of the elements in the array accepts with equal probability each permutation. The worst-case time-complexity (maximally-imbalanced partitioning: e.g. already-sorted array and leftmost element as pivot) is instead $O(n^2)$.

(f) `BUBBLE SORT` **on a array** A **whose length is** n **takes time** $\Omega(n)$.

The statement is true. Indeed such complexity is reached in the case of a pre-sorted array. The algorithm just needs to linearly-scan the array, verifying that it is sorted.

(g) **What is the complexity of** `BUBBLE SORT`**?**

As explained in previous answer, the best-case complexity of `BUBBLE SORT` is $\Theta(n)$. Average and worst case complexities equal to $O(n^2)$ due to the presence of nested looping over the entire array.

Exercise 4

Solve the following recursive equation:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 32 \\ 3T(\frac{n}{4}) + \Theta(n^{3/2}) & \text{otherwise} \end{cases}$$

Starting from the recursive equation proposed, we can build a recursion tree to ease its resolution. In this case, $cn^{3/2}$ for a given $c > 0$ fixed, has been chosen as a representative of the complexity class $\Theta(n^{3/2})$.

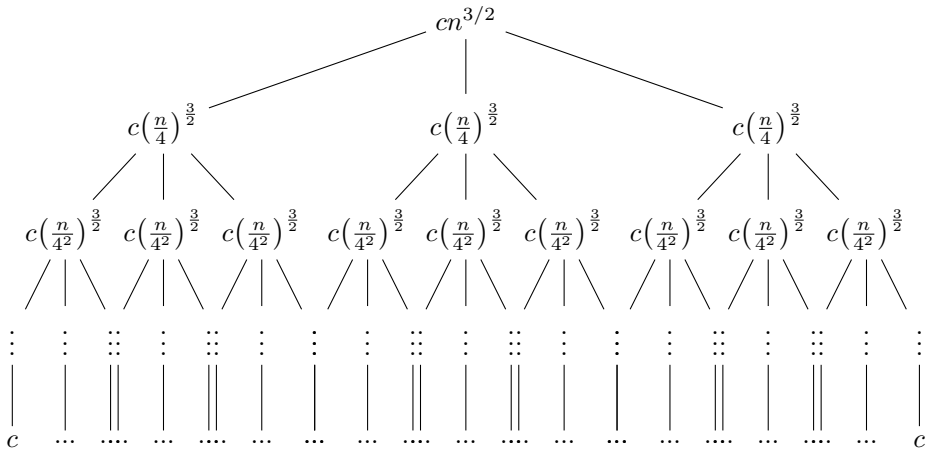


Figure 1: Recursion tree for the recursive equation above

As we can see (or read from the equation), by calling i the level of a node (starting from the root at $i = 0$), the complexity of each node in the tree is determined by $c(\frac{n}{4^i})^{3/2}$, whereas the number of nodes per level is 3^i.

The two expressions multiplied give the total cost per level, which is $c\ (\frac{3}{8})^i\ n^{3/2}$.

The last level of recursion is that, $\hat{i}$, such that $\left(\frac{n}{4^{\hat{i}}}\right) = 32 \rightarrow \hat{i} = \frac{log_2(n)-5}{2}$.
The total cost for such last level of the tree is obtained by substituting the $\hat{i}$ just found in the equations shown above, giving $\Theta(3^{-\frac{5}{2}}\ n^{\frac{log_2(3)}{2}}) \sim \Theta(n^{\frac{log_2(3)}{2}})$.
To conclude, we compute:

$$\sum_{i=0}^{\frac{log_2(3)-5}{2}} (3/8)^i\ cn^{3/2}+\Theta(n^{\frac{log_2(3)}{2}}) < \sum_{i=0}^{+\infty}(3/8)^i\ cn^{3/2}+\Theta(n^{\frac{log_2(3)}{2}}) = (8/5)\ cn^{3/2}+\Theta(n^{\frac{log_2(3)}{2}}) \sim O(n^{3/2}).$$