Weighted Graphs and Algorithms

Advanced Programming and Algorithmic Design

Alberto Casagrande
Email: `acasagrande@units.it`

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Weighted Graphs (Graph Theory)

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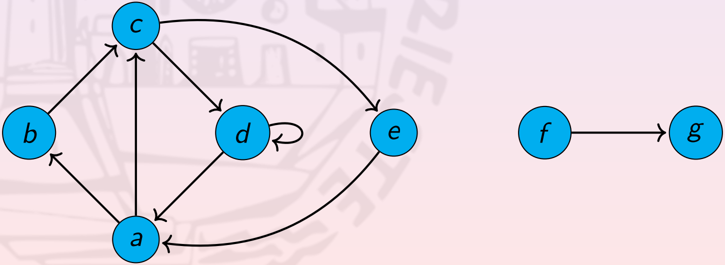
(V, E) is an (un)directed graph



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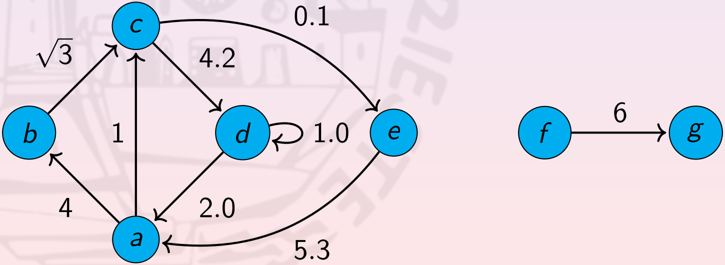

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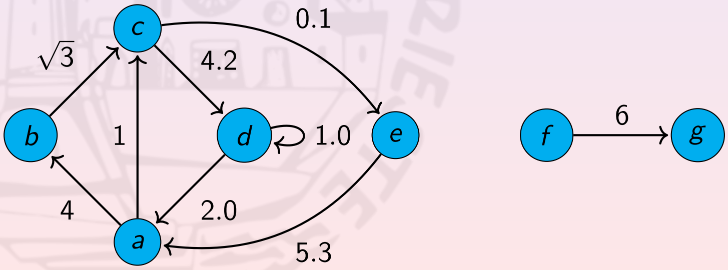
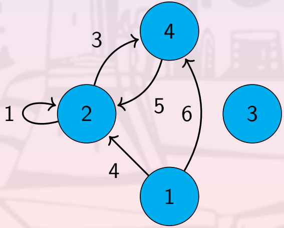
W is a function mapping edges into weights

The length of a path is the sum of all its edge labels



Representing Graphs

Two main ways:



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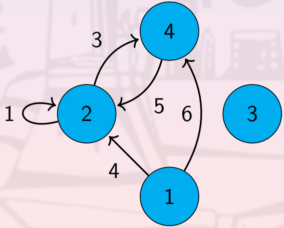

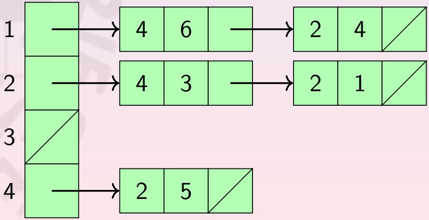

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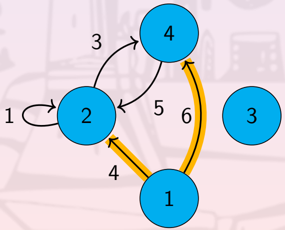

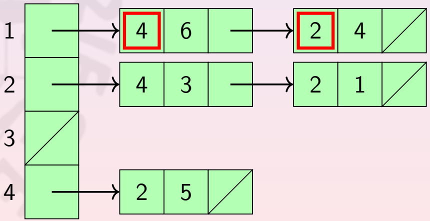

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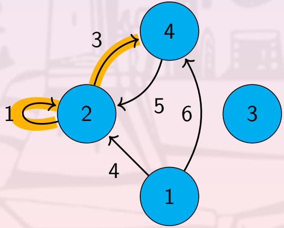

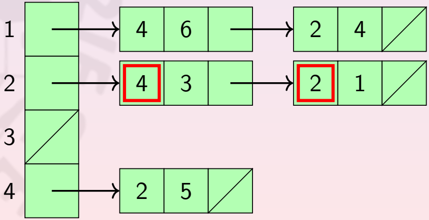

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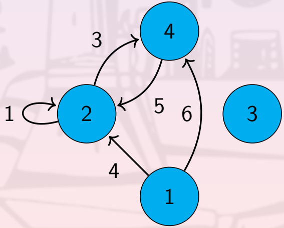

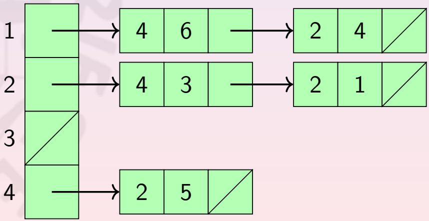

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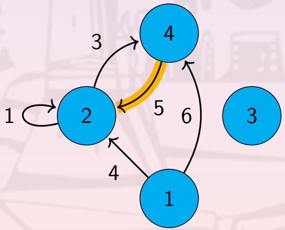

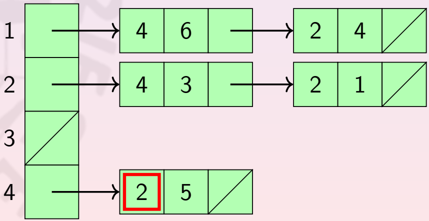

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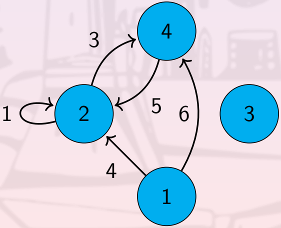

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1	N	4	N	6
2	N	1	N	3
3	N	N	N	N
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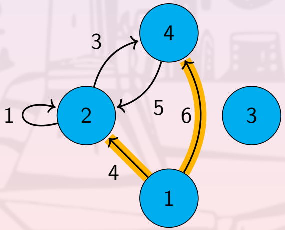

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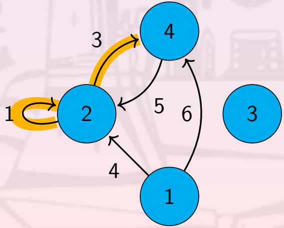

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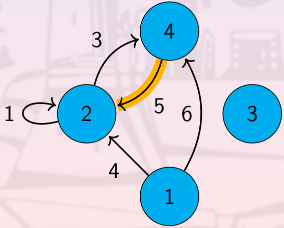

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Single Source Shortest Paths

Problem Definition and A Possible Strategy

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How to solve this problem?

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Let us have a look to BFS and try to adapt to SSSP

BFS Main Ingredients

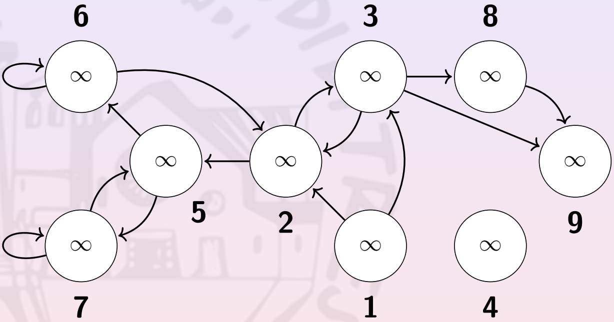
Nodes are WHITE, GRAY, or BLACK colored

- WHITE nodes have not been discovered yet
- GRAY nodes have been discovered, but some of their neighbors have not
- BLACK nodes have been discovered and all their neighbors have been discovered too

The search exclusively evolves from GRAY nodes

Their processing order is handled by a FIFO queue

BFS: A Working Example



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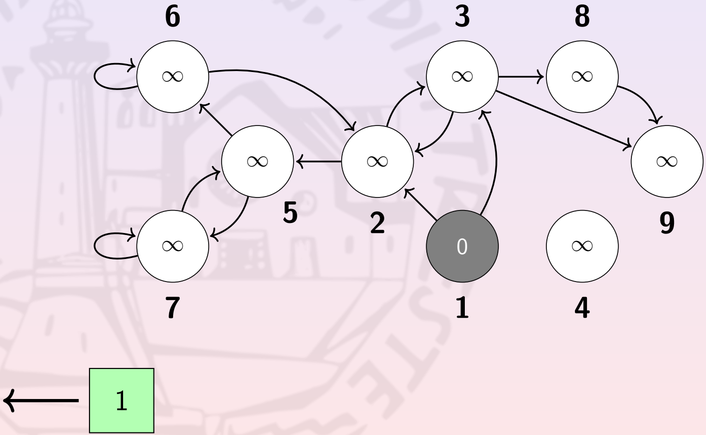

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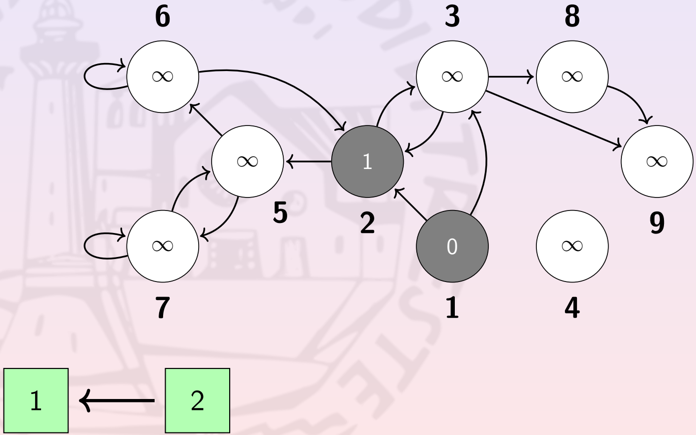

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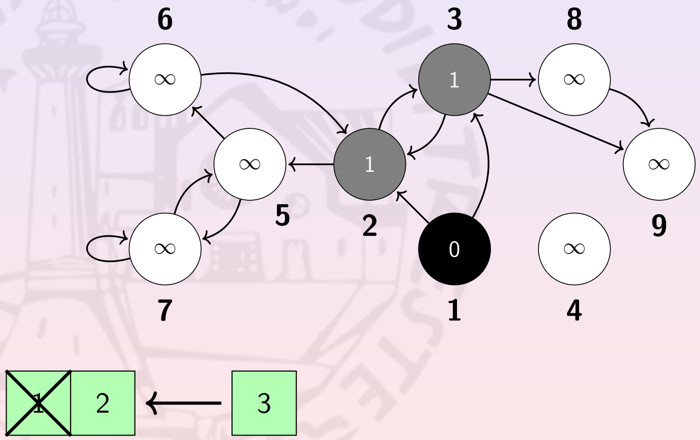

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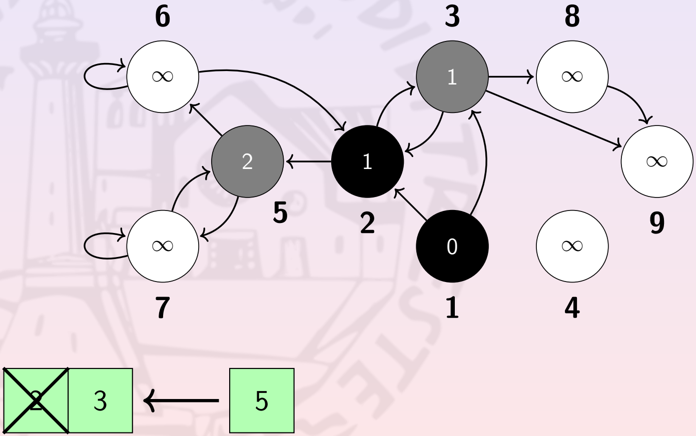

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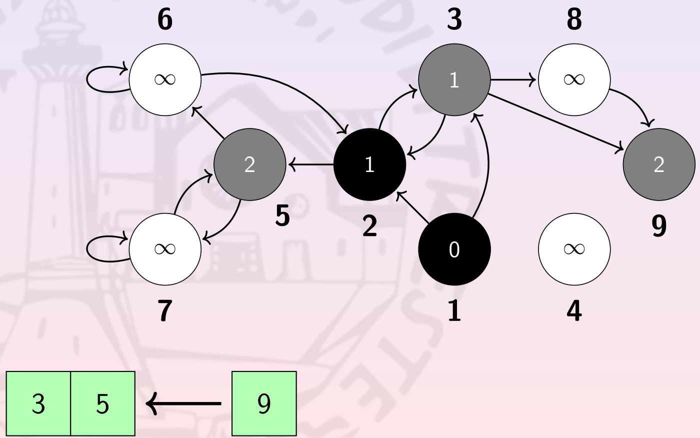

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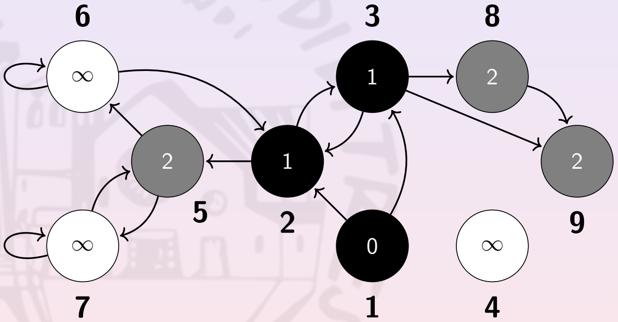

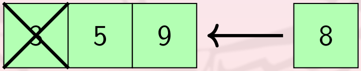

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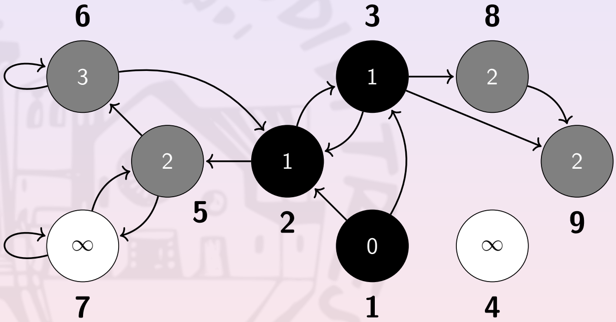

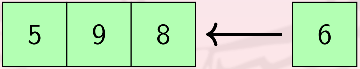

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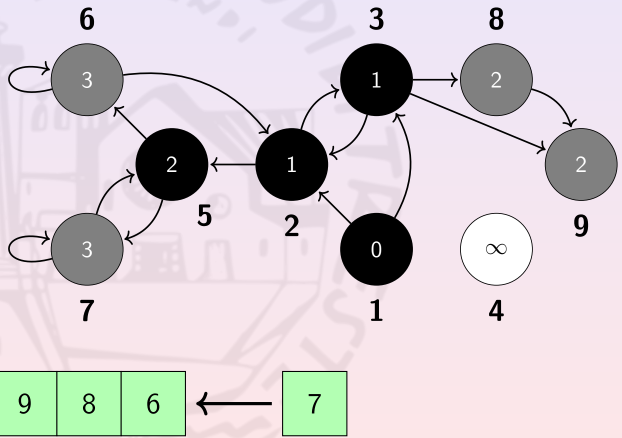

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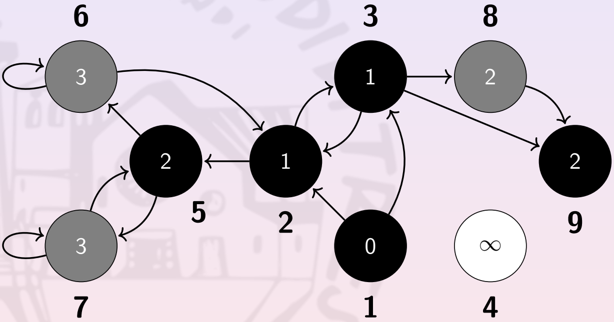

~~9~~	8	6	7

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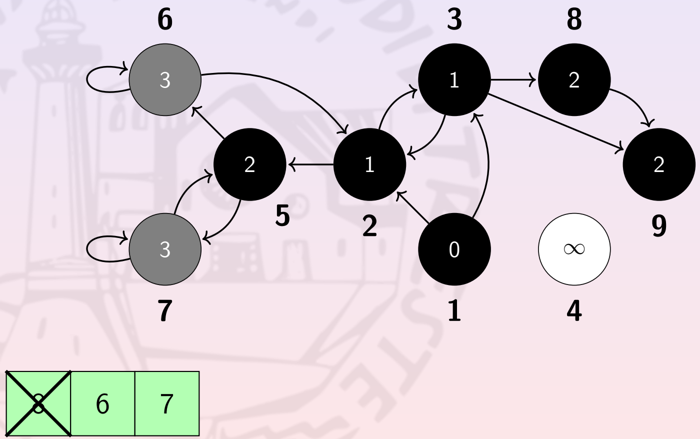

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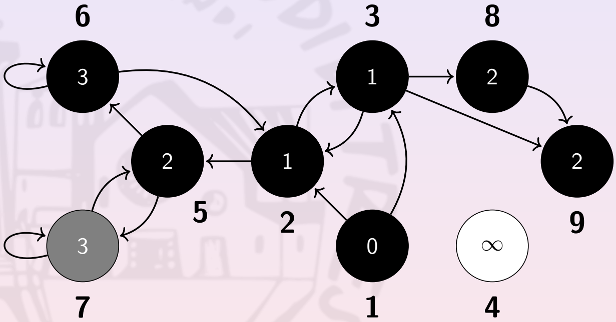

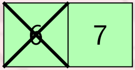

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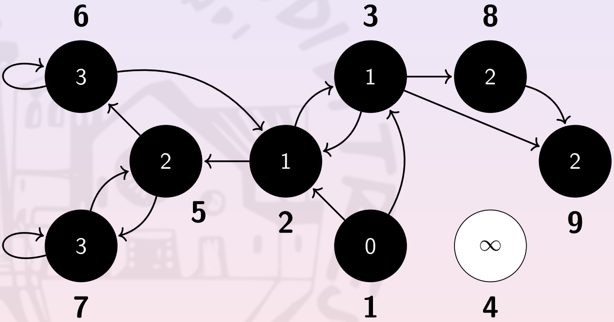

BFS: Pseudo-Code

```
def BFS_SET(v, color, d, pred):
    v.color ← color
    v.d ← d
    v.pred ← pred


def BFS_INIT(G,s):
    for v in G.V:
        BFS_SET(v, WHITE, ∞, NIL)
    endfor
    BFS_SET(s, GRAY, 0, s)

    return BUILD_QUEUE([s])
```

BFS: Pseudo-Code (Cont'd)

```
def BFS(G, s):
  Q ← BFS_INIT(G, s)
  while Q ≠ ∅:
    u ← DEQUEUE(Q)

    for v in G.Adj[u]:
      if v.color = WHITE:
        BFS_SET(v, GRAY, u.d+1, u)
        ENQUEUE(Q, v)
      endif
    endfor
    u.color ← BLACK
  endwhile
enddef
```

Upgrading BFS to Deal With Weights

BFS sets v's distance to $u.d + 1$ where u is queue head

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What if v's distance is set to $u.d + W[(u, v)]$?

It does NOT work!

Why Does BFS Work Properly?

BFS correctly sets v's distance because...

Lemma

Let $Q = [u_1, \ldots, u_n]$ be the queue during BFS. Then $u_{i-1}.d \leq u_i.d$ for all $i \in [2, n]$ and $u_n.d \leq u_1.d + 1$.

So, $u_1.d + 1 \leq u_i.d + 1$ for all $i \in [2, n]$

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So, $u_1.d + 1 \leq u_i.d + 1$ for all $i \in [2, n]$

If v is a successor of u_1, any other path reaching v through a node in Q is longer than $u_1.d + 1$

Why Does BFS Not Work on Weighted Graphs?

Even if $u_{i-1}.d \leq u_i.d$ for all $i \in [2, n]$, there may be $(u_k, \bar{v})$ s.t.

$$u_1.d + W[(u_1, v)] > u_k.d + W[(u_k, \bar{v})]$$

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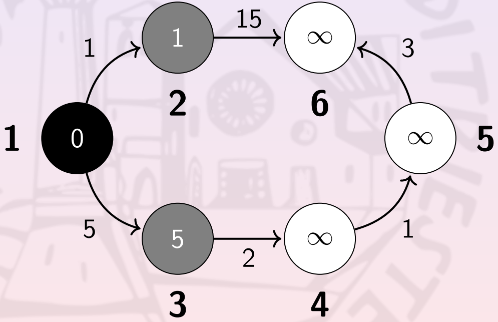

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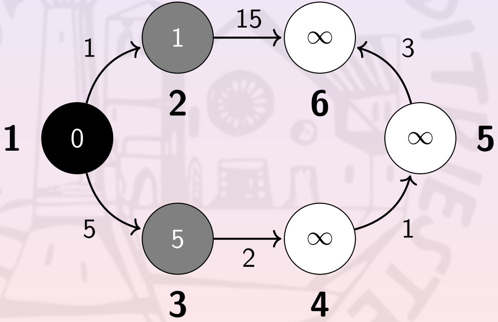

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$2.d + W[(2, 6)]$ $3.d + W[(3, 4)]$

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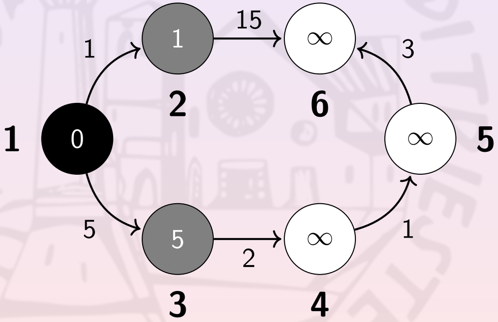

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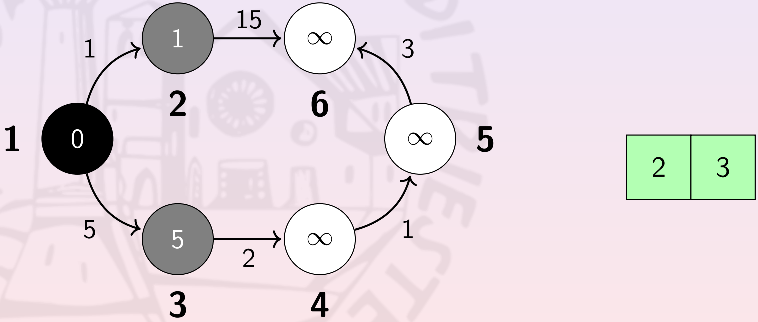

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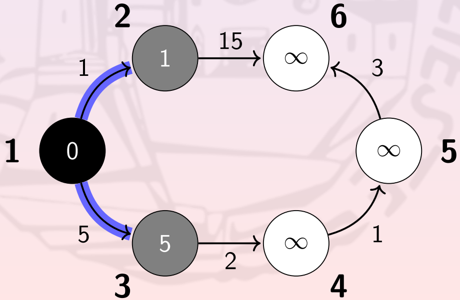

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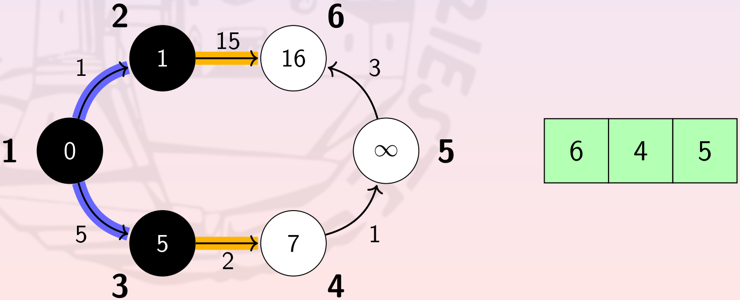

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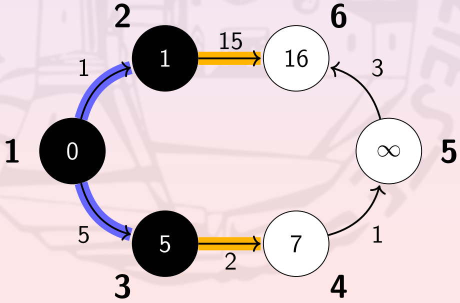

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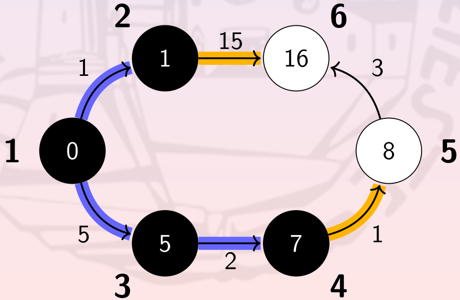

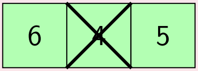

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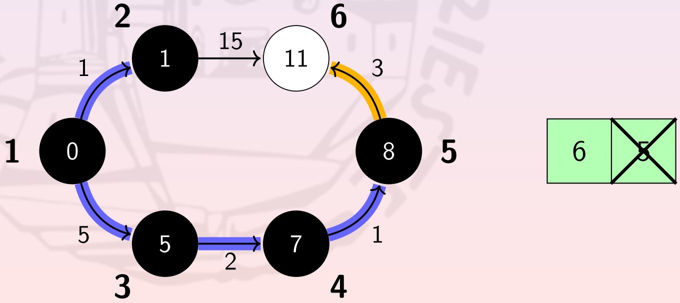

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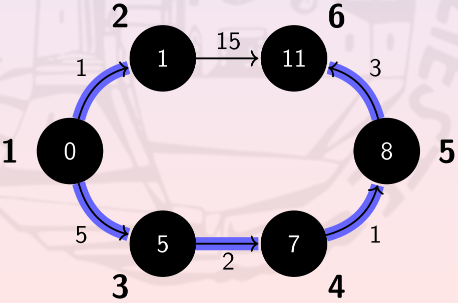

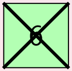

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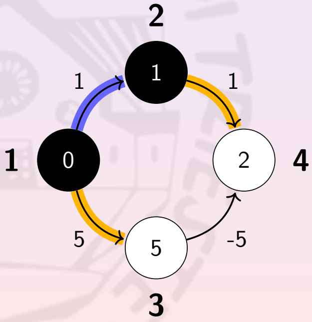
It is a kind of meta-algorithm: it does not specify how to handle the queue

Some Observations on Dijkstra's Algorithm (Cont'd)

No negative weights: if they were allowed, the minimal path to the node extracted from Q could be not discovered



Dijkstra's Algorithm: Pseudo-Code

```
def INIT_SSSP(G):
  for v in G.V:
    v.d ← ∞
    v.pred ← NIL
  endfor
enddef

def RELAX(Q, u, v, w):
  if u.d + w < v.d:
    UPDATE_DISTANCE(Q, v, u.d + w)
    v.pred ← u
  endif
enddef
```

Dijkstra's Algorithm: Pseudo-Code (Cont'd)

```
def DIJKSTRA(G, s):
  INIT_SSSP(G, s)
  s.d ← 0

  Q ← BUILD_QUEUE(G.V)
  while not IS_EMPTY(Q):
    u ← EXCTRACT_MIN(Q)

    for (v, w) in G.Adj[u]:
      RELAX(Q, u, v, w)
    endfor
  endwhile
enddef
```

Dijkstra's Algorithm: Complexity

All the nodes are in Q at the beginning of the computation

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The overall complexity of Dijkstra's algorithm is

$$T_D(G) = \Theta(|V|) + T_B(|V|) + |V| * T_E(|V|) + |E| * T_U(|V|)$$

where T_B, T_E, and T_U are the complexities of `BUILD_QUEUE`, `EXCTRACT_MIN`, `UPDATE_DISTANCE`

Dijkstra's Algorithm: Complexity (Cont'd)

Queue Data Structure	$\mathbf{T_B(n)}$	$\mathbf{T_E(n)}$	$\mathbf{T_U(n)}$	$\mathbf{T_D(G)}$
Arrays	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(\|V\|^2 + \|E\|)$

Dijkstra's Algorithm: Complexity (Cont'd)

Queue Data Structure	$T_B(n)$	$T_E(n)$	$T_U(n)$	$T_D(G)$
Arrays	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(\lvert V\rvert^2 + \lvert E\rvert)$
Binary Heaps	$\Theta(n)$	$O(\log n)$	$O(\log n)$	$O((\lvert V\rvert + \lvert E\rvert) * \log \lvert V\rvert)$

Dijkstra's Algorithm: Complexity (Cont'd)

Queue Data Structure	$T_B(n)$	$T_E(n)$	$T_U(n)$	$T_D(G)$
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Binary Heaps	$\Theta(n)$	$O(\log n)$	$O(\log n)$	$O((\lvert V\rvert + \lvert E\rvert) * \log \lvert V\rvert)$
Fibonacci Heaps[1]	$\Theta(n)$	$O(\log n)$	$\Theta(1)$	$O(\lvert E\rvert + \lvert V\rvert * \log \lvert V\rvert)$

[1] Amortized time

All Pairs Shortest Paths

Problem Definition and Possible Strategies

We want to compute the shortest paths between all the pairs of nodes

How to solve this problem?

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Run Dijkstra's algorithm and use each node as source

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Run Dijkstra's algorithm and use each node as source

No negative edges

Some Revealing Insights

Let us consider graphs whose nodes are natural numbers

Let $p = e_1, \ldots, e_h$ be the shortest path from i to j

Let k be the "greatest" internal node in the path

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So, $e_1, \ldots, e_{\bar{h}-1}$ and $e_{\bar{h}}, \ldots, e_h$ are shortest paths between i and k and between k and j

Their interal nodes are "smaller" than k

Floyd-Warshall's Algorithm: Intuition

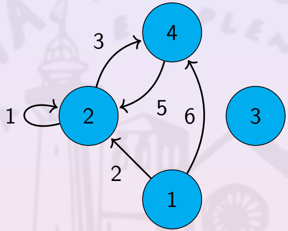
We can incrementally build shortest paths by admitting new untouched internal nodes at each step

If $D^{(k-1)}$ contains the lengths of the shortest paths whose internal nodes are smaller than k, we can compute $D^{(k)}$ as

$$D^{(k)}[i,j] = \min(D^{(k-1)}[i,k] + D^{(k-1)}[k,j], D^{(k-1)}[i,j])$$

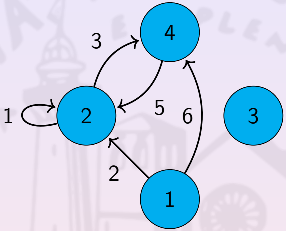
The same criterium applies to $\Pi^{(k)}$ where $\Pi^{(k)}[i,j]$ is the predecessor of j in the smallest path between i and j

Floyd-Warshall's Algorithm: A Working Example



	1	2	3	4
1	N	2	N	6
2	N	1	N	3
3	N	N	N	N
4	N	5	N	N

Floyd-Warshall's Algorithm: A Working Example

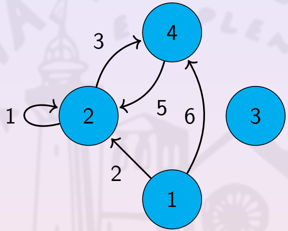


$$D^{(0)} = \begin{pmatrix} 0 & 2 & \infty & 6 \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

	1	2	3	4
1	N	2	N	6
2	N	1	N	3
3	N	N	N	N
4	N	5	N	N

$$\Pi^{(0)} = \begin{pmatrix} \mathrm{NIL} & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \end{pmatrix}$$

Floyd-Warshall's Algorithm: A Working Example



$$D^{(0)} = \begin{pmatrix} 0 & 2 & \infty & 6 \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

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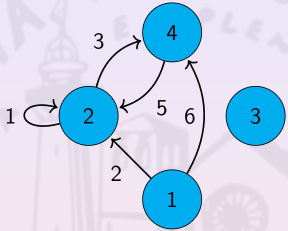

$$D^{(1)} = \begin{pmatrix} 0 & 2 & \infty & 6 \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

	1	2	3	4
1	N	2	N	6
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3	N	N	N	N
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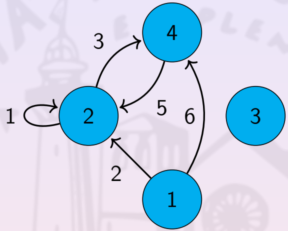
$$\Pi^{(1)} = \begin{pmatrix} \mathrm{NIL} & 1 & \mathrm{NIL} & 1 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \end{pmatrix}$$

Floyd-Warshall's Algorithm: A Working Example



$$D^{(2)} = \begin{pmatrix} 0 & 2 & \infty & \color{red}{5} \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

	1	2	3	4
1	N	2	N	6
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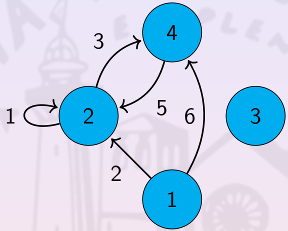
$$\Pi^{(2)} = \begin{pmatrix} \mathrm{NIL} & 1 & \mathrm{NIL} & \color{red}{2} \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \end{pmatrix}$$

Floyd-Warshall's Algorithm: A Working Example



$$D^{(3)} = \begin{pmatrix} 0 & 2 & \infty & 5 \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

	1	2	3	4
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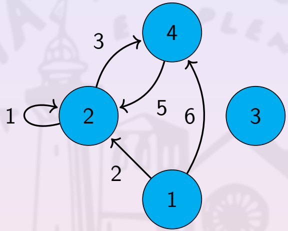
$$\Pi^{(3)} = \begin{pmatrix} \mathrm{NIL} & 1 & \mathrm{NIL} & 2 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & 2 \\ \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} & \mathrm{NIL} \\ \mathrm{NIL} & 4 & \mathrm{NIL} & \mathrm{NIL} \end{pmatrix}$$

Floyd-Warshall's Algorithm: A Working Example



$$D^{(4)} = \begin{pmatrix} 0 & 2 & \infty & 5 \\ \infty & 0 & \infty & 3 \\ \infty & \infty & 0 & \infty \\ \infty & 5 & \infty & 0 \end{pmatrix}$$

	1	2	3	4
1	N	2	N	6
2	N	1	N	3
3	N	N	N	N
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Floyd-Warshall's Algorithm: Pseudo-Code

```
def FLOYD_WARSHALL_STEP(old_D, old_P):
  D ← COPY_MATRIX(old_D)
  P ← COPY_MATRIX(old_P)

  for i ← 1 upto |G.V|:
    for j ← 1 upto |G.V|:
      if old_D[i][j] > old_D[i][k] + old_D[k][j]:
        D[i][j] ← old_D[i][k] + old_D[k][j]
        P[i][j] ← old_P[k][j]
      endif
    endfor
  endfor

  return (D, P)
enddef
```

Floyd-Warshall's Algorithm: Pseudo-Code

```
def FLOYD_WARSHALL(G):
  D[0] ← INIT_MATRIX_D0(G.W)
  P[0] ← INIT_MATRIX_P0(G.W)

  for k ← 1 upto |G.V|:
    D[k], P[k] ← FLOYD_WARSHALL_STEP(D[k−1],P[k−1])
  endfor

  return (D[|G.V|], P[|G.V|])
enddef
```

Routing

The Routing Problem

Given a weighted graph G,

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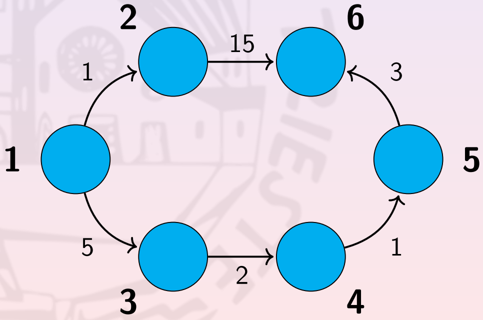
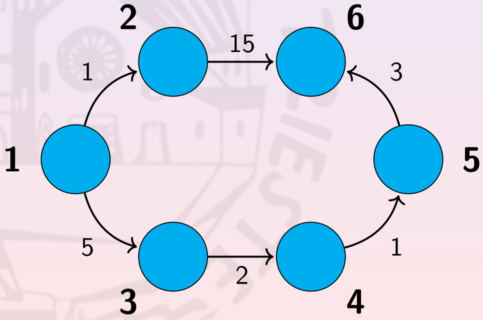


The Routing Problem

Given a weighted graph G, source s,



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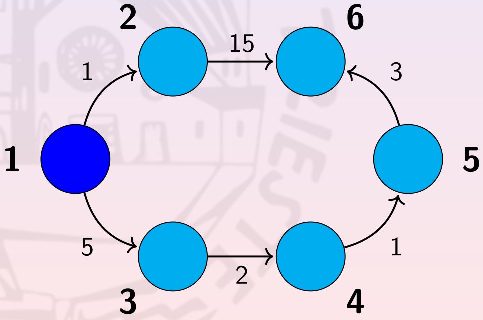
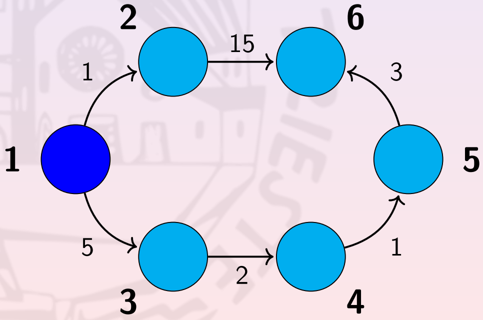


The Routing Problem

Given a weighted graph G, source s, and a destination d...



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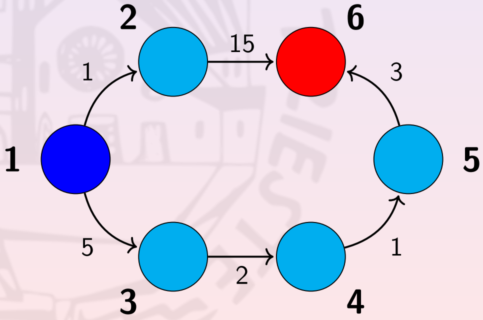

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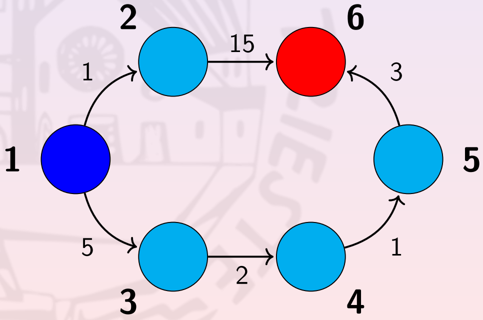
Given a weighted graph G, source s, and a destination d...

we aim for the shortest path in G from s to d



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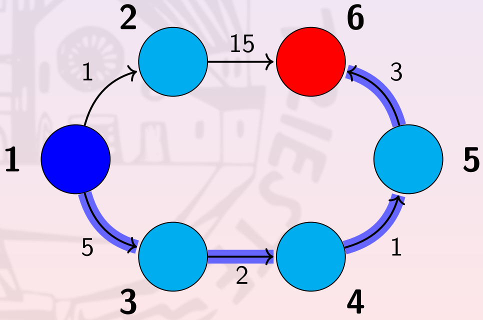


Routing By Using Dijkstra

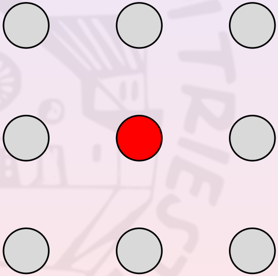
The routing problem is similar to SSSP

Let us try to use a “light” version of Dijkstra’s algorithm

The algorithm ends as soon as d has been finalized

Routing By Using Dijkstra: What Is Going On?

Let us consider a strongly structured graph. . .



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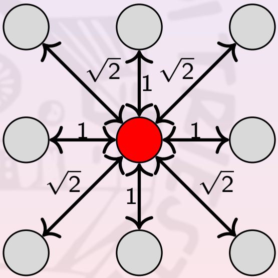

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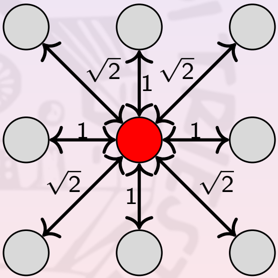

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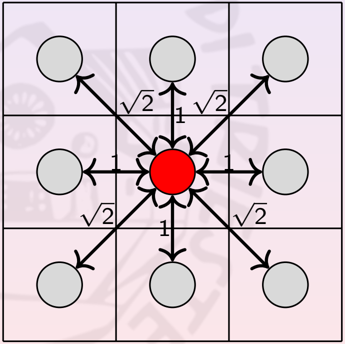

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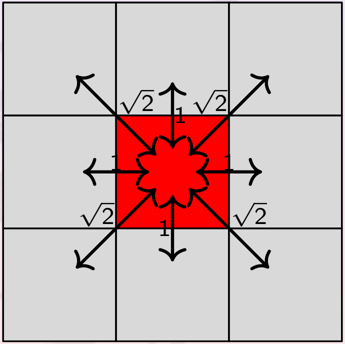
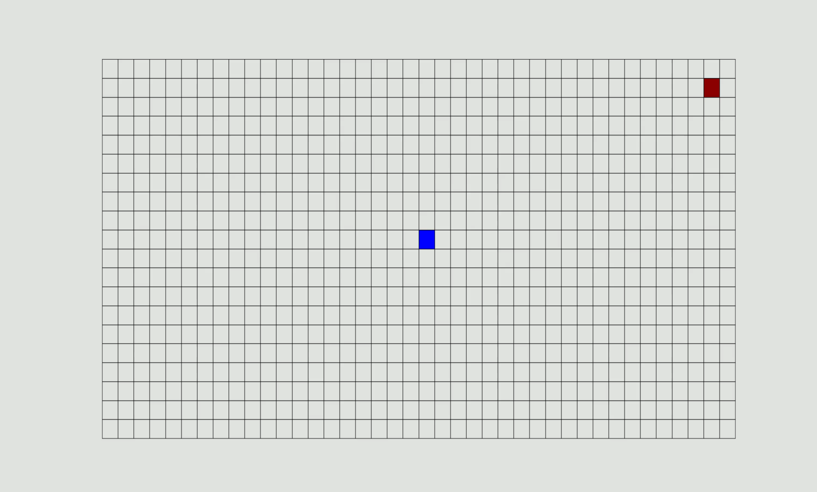


Routing By Using Dijkstra: A “Large” Example



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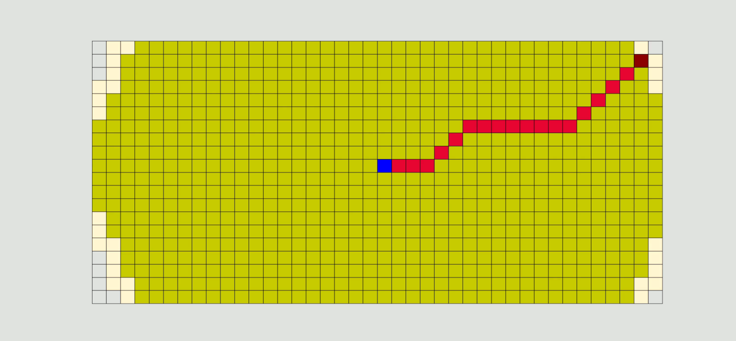

- $\|s - d\|_2 \approx 19.70$
- Queue extractions: 763
- Route length: ≈ 21.31

Anything Wrong With This Solution?

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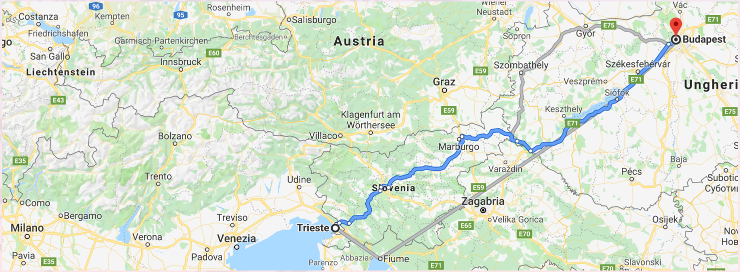

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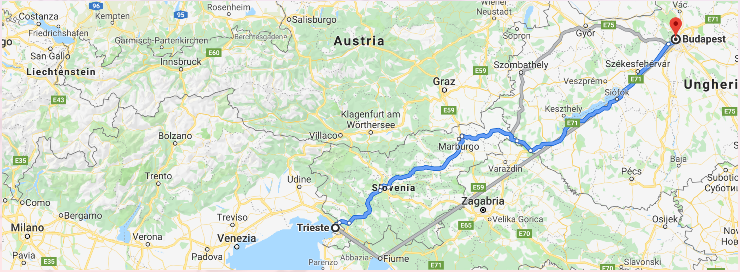

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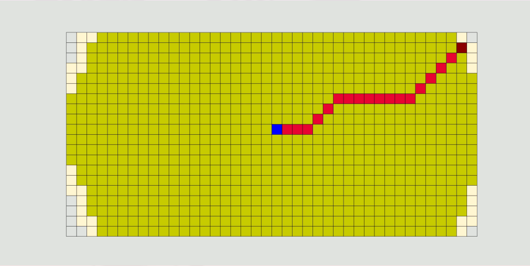

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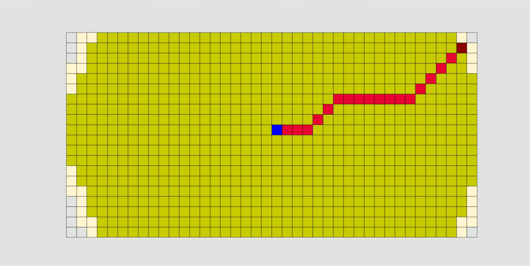

Heuristic Distance

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If the shortest path length from s to u is $u.d$, then

$$u.d + W(u, v) + h(v, d)$$

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If the shortest path length from s to u is $u.d$, then

$$u.d + W(u, v) + h(v, d)$$

estimates the length of the path between s and d

h can be any distance e.g., Euclidean, Manhattan, etc.

Estimation accuracy depends on both h and G topology

A* Algorithm: Dijkstra + Heuristic Distance

The A* algorithm has the same structure of the Dijkstra's algorithm

A* Algorithm: Dijkstra + Heuristic Distance

The A* algorithm has the same structure of the Dijkstra's algorithm, but Q is sorted according

$$u.d + W(u, v) + h(v, d)$$

where $u.d + W(u, v)$ is the guessed shortest path length to v

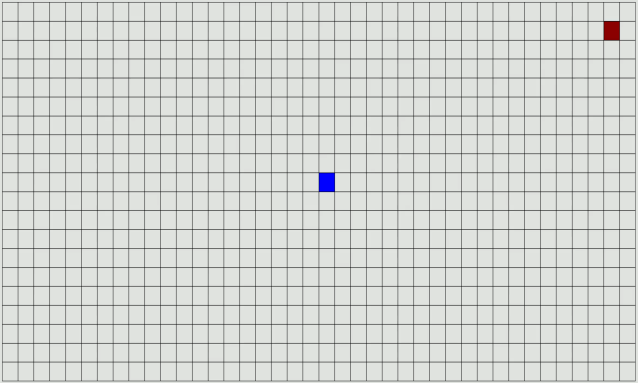
A* Algorithm: Pseudo-Code

```
def INIT_SSSP(G):
  for v in G.V:
    v.d ← ∞
    v.pred ← NIL
  endfor
enddef

def RELAX_ASTAR(Q, u, v, w, d, h):
  if u.d + w < v.d:
    UPDATE_DISTANCE(Q, v, u.d + w + h(v,d))
    v.pred ← u
  endif
enddef
```

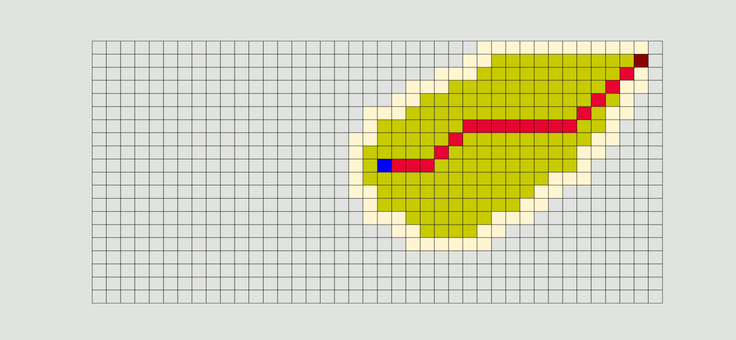
A* Algorithm: Pseudo-Code (Cont'd)

```
def ASTAR(G,s,d,h):
  INIT_SSSP(G,s)
  s.d ← h(s,d)

  Q ← BUILD_QUEUE(G.V)
  while not IS_EMPTY(Q):
    u ← EXCTRACT_MIN(Q)

    for (v, w) in G.Adj[u]:
      RELAX_ASTAR(Q,u,v,w,d,h)
    endfor
  endwhile
enddef
```

Using A* with Chebyshev Distance on Grid Example

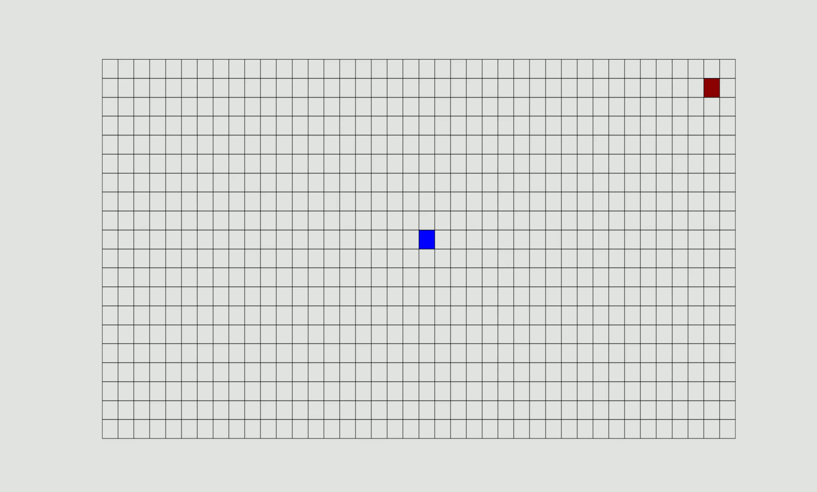


Using A* with Chebyshev Distance on Grid Example



- Queue extractions: 167 (were 763 using Dijkstra's light algorithm)
- Shortest path length: ≈ 21.31
- Route length: ≈ 21.31

Using A* with Euclidean Distance on Grid Example



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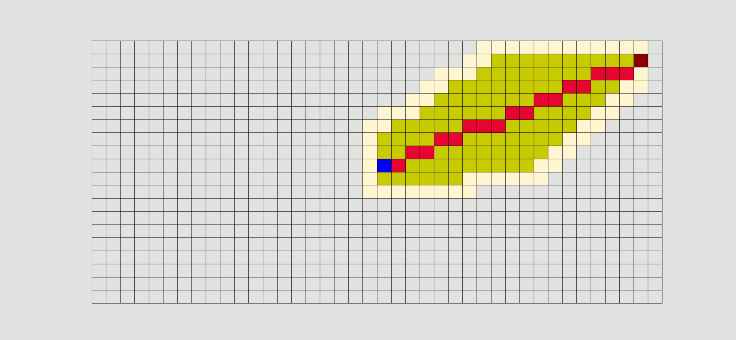
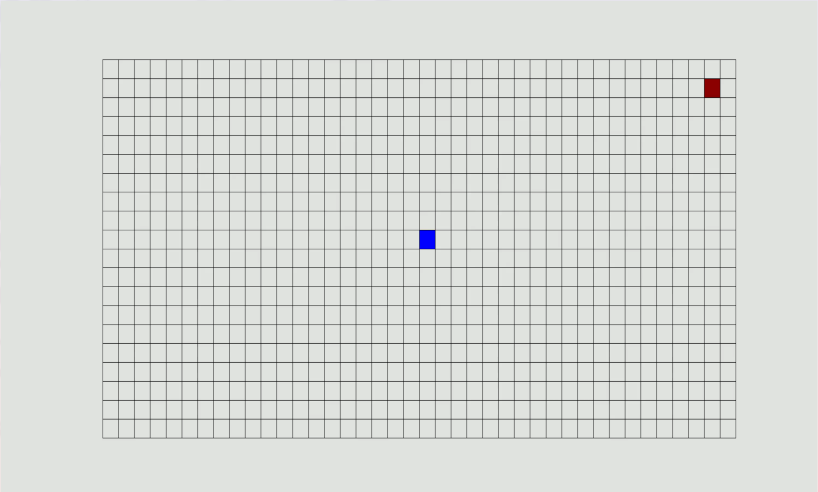


- Queue extractions: 113 (were 167 for A* with Chebyshev Distance)
- Shortest path length: ≈ 21.31
- Route length: ≈ 21.31

Using A* with Manhattan Distance on Grid Example



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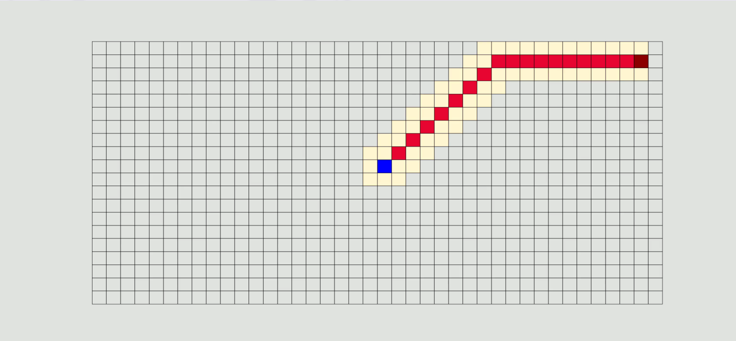

- Queue extractions: 19 (were 113 for A* with Euclidean Distance)
- Shortest path length: ≈ 21.31
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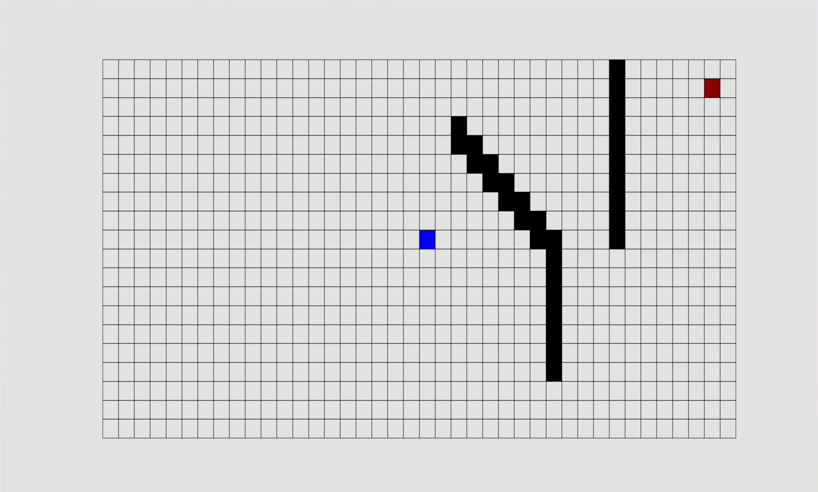
Dijkstra's Algorithm on a Different Grid Example



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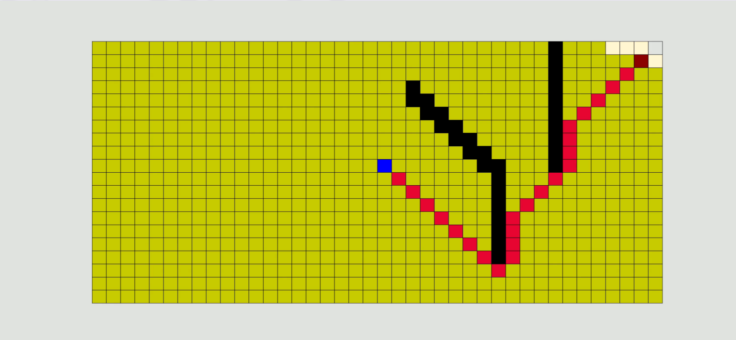
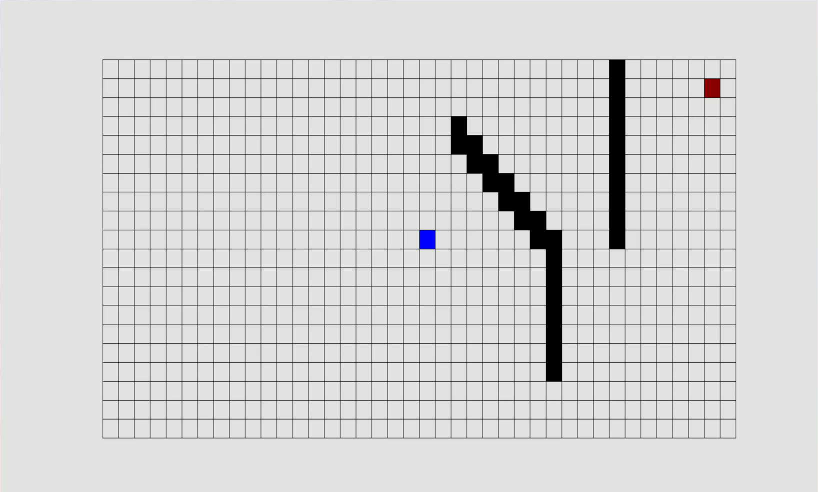


- Queue extractions: 765 (were 763 using Dijkstra's light algorithm on the other example)
- Route length: ≈ 31.46

Using Manhattan Distance on a Different Grid Example



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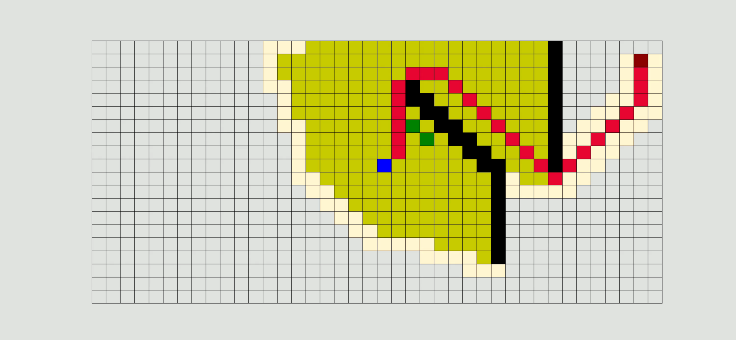
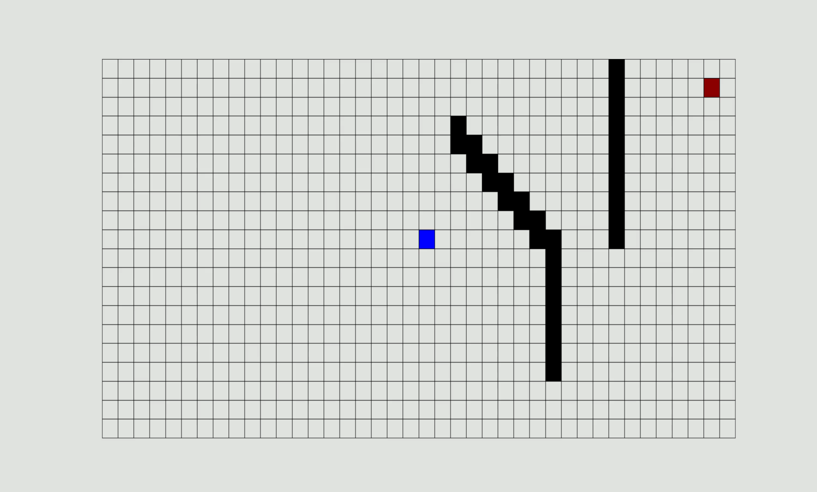
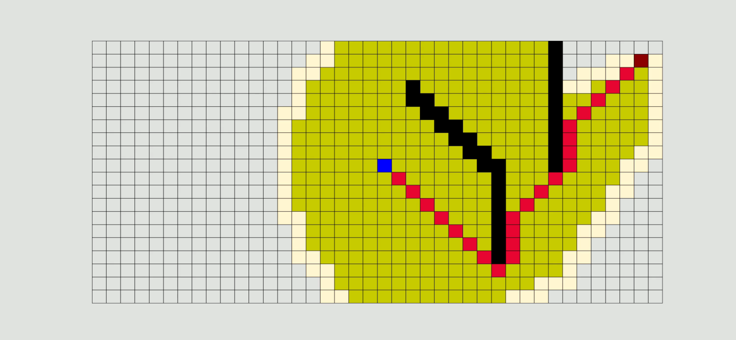


- Queue extractions: 231 (were 765 using Dijkstra's light algorithm)
- Shortest path length: ≈ 31.46
- Route length: ≈ 32.62

Using Chebyshev Distance on a Different Grid Example

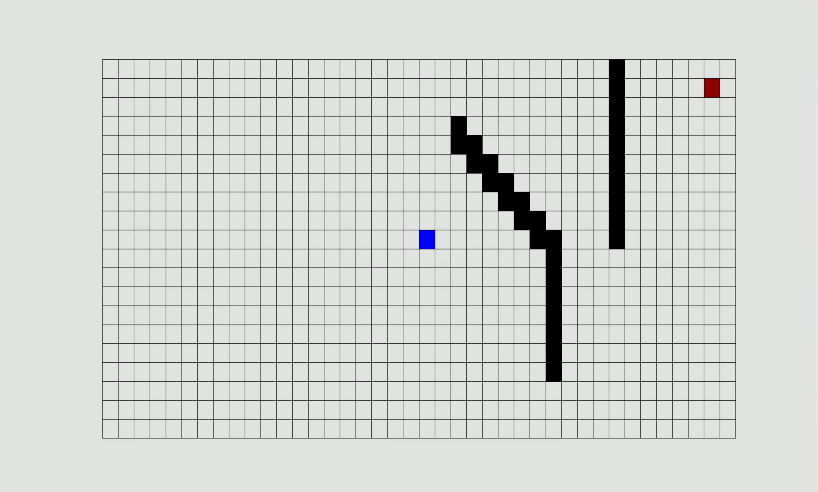


Using Chebyshev Distance on a Different Grid Example



- Queue extractions: 372 (were 231 for A* with Manhattan Distance)
- Shortest path length: ≈ 31.46
- Route length: ≈ 31.46

Using Euclidean Distance on a Different Grid Example



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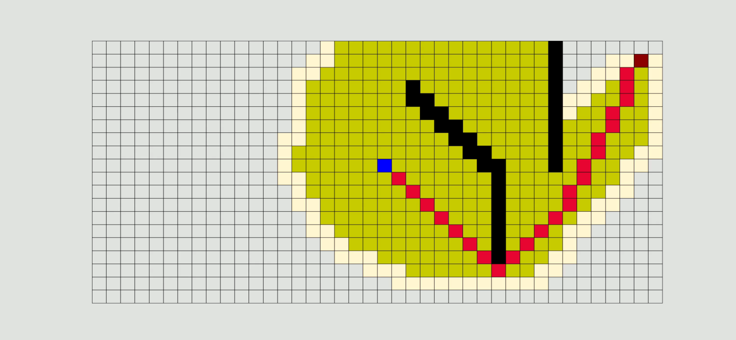

- Queue extractions: 315 (were 372 for A* with Chebyshev Distance)
- Shortest path length: ≈ 31.46
- Route length: ≈ 31.46

Routing on World Scale Graphs

We aim to apply routing techniques also to handle

- internet packets moving between severs
- parcels delivered by multiple couriers
- travelers commuting between airplanes
- cars moving along a continent-wide route system

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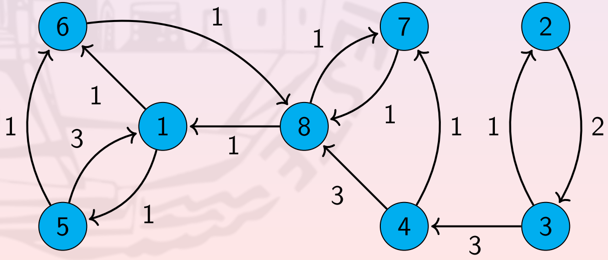
The graphs are too huge to be completely store in memory

neither Dijkstra nor A* can be applied

Shrinking Graphs Preserving Important Nodes

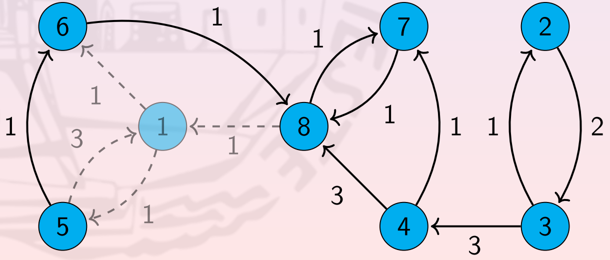
Let $V = \{1, \ldots, n\}$ be sorted by ascending "importance"



Shrinking Graphs Preserving Important Nodes

Let $V = \{1, \ldots, n\}$ be sorted by ascending "importance"

We can remove 1 preserving more important nodes



Shrinking Graphs Preserving Important Nodes

Let $V = \{1, \ldots, n\}$ be sorted by ascending "importance"

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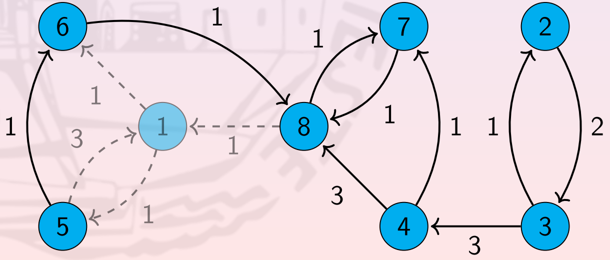
The shortest paths $(i, 1), (1, j)$ are replaced by shortcuts (i, j) s.t.

$$W(i, j) = W(i, 1) + W(1, j)$$



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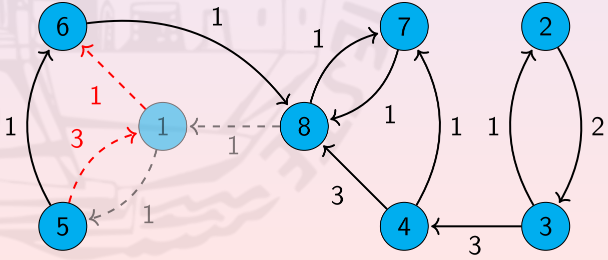

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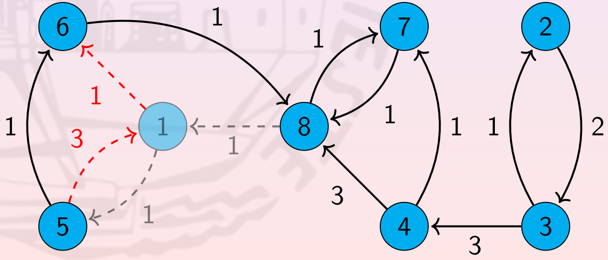

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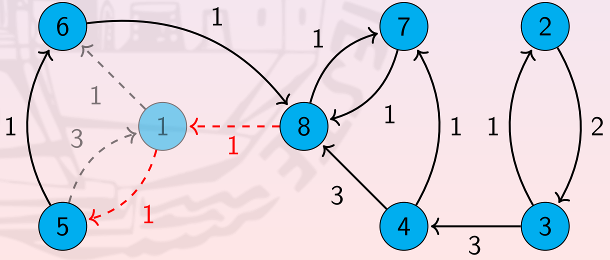

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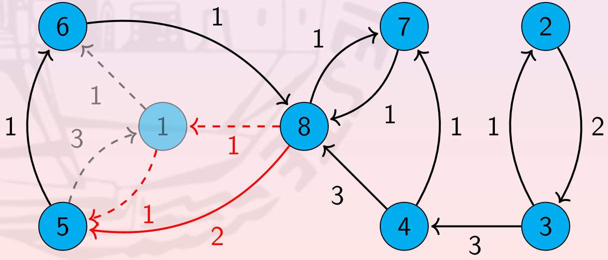

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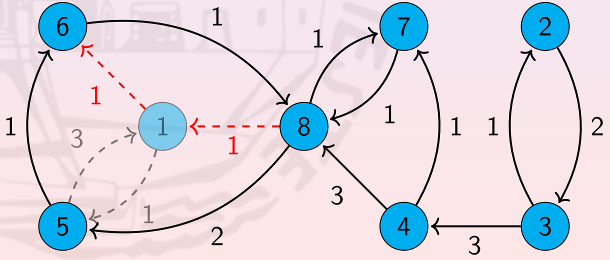

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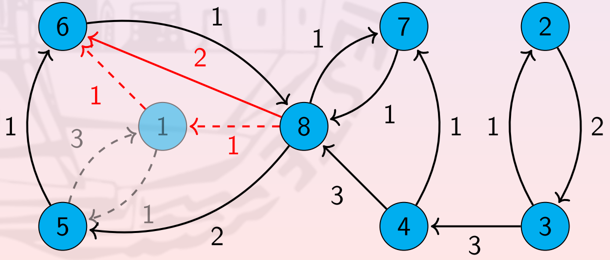

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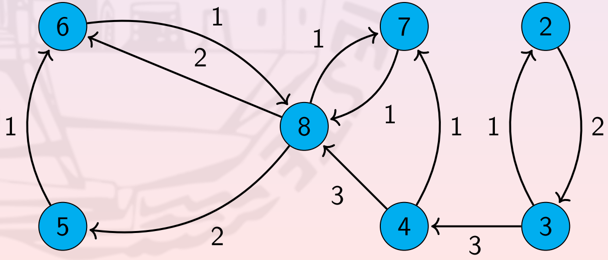

Contractions and Overlay Graphs

The contraction of node k consists in:

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- removing k

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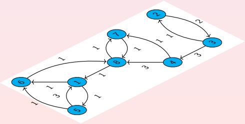
The resulting graph is the overlay graph

Contraction Hierarchy

The sequence of the overlay graphs is a contraction hierarchy



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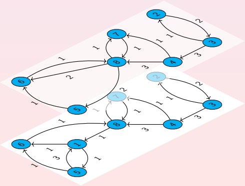

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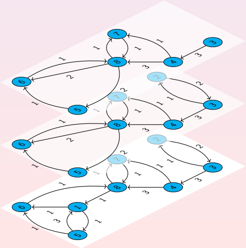

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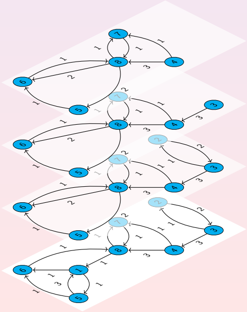

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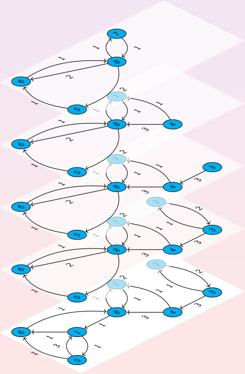

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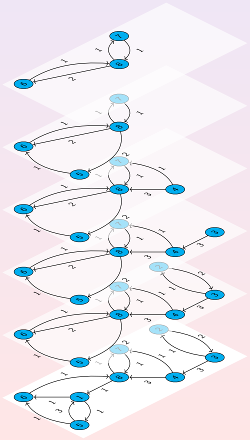

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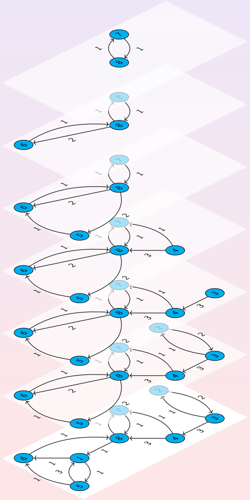

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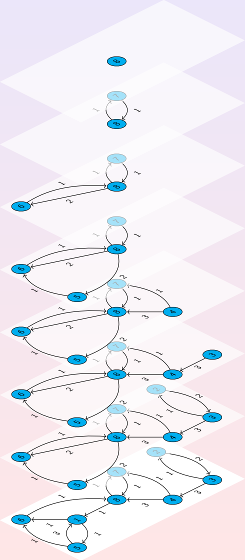
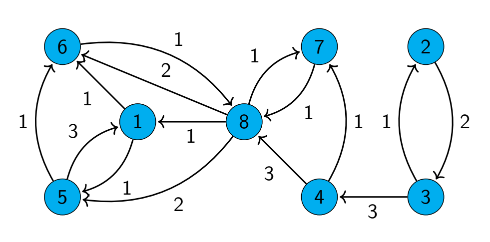
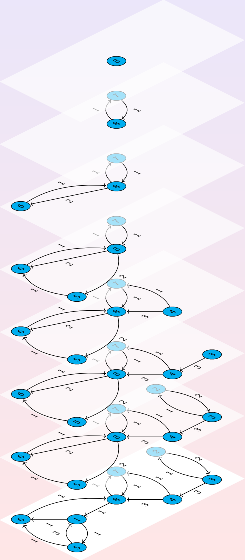


Contraction Hierarchy

The sequence of the overlay graphs is a contraction hierarchy

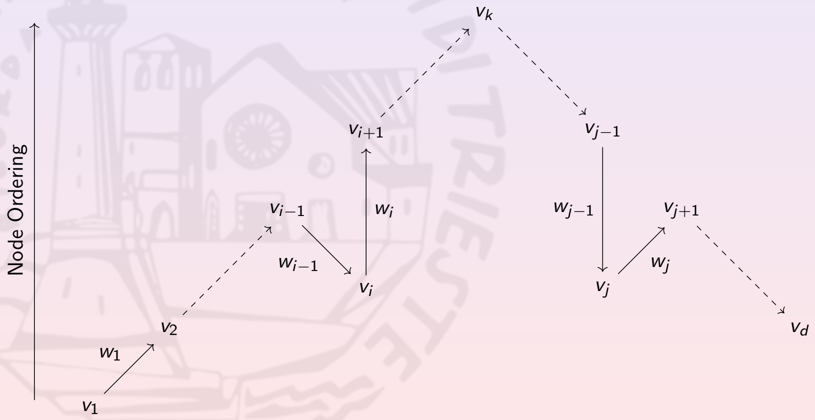
Looking it from the top, we get





Shortest Paths on Contraction Hierarchy

Let $v_1, \dots v_d$ be a shortest path on the CH with $w_i = W[v_1, v_d]$



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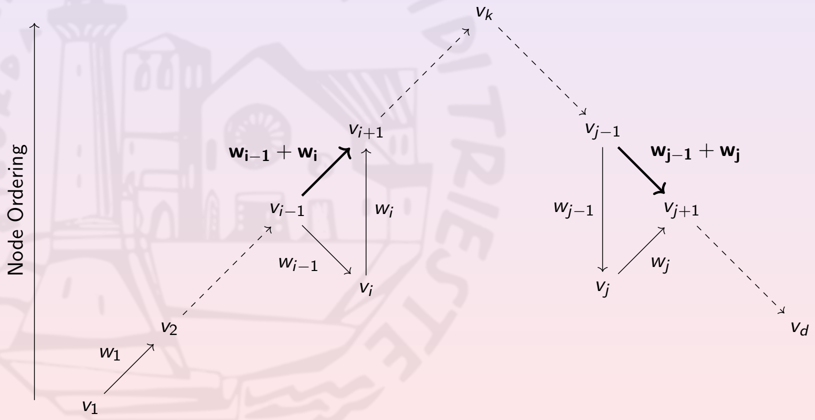

Shortest Paths on Contraction Hierarchy (Cont'd)

The shortest paths on CH have the form $v_1, \ldots, v_k, \ldots, v_d$ where:

- $v_{i-1} < v_i$ for all $i \leq k$
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Find them by building $G \uparrow = (V, E \uparrow)$ and $G \downarrow = (V, E \downarrow)$ where:

- $E \uparrow = \{(v, w) \in E \mid v < w\}$
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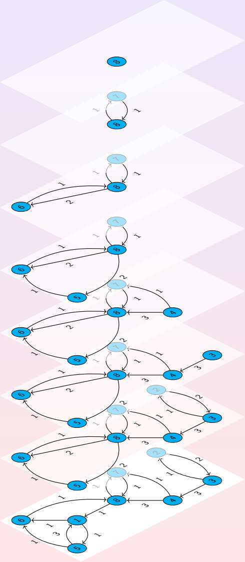
and using a bidirectional version of Dijkstra on $G\uparrow$ and $G\downarrow$

It search forward on $G\uparrow$ and backward on $G\downarrow$ until the two searches finalize the same node

Contraction Hierarchy and Graph Size

Many overlay graphs shares a large set of edges



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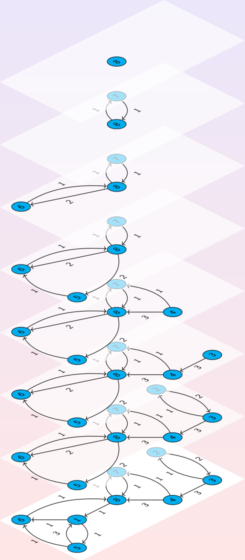

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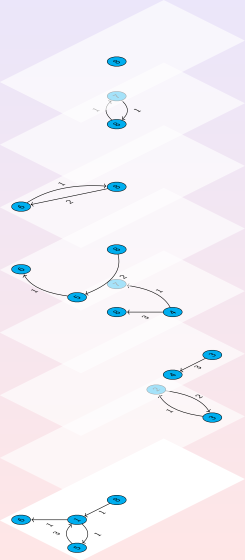

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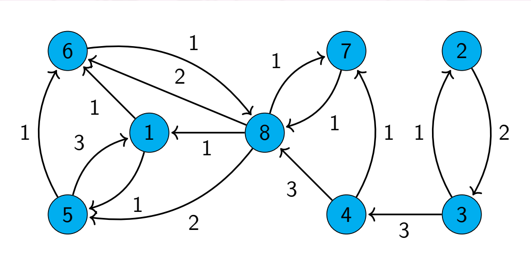

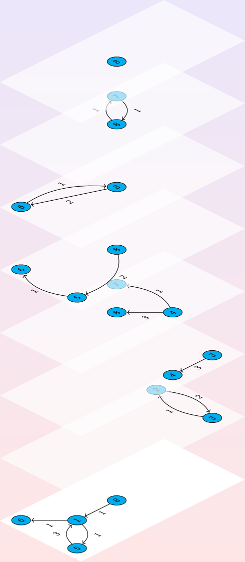

Partition Huge Graphs

By merging subsequent layers, we endup with graphs which have high probability to be disconnected

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Huge graphs can be parted into subgraphs at the lowest levels and connect them by using highest levels