Distance

zcl.space

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The **distance** $d(p, q)$ between two points $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is defined by[1] In Chapters I-V Buck writes $|p - q|$ instead of $d(p, q)$ but uses the notation $d(p, q)$ starting in Chapter˜VI (page˜304) in a more general setting.}

$$d(p, q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

The distance $d(v, 0)$ from a vector $v \in \mathbb{R}^n$ to the origin is called the **norm** of v denoted by $|v|$ so

$$d(p, q) := |p - q|.$$

The norm satisfies the following laws for $v, w \in \mathbb{R}^n$:

1. **zero norm** $|v| = 0 \iff v = 0$,
2. **homogeneity** $|av| = a\,|v|$ if $a > 0$,
3. **symmetry** $|-v| = |v|$,
4. **triangle inequality** $|v + w| \le |v| + |w|$.

The zero norm law holds because a sum of squares vanishes

The laws for the norm imply that the distance function satisfies the following:

1. **zero distance** $d(p, q) = 0 \iff p = q$,

[1] {

2. **symmetry** $d(p, q) = d(q, p)$,

3. **triangle inequality** $d(p, r) \leq d(p, q) + d(q, r)$.

These are proved by reading $v = p - q$ and $w = q - r$ in the corresponding law for the norm.

Define the **inner product** of two vectors in $\mathbb{R}^n$ by

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$. In freshman calculus you called this the **dot product** and learned that the angle θ between the two vectors u and v satisfies

$$\langle u, v\rangle = \cos\theta\,|u|\,|v|.$$

Since the cosine takes values between -1 and +1, it follows that

$$|\langle u, v\rangle| \leq |u|\,|v|. \tag{0.1}$$

This inequality is called the **Schwarz inequalityr**.

For $p \in \mathbb{R}^n$ and $\delta > 0$ the set

$$B(p, \delta) := \{q \in \mathbb{R}^n : d(p, q) < \delta\}$$

is called the **open ball** centered at p with radius δ. When $X \subseteq \mathbb{R}^n$ we often use the abbreviation

$$B_X(p, \delta) := B(p, \delta) \cap X := \{q \in X : d(p, q) < \delta\}$$

When $n = 1$ the open ball is an open interval:

$$\begin{aligned} B(a, \delta) &= \{x \in \mathbb{R} : |x - a| < \delta\} \\ &= \{x \in \mathbb{R} : a - \delta < x < a + \delta\} \\ &= (a - \delta, a + \delta) \end{aligned}$$

for $a \in \mathbb{R}$

A set S is **bounded** iff it is contained in some large ball, i.e. iff there exists $M > 0$ such that $|p| < M$ for all $p \in S$. Thus a set of real numbers is bounded if and only if it is bounded above and bounded below.