Connected sets

zcl.space

Contents

A set S is **disconnected** iff there are disjoint open sets U and V such that $S \subseteq U \cup V$ and both $S \cap U$ and $S \cap V$ are nonempty.

A set is **connected** iff it is not disconnected. A subset $S \subseteq \mathbb{R}$ of the real line is connected if and only if S is an interval, i.e. $[a, b] \subseteq S$ whenever $a, b \in S$. Assume S is not an interval, i.e. that there exist $a, b \in S$ with $[a, b] \nsubseteq S$. Then there is a $c \in [a, b]$ with $c \notin S$.

Let $U = (-\infty, c)$ and $V = (c, \infty)$. The point c lies in the open interval (a, b) as $a, b \in S$ so $a \in U$ and $b \in V$. Hence both $S \cap U$ and $S \cap V$ are nonempty and clearly $S \subseteq U \cup V$ (as $c \notin S$). Hence the open sets U and V separate S so S is disconnected as required.

Assume that S is disconnected, i.e. that there exist open sets $U, V \subseteq \mathbb{R}$ with $S \subseteq U \cup V$, $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$, and $U \cap V = \emptyset$. We must show that S is not an interval. Choose $a \in S \cap U$ and $b \in S \cap V$. Then $a \neq b$ as $U \cap V = \emptyset$. Assume without loss of generality that $a < b$. (The case $b < a$ is the same.)

The set $[a, b] \cap U$ is nonempty (it contains a) and bounded above (b is an upper bound). Let $c = \sup([a, b] \cap U)$. Since $a \in U$ there is an $\epsilon > 0$ with $(a - \epsilon, a + \epsilon) \subseteq U$. Making ϵ smaller we also have $a + \epsilon < b$. Therefore $[a, a + \epsilon) \subseteq [a, b] \cap U)$ so $a + \epsilon = \sup [a, a + \epsilon) \leq \sup [a, b] \cap U = c$. Since $b \in V$ there is an(other) $\epsilon > 0$ with $(b - \epsilon, b + \epsilon) \subseteq V$. Making ϵ smaller we also have $a < b - \epsilon$. Therefore $(b - \epsilon, b] \subseteq [a, b] \cap V)$ so $[b - \epsilon, b] \cap V = \emptyset$ so $b - \epsilon$ is an upperbound for $[a, b] \cap U$, so $c \leq b - \epsilon$. We have proved that $a < c < b$. If $c \in U$ there is an $\epsilon > 0$ with $a < c - \epsilon < c < c + \epsilon < b$ and $(c - \epsilon, c + \epsilon) \subseteq U$ contradicting the fact that c is an upper bound of $[a, b] \cap U$. If $c \in V$ there is an $\epsilon > 0$ with $a < c - \epsilon < c < c + \epsilon < b$ and $(c - \epsilon, c + \epsilon) \subseteq V$ so $c - \epsilon$ is an upperbound for $[a, b] \cap U$ contradicting the fact that c is the least upper bound of $[a, b] \cap U$. Hence $c \notin U \cup V$ so (as $S \subseteq U \cup V$) $c \notin S$. Thus $a < c < b$, $a \in S$, $b \in S$, $c \notin S$, so S is not an interval.

The continuous image of a connected set is connected: If $f : X \to \mathbb{R}^m$ is continuous and X is connected, then $f(X)$ is connected. Assume that S is connected and that $f : S \to \mathbb{R}$ is continuous. Suppose that $a, b \in f(S)$ and that $a < c < b$. Then $c \in f(S)$.

The Intermediate Value Theorem from calculus is a special case. It says that if $f : [\alpha, \beta] \to \mathbb{R}$ is a real valued continuous function on the closed interval $[\alpha, \beta] \subseteq \mathbb{R}$, $\{a, b\} = \{f(\alpha), f(\beta)\}$, and $a \leq c \leq b$, then the equation $f(x) = c$ has a solution $x \in [\alpha, \beta]$. A continuous function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is injective if and only if it is strictly monotonic. When these equivalent conditions hold, the image $J = f(I)$ is again an interval and the inverse function is continuous.

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be f is continuous. Then the set

$$\operatorname{graph}(f) := \{(x, y) \in I \times \mathbb{R} : y = f(x)\}$$

is connected.

Define $F : I \to \mathbb{R}$ by $F(x) = (x, f(x))$ so that $F(I) = \operatorname{graph}(f)$. Clearly f is continuous if and only if F is continuous. We will assume that I is an open interval; the case where I contains one of its endpoints is similar. Assume that $F(I)$ is not connected. Then there are open sets $U, V \subseteq \mathbb{R}^2$ with $F(I) \subseteq U \cup V$, $U \cap V = \emptyset$, $F(I) \cap U \neq \emptyset$, $F(I) \cap V \neq \emptyset$. Then $F^{-1}(U), F^{-1}(V) \subseteq \mathbb{R}^2$ are open, $I \subseteq F^{-1}(U) \cup F^{-1}(V)$, and $F^{-1}(U) \cup F^{-1}(V) = F^{-1}(U \cap V) = \emptyset$. This contradicts the fact that I is an interval and therefore connected.

The converse is false. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

This function is not continuous as follows. Let $x_n = (2n\pi + \pi/2)^{-1}$. Then $f(x_n) = 1$, $\lim_{n\to\infty} x_n = 0$, but $\lim_{n\to\infty} f(x_n) = 1 \neq 0 = f(0)$. However, the graph of f is connected. To see this suppose U and V are open subsets of $\mathbb{R}^2$ and $\operatorname{graph}(f) \subseteq U \cup V$ with $U \cup V = \emptyset$. Suppose that $(0, 0) \in U$. Then $(x, f(x)) \in U$ for $x \leq 0$ as f is continuous on $(-\infty, 0]$ and $(x, f(x)) \in U$ in U for $x > 0$ as f is continuous on $(0, \infty)$. But then $\operatorname{graph}(f) \subset U$ so $\operatorname{graph}(f) \cap V = \emptyset$.