Composition and Inverse

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If $f : X \to Y$ and $g : Y \to Z$, then the **composition** of f and g is the map $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x))$$

for $x \in X$. For any set X the **identity map** of X is the map $\mathrm{id}_X : X \to X$ defined by

$$\mathbb{I}_X(x) = x$$

for $x \in X$. Clearly

$$\mathbb{I}_Y \circ f = f \text{ and } f \circ \mathbb{I}_X = f$$

for $f : X \to Y$.

A map $g : Y \to X$ is said to be a **left inverse** for the map $f : X \to Y$ iff $g \circ f = \mathbb{I}_X$, i.e. iff $g(f(x)) = x$ for all $x \in X$. A map $g : Y \to X$ is said to be a **right inverse** for the map $f : X \to Y$ iff $f \circ g = \mathbb{I}_Y$, i.e. iff $f(g(y)) = y$ for all $y \in Y$. A map $g : Y \to X$ is said to be a (two sided) **inverse** to the map $f : X \to Y$ iff it is both a left inverse and a right inverse to f. If g is a left inverse to f and g' is a right inverse to f then $g = g'$. (Proof: $g = g \circ \mathbb{I}_X = g \circ f \circ g' = \mathbb{I}_Y \circ g' = g'$.) In this case there is a unique inverse and it is denoted f^{-1}. So if $f : X \to Y$ has an inverse $f^{-1} : Y \to X$, then

$$y = f(x) \iff x = f^{-1}(y)$$

for $x \in X$ and $y \in Y$.

A map $f : X \to Y$ is said to be **injective** iff

$$\forall x_1, x_2 \in X \ [f(x_1) = f(x_2) \implies x_1 = x_2]$$

and it is said to be **surjective** iff $Y = f(X)$, i.e.

$$\forall y \in Y \ \exists x \in X \ y = f(x).$$

A map is **bijective** iff it is both injective and surjective. Thus

1. A map is injective if and only if it has a left inverse;

2. A map is surjective if and only if it has a right inverse;

3. A map is bijective if and only if it has a (two-sided) inverse.

   a map is surjective if and only if its range equals its target.

Both Lang and Morgan use the more modern terms injective, surjective, bijective, but Buck and Rudin use the older terminology **one-one** instead of injective, **onto** instead of **surjective**, and **one-one onto** instead of bijective. The older terminology is used in teaching calculus.

The 'only if' part of item~(2) is called the **Axiom of Choice** . It was once controversial because one can imagine a situation where one can prove that a map f is surjective but where one cannot give an explicit formula for a right inverse.

The assertions (1-3) are false if continuity (defined later) is required: There is a continuous bijective map whose inverse is not continuous (and hence continuous injective map which does not have a continuous left inverse, and a continuous surjective map which does not have a continuous right inverse). Let $S := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denote the unit circle in $\mathbb{R}^2$ and define $f : [0, 2\pi) \to S$ by $f(\theta) = (\cos\theta, \sin\theta)$. Then f bijective and continuous but f^{-1} is not continuous. We will see that the inverse if a continuous bijective map is continuous if its domain is compact (defined later), but $[0, 2\pi)$ is not compact.

Consider the four maps

$$f_1 : \mathbb{R} \to \mathbb{R}, \quad f_2 : [0, \infty) \to \mathbb{R}, \quad f_3 : \mathbb{R} \to [0, \infty), \quad f_4 : [0, \infty) \to [0, \infty)$$

defined by $f_i(x) = x^2$. Then f_1 is not injective and not bijective, f_2 is injective but not surjective, f_3 is surjective but not injective, and f_4 is bijective. Any map $g_2 : \mathbb{R} \to [0, \infty)$ such that $g_2(y) = \sqrt{y}$ for $y \geq 0$ is a left inverse to f_2, and any map $g_3 : [0, \infty) \to \mathbb{R}$ such that $g_3(y) = \pm\sqrt{y}$ (the $\pm$ can depend on y) is a right inverse to f_3 The inverse map to f_4 is $f_4^{-1}(y) = \sqrt{y}$.