Functions and Maps

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A `function` is a rule which assigns a value $f(x)$ to every point x from a set called the `domain` of the function. The set

$$\mathrm{graph}(f) := \{(x, y) : y = f(x)\}$$

of all pairs (x, y) such that $y = f(x)$ is called the `graph` of the function f. Two functions are equal iff they have the same graph.

Let X and Y be sets. We say that f is a `map` from X to Y and write $f : X \to Y$ when f is a function which assigns a point $y = f(x) \in Y$ to each point $x \in X$. Two maps $f : X \to Y$ and $f' : X' \to Y'$ are said to be `equal` when $X = X'$, $Y = Y'$, and $f(x) = f'(x)$ for all $x \in X$. Thus if f and f' equal maps, then $\mathrm{graph}(f) = \mathrm{graph}(f')$ but not conversely (because $Y = Y'$ is part of the definition of equality for maps). However most authors would say that two functions are equal iff they have the same graph.

Some authors use the notation $x \mapsto f(x)$ to define a map. This allows them to avoid introducing a name for the map. Thus instead of writing

> Consider the map $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^5 + x$.

they may write

> Consider the map $\mathbf{R} \to \mathbf{R} : x \mapsto x^5 + x$.

When $A \subseteq X$, $B \subseteq Y$, and $f : X \to Y$, the sets

$f(A) := \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}$,

$f^{-1}(B) := \{x \in X : f(x) \in B\}$, are called respectively the `image` of A by f and `inverse image` of B by f.

The sets X and Y are sometimes called the `source` and `target` of a map $f : X \to Y$. The image $f(X)$ of the source is what is called the {range} of the function f in calculus. Thus the domain of a map is the same as its source while the range is a subset of its target.

There are slight variations in terminology among authors. Morgan avoids the word `map` and Lang sometimes uses the word `mapping`. Buck uses the term the `preimage` instead of `inverse image`. Lang uses the $\mapsto$ notation but the other authors apparently avoid it. For Lang a function is a map whose target is $\mathbf{R}$. In precalculus courses a function is usually defined by an expression and the domain is implicitly taken to be the largest set of numbers for which the expression is meaningful, but in advanced mathematics authors usually make the domain explicit.