Continuity

zcl.space

Contents

Throughout this chapter $f : X \to Y$ where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. The map f is said to be continuous at a point $p \in X$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_X(p, \delta)) \subseteq B(f(p), \epsilon)$. The map f is continuous at $p \in X$ if and only if for every sequence $\{p_n\}$ of points in X we have

$$\lim_{n \to \infty} p_n = p \implies \lim_{n \to \infty} f(p_n) = f(p). \tag{1}$$

Assume f is continuous at p. Choose sequence $\{p_n\}$ of points in X. Assume

$$\lim_{n \to \infty} p_n = p. \tag{2}$$

Choose $\epsilon > 0$. Because f is assumed to be continuous at p there is a $\delta > 0$ such that for all $q \in X$ %

$$|q - p| < \delta \implies |f(q) - f(p)| < \epsilon. \tag{3}$$

% there is an N such that $|p_n - p| < \delta$ for $n > N$. Hence $|f(p_n) - f(q)| < \epsilon$ for $n > N$. This proves %%

$$\lim_{n \to \infty} f(p_n) = f(p). \tag{4}$$

as required.

Assume that f is not continuous at $p \in X$. Then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a $q \in X$ such that

$$|q - p| < \delta \text{ but } |f(q) - f(p)| \geq \epsilon$$

.

In particular, for each $n \in \mathbb{Z}^+$ there is a q_n such that

$$|q_n - p| < \frac{1}{n} \text{ but } |f(q_n) - f(p)| \geq \epsilon$$

. The map f is said to be continuous iff it is continuous at every point of X, i.e. iff

$$\forall p \in X \ \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } f(B(p, \delta)) \subseteq B(f(p), \epsilon).$$

The map f is said to be **uniformly continuous** iff

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \text{ such that } \forall p \in X \text{ we have } f(B(p, \delta)) \subseteq B(f(p), \epsilon).$$

(For continuity $\delta = \delta(p, \epsilon)$; for uniform continuity $\delta = \delta(\epsilon)$.) (Buck Theorem 23 and its corollary on pages 62-63.)

If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then so is the composition $g \circ f : X \to Z$.

Choose $p_0 \in X$ and $\epsilon > 0$. As g is continuous there exists $\eta > 0$ such that $|g(q) - g(f(p_0))| < \epsilon$ whenever $q \in Y$ and $|q - f(p_0)| < \eta$. As f is continuous, there exists $\delta > 0$ such that $|f(p) - f(p_0)| < \eta$ whenever $p \in X$ and $|p - p_0| < \delta$. For $p \in X$ we have $q = f(p) \in Y$ so

$$|p - p_0| < \delta \implies |f(p) = f(p_0)| < \eta \implies |(g \circ f)(p) - (g \circ f)(p_0)| < \epsilon$$

as required.

The map f is said to be **Lipschitz** iff there is a constant M such that

$$|f(p) - f(q)| \leq M|p - q|$$

for all $p, q \in X$. A Lipschitz function is uniformly continuous. (Proof: $\delta = \epsilon/M$.)