Open sets and closed sets

zcl.space

Contents

In all the following definitions the term **set** means **subset of** $\mathbb{R}^m$.

A set U is **open** iff for every $p \in U$ there exists a $\delta > 0$ such that $B(p, \delta) \subseteq U$.

The collection of all open sets in $\mathbb{R}^m$ satisfies the following conditions:

1. The set $\mathbb{R}^m$ and the empty set $\emptyset$ are both open.
2. The intersection of a finite collection of open sets is open.
3. The union of an arbitrary collection of open sets is open.

The set $\mathbb{R}^m$ is open because $B(p, 1) \subseteq \mathbb{R}^m$ for $p \in \mathbb{R}^m$. The empty set is open because for every $p \in \emptyset$ satisfies the required condition – or any other condition – since **'false** *implies* **anything' is true**. To prove~(2) assume U is open. Then for every point $p \in U$ there is a $\delta = \delta_p$ such that $B(p, \delta_p) \subseteq U$.[1]

It follows that

$$U = \bigcup_{p \in U} B(p, \delta_p),$$

i.e. that U is an union of balls. A union of unions is a union:

$$\bigcup_{i \in I} \bigcup_{j \in I_j} B_{ij} = \bigcup_{(i,j) \in K} B_{ij}, \qquad K := \{(i, j) : i \in I,\ j \in I_j\}$$

so~(2) follows.

To prove~(3) assume that $U_1, U_2, \ldots U_m$ are open and choose an arbitrary point $p \in \bigcap_{i=1}^m U_i$. Then $p \in U_i$ so there is a $\delta_i > 0$ with $B(p, \delta_i) \subseteq U_i$.

Let $\delta = \min(\delta_1, \ldots, \delta_m)$. Then

$$B(p, \delta) \subseteq \bigcap_{i=1}^{m} B(p, \delta_i) \subseteq \bigcap_{i=1}^{m} U_i$$

as required. A set $W \subseteq X$ is called **relatively open** in X iff for every $p \in W$ there exists a $\delta > 0$ such that $B_X(p, \delta) \subseteq W$.

A set $U \subseteq \mathbb{R}^m$ is open if and only if it is relatively open in $\mathbb{R}^m$. For this reason many theorems can be generalized by systematically replacing $\mathbb{R}^m$ by X, $B(p, \delta)$ by $B_X(p, \delta)$, and the word {*open*} *by the phrase* {relatively open in X}.

A set W is relatively open in X if and only if $W = X \cap U$ for some open set $U \subset \mathbb{R}^n$.

[1] This is actually an example of an application of the Axiom of Choice.

A map \$f:X→ \$Y is continuous if and only if the inverse image $f^{-1}(V)$ of every relatively open subset V of Y is a relatively open subset of X.

Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Then $g \circ f : X \to Z$ is continuous.

A set X is closed iff its complement $\mathbb{R}^n \setminus X$ is open.

The collection of all closed sets in $\mathbb{R}^n$ satisfies the following conditions:

1. The set $\mathbb{R}^n$ and the empty set $\emptyset$ of both closed.
2. The intersection of an arbitrary collection of closed sets is closed.
3. The union of a finite collection of closed sets is closed.

A set S is closed if and only if it is closed under limits of sequences, i.e. whenever $\lim_{n\to\infty} p_n = p$ and each $p_n \in S$ we have $p \in S$. To prove {*only if*} *assume that S is closed, that $\lim_{n\to\infty} p_n = p$, and that each $p_n \in S$. If $p \notin S$ then $p \in \mathbb{R}^m \setminus S$.*

As this set is open there is a $\delta > 0$ such that $B(p, \delta) \subset \mathbb{R}^m \setminus S$. As the sequence converges to p there is an N such that $p_n \in B(p, \delta)$ for $n > N$ contradicting the hypothesis that $p_n \in S$. To prove {if} assume that S is not closed. Then $\mathbb{R}^m \setminus S$ is not open so there is a point $p \in \mathbb{R}^m \setminus S$ such that $B(p, \delta) \not\subseteq \mathbb{R}^m \setminus S$ for every $\delta > 0$. In particular for $\delta = 1/n$ there is a point $p_n \in B(p, 1/n)$ (i.e. $|p_n - p| < 1/n$) such that $p_n \notin \mathbb{R}^m \setminus S$, i.e. $p_n \in S$. Thus $\lim_{n\to\infty} p_n = p$ and $p \notin S$ as desired.

Let $S \subseteq \mathbb{R}^n$. For any point $p \in \mathbb{R}^n$ exactly one of the following alternatives holds:

1. $B(p, \delta) \subseteq S$ for some $\delta > 0$.
2. $B(p, \delta) \subseteq \mathbb{R}^n \setminus S$ for some $\delta > 0$.
3. $B(p, \delta) \cap S \neq \emptyset$ and $B(p, \delta) \cap (\mathbb{R}^n \setminus S) \neq \emptyset$ for all $\delta > 0$.

The **interior** of S is the set $\text{int}(S)$ of all points p where˜(1) holds, the **exterior** of S is the set $\text{ext}(S)$ of all points p where˜(2) holds, and the **boundary** of a set S is the set $\text{bdry}(S)$ of all points p where˜() holds. The ambient space $\mathbb{R}^n$ may be written as the pairwise disjoint union

$$\mathbb{R}^n = \text{int}(S) \cup \text{ext}(S) \cup \text{bdry}(S).$$

The notations

$$\overset{\circ}{S} := \text{int}(S), \qquad \partial S := \text{bdry}(S)$$

are commonly used.

For the half open interval $S = [a, b) \subseteq \mathbb{R}$ we have

$$\text{int}(S) = (a, b), \qquad \text{ext}(S) = (-\infty, a) \cup (b, \infty), \qquad \text{bdry}(S) = \{a, b\}.$$

A set $S \subseteq \mathbb{R}^n$ is closed iff its complement $\mathbb{R}^n \setminus S$ is open. The closure of the set S is the set

$$\text{cl}(S) := \bar{S} := S \cup \text{bdry}(S).$$

The interior $\text{int}(S)$ of S is the largest open set contained in S and closure $\bar{S}$ of S is the smallest closed set containing S.