Limits

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目录

The intuitive idea of the notation

$$\lim_{p \to p_0} F(p) = L$$

is that $F(p)$ is very close to L when p is very close to p_0. Some authors write $F(p) \to L$ as $p \to p_0$; others write $F(p) \approx L$ when $p \approx p_0$. In this chapter we give a more precise definition. The following lingo is helpful.

A set U is called a **neighborhood** of the point p if U contains some open ball $B(p, \delta)$ centered at p. A **punctured neighborhood** of p is a set of form $U \setminus \{p\}$ where U is a neighborhood of p.

A point p is a **accumulation point** of a set S iff every punctured neighborhood of p contains a point of S. The following equivalent definition appears in some books.

定理 0.1

A point p is an accumulation point of the set S if and only if every neighborhood of p contains infinitely many points of S.

证 *"If" is easy: an infinite set is nonempty and at most one of the points in an infinite set is p. For "only if" assume p is an accumulation point of the set S and choose a neighborhood U of p. By definition there is a point $p_0 \in U \cap S \setminus \{p\}$. Let $\delta_0 = |p - p_0|$. For $n > 0$ define $\delta_n > 0$ and $p_n \in S$ inductively by $\delta_n = \min(|p_n - p|, 1/n)$ and $p_{n+1} \in B(p, \delta_n) \cap S$. The map $n \mapsto p_n$ is injective as $|p_n - p| < |p_m - p|$ for $n > m$. Choose $\delta > 0$ so that*

$B(p,\delta) \subset U$ (by the definition of neighborhood). Then $\delta_n < \delta$ for $1/n < \delta$ so $p_n \in B(p,\delta) \cap S \subseteq U \cap S$. Hence U contains the infinite set $\{p_n : n > 1/\delta\}$. $\square$

Let p_0 be a accumulation point of a set S and F be a function defined on S (but possibly not at p_0). The notation

$$\lim_{p \to p_0} F(p) = L$$

means that for every neighborhood V of L of there is a punctured neighborhood $U \setminus \{p\}$ of L such that $f(S \cap U \setminus \{p\}) \subset V$. When $p_0 \in S$ and p_0 is a accumulation point of S we have that a function f defined on S is continuous at p_0 if and only if

$$\lim_{p \to p_0} f(p) = f(p_0)$$

(and the function is trivially continuous at a point $p_0 \in S$ which is not an accumulation point of S). However, the limit notation is usually used in situations where (p_0 is a accumulation point of S but) $p_0 \notin S$. For example, the derivative of a real valued function $f : I \to \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$ is defined by

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The ratio in the limit is undefined when $x = x_0$ but is defined for nearby values of x.

For a real valued function f defined on a subset of $\mathbb{R}$ we can extend the definition of the notation $\lim_{x \to a} F(x) = L$ to include the cases where $a = \pm\infty$ and/or $L = \pm\infty$ as follows. Let

$$\hat{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

consist of the set of real numbers together with two additional points which we think of as located at infinity. The set $\hat{\mathbb{R}}$ is sometimes called the set of **extended real numbers**. Extend the usual order relation on $\mathbb{R}$ to $\hat{\mathbb{R}}$ in the obvious way. For $a \in \hat{\mathbb{R}}$, a set $U \subseteq \hat{\mathbb{R}}$ is called **neighborhood** of a iff

1. either $a \in \mathbb{R}$ and U contains an open interval $(a-\delta, a+\delta)$ for some $\delta > 0$,
2. or else $a = \infty$ and U contains an interval $(M, \infty]$ for some $M > 0$,
3. or else $a = -\infty$ and U contains an interval $[-\infty, -M)$ for some $M > 0$.

Because $B(a,\delta) = (a-\delta, a+\delta)$ this definition agrees with the definition of **limit** for $a \in \mathbb{R}$. A point $a \in \hat{\mathbb{R}}$ is called a **accumulation point** of a subset $S \subseteq \mathbb{R}$ iff every punctured neighborhood of a intersects S. If $f : S \to \mathbb{R}$ and a is a accumulation point of S, then the notation

$$\lim_{x \to a} F(x) = L$$

means that for every neighborhood V of L there is a punctured neighborhood U of a such that $f(U \cap S) \subseteq V$.

Unraveling the above definitions we see that for $a, L \in \mathbb{R}$ we have that $\lim_{x \to a} F(x) = L$ iff $\forall \epsilon > 0 \, \exists \delta > 0$ such that $\forall x \in S$ we have that $0 < |x-a| < \delta \implies |F(x) - L| < \epsilon$. Also $\lim_{x \to \infty} F(x) = L$ iff $\forall \epsilon > 0 \, \exists M > 0$ such that $\forall x \in S$. we have that $M < x \implies |F(x) - L| < \epsilon$ with similar definitions for the other cases where $a, L \in \{\pm\infty\}$.

The various definitions given in **limit** and **extended_reals** are easier to understand because the lingo makes them look the same and because there aren't so many symbols. This is why the terminology was invented.