Convergence of Sequence

zcl.space

目录

A **sequence** is a function defined on a subset of the integers. (Usually this subset is the set $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n > 0\}$ of positive integers or the set $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$ of nonnegative inregers.)

It is customary to denote the value of a sequence at an integer n with a subscript rather than with parentheses and to denote a sequence with a notation like $(p_n)_n$ or $(p_n)_{n\in\mathbb{Z}^+}$.

定义 *0.1* *The sequence $(p_n)_n$ of points of $\mathbb{R}^m$ is said to* ***converge*** *to the point $p \in \mathbb{R}^m$ iff*

$$\lim_{n\to\infty} p_n = p$$

This is sometimes abbreviated as $p_n \to p$ as $n \to \infty$. We say a sequence ***converges*** *or is* ***convergent*** *iff it converges to p for some $p \in \mathbb{R}^m$. A sequence is said to* ***diverge*** *when it does not converge. (A sequence in $\mathbb{R}$ whose limit is infinite is also said to diverge.)*

Using the lingo introduced in˜**??** this may be stated as

$$\lim_{n\to\infty} p_n = p$$

iff for every $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that $n \geq N \implies |p_n - p| < \epsilon$.

定理 *0.1* *Assume that $(a_n)_n$ and $(b_n)_n$ are convergent sequences of real numbers:*

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

Then

$$\lim_{n\to\infty} a_n + b_n = a + b, \qquad \lim_{n\to\infty} a_n b_n = ab.$$

Moreover, if $b \neq 0$ then $b_n \neq 0$ for sufficiently large n and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

证 *We prove $\lim_{n\to\infty} a_n + b_n = a + b$. Choose $\epsilon > 0$. By hypothesis there exists N_1 and N_2 such that*

$$n > N_1 \implies |a_n - a| < \frac{\epsilon}{2}, \qquad n > N_2 \implies |b_n - b| < \frac{\epsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then for $n > N$ we have

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

as required.

We prove $\lim_{n\to\infty} a_n b_n = ab$. Choose $\epsilon > 0$. By hypothesis there exists N_0,N_1,N_2 such that

$$\begin{aligned} n > N_0 &\implies |a_n - a| < 1, \\ n > N_1 &\implies |b_n - b| < \frac{\epsilon}{2(|a| + 1)}, \\ n > N_2 &\implies |a_n - a| < \frac{\epsilon}{2|b|}. \end{aligned}$$

Let $N = \max(N_0, N_1, N_2)$. Then for $n > N$ we have

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n|\,|b_n - b| + |a_n - a|\,|b| \\ &< (|a| + 1)|b_n - b| + |a_n - a|\,|b| \\ &< (|a| + 1)\frac{\epsilon}{2(|a| + 1)} + \frac{\epsilon}{2|b|}|b| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We prove $\lim_{n\to\infty} 1/b_n = 1/b$ if $b \neq 0$. Choose $\epsilon > 0$. By hypothesis there exists N_1 such that for $n > N_1$ we have that $|b_n - b| < \frac{1}{2}|b|$ (and hence that $\frac{1}{2}|b| < |b_n|$) and there exists N_2 such that $|b_n - b| < \epsilon\frac{1}{2}|b|^2$. Then for $n > \max(N_1, N_2)$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|bb_n|} < \frac{|b_n - b|}{\frac{1}{2}|b|^2} < \epsilon.$$

That $\lim_{n\to\infty} a_n/b_n = a/b$ follows immediately from˜(II) and˜(III) by substituting $1/b_n$ for b_n and $1/b$ for b □

定义 *0.2* *A real valued function f defined on a subset of the real numbers $\mathbb{R}$ is called*

1. ***increasing*** *iff $x_1 < x_2 \implies f(x_1) < f(x_2)$,*

2. ***decreasing*** *iff $x_1 > x_2 \implies f(x_1) > f(x_2)$,*

3. ***monotonic*** *iff it is either increasing or decreasing.*

If $<$ is replaced by $\leq$ in this definition, the meaning changes: a constant function satisfies the modified definition. When we want to use the weaker form we use the term **nondecreasing** instead of **increasing**, the term **nonincreasing** instead of **decreasing**, and the term **weakly monotonic** instead of **monotonic**. Thus in freshman calculus it is correct to say that **a differentiable function is nondecreasing if and only if its derivative is everywhere nonnegative** but incorrect to say that **a differentiable function is increasing if and only if its derivative is everywhere positive**. (If $f(x) = x^3$, then f is increasing but $f'(0) = 0$.) To avoid this confusion some authors insert the word **strictly** .

定理 *0.2* *A bounded weakly monotonic sequence is convergent. In fact*

$$\lim_{n\to\infty} a_n = \sup_n a_n$$

if the sequence $(a_n)_n$ is nondecreasing, and

$$\lim_{n\to\infty} a_n = \inf_n a_n$$

if the sequence $\{a_n\}$ is nonincreasing.

证 *In this proof we use the completeness axiom for the first time.*

Assume that the sequence $(a_n)_n$ is nondecreasing and let $a = \sup\{a_n : n \in \mathbb{N}\}$. Then $a_n \leq a$ for all n as a is an upperbound for the set $\{a_n : n \in \mathbb{N}\}$. Choose $\epsilon > 0$. Then $a - \epsilon < a$ so $a - \epsilon$ is not an upperbound for the set $\{a_n : n \in \mathbb{N}\}$. Hence there is an N with $a - \epsilon < a_N$. For $n > N$ we have $a_N \leq a_n$ as the sequence $(a_n)_n$ is nondecreasing so

$$a - \epsilon < a_N \leq a_n \leq a < a + \epsilon$$

so $|a_n - a| < \epsilon$ for $n > N$ as required. The nonincreasing case is proved by an analogous argument, or alternatively by applying the nondecreasing case to the seqeunce $(-a_n)_n$. □

We introduce some handy notation. For any sequence $(a_k)_k$ of real numbers we have $\{a_k : k \geq n\} \subseteq \{a_k : k \geq m\}$ for $m < n$. If the sequence $(a_k)_k$ is bounded above, then the sequence $s_n := \sup\{a_k : k \geq n\}$ is nonincreasing. The limit of the latter sequence is denoted

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

Similarly for a sequence which is bounded below,

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

定义 *0.3* *When $n_1 < n_2 < n_3 < \cdots$ is an increasing sequence of positive integers, the sequence $(p_{n_k})_k$ is called a* ***subsequence*** *of the sequence $(p_n)_n$.*

If the sequence $(p_n)_n$ converges to p, then every subsequence $(p_{n_k})_k$ also converges to p. This follows immediately from the definition of convergence: $n_k \geq k$ so if $N = N(\epsilon)$ satisfies $|p_n - p| < \epsilon$ for $n > N(\epsilon)$ then in particular we have $|p_{n_k} - p| < \epsilon$ for $k > N(\epsilon)$.

定理 *0.3* *Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.*

证 *We first do the case $m = 1$. Let $(a_n)_n$ be a bounded sequence of real numbers. Then there is a number M such that $-m \leq a_n \leq M$ for all M. For each n define*

$$A_n := \{a_{n+1}, a_{n+2}, \ldots, \}, \qquad b_n := \inf A_n.$$

Since $A_n \subset A_{n+1}$ we have that $b_n \leq b_{n+1}$ so the sequence $(b_n)_n$ converges to its supremum $b := \sup\{b_n\} := \liminf_n a_n$. We will show that a subsequence of the sequence $(a_n)_n$ also converges to b. For $n \in \mathbb{N}$ we have that $b_n < b_n + n^{-1}$ and b_n is the greatest lower bound for A_n so $b_n + n^{-1}$ is not a lower bound for A_n so there is a $c_n \in A_n$ with $b_n \leq c_n < b_n + n^{-1}$. As b_n converges to b We have that fore every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $|b_n - b| < \epsilon/2$ for $n > N(\epsilon)$. Hence for $n > \max(2/\epsilon, N(\epsilon)$ we have that

$$|c_n - b| \leq |c_n - b_n| + |b + n - b| < n^1 + \frac{\epsilon}{2} < \epsilon \tag{0.1}$$

which shows that $(c_n)_n$ converges to b_n.

Now $c_n \in A_n$ so $c_n = a_j$ for some $j = j(n) \geq n+1$, but we aren't quite done because the definition of subsequence requires that the subscripts $j(n)$ increase and there is no reason for that to be true. However we can extract a further *subsequence by induction. Namely if $n_1 < n_2 < \cdots < n_k$ have been defined, define n_{k+1} by $n_{k+1} = j(n_k)$. Then $n_{k+1} = j(n_k) \geq n_k + 1 > n_k$ as required. (The further subsequence still converges .*

Now we prove the theorem for a sequence of points in $\mathbb{R}^m$ by induction on m. Assume the theorem holds for $\mathbb{R}^m$ and choose a bounded sequence $(p_n)_n$ of points in $\mathbb{R}^{m+1}$. Then $p_n = (q_n, a_n)$ where $q_n \in \mathbb{R}^m$ and $a_n \in \mathbb{R}$. That the sequence $(p_n)_n$ is bounded means that there is an M such that $|p_n| \leq M$ for all n, As $|p_n|^2 = |q_n|^2 + a_n^2$ it follows that $|q_n| \leq M$ and $|a_n| \leq M$ for all n, i.e. the sequence $(q_n)_n$ and $(a_n)_n$ are also bounded. By the inductive hypothesis the sequence $(q_n)_n$ has a subsequence $(q_{n_k})_k$ converging to q. By replacing the sequence $(p_n)_n$ by the sequence $(p_{n_k})_k$ we may assume that the sequence $(q_n)_n$ converges.) Now by the case $m = 1$ (already proved) the sequence $(a_n)_n$ contains a convergent subsequence $(a_{n_k})_k$. Hence

$$\lim_{k\to\infty} q_{n_k} = q, \qquad \lim_{k\to\infty} a_{n_l} = a.$$

By the triangle inequality $|(q', a') - (q, a)| \leq |q' - a| + |a' - a|$ so

$$\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} (q_{n_k}, a_{n_k}) = (q, a)$$

as required. □

推论 *0.1 For a subset $S \subseteq \mathbb{R}^m$ the following conditions are equivalent.*

1. *For every sequence $(p_n)_n$ of points of S there is a subsequence $(p_{n_k})_k$ which converges to $p \in S$.*

2. *The set S closed and bounded.*

推论 *0.2 Every bounded infinite subset of $\mathbb{R}^m$ has an accumulation point.*

The image[1] Buck calls the set S the **trace** of the sequence, but that terminology is uncommon. } of the sequence $(p_n)_{n\in\mathbb{N}}$ (when viewed as a map $n \mapsto p_n$) is the set

$$S = \{p_n : n \in \mathbb{N}\}.$$

The set S can be finite. For example for the sequence $p_n = (-1)^n$, the set S is the two element set $S = \{-1, 1\}$. If the image of a sequence is finite then

[1]{

there must be at least one constant subsequence and a constant subsequence is trivially convergent. By definition only an infinite set can have an accumulation point.

定义 *0.4* *A sequence* $\{p_n\}$ *is called* ***Cauchy*** *iff*

$$\lim_{m,n\to\infty} |p_n - p_m| = 0$$

i.e. iff for every $\epsilon > 0$ *there exists* $N > 0$ *such that* $|p_n - p_m| < \epsilon$ *for* $n, m \geq N$.

定理 *0.4* *A sequence in* $\mathbb{R}^n$ *converges if and only if it is a Cauchy sequence.*