Quantum Information

Problem session 6

November 18th, 2022

Task 1:

Here is a 2-qubit state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle .$$

1. Suppose you measure the first qubit and obtain a measurement of $|0\rangle$.

   (a) What is the state after this measurement?

   Solution
   The classical state $|11\rangle$ drops off, and we obtain, after normalizing by the length

   $$\frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle\right) = \frac{\sqrt{2}}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle .$$

   (b) If you now measure the second qubit, what are the possible results and probabilities of those results?

   Solution
   We have probability $2/3$ that we obtain 00 and probability $1/3$ that we obtain 01.
   Fortunately, these add up to 1 (always a good check to do).

2. Suppose instead you measure the first qubit and obtain a measurement of $|1\rangle$.

   (a) What is the state after this measurement?

   Solution
   The classical state $|11\rangle$ is the only one surviving, and we obtain, after normalizing by the length

   $$|11\rangle .$$

   (b) If you now measure the second qubit, what are the possible results and probabilities of those results?

   Solution
   We have probability 1 that we obtain 11.

3. Is this an entangled state? Justify.

   Solution

   Yes.

   One way to see this is that in the two different outcomes to measuring the first qubit, we get differing probabilities for the outcome of the second. Hence they are not independent.

Task 2:

Suppose you have $n+1$ qubits. We will write $|\vec{x}\rangle$ to mean the n-qubit classical state given by the number x in binary. So for example, if $n=2$ then $\left|\vec{0}\right\rangle = |00\rangle$, $\left|\vec{3}\right\rangle = |11\rangle$ etc. Suppose the qubits are in this state:

$$|\Phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |\vec{x}\rangle\, |x \mod 2\rangle$$

1. What is the resulting state if we measure the last qubit and obtain $|0\rangle$? Make sure you have normalized your state.

   Solution

   All the states with last qubit $|1\rangle$ disappear from the summation. That leaves exactly half of them, i.e. those where x is even. Therefore we have

   $$\frac{1}{\sqrt{2^{n-1}}} \sum_{x=0}^{2^{n-1}-1} \left|\overrightarrow{2x}\right\rangle |0\rangle\,.$$

   Notice that the length of the vector has changed (it has half as many non-zero entries), so the scalar out front has altered. That's the normalization.

2. What is the resulting state if we measure the last qubit and obtain $|1\rangle$? Make sure you have normalized your state.

   Solution All the states with last qubit $|0\rangle$ disappear from the summation. That leaves exactly half of them, i.e. those where x is odd. Therefore we have

   $$\frac{1}{\sqrt{2^{n-1}}} \sum_{x=0}^{2^{n-1}-1} \left|\overrightarrow{2x+1}\right\rangle |1\rangle\,.$$

Task 3:

Exercise 3. Entanglement purification

The purpose of the exercise is to show from non-fully entangled states we can create with finite probability fully entangled states. We will come back to this later in the course when we will treat "entanglement purification". We have four Qbits in the state $|\Psi_\alpha\rangle \otimes |\Psi_\alpha\rangle$ where

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1-\alpha^2)^{1/2}|11\rangle$$

We measure the observable

$$Z \otimes I \otimes I \otimes I + I \otimes I \otimes Z \otimes I$$

Give all the possible outcomes of this measurement together with their respective probabilities. What is the probability that we obtain a fully entangled state ?

Hint: write the observable in the Dirac notation.

Task 4:

Exercise 4. Bell inequality for a non-maximally entangled state.

Calculate the QM prediction for the CHSH quantity (we called it X in the lecture on Bell's inequality) when the EPR pair is produced in the state

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1-\alpha^2)^{1/2}|11\rangle$$

Consider first the extreme case of separable pure states, for which

$$\langle \boldsymbol{ab}\rangle = \langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle. \tag{4.60}$$

In this case, it is clear that no Bell inequality violation can occur, because we have already seen that a (local) hidden variable theory *does* exist that correctly reproduces the predictions of quantum theory for a pure state of a single qubit. Returning to the spin-$\frac{1}{2}$ notation, suppose that we measure the spin of each particle along an axis $\hat{n} = (\sin\theta, 0, \cos\theta)$ in the xz plane. Then

$$\begin{aligned} \boldsymbol{a} &= (\boldsymbol{\sigma}^{(A)}\cdot\hat{n}_1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{pmatrix}^{(A)}, \\ \boldsymbol{b} &= (\boldsymbol{\sigma}^{(B)}\cdot\hat{n}_2) = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{pmatrix}^{(B)}, \end{aligned} \tag{4.61}$$

so that quantum mechanics predicts

$$\begin{aligned} \langle \boldsymbol{ab}\rangle &= \langle\phi|\boldsymbol{ab}|\phi\rangle \\ &= \cos\theta_1\cos\theta_2 + 2\alpha\beta\sin\theta_1\sin\theta_2 \end{aligned} \tag{4.62}$$

(and we recover $\cos(\theta_1-\theta_2)$ in the maximally entangled case $\alpha=\beta=1/\sqrt{2}$). Now let us consider, for simplicity, the (nonoptimal!) special case

$$\theta_A = 0, \quad \theta'_A = \frac{\pi}{2}, \quad \theta'_B = -\theta_B, \tag{4.63}$$

so that the quantum predictions are:

$$\begin{aligned} \langle \boldsymbol{ab}\rangle &= \cos\theta_B = \langle \boldsymbol{ab}'\rangle \\ \langle \boldsymbol{a}'\boldsymbol{b}\rangle &= 2\alpha\beta\sin\theta_B = -\langle \boldsymbol{a}'\boldsymbol{b}'\rangle \end{aligned} \tag{4.64}$$

Plugging into the CHSH inequality, we obtain

$$|\cos\theta_B - 2\alpha\beta\sin\theta_B| \le 1, \tag{4.65}$$

and we easily see that violations occur for θ_B close to 0 or π. Expanding to linear order in θ_B, the left hand side is

$$\simeq 1 - 2\alpha\beta\theta_B, \tag{4.66}$$

which surely exceeds 1 for θ_B negative and small.

We have shown, then, that *any* entangled pure state of two qubits violates some Bell inequality. It is not hard to generalize the argument to an arbitrary bipartite pure state. For bipartite pure states, then, "entangled" is equivalent to "Bell-inequality violating." For bipartite mixed states, however, we will see shortly that the situation is more subtle.

Task 5:

Exercises for circuit ***approach***
For example, H CNOT H = Z \otimes I