Quantum Information

Problem session 4

October 28st, 2022

Task 1: Ensemble Interpretation

It is a tempting (and surprisingly common) fallacy to suppose that the eigenvalues and eigenvectors of a density matrix have some special significance with regard to the ensemble of quantum states represented by that density matrix. For example, one might suppose that a quantum system with density matrix

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| . \tag{2.162}$$

must be in the state $|0\rangle$ with probability $3/4$ and in the state $|1\rangle$ with probability $1/4$. However, this is not necessarily the case. Suppose we define

$$|a\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle \tag{2.163}$$

$$|b\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle, \tag{2.164}$$

and the quantum system is prepared in the state $|a\rangle$ with probability $1/2$ and in the state $|b\rangle$ with probability $1/2$. Then it is easily checked that the corresponding density matrix is

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|. \tag{2.165}$$

That is, these two *different* ensembles of quantum states give rise to the *same* density matrix. In general, the eigenvectors and eigenvalues of a density matrix just indicate *one* of many possible ensembles that may give rise to a specific density matrix, and there is no reason to suppose it is an especially privileged ensemble.

Task 2: Measurement Process

Assume that a system S is coupled to a measuring device M. The state before the measurement is

$$|\,\Psi_0\rangle = |\,s\rangle \otimes |\,M_0\rangle\,, \qquad |\,s\rangle = \sum_n a_n\,|\,s_n\rangle \tag{32}$$

and the state after the measurement is

$$|\,\Psi\rangle = \sum_n a_n\,|\,s_n\rangle \otimes |\,M_n\rangle \tag{33}$$

where the device states $|\,M_n\rangle$ are orthogonal. Find the density operator $\hat{\rho}$ for the subsystem S after the measurement.

Solution:

After measurement the density matrix of the whole system is:

$$\hat{\rho}_{S+M} = |\,\Psi\rangle\,\langle\Psi\,| = \sum_{m,n} a_m^* a_n\,|\,s_n\rangle \otimes |\,M_n\rangle\,\langle M_m\,| \otimes \langle s_m\,|$$

The density matrix of the subsystem S is is obtained by tracing over the subsytem M:

$$\begin{aligned}\hat{\rho}_S = \mathrm{tr}_M(\hat{\rho}_{M+S}) &= \sum_k \langle M_k\,| \left(\sum_{m,n} a_m^* a_n\,|\,s_n\rangle \otimes |\,M_n\rangle\,\langle M_m\,| \otimes \langle s_m\,|\right) |\,M_k\rangle \\ &= \sum_k \sum_{m,n} a_m^* a_n\,|\,s_n\rangle\,\langle s_m\,|\,\delta_{kn}\delta_{km} = \sum_{m,n} a_m^* a_n \delta_{nm}\,|\,s_n\rangle\,\langle s_m\,| = \sum_n |a_n|^2\,|\,s_n\rangle\,\langle s_n\,|\end{aligned}$$

Task 3: Unambiguous State Discrimination

He have shown that it is not possible to discriminate non-othogonal states. But let's try anyway☺

Suppose Alice gives Bob a qubit prepared in one of two non-orthogonal states, $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. As we have seen, it is impossible for Bob to determine whether he has been given $|\psi_1\rangle$ or $|\psi_2\rangle$. with perfect reliability. However, it is possible for him to perform a measurement which distinguishes the states some of the time, but *never* makes an error of mis-identification. Consider a POVM containing three elements,

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|,$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2},$$

$$E_3 = I - E_1 - E_2.$$

It is straightforward to verify that these are positive operators which satisfy the completeness relation $\sum_m E_m = I$, and therefore form a legitimate POVM.

Task 3: cont'd

Suppose Bob is given the state $|\psi_1\rangle = |0\rangle$. He performs the measurement described by the POVM $\{E_1, E_2, E_3\}$. There is zero probability that he will observe the result E_1, since E_1 has been cleverly chosen to ensure that

$$\langle\psi_1|E_1|\psi_1\rangle = 0 \ .$$

Therefore, if the result of his measurement is E_1 then Bob can safely conclude that the state he received must have been $|\psi_2\rangle$.

A similar line of reasoning shows that if the measurement outcome E_2 occurs then it must have been the state $|\psi_1\rangle$ that Bob received. Some of the time, however, Bob will obtain the measurement outcome E_3, and he can infer nothing about the identity of the state he was given. The key point, however, is that Bob **never makes a mistake** identifying the state he has been given. This infallibility comes at the price that sometimes Bob obtains no information about the identity of the state.

Conclusion: although unambiguous state discrimination of two non-orthogonal states cannot be achieved deterministically, it can be done probabilistically.

Task 4: POVM

Exercise 7.1 Generalized Measurement by Direct (Tensor) Product

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle$ and the two systems interact via the unitary U_{AB} as follows:

$$\begin{aligned}|0\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B)\\ |1\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{6}}(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B)\end{aligned}$$

1. *Calculate the measurement operators acting on A corresponding to a measurement on B in the canonical basis $|0\rangle, |1\rangle, |2\rangle$.*

Name the output states $|\phi_{00}\rangle_{AB}$ and $|\phi_{01}\rangle_{AB}$, respectively. Although the specification of U is not complete, we have the pieces we need, and we can write $U_{AB} = \sum_{jk} |\phi_{jk}\rangle\langle jk|$ for some states $|\phi_{10}\rangle$ and $|\phi_{11}\rangle$. The measurement operators A_k are defined implicitly by

$$U_{AB}|\psi\rangle_A|0\rangle_B = \sum_k (A_k)_A|\psi\rangle_A|k\rangle_B.$$

Thus $A_k = {}_B\langle k|U_{AB}|0\rangle_B = \sum_j {}_B\langle k|\phi_{j0}\rangle_{AB}\langle j|_A$, which is an operator on system A, even though it might not look like it at first glance. We then find

$$A_0 = \frac{2}{\sqrt{6}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix}.$$

Task 4: cont'd

2. *Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?*

   The corresponding POVM elements are given by $E_j = A_j^\dagger A_j$:

$$E_0 = \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \frac{1}{6}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad E_2 = \frac{1}{6}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

   They are each rank one (which can be verified by calculating the determinant). The POVM elements project onto trine states $|1\rangle, (\sqrt{3}|0\rangle \pm |1\rangle)/2$.

3. *Suppose A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the state after a measurement, averaging over the measurement result?*

   The averaged post-measurement state is given by $\rho' = \sum_j A_j \rho A_j^\dagger$. In this case we have $\rho' = \mathrm{diag}(2/3, 1/3)$.