Quantum Information

Problem session 8

December 2nd, 2022

Task 3: Quantum Compression

10.3.1 Quantum compression: an example

Before discussing Schumacher's quantum compression protocol in full generality, it is helpful to consider a simple example. Suppose that each letter is a single qubit drawn from the ensemble

$$|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad p = \frac{1}{2}, \tag{10.121}$$

$$|\uparrow_x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad p = \frac{1}{2}, \tag{10.122}$$

so that the density operator of each letter is

$$\begin{aligned} \boldsymbol{\rho} &= \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \end{aligned} \tag{10.123}$$

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As is obvious from symmetry, the eigenstates of $\boldsymbol{\rho}$ are qubits oriented up and down along the axis $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$,

$$|0'\rangle \equiv |\uparrow_{\hat{n}}\rangle = \begin{pmatrix} \cos \frac{\pi}{8} \\ \sin \frac{\pi}{8} \end{pmatrix},$$
$$|1'\rangle \equiv |\downarrow_{\hat{n}}\rangle = \begin{pmatrix} \sin \frac{\pi}{8} \\ -\cos \frac{\pi}{8} \end{pmatrix}; \tag{10.124}$$

the eigenvalues are

$$\lambda(0') = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2 \frac{\pi}{8},$$
$$\lambda(1') = \frac{1}{2} - \frac{1}{2\sqrt{2}} = \sin^2 \frac{\pi}{8}; \tag{10.125}$$

evidently $\lambda(0') + \lambda(1') = 1$ and $\lambda(0')\lambda(1') = \frac{1}{8} = \det \boldsymbol{\rho}$. The eigenstate $|0'\rangle$ has equal (and relatively large) overlap with both signal states

$$|\langle 0'|\uparrow_z\rangle|^2 = |\langle 0'|\uparrow_x\rangle|^2 = \cos^2 \frac{\pi}{8} = .8535, \tag{10.126}$$

while $|1'\rangle$ has equal (and relatively small) overlap with both,

$$|\langle 1'|\uparrow_z\rangle|^2 = |\langle 1'|\uparrow_x\rangle|^2 = \sin^2 \frac{\pi}{8} = .1465. \tag{10.127}$$

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Thus if we don't know whether $|\uparrow_z\rangle$ or $|\uparrow_x\rangle$ was sent, the best guess we can make is $|\psi\rangle = |0'\rangle$. This guess has the maximal *fidelity* with $\boldsymbol{\rho}$

$$F = \frac{1}{2}|\langle\uparrow_z|\psi\rangle|^2 + \frac{1}{2}|\langle\uparrow_x|\psi\rangle|^2, \tag{10.128}$$

among all possible single-qubit states $|\psi\rangle$ ($F = .8535$).

Now imagine that Alice needs to send three letters to Bob, but she can afford to send only two qubits. Still, she wants Bob to reconstruct her state with the highest possible fidelity. She could send Bob two of her three letters, and ask Bob to guess $|0'\rangle$ for the third. Then Bob receives two letters with perfect fidelity, and his guess has $F = .8535$ for the third; hence $F = .8535$ overall. But is there a more clever procedure that achieves higher fidelity?

Yes, there is. By diagonalizing $\boldsymbol{\rho}$, we decomposed the Hilbert space of a single qubit into a "likely" one-dimensional subspace (spanned by $|0'\rangle$) and an "unlikely" one-dimensional subspace (spanned by $|1'\rangle$). In a similar way we can decompose the Hilbert space of three qubits into likely and unlikely subspaces. If $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$ is any signal state, where the state of each qubit is either $|\uparrow_z\rangle$ or $|\uparrow_x\rangle$, we have

$$\begin{aligned}
|\langle 0'0'0'|\psi\rangle|^2 &= \cos^6\left(\frac{\pi}{8}\right) = .6219, \\
|\langle 0'0'1'|\psi\rangle|^2 = |\langle 0'1'0'|\psi\rangle|^2 = |\langle 1'0'0'|\psi\rangle|^2 &= \cos^4\left(\frac{\pi}{8}\right)\sin^2\left(\frac{\pi}{8}\right) = .1067, \\
|\langle 0'1'1'|\psi\rangle|^2 = |\langle 1'0'1'|\psi\rangle|^2 = |\langle 1'1'0'|\psi\rangle|^2 &= \cos^2\left(\frac{\pi}{8}\right)\sin^4\left(\frac{\pi}{8}\right) = .0183, \\
|\langle 1'1'1'|\psi\rangle|^2 &= \sin^6\left(\frac{\pi}{8}\right) = .0031.
\end{aligned} \tag{10.129}$$

Thus, we may decompose the space into the likely subspace Λ spanned by $\{|0'0'0'\rangle, |0'0'1'\rangle, |0'1'0'\rangle, |1'0'0'\rangle\}$, and its orthogonal complement $\Lambda^\perp$. If we make an

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incomplete orthogonal measurement that projects a signal state onto Λ or $\Lambda^{\perp}$, the probability of projecting onto the likely subspace Λ is

$$p_{\text{likely}} = .6219 + 3(.1067) = .9419, \tag{10.130}$$

while the probability of projecting onto the unlikely subspace is

$$p_{\text{unlikely}} = 3(.0183) + .0031 = .0581. \tag{10.131}$$

To perform this measurement, Alice could, for example, first apply a unitary transformation $\boldsymbol{U}$ that rotates the four high-probability basis states to

$$|\cdot\rangle \otimes |\cdot\rangle \otimes |0\rangle, \tag{10.132}$$

and the four low-probability basis states to

$$|\cdot\rangle \otimes |\cdot\rangle \otimes |1\rangle; \tag{10.133}$$

then Alice measures the third qubit to perform the projection. If the outcome is $|0\rangle$, then Alice's input state has in effect been projected onto Λ. She sends the remaining two unmeasured qubits to Bob. When Bob receives this compressed two-qubit state $|\psi_{\text{comp}}\rangle$, he decompresses it by appending $|0\rangle$ and applying $\boldsymbol{U}^{-1}$, obtaining

$$|\psi'\rangle = \boldsymbol{U}^{-1}(|\psi_{\text{comp}}\rangle \otimes |0\rangle). \tag{10.134}$$

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If Alice's measurement of the third qubit yields $|1\rangle$, she has projected her input state onto the low-probability subspace $\Lambda^\perp$. In this event, the best thing she can do is send the state that Bob will decompress to the most likely state $|0'0'0'\rangle$ – that is, she sends the state $|\psi_{\text{comp}}\rangle$ such that

$$|\psi'\rangle = \boldsymbol{U}^{-1}(|\psi_{\text{comp}}\rangle \otimes |0\rangle) = |0'0'0'\rangle. \tag{10.135}$$

Thus, if Alice encodes the three-qubit signal state $|\psi\rangle$, sends two qubits to Bob, and Bob decodes as just described, then Bob obtains the state $\boldsymbol{\rho}'$

$$|\psi\rangle\langle\psi| \to \boldsymbol{\rho}' = \boldsymbol{E}|\psi\rangle\langle\psi|\boldsymbol{E} + |0'0'0'\rangle\langle\psi|(\boldsymbol{I}-\boldsymbol{E})|\psi\rangle\langle 0'0'0'|, \tag{10.136}$$

where $\boldsymbol{E}$ is the projection onto Λ. The fidelity achieved by this procedure is

$$\begin{aligned} F = \langle\psi|\boldsymbol{\rho}'|\psi\rangle &= (\langle\psi|\boldsymbol{E}|\psi\rangle)^2 + (\langle\psi|(\boldsymbol{I}-\boldsymbol{E})|\psi\rangle)(\langle\psi|0'0'0'\rangle)^2 \\ &= (.9419)^2 + (.0581)(.6219) = .9234. \end{aligned} \tag{10.137}$$

This is indeed better than the naive procedure of sending two of the three qubits each with perfect fidelity.

As we consider longer messages with more letters, the fidelity of the compression improves, as long as we don't try to compress too much. The Von-Neumann entropy of the one-qubit ensemble is

$$H(\boldsymbol{\rho}) = H\left(\cos^2\frac{\pi}{8}\right) = .60088\ldots \tag{10.138}$$

Therefore, according to Schumacher's theorem, we can shorten a long message by the factor, say, .6009, and still achieve very good fidelity.

Task 4: Properties of von Neumann Entropy