The EPR paradox and Bell's inequalities

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1 Bell's inequalities

The Bell inequality test is best understood through a cooperative game involving two players, Alice and Bob, that make decisions based on input from a referee. Alice and Bob are separated (sitting in different rooms, say) and cannot communicate during the game. At each round of the game, the referee sends one bit to Alice, call it x, and one bit to Bob, call it y. Depending on the value of the bit, Alice sends a bit of her own, $a(x)$, back to the referee. Similarly, Bob sends a bit of his own, $b(y)$, back to the referee. looks at both bits and decides if Alice and Bob win or lose that round. The condition for winning the round is

$$a(x) \otimes b(y) = xy \tag{1}$$

where $\otimes$ denotes addition modulo-2 (or, equivalently, XOR).

Alice and Bob's goal is to win as many rounds as possible. Although they cannot communicate during the game, they are allowed to meet before the game and set up a strategy. An example strategy might be "Alice always sends back $a(x) = x$, and Bob always sends back $b(y) = 0$." Since each of $a(x)$ and $b(y)/$ can have two possible values, there are four possible deterministic strategies that Alice and Bob can implement. Additionally, since there are only four bits involved in the entire game, it's not difficult to enumerate all possible outcomes and see which strategy allows Alice and Bob to win the most rounds (or, equivalently, win each round with the highest probability).

x	y	$a(x)$	$b(y)$	$a(x) \oplus b(y)$	xy	Win?
0	0	0	0	0	0	Yes
0	1	0	0	0	0	Yes
1	0	0	0	0	0	Yes
1	1	0	0	0	1	No

Table 1: #1 Startegy. Win rate: %75

x	y	$a(x)$	$b(y)$	$a(x) \oplus b(y)$	xy	Win?
0	0	0	0	0	0	Yes
0	1	0	1	1	0	No
1	0	0	0	0	0	Yes
1	1	0	1	1	1	Yes

Table 2: #2 Startegy. Win rate: %75

All possible outcomes of the Bell inequality test game: the first two columns show bits the referee sends to Alice (x) and Bob (y). The next two columns show Alice's response ($a(x)$) and Bob's response

x	y	$a(x)$	$b(y)$	$a(x) \oplus b(y)$	xy	Win?
0	0	1	0	1	0	No
0	1	1	1	0	0	Yes
1	0	1	0	1	0	No
1	1	1	1	0	1	No

Table 3: #3 Startegy. Win rate: %25

x	y	$a(x)$	$b(y)$	$a(x) \oplus b(y)$	xy	Win?
0	0	1	1	0	0	Yes
0	1	1	1	0	0	Yes
1	0	1	1	0	0	Yes
1	1	1	1	0	1	No

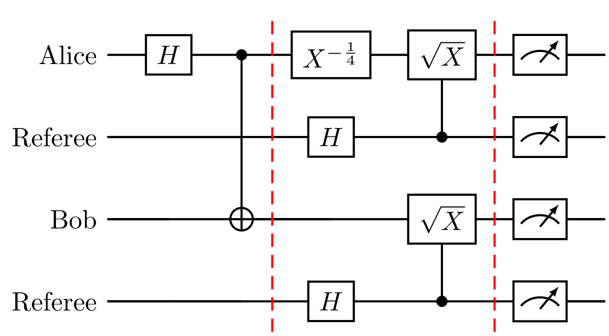
Table 4: #4 Startegy. Win rate: %75

$(b(y))$. The next column computes $a(x) \oplus b(y)$, and the next column shows the product xy. If these are equal, Alice and Bob win. The final column shows which strategy Alice and Bob are using in each row, numbered #1 - #4.

Tables (1), (2), (3) and (4) enumerates all possible outcomes of the game. By analyzing each strategy, one can see that Alice and Bob win at most 75strategies that achieve this win percentage are #1, in which $a(x) = b(y) = 0$, and #4 in which $a(x) = b(y) = 1$. So the best that Alice and Bob can do is to agree on either strategy #1 or strategy #4 before the game begins to achieve the the best possible classical win rate of 75%.

An interesting phenomena happens when we allow a quantum strategy between Alice and Bob. By a quantum strategy, we mean Alice and Bob are allowed to use entanglement as a resource in their strategy. As we have seen in this book, entanglement allows for stronger than classical correlations in physical systems. If Alice and Bob are allowed to share entangled qubits, they can remarkably win the Bell inequality test game with a higher probability! The best quantum strategy achieves a winning probability of $\cos^2(\pi/2)$, or about 85%.

The quantum strategy for this game is shown in the circuit diagram below. Here, the top (first) qubit belongs to Alice, and the third qubit from the top belongs to Bob. The first part of the circuit creates entanglement between Alice and Bob's qubits. Then, the referee "sends" in a random bit each to Alice and Bob. In the circuit, this is done by performing a Hadamard operation on a "fresh qubit" (one in the $|0\rangle$ state) to produce equal superposition. Alice and Bob then perform a control-$\sqrt{X}$ operation on their qubits and measure to record the results



2 The EPR paradox and Bell's inequalities

The most spectacular and counter-intuitive manifestation of quantum mechanics is the phenomenon of entanglement, observed in composite quantum systems. Let us now discuss the problem. The Hilbert space $\mathcal{H}$ associated with a composite system is the tensor product of the Hilbert spaces $\mathcal{H}_i$ associated with the system's components i. In the simplest case of a bipartite quantum system, we have

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{2}$$

The most natural basis for the Hilbert space $\mathcal{H}$ is constructed from the tensor products of the basis vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$. If, for example, the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are two-dimensional and

$$\{|0\rangle_1, |1\rangle_1\}, \qquad \{|0\rangle_2, |1\rangle_2\} \tag{3}$$

denote their basis vectors, then a basis for the Hilbert space H is given by the four vectors

$$\{|0\rangle_1 |0\rangle_2, |0\rangle_1 |1\rangle_2, |1\rangle_1 |0\rangle_2, |1\rangle_1 |1\rangle_2\}. \tag{4}$$

The superposition principle tells us that the most general state in the Hilbert space $\mathcal{H}$ is not a tensor product of states residing in $\mathcal{H}_1$ and $\mathcal{H}_2$, but an arbitrary superposition of such states, which we can write as follows:

$$|\psi\rangle = \sum_{i,j=0}^{1} c_{ij} |i\rangle_1 |j\rangle_2 \tag{5}$$

$$= \sum_{i,j=0}^{1} c_{ij} |ij\rangle \tag{6}$$

By definition, a state in $\mathcal{H}$ is said to be *entangled*, or *non-separable*, if it cannot be written as a simple tensor product of a state $|\alpha\rangle_1$ belonging to $\mathcal{H}_1$ and a state $|\beta\rangle_2$ belonging to $\mathcal{H}_2$. In contrast, if we can write

$$|\psi\rangle = |\alpha\rangle_1 \otimes |\beta\rangle_2 \tag{7}$$

we say that the state $|\psi\rangle$ is *separable*. As simple examples, let us consider the state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{8}$$

which is entangled, and the state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \tag{9}$$

which is separable, since we can write

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \tag{10}$$

When two systems are entangled, it is not possible to assign them individual state vectors $|\alpha\rangle_1$ and $|\beta\rangle_2$. The intriguing non-classical properties of entangled states were clearly illustrated by Einstein, Podolsky and Rosen (EPR) in 1935. These authors showed that quantum theory leads to a contradiction, provided that we accept the following two, seemingly natural, assumptions:

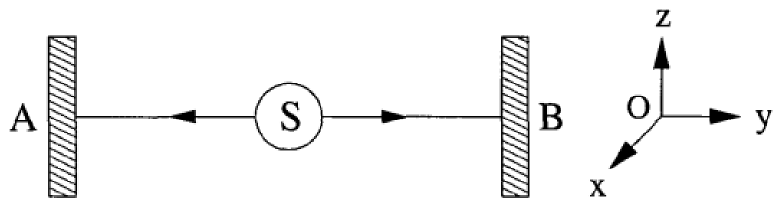


Figure 1: Schematic drawing of the EPR gedanken experiment.

1. *Reality principle*: If we can predict with certainty the value of a physical quantity, then this value has physical reality, independently of our observation. For example, if a system's wave function $|\psi\rangle$ is an eigenstate of an operator A, namely,
$$A\,|\psi\rangle = a\,|\psi\rangle \tag{11}$$
then the value a of the observable A is an element of physical reality.

2. *Locality principle*: If two systems are causally disconnected, the result of any measurement performed on one system cannot influence the result of a measurement performed on the second system. Following the theory of relativity, we say that two measurement events are disconnected if $(\Delta x)^2 > c^2(\Delta t)^2$, where Δx; and Δt are the space and time separations of the two events in some inertial reference frame and c is the speed of light (the two events take place at space-time coordinates (x_1, t_1) and (x_2, t_2), respectively, and $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$.

In quantum mechanics, if an operator B does not commute with A, then the two physical quantities corresponding to the operators A and B cannot have simultaneous reality since we cannot predict with certainty the outcome of the simultaneous measurement of both A and B. Following Heisenberg's principle, a measurement of A destroys knowledge of B.

Let us illustrate the EPR paradox by means of the following simple example, first introduced by *Bohm*. Consider a source S that emits a pair of spin-$\frac{1}{2}$ particles in the entangled state
$$|\psi\rangle = \frac{1}{2}(|01\rangle - |10\rangle) \tag{12}$$
This state is called an *EPR* or *Bell state*. We also say that the system is in a *spin-singlet state*. One spin-$\frac{1}{2}$ particle is sent to an observer called **Alice** and the second to another observer called **Bob** (see Fig. 1). Note that Alice and Bob may be located arbitrarily far away from each other. The only requirement is that the measurements performed by Alice and Bob be causally disconnected. If Alice measures the z component of the spin of the particle in her possession and obtains, for instance, $\sigma_z^{(A)} = +1$, then the EPR state collapses onto the state $|01\rangle$. Subsequently, if Bob measures the z component of the spin for his particle, he will obtain $\sigma_z^{(B)} = -1$ with unit probability. Therefore, the results of the measurements of Alice and Bob are perfectly anticorrelated. This result is not surprising according to our intuition, since it is easy to find analogous classical situations. As an example, let us consider two balls, one black and the other white. One ball is sent to Alice and the other to Bob. If Alice finds that her ball is black, then Bob will find with certainty that his ball is white. The surprising point comes from the observation that the spin-singlet state (2.175) can be also written as
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle) \tag{13}$$
where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ are eigenstates of σ_x with eigenvalues σ_x $+1$ and -1, respectively.

Exercise:
Prove that the spin-singlet state (2.175) is rotationally invariant, that is, that it takes the same form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_u |-\rangle_u - |-\rangle_u |+\rangle_u) \tag{14}$$

for any direction $\boldsymbol{u}$, the states $|+\rangle_u$ and $|-\rangle_u$ being eigenstates of $\boldsymbol{\sigma}.\boldsymbol{u}$.

This result is actually rather obvious a priori: a spin-singlet state corresponds to zero total spin; thus, no spin vector and no preferred direction can be associated with such a state.

If Alice measures $\sigma_x^{(A)}$ and obtains, for example, the outcome $\sigma_x^{(A)} = +1$, then the EPR state collapses onto $|+-\rangle$ and Bob will obtain with certainty from the measurement of $\sigma_x^{(B)}$ the outcome $\sigma_x^{(B)} = -1$. Therefore, the state of one particle depends upon the nature of the observable measured on the other particle. If Alice measures $\sigma_z^{(A)}$, then the state of Bob's particle collapses onto an eigenstate of $\sigma_z^{(B)}$. In contrast, if Alice measures $\sigma_x^{(A)}$, then the state of Bob's particle collapses onto an eigenstate of $\sigma_x^{(B)}$.

Remarks:
Using the EPR language, we say that in the first case we associate an element of physical reality with $\sigma_z^{(B)}$, in the latter with $\sigma_x^{(B)}$. It is impossible to assign simultaneous physical reality to both observables since they do not commute, $[\sigma_x, \sigma_z] = 0$.

The main point is that Alice can choose which observable to measure even after the particles have separated. Therefore, according to the locality principle, any measurement performed by Alice cannot modify the state of Bob's particle. Thus, quantum theory leads to a contradiction if we accept the principles both of realism and locality described above. The EPR conclusion was that quantum mechanics is an incomplete theory. It was later proposed that quantum theory be completed by introducing so-called *hidden variables*. The suggestion was that measurement is in reality a deterministic process, which merely appears probabilistic since some degrees of freedom (hidden variables) are not precisely known. We point out that the standard interpretation of quantum mechanics does not accept Einstein's local realism. The wave function is not seen as a physical object, but just a mathematical tool, useful to predict probabilities for the outcome of experiments.

The debate on the physical reality of quantum systems became the subject of experimental investigation after the formulation, in 1964, of Bell's inequalities. These inequalities are obtained assuming the principles of realism and locality. Since it is possible to devise situations in which quantum mechanics predicts a violation of these inequalities, any experimental observation of such a violation excludes the possibility of a local and realistic description of natural phenomena. In short, Bell showed that the principles of realism and locality lead to experimentally testable inequality relations in disagreement with the predictions of quantum mechanics.

It is instructive to derive Bell's inequalities following a simple model proposed by Wigner. Here we follow the presentation of Sakurai (1994). We assume that a source emits a large number of spin pairs in the singlet state Eq. (12). Alice and Bob each receive a member of each pair and can measure its polarization along any of three axes $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. We divide the particles in groups as follows. If Alice obtains, for instance, outcome $+1$ when she measures $\sigma_{\boldsymbol{a}}^{(A)}$, $+1$ when she measures $\sigma_{\boldsymbol{b}}^{(A)}$ and -1 when she measures $\sigma_{\boldsymbol{c}}^{(A)}$, then we say that the particle belongs to group $(\boldsymbol{a}+, \boldsymbol{b}+, \boldsymbol{c}-)$.

Remarks:
We should stress that we are not saying that Alice measures $\sigma_{\boldsymbol{a}}^{(A)}$, $\sigma_{\boldsymbol{b}}^{(A)}$ and $\sigma_{\boldsymbol{c}}^{(A)}$ simultaneously.

Population	Alice's particle	Bob's particle
N_1	$(\boldsymbol{a}+, \boldsymbol{b}+, \boldsymbol{c}+)$	$(\boldsymbol{a}-, \boldsymbol{b}-, \boldsymbol{c}-)$
N_2	$(\boldsymbol{a}+, \boldsymbol{b}+, \boldsymbol{c}-)$	$(\boldsymbol{a}-, \boldsymbol{b}-, \boldsymbol{c}+)$
N_3	$(\boldsymbol{a}+, \boldsymbol{b}-, \boldsymbol{c}+)$	$(\boldsymbol{a}-, \boldsymbol{b}+, \boldsymbol{c}-)$
N_4	$(\boldsymbol{a}+, \boldsymbol{b}-, \boldsymbol{c}-)$	$(\boldsymbol{a}-, \boldsymbol{b}+, \boldsymbol{c}+)$
N_5	$(\boldsymbol{a}-, \boldsymbol{b}+, \boldsymbol{c}+)$	$(\boldsymbol{a}+, \boldsymbol{b}-, \boldsymbol{c}-)$
N_6	$(\boldsymbol{a}-, \boldsymbol{b}+, \boldsymbol{c}-)$	$(\boldsymbol{a}+, \boldsymbol{b}-, \boldsymbol{c}+)$
N_7	$(\boldsymbol{a}-, \boldsymbol{b}-, \boldsymbol{c}+)$	$(\boldsymbol{a}+, \boldsymbol{b}+, \boldsymbol{c}-)$
N_8	$(\boldsymbol{a}-, \boldsymbol{b}-, \boldsymbol{c}-)$	$(\boldsymbol{a}+, \boldsymbol{b}+, \boldsymbol{c}+)$

Table 5: Division of the spin-singlet states into mutually exclusive groups

She may only measure any one of the spin components. For instance, if she measures $\sigma_{\boldsymbol{a}}^{(A)}$, then she measures neither $\sigma_{\boldsymbol{b}}^{(A)}$ nor $\sigma_{\boldsymbol{c}}^{(A)}$. However, according to the reality principle, we may assign well-defined values to the spin components along the three axes, that is, we assume that these values have physical reality independently of our observation.

Now remember that the results of Alice's and Bob's measurements must be perfectly anti-correlated for a spin-singlet state. Thus, if Alice's particle belongs to group $(\boldsymbol{a}-, \boldsymbol{b}+, \boldsymbol{c}-)$, then Bob's particle has to be in group $(\boldsymbol{a}-, \boldsymbol{b}-, \boldsymbol{c}+)$. The eight mutually exclusive possibilities are shown in Table. (5)

Let $p(\boldsymbol{a}+, \boldsymbol{b}+)$ denote the probability that Alice obtains $\sigma_{\boldsymbol{a}}^{(A)} = +1$ and Bob obtains $\sigma^{(B)_{\boldsymbol{b}}} = +1$. It is clearly seen from Table (5) that

$$p(\boldsymbol{a}+, \boldsymbol{b}+) = \frac{N_3 + N_4}{N_t} \tag{15}$$

where $N_t = \sum_{i=1}^{8} N_i$.

Remarks:
Why $p(\boldsymbol{a}+, \boldsymbol{b}+)$ is equal to the $(N_3 + N_4)/N_t$?
hint: Check the Table. 5

Similarly, we obtain

$$p(\boldsymbol{a}+, \boldsymbol{c}+) = \frac{N_2 + N_4}{N_t}, \qquad p(\boldsymbol{c}+, \boldsymbol{b}+) = \frac{N_3 + N_7}{N_t} \tag{16}$$

Since $N_i > 0$, we have $N_3 + N_4 < (N_2 + N_4) + (N_3 + N_7)$ and therefore we obtain the following Bell inequality:

$$p(\boldsymbol{a}+, \boldsymbol{b}+) \leq p(\boldsymbol{a}+, \boldsymbol{c}+) + p(\boldsymbol{c}+, \boldsymbol{b}+) \tag{17}$$

Attention:
We point out that we have assumed the *locality principle to derive this inequality.* Indeed, if a pair belongs to group 1 and Alice chooses to measure $\sigma_{\boldsymbol{a}}^{(A)}$, then she will certainly obtain outcome 1, independently of the fact that Bob might choose to perform a measurement along the axes $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{c}$.

We now evaluate the probabilities appearing in Bell's inequality (17) following quantum theory. Let us consider $p(a+, b+)$. If Alice finds $\sigma_{\boldsymbol{a}} = +1$, then the state of Bob's particle collapses onto the eigenstate $|-\rangle_{\boldsymbol{a}}$ of $\sigma_{\boldsymbol{a}}^{(B)}$ with eigenvalue -1. Thus, provided that $\sigma_{\boldsymbol{a}}^{(A)} = +1$, it is easy to check that Bob obtains $\sigma_{\boldsymbol{b}}^{(B)} = +1$ with probability

$$|_{\boldsymbol{b}} \langle +|-\rangle_{\boldsymbol{a}}|^2 = \sin^2(\theta_{ab}/2), \tag{18}$$

where θ_{ab} is the angle between the axes $\boldsymbol{a}$ and $\boldsymbol{b}$. Since Alice obtains $\sigma_{\boldsymbol{a}}^{(A)} = +1$ with probability one half, we obtain

$$p(\boldsymbol{a}+, \boldsymbol{b}+) = \frac{1}{2}\sin^2(\frac{\theta_{ab}}{2}) \tag{19}$$

In the same way we can compute $p(a+, c+)$ and $p(c+, b+)$. Hence, Bell's inequality (17) gives

$$\sin^2\left(\frac{\theta_{ab}}{2}\right) \leq \sin^2\left(\frac{\theta_{ac}}{2}\right) + \sin^2\left(\frac{\theta_{bc}}{2}\right) \tag{20}$$

Remarks:
If we choose the axes $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ such that

$$\theta_{ab} = 2\theta, \quad \theta_{ac} = \theta_{cb} = \theta, \tag{21}$$

then this inequality is violated for $0 < \theta < \frac{\pi}{2}$. Therefore, quantum mechanics leads to an experimentally testable violation of Bell's inequalities

As an Example, for $\theta = \dfrac{\pi}{4}$, we observe that

$$\begin{aligned} \sin^2\left(\theta\right) &\leq 2\sin^2\left(\frac{\theta}{2}\right) \\ 0.5 &\leq 0.292 \end{aligned} \tag{22}$$

This results in a violation of the inequality 17.

We now give an alternative derivation of Bell's inequalities. Let us assume that there exists a ***hidden variable*** λ such that, for any value of λ, a well-defined (*deterministic*) result $O(\lambda)$ is obtained from the measurement of a physical observable O. We require that the distribution probability $\mathcal{P}(\lambda)$ of the variable λ be such that the average values predicted by quantum mechanics are recovered; that is,

$$\langle O \rangle = \int O(\lambda)\mathcal{P}(\lambda)d\lambda \tag{23}$$

Let us consider the EPR *gedanken* experiment drawn schematically in Fig. 1. We call $A(\boldsymbol{a}, \lambda)$ and $B(\boldsymbol{b}, \lambda)$ the results of the measurements of the (causally disconnected) spin polarizations $\boldsymbol{\sigma}^{(A)}.\boldsymbol{a}$, $\boldsymbol{\sigma}^{(B)}.\boldsymbol{b}$ and along the directions $\boldsymbol{a}$ and $\boldsymbol{b}$, performed by Alice and Bob, respectively. Assuming the locality principle, the outcome of Alice's measurements cannot depend on the outcome of Bob's measurements. Therefore, the mean value of the correlations between their polarization measurements is given by

$$C(\boldsymbol{a}, \boldsymbol{b}) = \int A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda)\mathcal{P}(\lambda)d\lambda \tag{24}$$

For example, as we have seen above, quantum mechanics predicts perfect anticorrelation for the EPR state (2.175) when $\boldsymbol{a} = \boldsymbol{b}$ and therefore

$$C(\boldsymbol{a}, \boldsymbol{a})_{\text{quantum}} = -1 \tag{25}$$

Attention:
We know

$$C(\boldsymbol{a}, \boldsymbol{a}) = \int A(\boldsymbol{a}, \lambda)B(\boldsymbol{a}, \lambda)\mathcal{P}(\lambda)d\lambda \tag{26}$$

which means Alice and Bob are applying same observable $\boldsymbol{a}$ on their particle separately. Due to the attribute of the singlet-state we now the results of measuring $\boldsymbol{a}$ for Alice and Bob is the same with different sign.

Let us compute

$$C(\boldsymbol{a}, \boldsymbol{b}) - C(\boldsymbol{a}, \boldsymbol{b}') = \int \Big[A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda) - A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}', \lambda)\Big]\mathcal{P}(\lambda)d\lambda \tag{27}$$

By adding and subtracting $\pm A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda)A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}', \lambda)$ and $\pm A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}', \lambda)A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}, \lambda)$ from the first and second terms respectively, we will have

$$\begin{aligned} &= \int \Big[A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda) \pm A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda)A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}', \lambda) \\ &\quad -B(\boldsymbol{a}, \lambda)A(\boldsymbol{b}', \lambda) \pm A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}', \lambda)A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}, \lambda)\Big]\mathcal{P}(\lambda)d\lambda \\ &= \int A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}, \lambda)\Big[1 \pm A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}', \lambda)\Big]\mathcal{P}(\lambda)d\lambda - \int A(\boldsymbol{a}, \lambda)B(\boldsymbol{b}', \lambda)\Big[1 \pm A(\boldsymbol{a}', \lambda)B(\boldsymbol{b}, \lambda)\Big]\mathcal{P}(\lambda)d\lambda \end{aligned} \tag{28}$$

Since $A(\boldsymbol{a}, A)$ and $B(\boldsymbol{b}, A)$ are polarization measurements, we have

$$|A(\boldsymbol{a}, \lambda)| = 1, \qquad |B(\boldsymbol{b}, \lambda)| = 1 \tag{29}$$

Moreover, $\mathcal{P}(\lambda)$ is a probability distribution and therefore is non-negative for any λ. Thus, we have

$$\begin{aligned}
\left|C(\boldsymbol{a},\boldsymbol{b}) - C(\boldsymbol{a},\boldsymbol{b}')\right| &\leq \int \Big[1 \pm A(\boldsymbol{a}',\lambda)B(\boldsymbol{b}',\lambda)\Big]\mathcal{P}(\lambda)d\lambda + \int \Big[1 \pm A(\boldsymbol{a}',\lambda)B(\boldsymbol{b},\lambda)\Big]\mathcal{P}(\lambda)d\lambda \\
&\leq \pm \int A(\boldsymbol{a}',\lambda)B(\boldsymbol{b}',\lambda)\mathcal{P}(\lambda)d\lambda \pm \int A(\boldsymbol{a}',\lambda)B(\boldsymbol{b},\lambda)\mathcal{P}(\lambda)d\lambda + 2\int \mathcal{P}(\lambda)d\lambda \\
&\leq \pm C(\boldsymbol{a}',\boldsymbol{b}') \pm C(\boldsymbol{a}',\boldsymbol{b}) + 2\int \mathcal{P}(\lambda)d\lambda \\
&\leq -\Big|C(\boldsymbol{a}',\boldsymbol{b}' + C(\boldsymbol{a}',\boldsymbol{b})\Big| + 2\int \mathcal{P}(\lambda)d\lambda
\end{aligned} \tag{30}$$

and therefore

$$\Big|C(\boldsymbol{a},\boldsymbol{b}) - C(\boldsymbol{a},\boldsymbol{b}')\Big| \leq -\Big|C(\boldsymbol{a}',\boldsymbol{b}' + C(\boldsymbol{a}',\boldsymbol{b})\Big| + 2\int \mathcal{P}(\lambda)d\lambda \tag{31}$$

we finally obtain

$$\Big|C(\boldsymbol{a},\boldsymbol{b}) - C(\boldsymbol{a},\boldsymbol{b}')\Big| + \Big|C(\boldsymbol{a}',\boldsymbol{b}) + C(\boldsymbol{a}',\boldsymbol{b}')\Big| \leq 2 \tag{32}$$

where we have used the normalization of the probability distribution $\mathcal{P}(\lambda)$, that is, $\int \mathcal{P}(\lambda)d\lambda = 1$. Inequality (32) is known as the *CHSH inequality*, after its four discovers (**Clauser, Home, Shimony and Holt**). It is an example of a larger set known as Bell's inequalities.

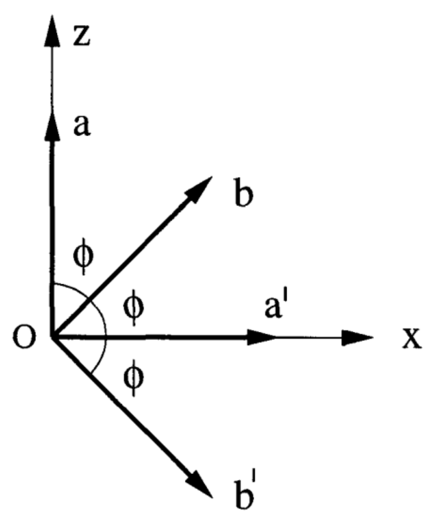


Figure 2: Choice of directions leading to a violation of the CHSH inequality (32). The angles labelled ϕ are taken equal to $\frac{\pi}{4}$.

Attention:
The main point is that there exist directions $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}',\boldsymbol{b}')$ such that, considering entangled states, quantum mechanics violates the CHSH inequality. For instance, we may consider the set of directions $\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}'$ shown in Fig. 2.

Exercise:
Show that the quantum-mechanical mean value of the correlation

$$C(\boldsymbol{a},\boldsymbol{b})_{\mathrm{quantum}} = \langle\psi|(\boldsymbol{\sigma}^{(A)}.\boldsymbol{a})(\boldsymbol{\sigma}^{(B)}.\boldsymbol{b})|\psi\rangle \tag{33}$$

is equal to $-\boldsymbol{a}.\boldsymbol{b}$ when ψ is the EPR state (12)

For the spin-singlet state (12), quantum mechanics predicts that

$$C(\boldsymbol{a},\boldsymbol{b}) = -\boldsymbol{a}.\boldsymbol{b} = -\cos(\theta_{\boldsymbol{ab}}) \tag{34}$$

where $\theta_{\boldsymbol{ab}}$, is the angle between the directions a and b, thus we have

$$\begin{aligned}\left\{\left|C(\boldsymbol{a},\boldsymbol{b}) - C(\boldsymbol{a},\boldsymbol{b}')\right| + \left|C(\boldsymbol{a}',\boldsymbol{b} + C(\boldsymbol{a}',\boldsymbol{b}')\right|\right\}_{quantum} &= \left|-\cos(\phi) + \cos(3\phi)\right| + \left|-\cos(\phi) - \cos(\phi)\right| \\ &= 2\sqrt{2} \geq 2 \end{aligned} \tag{35}$$

when $\phi = \dfrac{\pi}{4}$.

Bell's inequalities represent, first of all, an experimental test of the consistency of quantum mechanics. Many experiments have been performed in order to check Bell's inequalities; the most famous involved EPR pairs of photons and was performed by Aspect and co-workers in 1982. This experiment displayed an unambiguous violation of the CHSH inequality by tens of standard deviations and an excellent agreement with quantum mechanics. More recently, other experiments have come closer to the requirements of the ideal EPR scheme and again impressive agreement with the predictions of quantum mechanics has always been found. Nonetheless, there is no general consensus as to whether or not these experiments may be considered conclusive, owing to the limited efficiency of detectors. If, for the sake of argument, we assume that the present results will not be contradicted by future experiments with high-efficiency detectors, we must conclude that Nature does not experimentally support the EPR point of view. In summary, the World is not locally realistic. We should stress that there is more to learn from Bell's inequalities and Aspect's experiments than merely a consistency test of quantum mechanics. These profound results show us that entanglement is a fundamentally new resource, beyond the realm of classical physics, and that it is possible to experimentally manipulate entangled states. As we shall see in the following chapters, a major goal of quantum-information science is to exploit this resource to perform computation and communication tasks beyond classical capabilities.