Quantum Measurements

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1 Measurement of the density matrix for a single qubit

We saw that the coordinates (x, y, z) singling out a pure state on the Bloch sphere can be measured, provided a large number of states prepared in the same manner are available. We now show that the same conclusions hold for mixed states. Consider the generic state for q qubit

$$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin|1\rangle \tag{1}$$

The corresponding density operator is given by

$$\begin{aligned}\rho(\theta,\phi) &= |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| \\ &= \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}e^{-i\phi} \\ \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}e^{i\phi} & \cos^2\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{2}$$

It is easy to check that $\rho^2(\theta,\phi) = \rho(\theta,\phi)$, as it must be for a pure state. We can also extend the $\rho(\theta,\phi)$ based on the Pauli operators and identity matrix which are the generators of SU(2).

$$\rho(\theta,\phi) = aI + b\sigma_x + c\sigma_y + b\sigma_z \tag{3}$$

$$\begin{aligned} &= \frac{1}{2}\Big(I + x\sigma_x + y\sigma_y + z\sigma_z\Big) \\ &= \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \end{aligned} \tag{4}$$

which $x = 2b$, $y = 2c$, $z = 2d$.

The measurement procedure is shown in Fig. (1) a unitary transformation U maps ρ into a new density matrix $\rho^{'} = U\rho U^\dagger$ and the detector D measures σ_z. The possible outcomes of this measurement are $i = 0, 1$ (we associate $i = 0$ with $\sigma_z = +1$ and $i = 1$ with $\sigma_z = -1$), obtained with probabilities

$$p_i = \mathrm{Tr}(\rho' P_i) \tag{5}$$

where the projector operators P_i read in the $\{|0\rangle, |1\rangle\}$ basis as follows:

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

We can also write

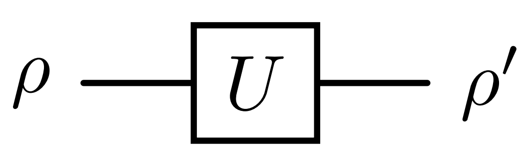

Figure 1: A schematic drawing of the measurement of the density matrix. The unitary transformation U comes before a standard measurement performed by the detector D.

$$p_i = \mathrm{Tr}(U\rho U^\dagger P_i) = \mathrm{Tr}(\rho U^\dagger P_i U) = \mathrm{Tr}(\rho Q_i) \tag{7}$$

where we have defined new operatin

$$Q_i = U^\dagger P_i U \tag{8}$$

In order to measure the coordinate z, we take $U = I$, so that $Q_0 = P_0$ and $Q_1 = P_1$. It is easy to compute p_0, p_1 and to check that

$$p_0 = \mathrm{Tr}(\rho P_0) = \mathrm{Tr}(\frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \frac{1}{2}(1-z) \tag{9}$$

$$p_1 = \mathrm{Tr}(\rho P_1) = \mathrm{Tr}(\frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \frac{1}{2}(1-z) \tag{10}$$

$$p_0 - p_1 = \frac{1}{2}\Big(1 + z - (1-z)\Big) = z \tag{11}$$

Remarks:

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{12}$$

To compute x, we take $U = R_y(-\frac{\pi}{2})$; that is, the Bloch sphere is rotated clockwise through an angle $\frac{\pi}{2}$ about the y-axis. In this manner, the x-axis is transformed into the z-axis and the coordinate x can be computed by measuring σ_z. Hence, we consider

$$R_y(-\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{13}$$

and therefore

$$Q_0 = U^\dagger P_0 U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Q_1 = U^\dagger P_1 U = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{14}$$

$$p_0 = \mathrm{Tr}(\rho Q_0) = \mathrm{Tr}(\frac{1}{4}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) = \frac{1}{2}(1+x) \tag{15}$$

$$p_1 = \mathrm{Tr}(\rho Q_1) = \mathrm{Tr}(\frac{1}{4}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}) = \frac{1}{2}(1-x) \tag{16}$$

$$p_0 - p_1 = \frac{1}{2}\Big(1 + x - (1-x)\Big) = x \tag{17}$$

Remarks:

$$R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{18}$$

Likewise, we can compute y. We take

$$R_x(-\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{19}$$

and therefore

$$Q_0 = U^\dagger P_0 U = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad Q_1 = U^\dagger P_1 U = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{20}$$

$$p_0 = \mathrm{Tr}(\rho Q_0) = \mathrm{Tr}(\frac{1}{4}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}(1-y) \tag{21}$$

$$p_1 = \mathrm{Tr}(\rho Q_1) = \mathrm{Tr}(\frac{1}{4}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}) = \frac{1}{2}(1+y) \tag{22}$$

$$p_0 - p_1 = \frac{1}{2}\Big(1 - y - (1+y)\Big) = -y \tag{23}$$

2 Generalized Measurement

A generalized measurement is described by a set $\{Mi\}$ of measurement operators, not necessarily self-adjoint, that satisfy the completeness relation

$$\sum_i M_i M_i^\dagger = I \tag{24}$$

If the state vector of the system before the measurement is $|\psi\rangle$, then with probability

$$p_i = \langle\psi| M_i M_i^\dagger |\psi\rangle \tag{25}$$

the measurement gives outcome i and the post-measurement state of the system is

$$|\psi'\rangle \quad = \quad \frac{M_i |\psi\rangle}{\sqrt{\langle\psi_i| M_i M_i^\dagger |\psi_i\rangle}} = \frac{M_i |\psi\rangle}{\sqrt{p_i}} \tag{26}$$

We note that the completeness equation (25) assures the fact that the probabilities sum to unity; that is,

$$\begin{aligned} \sum_i p_i &= \sum_i \langle\psi| M_i M_i^\dagger |\psi\rangle \\ &= \langle\psi| \sum_i M_i M_i^\dagger |\psi\rangle = \langle\psi|\psi\rangle \\ &= 1 \end{aligned} \tag{27}$$

We note that the projective measurements described in Sec. 2.4 are a special case of generalized measurements, in which the operators M_i are orthogonal projectors; that is

$$M_i = M_i^\dagger \tag{28}$$

$$M_i M_j^\dagger = \delta_{ij} M_i \tag{29}$$

Therefore, in this case, the completeness relation becomes

$$\sum_i M_i = I \tag{30}$$

It turns out that projective measurements together with unitary operations are equivalent to generalized measurements, provided ancillary qubits are added. This simply means that generalized measurements are equivalent to projective measurements on a larger Hilbert space. This statement is known as Neumark's theorem and is discussed, e.g., in Peres (1993).

In the following, we show that, if we restrict our attention to a subsystem of a given system, a projective measurement performed on the system cannot in general be described as a projective measurement on the subsystem. Let us consider the unitary evolution of a composite system 1 + 2, initially in the state $|\psi\rangle_1 |0\rangle_2$:

$$U |\psi\rangle_1 |0\rangle_2 \equiv \sum_k E_k |\psi\rangle_1 |k\rangle_2 \tag{31}$$

where $\{|k\rangle_2\}$ is an orthonormal basis for subsystem 2 and Kraus operators E_k satisfy the condition

$$\sum_k E_i E_k^\dagger = I_1, \tag{32}$$

and, in general, are not projectors. A projective measurement, described by the projectors

$$P_i = I \otimes |i\rangle_2 \langle i|_2 \tag{33}$$

with

$$\sum_i P_i = I_1 \otimes I_2 \tag{34}$$

gives outcome i with probability

$$
\begin{aligned}
p_i &= \mathrm{Tr}(\rho_{12} P_i) = \mathrm{Tr}\left(\sum_{k,k'} E_k \left|\psi\right\rangle_1 \left|k\right\rangle_2 \left\langle\psi\right|_1 \left\langle k'\right|_2 E_{k'}^\dagger \Big(I \otimes \left|i\right\rangle_2 \left\langle i\right|_2 \Big) \right) &(35)\\
&= {}_1\left\langle \psi | E_k E_k^\dagger | \psi \right\rangle_1 &(36)
\end{aligned}
$$

Therefore, a standard projective measurement performed on the system can be described as a generalized measurement on subsystem 1.

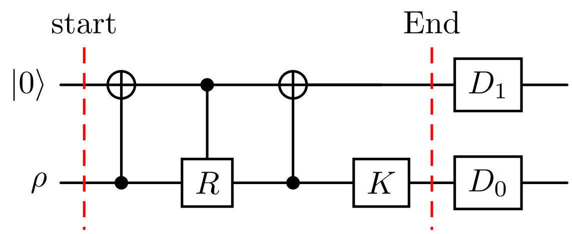

Figure 2: A quantum circuit implementing a POVM measurement. The letters D_0 and D_1 denote two detectors performing standard projective single qubit measurements.

3 POVM Measurements

POVM's ("Positive Operator-Valued Measurements") are well suited to describing experiments where the system is measured **only once** and therefore we are not interested in the **state of the system after the measurement**. This is, for instance, the case of a photon detected by a photomultiplier: the photon is destroyed in the measurement process and therefore the measurement cannot be repeated. A POVM is described by a set of positive (more precisely, non-negative) operators F_i (POVM elements), such that

$$\sum_i F_i = I \tag{37}$$

If the measurement is performed on a system described by the state vector $|\psi\rangle$, the probability of obtaining outcome i is

$$p_i = \langle\psi|F_i|\psi\rangle \tag{38}$$

POVM's can be seen as generalized measurements, provided we define

$$F_i = M_i M_i^\dagger \tag{39}$$

Indeed, it is evident that this definition assures that F_i is a non-negative operator. It is also clear that projective measurements are POVM's since in this case

$$F_i = M_i M_i^\dagger = M_i \tag{40}$$

$$\sum_i F_i = \sum_i M_i = I \tag{41}$$

However, we stress that in the POVM formalism we do not make any assumption on the post-measurement state of the system. An example of POVM is shown in Fig. 2. The system qubit is initially in the state ρ, the environment qubit in the state $|0\rangle$. In Fig. 2, R denotes the rotation matrix

$$R = \begin{pmatrix} r & t \\ -t & r \end{pmatrix} \tag{42}$$

(we assume $0 < r < 1$ and $t = \sqrt{1 - r_2}$) and K a modified Hadamard matrix:

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{43}$$

It is easy to check by direct matrix multiplication that the circuit in Fig. 2 implements the unitary transformation

$$\begin{aligned}
U &= (I \otimes K) \times U^{21}_{CNOT} \times U_{CR} \times U^{21}_{CNOT} &(44)\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & t \\ 0 & 0 & -t & r \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} &(45)\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & t \\ 0 & 0 & -t & r \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} &(46)
\end{aligned}$$

and eventually we reach to

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & r & -t & 0 \\ -1 & r & -t & 0 \\ 0 & t & r & 1 \\ 0 & -t & -r & 1 \end{pmatrix} \tag{47}$$

The two detectors D_1 and D_0 drawn in Fig. 2 perform a standard projective measurement with possible outcomes 0 and 1. In general, we have four possible outcomes: $00, 01, 10$ and 11 (in integer notation, $0, 1, 2$ and 3), associated with the projectors

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

The probability of obtaining outcome i is given by

$$\begin{aligned} p_i &= \mathrm{Tr}(U \rho_{in}^{(tot)} U^\dagger P_i) \\ &= \mathrm{Tr}(\rho_{in}^{(tot)} Q_i) \end{aligned} \tag{50}$$

where $\rho_{in}^{(tot)}$ is the initial two-qubit state

$$\rho_{in}^{(tot)} = |0\rangle \langle 0| \otimes \rho \tag{51}$$

with matrix representation

$$\rho_{in}^{(tot)} = \begin{pmatrix} \rho & \hat{0} \\ \hat{0} & \hat{0} \end{pmatrix}_{4\times 4} \tag{52}$$

where ρ and $\hat{0}$ are 2×2 submatrices and $\hat{0}$ has all matrix elements equal to 0.

$$\hat{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \tag{53}$$

Given this initial state, we have

$$p_i = \mathrm{Tr}(\rho_{in}^{(tot)} Q_i) \tag{54}$$

which Q_i are equal to

$$Q_0 = U^\dagger P_0 U = \frac{1}{2} \begin{pmatrix} 1 & r & -t & 0 \\ -r & r^2 & -tr & 0 \\ -t & -tr & t^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{55}$$

$$Q_1 = U^\dagger P_1 U = \frac{1}{2} \begin{pmatrix} -1 & -r & t & 0 \\ -r & r^2 & -tr & 0 \\ t & -tr & t^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{56}$$

$$Q_2 = U^\dagger P_2 U = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t^2 & tr & t \\ 0 & tr & r^2 & r \\ 0 & t & r & 1 \end{pmatrix} \tag{57}$$

$$Q_3 = U^\dagger P_3 U = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t^2 & tr & -t \\ 0 & tr & r^2 & -r \\ 0 & -t & -r & 1 \end{pmatrix} \tag{58}$$

For POVM operators we can write

$$p_i = \mathrm{Tr}(\rho_{in}^{(tot)} Q_i) = \mathrm{Tr}(\rho F_i) \tag{59}$$

where F_i is the 2×2 submatrix of Q_i corresponding to the value 0 of the most significant qubit. In particular, if $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $p_i = \langle\psi|F_i|\psi\rangle$.

We obtain

$$F_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & r^2 \end{pmatrix}, \qquad F_1 = \frac{1}{2}\begin{pmatrix} 1 & -r \\ -r & r^2 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1-r^2 \end{pmatrix} \tag{60}$$

where we have added in F_2 the contributions coming from Q_2 and Q_3 since they are identical. The F_i constitute a POVM. Indeed, they are nonnegative operators and fulfill the condition $\sum_i F_i = I$

$$\sum_i F_i = F_0 + F_1 + F_2 \tag{61}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & r \\ r & r^2 \end{pmatrix} + \begin{pmatrix} 1 & -r \\ -r & r^2 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1-r^2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{62}$$

POVM measurements are useful, for instance, to avoid misidentification of **non-orthogonal states**. Let us consider the following example: **Alice** sends **Bob** one of the following two states:

$$|\psi_1\rangle = \sin\theta\,|0\rangle + \cos\theta\,|1\rangle \qquad |\psi_2\rangle = \sin\theta\,|0\rangle - \cos\theta\,|1\rangle \tag{63}$$

where we assume $0 < \theta < \pi/4$. Then Bob performs on the received state a measurement described by the POVM elements F_0, F_1 and F_2 defined by Eq. 60. Bob's probability of obtaining outcome i, provided he received the state $|\psi_k\rangle$ $(k = 1, 2)$, is

$$p(i|k) = \langle\psi_k|F_i|\psi_k\rangle \tag{64}$$

We choose $r = \tan\theta$. We have $p(1|1) = 0$ and $p(0|2) = 0$. Therefore, the outcome $i = 1$ excludes that the state $|\psi_1\rangle$, was sent, whereas $i = 0$ excludes $|\psi_2\rangle$. Finally, if we obtain outcome $i = 2$, we cannot conclude anything. Bob cannot always distinguish which one of the two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ was sent. However, taking advantage of POVM measurements, he can avoid misidentification.