Quantum Information Problem Session

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1 Problem Session One

1.1 Question:

Determine if each of the following function is acceptable as a wavefunction over the indicated regions:

1. Cos(x) over $x \in (0, \infty)$
2. Exp(x) over over $x \in (-\infty, \infty)$
3. Exp($-x$) over over $x \in (0, \infty)$
4. Tan(x) over over $x \in (0, 2\pi)$

1.2 Question:Proof

Proof that two orthogonal states can be fully distinguished.

Consider two parties, Alice and Bob. Alice choose a state $|\psi_i\rangle$ $(1 \leq i \leq n)$ from a fixed set of states known to both parties. Bob uses the following set of measurement operators $\{M_i\}_{i=0}^{n}$, $M_i = |\Psi_i\rangle\langle\Psi_i|$.

1.3 Question:Proof

Proof that two non-orthogonal states cannot be reliably distinguished.

Consider Alice prepares one of two non-orthonormal states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Bob uses the set of measurement operators $\{M_i\}$ and depending on the outcome of his measurement he tries to guess the index i of Alice's state using some rule $i = f(i)$, *e.g.* $f(1) = 1, f(2) = 2$.

1.4 Question:

A system is described by a 2-dimensional space of states. Let $|1\rangle$ and $|2\rangle$ be the normalized eigenstates of an observable $\mathcal{O}$ with eigenvalues $\lambda = 1$ and $\lambda = 2$ respectively. Let $|v\rangle = |1\rangle + i\sqrt{3}\,|2\rangle$ be a state of the system.

1. What is the probability that the measure of $\mathcal{O}$ on $|v\rangle$ gives the result $\lambda = 2$?
2. Write the 2×2 matrix which represents $\mathcal{O}$ in the basis $\{|1\rangle, |2\rangle\}$
3. Let $\mathcal{O}'$ be another observable, such that

$$\mathcal{O}'\,|1\rangle = |2\rangle \tag{1}$$

$$\mathcal{O}'\,|2\rangle = |1\rangle \tag{2}$$

4. What are the eigenvalues of $\mathcal{O}'$?
5. What is the average of $\mathcal{O}'$ on the state $|v\rangle$

1.5 Question:

On a spin 1/2 system the following 3 observables are defined

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

1. Find the eigenvalues and the normalized eigenvectors of the 3 observables $\{S_x, S_y, S_z\}$

2. Suppose the system is in the state described by the vector

$$|v\rangle = \begin{pmatrix} 1 \\ 3i \end{pmatrix} \tag{4}$$

   Find the probabilities to obtain the various possible values of $\{S_x, S_y, S_z\}$ on $|v\rangle$.

3. Compute the 2×2 unitary matrix which connects the basis of eigenvectors of S_z with the basis of eigenvectors of S_y.

4. Find the eigenvalues and the normalized eigenvectors of the observable

$$\vec{S}.\vec{n} = n_x S_x + n_y S_y + n_z S_z \tag{5}$$

   where

$$\vec{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{6}$$

   θ and ϕ are polar coordinates:

$$0 \leq \theta \leq 2\pi \tag{7}$$
$$0 \leq \phi \leq \pi \tag{8}$$
$$\tag{9}$$

5. Compute the 2×2 unitary matrix which connects the basis of eigenvectors of S_z with the basis of eigenvectors of $\vec{S}.\vec{n}$

2 Problem Session Two

2.1 Question:Commutation relation for the Pauli matrices

Verify the commutation relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z; \qquad [\sigma_y, \sigma_z] = 2i\sigma_x; \qquad [\sigma_z, \sigma_x] = 2i\sigma_y. \tag{10}$$

There is an elegant way of writing this using $\epsilon_{ijk} = 0$ expect for $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$:

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk}\sigma_k \tag{11}$$

2.2 Question: Heisenberg Uncertainty Relation

Perhaps the best known result of quantum mechanics is the *Heisenberg uncertainty principle*. Suppose A, B are two Hermitian operators, and $|\psi\rangle$ is a quantum state. Suppose

$$\langle\psi|AB|\psi\rangle = x + iy, \tag{12}$$

where x and y are real. Note that $\langle\psi|[A,B]|\psi\rangle = 2iy$ and $\langle\psi|\{A,B\}|\psi\rangle = 2x$. this implies that

$$|\langle\psi|[A,B]|\psi\rangle|^2 + |\langle\psi|\{A,b\}|\psi\rangle|^2 = 4|\langle\psi|AB|\psi\rangle|^2. \tag{13}$$

By the Cauchy-Schwarz inequality

$$|\langle\psi|AB|\psi\rangle|^2 \leq 4\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \tag{14}$$

Suppose C and D are two observables. Substituting $A = C - \langle C\rangle$ and $B = D - \langle D\rangle$ into the last equation, we obtain Heisenberg's uncertainty principle as it is usually stated:

$$\Delta(C)\Delta(D) \geq \frac{\langle\psi|[C,D]|\psi\rangle}{2} \tag{15}$$

2.3 Question: Tensor Product

1. let $\psi = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out:
   - $|\psi\rangle^{\otimes 2}$
   - $|\psi\rangle^{\otimes 3}$
2. Calculate the matrix representation of the tensor products of the Pauli operators

2.4 Question: Bell States

2.5 Question: Characterization of Density Operators

1. An operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies the condition:
   - (Trace condition) ρ has trace equal to one
   - (Positivity condition) ρ has trace equal to one
2. How can we say a density matrix is pure or mixed state?
3. Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{I + \vec{r}.\vec{\sigma}}{2} \tag{16}$$

where $\vec{r}$ is a real three-dimensional vector such that $||\vec{r}|| \leq 1$. This vector is known as the *Bloch vector* for the state ρ

4. What is the Bloch vector representation for the state $\rho = I/2$?
5. Show that a state ρ is pure if and only if $||\vec{r}|| = 1$.
6. Suppose a quantum system is in the product state $\rho^{AB} = \rho \otimes \sigma$, where ρ is a density operator for system A, and σ is a density operator for system B. then

$$\rho_A = \mathrm{tr}_B(\rho \otimes \sigma) = \rho\mathrm{tr}(\sigma) = \rho \tag{17}$$

which is the result we intuitively expect.Similarly, $\rho^B = \sigma$ for this state. A less trivial example is the Bell state $(|00\rangle + |11\rangle / \sqrt{2})$.

3 Problem Session Three

3.1 Question: Singular value decomposition

Let A be a square matrix. Then there exist unitary matrices U, and V, and a diagonal matrix D with non-negative entries such that

$$A = UDV \tag{18}$$

The diagonal elements of D are called the singular values of A.

3.2 Question: Singlet State

Suppose we prepare the two qubit state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

a state sometimes known as the spin singlet for historical reasons. It is not difficult to show that this state is an entangled state of the two qubit system. Suppose we perform a measurement of spin along the $\vec{v}$ axis on both qubits, that is, we measure the observable $\vec{v}.\vec{\sigma}$ on each qubit, getting a result of +1 or -1 for each qubit. it turns out that no matter what choice or $\vec{v}$ we make, the results of the two measurements are always opposite to one another. That is, if the measurement on the first qubit yields +1, then the measurement on the second qubit will yield -1, and vice versa. It is as though the second qubit knows the result of the measurement on the first, no matter how the first qubit is measured. To see why this is true, suppose $|\alpha\rangle$ and $|b\rangle$ are the eigenstates of $\vec{v}.\vec{\sigma}$. Then there exist complex numbers α, β, γ, δ such that

3.3 Question: Bell Inequalities

3.4 Question: Density Operator

1. A harmonic oscillator has an equal classical probability 1/3 to be found in each of the states $|0\rangle$, $|1\rangle$ and $4\,|0\rangle + 3\,|1\rangle$. Write down the corresponding density matrix $\hat{\rho}$ explicitly.

2. Consider a system $\sum$ consisting of two subsystem I and II. $\sum$ is in a pure state $|\psi_{\sum}\rangle$, where $|\psi_{\sum}\rangle$ is a vector in the product space. The density matrices $\hat{\rho}_{I,II}$ and $\hat{\rho}_{\sum}$ are defined as in the lecture. Remembering that

$$\hat{\rho}_I = \mathrm{tr}_{II}\hat{\rho}_{\sum} \equiv \langle^{II}\delta^i|\hat{\rho}_{\sum}|^{II}\delta^i\rangle$$

   and

$$\hat{\rho}_{II} = \mathrm{tr}_{I}\hat{\rho}_{\sum} \equiv \langle^{I}\delta^i|\hat{\rho}_{\sum}|^{I}\delta^i\rangle$$

   and given that $\{n\}_{n\in N}$ is a set of orthonormal vectors, calculate the density matrix for subsystems I and II, for

$$|\psi_{\sum}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \times |1\rangle + |1\rangle \times |2\rangle)$$

3. consider $\hat{\rho}_I$ of the previous point. Does it correspond to a pure or to a mixed state?

4. Given a density matrix, the entropy of the described state is given by $S = -\mathrm{tr}(\hat{\rho}\log(\hat{\rho}))$. Express the entropy as a function of the eigenvalues of $\hat{\rho}$.
   It can be shown that the maximum entropy is reached when all eigenvalues are equal, i.e. when every state is possible with the same probability; what is the maximum entropy of $\hat{\rho}_I$ and $\hat{\rho}_I I$ of the previous part.

3.5 Question: Schmidt Decomposition

1. Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that ψ is a product state if and only if ρ^A(and thus ρ^B) are pure states.

2. Consider a composite system consisting of two qubits. Find the Schmidt decomposition of the states

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}; \quad \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{\sqrt{2}}; \quad \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{2}}; \tag{19}$$

4 Problem Session Four

4.1 Question

4.2 Question

Assume that a system S is coupled to a measuring device M. The state before the measurement is

$$|\Psi_0\rangle = |s\rangle \otimes |M_0\rangle, \qquad |s\rangle = \sum_n a_n |s_n\rangle$$

and the state after the measurement is

$$|\Psi\rangle = \sum_n a_n |s_n\rangle \otimes |M_n\rangle$$

where the device states $|M_n\rangle$ are orthogonal. Find the density operator $\tilde{\rho}$ for the subsystem S after the measurement.

4.3 Question

He have shown that it is not possible to discriminate non-orthogonal states.But let's try anyway

4.4 Question: Generalized Measurement by Direct (Tensor) Product

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle$ and the tow systems interact via the unitary U_{AB} as follows:

$$\begin{aligned} |0\rangle_A |0\rangle_B &\longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B + |0\rangle_A |2\rangle_B) \\ |1\rangle_A |0\rangle_B &\longrightarrow \frac{1}{\sqrt{6}}(2\,|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B - |0\rangle_A |2\rangle_B) \end{aligned} \tag{20}$$

1. *Calculate the measurement operators acting on A corresponding to a measurement on B in the canonical basis $|0\rangle$, $|1\rangle$,$|2\rangle$.*

   Name the output states $|\phi_{00}\rangle_{AB}$ and $|\phi_{01}\rangle_{AB}$, respectively. Although teh specification of U is not complete, we have the pieces we need, and we can write $U_{AB} = \sum_{jk} |\phi_{jk}\rangle \langle jk|$ for some states $|\phi_{10}\rangle$ and $|\phi_{11}\rangle$. The measurement operators A_k are defined implicitly by

   $$U_{AB} |\psi\rangle_A |0\rangle_B = \sum_k (A_k)_A |\phi_A\rangle \langle k|_B\,. \tag{21}$$

   Thus $A_k = \langle k|U_{AB}|0\rangle = \sum_j {}_B\langle k|\phi_{j0}\rangle_{AB} \langle j|_A$, which is an operator on system A, even though it might not look like it at first glance. We then find

   $$A_0 = \frac{2}{\sqrt{6}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 0 \end{pmatrix}, A_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

2. *Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?*

   The corresponding POVM elements are given by $E_j = A_j^\dagger A_j$:

   $$E_0 = \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_1 = \frac{1}{6}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, E_2 = \frac{1}{6}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}. \tag{23}$$

   They are each rank one (which can be verified by calculating the determinant). The POVM elements project onto trine states $|1\rangle, (\sqrt{3}\,|0\rangle\,|1\rangle)/2$.

3. *Suppose A is in the state* $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the state after a measurement, averaging over the measurement result?

   The averaged post-measurement state is given by $\rho' = \sum_j A_j \rho A_j^\dagger$. In this case we have $\rho' = diag(2/3.1/3)$

5 Problem Session Five