Quantum Information

Problem session 1

October 7th, 2022

The Goal

- Get acquainted with the **Dirac notation**
- Get acquainted with **Axioms 1-3**
  - Wave function (associate the Dirac notation with integral expressions)
  - Observables
  - Quantum measurement
- Get acquainted with a **Qubit** and **Pauli operators**

Always start with a brief recollection of definitions and thms necessary for completing a particular task .

Task 1: Warm up: is this a wavefunction?

Determine if each of the following functions is acceptable as a wavefunction over the indicated regions:

a. $\cos x$ over $(0, \infty)$
b. e^x over $(-\infty, \infty)$
c. e^{-x} over $[0, \infty)$
d. $\tan \theta$ over $[0, 2\theta]$

Solution a

This is not an acceptable wavefunction. It is **single-valued** across the entire range. There is a single value for each value of x. It is **continuous** over the defined limits of integration, as we can see from a plot given below. However, it is not square-integrable.

$$\int_0^\infty |\cos(x)|^2 dx \times \infty$$

Solution b

This is not an acceptable wavefunction. Over the limits of integration from $-\infty$ to ∞, this function is not square-integrable. Note in the plot below, how the function is indefinite approaching the limits of ∞.

Solution c

This is an acceptable wavefunction over the given limits. It is **finite** over the given limits. It is **continuous** within given limits. It is **single-valued.** It is square-integrable with $\int_0^\infty |\Psi(x)|^2 dx = \frac{1}{2}$.

Solution d

This is not an acceptable wavefunction. It is discontinuous over the limits of integration.

Task 2: Distinguishability of quantum states

Proof that non-orthogonal states can't be reliably distinguished.

Suppose such a measurement is possible. If the state $|\psi_1\rangle$ ($|\psi_2\rangle$) is prepared, then the probability of measuring j such that $f(j) = 1$ ($f(j) = 2$) must be 1. Defining $E_i \equiv \sum_{j:f(j)=i} M_j^\dagger M_j$, these observations may be written as:

$$\langle\psi_1|E_1|\psi_1\rangle = 1; \quad \langle\psi_2|E_2|\psi_2\rangle = 1. \qquad (1)$$

Since $\sum_i E_i = I$ it follows that $\langle\psi_1|E_i|\psi_1\rangle = 1$, and since $\langle\psi_1|E_1|\psi_1\rangle = 1$ we must have $\langle\psi_1|E_2|\psi_1\rangle = 0$, and thus $\sqrt{E_2}|\psi_1\rangle = 0$. Suppose we decompose

$$|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\varphi\rangle,$$

where $|\varphi\rangle$ is orthonormal to $|\psi_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, and $|\beta| < 1$ since $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal. Then $\sqrt{E_2}|\psi_2\rangle = \beta\sqrt{E_2}|\varphi\rangle$, which implies a contradiction with (1), as

$$\langle\psi_2|E_2|\psi_2\rangle = |\beta|^2\langle\varphi|E_2|\varphi\rangle \leq |\beta|^2 < 1,$$

where the second last inequality follows from the observation that

$$\langle\varphi|E_2|\varphi\rangle \leq \sum_i\langle\varphi|E_i|\varphi\rangle = \langle\varphi|\varphi\rangle = 1.$$

Task 3: Simple measurements

Problem 10. A system is described by a 2-dimensional space of states. Let $|1\rangle$ and $|2\rangle$ be the normalized eigenstates of an observable $\mathcal{O}$ with eigenvalues $\lambda = 1$ and $\lambda = 2$ respectively. Let $v = |1\rangle + i\sqrt{3}|2\rangle$ be a state of the system.

a) What is the probability that the measure of $\mathcal{O}$ on v gives the result $\lambda = 2$?

b) Write the 2×2 matrix which represents $\mathcal{O}$ in the basis $\{|1\rangle, |2\rangle\}$.

c) Let $\mathcal{O}'$ be another observable, such that $\mathcal{O}'|1\rangle = |2\rangle$ and $\mathcal{O}'|2\rangle = |1\rangle$. Write the 2×2 matrix which represents $\mathcal{O}'$ in the basis $\{|1\rangle, |2\rangle\}$.

d) What are the eigenvalues of $\mathcal{O}'$?

e) What is the average of $\mathcal{O}'$ on the state v?

Solution:

- a) Let us compute the norm of v

$$|v|^2 = 1 + |i\sqrt{3}|^2 = 1 + 3 = 4 \tag{2.1}$$

Thus the normalized vector

$$v' = \frac{1}{2}\left(|1\rangle + i\sqrt{3}|2\rangle\right) \tag{2.2}$$

describes the same physical state as v. From (2.2) we read the probabilities

$$P_{\lambda=1}(v) = \left|\frac{1}{2}\right|^2 = \frac{1}{4} \qquad P_{\lambda=2}(v) = \left|\frac{i\sqrt{3}}{2}\right|^2 = \frac{3}{4} \tag{2.3}$$

- b) Since $\{|1\rangle, |2\rangle\}$ are the eigenstates of $\mathcal{O}$, in this basis $\mathcal{O}$ is represented by a diagonal matrix

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.4}$$

- c) The matrix representing $\mathcal{O}'$ in the basis $\{|1\rangle, |2\rangle\}$ is

$$\mathcal{O}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

Task 3: cont

- d) The eigenvalues of $\mathcal{O}'$ are obtained from the secular equation

$$\det\begin{pmatrix} -\lambda' & 1 \\ 1 & -\lambda' \end{pmatrix} = (\lambda')^2 - 1 = 0 \Leftrightarrow \lambda' = 1, -1 \tag{2.6}$$

- e) The average of $\mathcal{O}'$ on v is

$$\begin{aligned} \langle v', \mathcal{O}'\, v'\rangle &= \langle v', \mathcal{O}'\, \frac{1}{2}\left(|1\rangle + i\sqrt{3}|2\rangle\right)\rangle = \\ &= \langle v', \frac{1}{2}\left(|2\rangle + i\sqrt{3}|1\rangle\right)\rangle = \\ &= \langle \frac{1}{2}\left(|1\rangle + i\sqrt{3}|2\rangle\right), \frac{1}{2}\left(|2\rangle + i\sqrt{3}|1\rangle\right)\rangle = \\ &= \frac{1}{4}\left(i\sqrt{3} - i\sqrt{3}\right) = 0 \end{aligned} \tag{2.7}$$

Task 4: Pauli matrices

Problem 11. On a spin 1/2 system the following 3 observables are defined

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \tag{2.8}$$

a) Find the eigenvalues and the normalized eigenvectors of the 3 observables $\{S_x, S_y, S_z\}$.

b) Suppose the system is in the state described by the vector

$$v = \begin{pmatrix}1\\ 3\,i\end{pmatrix} \tag{2.9}$$

Find the probabilities to obtain the various possible values of $\{S_x, S_y, S_z\}$ on v. (You should in other words compute 6 probabilities, 2 for each of the 3 observables.)

c) Compute the 2x2 unitary matrix which connects the basis of eigenvectors of S_z with the basis of eigenvectors of S_y.

d) Find the eigenvalues and the normalized eigenvectors of the observable

$$S_{\vec{n}} \equiv n_x\, S_x + n_y\, S_y + n_z\, S_z \tag{2.10}$$

where

$$\vec{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.11}$$

Task 4: cont.

θ and ϕ are polar coordinates: $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$.

e) Compute the 2x2 unitary matrix which connects the basis of eigenvectors of S_z with the basis of eigenvectors of $S_{\vec{n}}$.

Solution:

a) The eigenvalues of any of $\{S_x, S_y, S_z\}$ are

$$\lambda_{\pm} = \pm\frac{\hbar}{2} \tag{2.12}$$

The eigenvectors of S_x are obtained from

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} = \pm \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} \tag{2.13}$$

that is

$$|S_x = \pm\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{2.14}$$

The eigenvectors of S_y are obtained from

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} = \pm \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} \tag{2.15}$$

that is

$$|S_y = \pm\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{2.16}$$

Finally, yhe eigenvectors of S_z are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} = \pm \begin{pmatrix} x_{\pm} \\ y_{\pm} \end{pmatrix} \tag{2.17}$$

that is

$$|S_z = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |S_z = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.18}$$

b) Let us consider the normalized vector

$$|v\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3\,i \end{pmatrix} \tag{2.19}$$

Task 4: cont.

The transition amplitudes are given by the scalar products

$$\begin{aligned}\langle S_x = \frac{\hbar}{2}|v\rangle &= \frac{1}{\sqrt{20}}(1+3\,i)\\ \langle S_y = \frac{\hbar}{2}|v\rangle &= \frac{1}{\sqrt{20}}(1+3) = \frac{1}{\sqrt{20}}4\\ \langle S_z = \frac{\hbar}{2}|v\rangle &= \frac{1}{\sqrt{10}}\end{aligned} \tag{2.20}$$

The corresponding probabilities are therefore

$$\begin{aligned}\left|\langle S_x = \frac{\hbar}{2}|v\rangle\right|^2 &= \frac{1}{20}|1+3\,i|^2 = \frac{1}{2}\\ \left|\langle S_y = \frac{\hbar}{2}|v\rangle\right|^2 &= \frac{16}{20} = \frac{4}{5}\\ \left|\langle S_z = \frac{\hbar}{2}|v\rangle\right|^2 &= \frac{1}{10}\end{aligned} \tag{2.21}$$

Hence the other probabilities are therefore

$$\begin{aligned}\left|\langle S_x = -\frac{\hbar}{2}|v\rangle\right|^2 &= 1-\frac{1}{2} = \frac{1}{2}\\ \left|\langle S_y = -\frac{\hbar}{2}|v\rangle\right|^2 &= 1-\frac{4}{5} = \frac{1}{5}\\ \left|\langle S_z = -\frac{\hbar}{2}|v\rangle\right|^2 &= 1-\frac{1}{10} = \frac{9}{10}\end{aligned} \tag{2.22}$$

c)

$$\begin{aligned}|S_y = \frac{\hbar}{2}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}|S_z = \frac{\hbar}{2}\rangle + \frac{i}{\sqrt{2}}|S_z = -\frac{\hbar}{2}\rangle\\ |S_y = -\frac{\hbar}{2}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}|S_z = \frac{\hbar}{2}\rangle - \frac{i}{\sqrt{2}}|S_z = -\frac{\hbar}{2}\rangle\end{aligned} \tag{2.23}$$

The unitary matrix is therefore

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} \tag{2.24}$$

d) The matrix

$$S_{\vec{n}} = S_x = \frac{\hbar}{2}\begin{pmatrix}n_z & n_x - in_y\\ n_z + in_y & n_z\end{pmatrix} \tag{2.25}$$

Task 4: cont.

has eigenvalues

$$\lambda_\pm = \pm\frac{\hbar}{2} \tag{2.26}$$

The eigenvectors are obtained from

$$\begin{pmatrix} n_z & n_x - i\,n_y \\ n_x + i\,n_y & -n_z \end{pmatrix} \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} = \pm \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} \tag{2.27}$$

that is

$$n_z\,x_\pm + (n_x - i\,n_y)y_\pm = \pm x_\pm \tag{2.28}$$

Hence

$$|S_{\vec{n}} = \pm\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2(1 \mp n_z)}} \begin{pmatrix} n_x - i\,n_y \\ \pm 1 - n_z \end{pmatrix} = \frac{1}{\sqrt{2(1 \mp \cos\theta)}} \begin{pmatrix} \sin\theta\,\mathrm{e}^{-i\,\phi} \\ \pm 1 - \cos\theta \end{pmatrix}$$

Therefore

$$|S_{\vec{n}} = \frac{\hbar}{2}\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\,\mathrm{e}^{-i\,\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad |S_{\vec{n}} = -\frac{\hbar}{2}\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\,\mathrm{e}^{-i\,\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \tag{2.29}$$

e)

$$U = \begin{pmatrix} \cos\frac{\theta}{2}\,\mathrm{e}^{-i\,\phi} & \sin\frac{\theta}{2}\,\mathrm{e}^{-i\,\phi} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \tag{2.30}$$

Task 5: Wavefunction carries all the information

6. At some time a triangular hat wave function is given by

$$\Psi(x,t) = \begin{cases} A\dfrac{x}{a}, & x \in [0,a]; \\ A\left(\dfrac{b-x}{b-a}\right), & x \in [a,b]; \\ 0 & \text{otherwise,} \end{cases}$$

where A, a, and b are constants.

a) Sketch Ψ and locate most probable location for a particle (i.e., the mode of the $|\Psi|^2$ probability distribution).

b) Determine the normalization constant A in terms of a and b. Recall the difference between wave function and probability distribution here and in the later parts of this question.

c) What are the probabilities of being found left and right of a, respectively?

d) What is $\langle x \rangle$?