Quantum Information

Problem session 5
November 4th, 2022

Task 1: **The Depolarizing Channel**

The quantum channel that we describe here is part of the so-called *noisy channels* in Quantum Information Theory. The noisy channels describe what occurs to a qubit which is transmitted to someone else and which is affected by the fact that the transmission is not perfect: the communication channel has to undergo some perturbations (some noise) coming from the environment. Hence, the noisy channel tries to describe the typical defects that the quantum bit may undergo during its transmission.

The noisy channel that we shall describe is the *depolarizing channel*. It describes the fact that the qubit may be left unchanged with probability $q = 1 - p \in [0, 1]$, or may undergo, with probability $p/3$, one the three following transformations:

– bit flip: $\begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_x|\psi\rangle\,,$

– phase flip: $\begin{cases} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto |-1\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_z|\psi\rangle\,,$

– both: $\begin{cases} |0\rangle \mapsto i|1\rangle \\ |1\rangle \mapsto -i|0\rangle\,, \end{cases}$ that is, $|\psi\rangle \mapsto \sigma_y|\psi\rangle\,.$

This channel can be represented through a unitary evolution U staking a four dimensional environment $\mathcal{H}_E$ with orthonormal basis denoted by $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. More precisely, the small system is $\mathcal{H}_A = \mathbb{C}^2$, which we identify to the subspace $\mathcal{H}_A \otimes |0\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_E$. The operator U acts on $\mathcal{H}_A \otimes \mathcal{H}_E$ by

$$\mathsf{U}\,(|\psi\rangle \otimes |0\rangle) = \sqrt{1-p}\;|\psi\rangle \otimes |0\rangle + \sqrt{\frac{p}{3}}\,\Big[\sigma_x|\psi\rangle \otimes |1\rangle + \sigma_y|\psi\rangle \otimes |2\rangle + \\ + \sigma_z|\psi\rangle \otimes |3\rangle\Big]$$

and U is completed in any way as a unitary operator on $\mathcal{H}_A \otimes \mathcal{H}_E$.

The effect of the transform U when seen only from the small system $\mathcal{H}_A$ is then

$$\mathcal{L}(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(\mathsf{U}(\rho \otimes |0\rangle\langle 0|)\mathsf{U}^*\right)\,.$$

An easy computation gives the Krauss representation

$$\mathcal{L}(\rho) = \sum_{i=0}^{3} \mathsf{M}_i\,\rho\,\mathsf{M}_i^*$$

with

$$\mathsf{M}_0 = \sqrt{1-p}\;\mathrm{I}\,,\ \mathsf{M}_1 = \sqrt{\frac{p}{3}}\;\sigma_x\,,\ \mathsf{M}_2 = \sqrt{\frac{p}{3}}\;\sigma_y\,,\ \mathsf{M}_3 = \sqrt{\frac{p}{3}}\;\sigma_z\,.$$

Task 2:

6.1.6.2 The Phase-Damping Channel

The second type of channel we shall describe is the *phase-damping channel*. It is advantageously defined through a unitary evolution involving the small system $\mathcal{H}_A = \mathbb{C}^2$ and the environment $\mathcal{H}_E$ which is now 3 dimensional. This unitary operator is given by

$$\mathsf{U}\left(|0\rangle \otimes |0\rangle\right) = \sqrt{1-p}\;|0\rangle \otimes |0\rangle + \sqrt{p}\;|0\rangle \otimes |1\rangle$$

and

$$\mathsf{U}\left(|1\rangle \otimes |0\rangle\right) = \sqrt{1-p}\;|1\rangle \otimes |0\rangle + \sqrt{p}\;|1\rangle \otimes |2\rangle\,.$$

The action of U on the other types of vectors of $\mathcal{H}_A \otimes \mathcal{H}_E$ is not necessary to describe.

In this interaction with the environment, the small system is not changed, only the environment, in contact with $\mathcal{H}_A$ may scatter (with probability p) from the ground state $|0\rangle$ to an excited state $|1\rangle$ or $|2\rangle$, depending on the state of $\mathcal{H}_A$. In some way, the environment "reads" the state of $\mathcal{H}_A$ and may be influenced by it.

The Krauss decomposition resulting from this unitary transform is then given by

$$\mathcal{L}(\rho) = \sum_{i=0}^{2} \mathsf{M}_i\,\rho\,\mathsf{M}_i^*$$

with

$$\mathsf{M}_0 = \sqrt{1-p}\;\mathrm{I}\,, \qquad \mathsf{M}_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{M}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

An easy computation shows that the associated action on the Bloch sphere is the transform:

$$(x, y, z) \longmapsto \left((1-p)x, (1-p)y, z\right)\,.$$

Task 3:

6.1.6.3 Spontaneous Emission

The third channel to be presented here is the *amplitude-damping channel* or *spontaneous emission*. Here the environment is 2 dimensional and the unitary evolution is given by

$$\mathsf{U}\,(|0\rangle \otimes |0\rangle\,) = |0\rangle \otimes |0\rangle$$
$$\mathsf{U}\,(|1\rangle \otimes |0\rangle) = \sqrt{1-p}\,|1\rangle \otimes |0\rangle + \sqrt{p}\,|0\rangle \otimes |1\rangle\,.$$

In other words, if the small system is in the ground state $|0\rangle$ then nothing happens, if it is in the excited state $|1\rangle$ then it may emit this energy into the environment with probability p. This is the simplest model of spontaneous emission of an excited particle: the excited particle goes down to the ground state, emiting a photon into the environment.

In this model there are only two Krauss operators for the associated completely positive map:

$$\mathsf{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad \mathsf{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Successive applications of the associated completely positive map $\mathcal{L}$ make any initial state ρ_0 converge exponentially fast to the ground state

$$\rho_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|\,.$$

That is, as we explained above, the system $\mathcal{H}_A$ ends up emitting all its energy into the environment and hence converges to the ground state.