Quantum Information

Problem session 3

October 21th, 2022

Task 1: Singular Value Decomposition

Corollary 2.4: (Singular value decomposition) Let A be a square matrix. Then there exist unitary matrices U and V, and a diagonal matrix D with non-negative entries such that

$$A = UDV\,. \tag{2.80}$$

The diagonal elements of D are called the *singular values* of A.

Proof

By the polar decomposition, $A = SJ$, for unitary S, and positive J. By the spectral theorem, $J = TDT^\dagger$, for unitary T and diagonal D with non-negative entries. Setting $U \equiv ST$ and $V \equiv T^\dagger$ completes the proof. $\square$

Task 2: Singlet State

Suppose we prepare the two qubit state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \tag{2.213}$$

a state sometimes known as the *spin singlet* for historical reasons. It is not difficult to show that this state is an entangled state of the two qubit system. Suppose we perform a measurement of spin along the $\vec{v}$ axis on both qubits, that is, we measure the observable $\vec{v} \cdot \vec{\sigma}$ (defined in Equation (2.116) on page 90) on each qubit, getting a result of $+1$ or -1 for each qubit. It turns out that no matter what choice of $\vec{v}$ we make, the results of the two measurements are always opposite to one another. That is, if the measurement on the first qubit yields $+1$, then the measurement on the second qubit will yield -1, and vice versa. It is as though the second qubit knows the result of the measurement on the first, no matter how the first qubit is measured. To see why this is true, suppose $|a\rangle$ and $|b\rangle$ are the eigenstates of $\vec{v} \cdot \vec{\sigma}$. Then there exist complex numbers $\alpha, \beta, \gamma, \delta$ such that

Task 2: cont’d

$$|0\rangle = \alpha|a\rangle + \beta|b\rangle \tag{2.214}$$

$$|1\rangle = \gamma|a\rangle + \delta|b\rangle. \tag{2.215}$$

Substituting we obtain

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\frac{|ab\rangle - |ba\rangle}{\sqrt{2}}. \tag{2.216}$$

But $\alpha\delta - \beta\gamma$ is the determinant of the unitary matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and thus is equal to a phase factor $e^{i\theta}$ for some real θ. Thus

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|ab\rangle - |ba\rangle}{\sqrt{2}}, \tag{2.217}$$

up to an unobservable global phase factor. As a result, if a measurement of $\vec{v} \cdot \vec{\sigma}$ is performed on both qubits, then we can see that a result of $+1$ (-1) on the first qubit implies a result of -1 $(+1)$ on the second qubit.

Task 3: Bell Inequalities

Nielsen & Chuang, p. 112-117

Task 3: Density Operator

(i) A harmonic oscillator has an equal classical probability 1/3 to be found in each of the states $|\,0\rangle$, $|\,1\rangle$ and $4\,|\,0\rangle + 3\,|\,1\rangle$. Write down the corresponding density matrix $\hat{\rho}$ explicitly.

(ii) Consider a system Σ consisting of two subsystems I and II. Σ is in a pure state $|\,\psi_\Sigma\rangle$, where $|\,\psi_\Sigma\rangle$ is a vector in the product space. The density matrices $\hat{\rho}_{I,II}$ and $\hat{\rho}_\Sigma$ are defined as in the lecture. Remembering that

$$\hat{\rho}_I = \mathrm{tr}_{II}\,\hat{\rho}_\Sigma \equiv \left\langle {}^{II}\delta^i \,|\, \hat{\rho}_\Sigma | {}^{II}\delta_i \right\rangle \quad \text{and} \quad \hat{\rho}_{II} = \mathrm{tr}_I\,\hat{\rho}_\Sigma \equiv \left\langle {}^{I}\delta^i \,|\, \hat{\rho}_\Sigma | {}^{I}\delta_i \right\rangle \tag{15}$$

and given that $\{|\,n\rangle\}_{n\in\mathbb{N}}$ is a set of orthonormal vectors, calculate the density matrix for subsystems I and II, for

$$|\,\psi_\Sigma\rangle = \frac{1}{\sqrt{2}}\left(|\,0\rangle \otimes |\,1\rangle + |\,1\rangle \otimes |\,2\rangle\right) \tag{16}$$

(iii) Consider $\hat{\rho}_I$ of the previous point. Does it correspond to a pure or to a mixed state?

(iv) Given a density matrix, the entropy of the described state is given by $S = -\,\mathrm{tr}(\hat{\rho}\log(\hat{\rho}))$. Express the entropy as a function of the eigenvalues of $\hat{\rho}$.
It can be shown that the maximum entropy is reached when all eigenvalues are equal, i.e. when every state is possible with the same probability; what is the maximum entropy for a Hilbert space of dimension d? What is the entropy of a pure state? Calculate the entropy of $\hat{\rho}_I$ and $\hat{\rho}_{II}$ of the previous part.

Task 3: cont'd

Solution

(i) First of all, we need all states to be normalized. In particular, we have to normalize the third state

$$4\,|\,0\rangle + 3\,|\,1\rangle \quad \rightarrow \quad \frac{4}{5}\,|\,0\rangle + \frac{3}{5}\,|\,1\rangle \tag{17}$$

Now we can build the density matrix of the harmonic oscillator as follows

$$\begin{aligned}
\hat{\rho} &= \sum_i p_i\,|\,\psi_i\rangle\,\langle\psi_i\,| \overset{p_i=1/3}{=} \frac{1}{3}\sum_i |\,\psi_i\rangle\,\langle\psi_i\,| = \\
&= \frac{1}{3}\left[|\,0\rangle\,\langle 0\,| + |\,1\rangle\,\langle 1\,| + \left(\frac{4}{5}\,|\,0\rangle + \frac{3}{5}\,|\,1\rangle\right)\left(\frac{4}{5}\,\langle 0\,| + \frac{3}{5}\,\langle 1\,|\right)\right] \\
&= \frac{41}{75}\,|\,0\rangle\,\langle 0\,| + \frac{34}{75}\,|\,1\rangle\,\langle 1\,| + \frac{12}{75}\left(|\,0\rangle\,\langle 1\,| + |\,1\rangle\,\langle 0\,|\right)
\end{aligned} \tag{18}$$

(ii) For the sake of legibility we adopt the following notation:

$$|\,n\rangle \otimes |\,m\rangle \equiv |\,nm\rangle \quad , \quad \langle n\,| \otimes \langle m\,| \equiv \langle nm\,| \tag{19}$$

In this notation the state is

$$|\,\psi_\Sigma\rangle = \frac{1}{\sqrt{2}}(|\,01\rangle + |\,12\rangle) \tag{20}$$

Task 3: cont'd

Therefore

$$\hat{\rho}_\Sigma = |\,\psi_\Sigma\rangle\,\langle\psi_\Sigma\,| = \frac{1}{2}(|\,01\rangle + |\,12\rangle)(\langle 01\,| + \langle 12\,|) = \frac{1}{2}(|\,01\rangle\,\langle 01\,| + |\,12\rangle\,\langle 12\,| + |\,01\rangle\,\langle 12\,| + |\,12\rangle\,\langle 01\,|) \quad (21)$$

Tracing out the subsystem II and I we find

$$\hat{\rho}_I = \mathrm{tr}_{II}\,\hat{\rho}_\Sigma = \frac{1}{2}(|\,0\rangle\,\langle 0\,| + |\,1\rangle\,\langle 1\,|) \quad (22)$$

$$\hat{\rho}_{II} = \mathrm{tr}_{I}\,\hat{\rho}_\Sigma = \frac{1}{2}(|\,1\rangle\,\langle 1\,| + |\,2\rangle\,\langle 2\,|) \quad (23)$$

(iii) We have from the previous point $\hat{\rho}_I = \mathrm{tr}_{II}\,\hat{\rho}_\Sigma = \frac{1}{2}(|\,0\rangle\,\langle 0\,| + |\,1\rangle\,\langle 1\,|)$. It corresponds to a mixed state, and we have two ways of proving it.

Task 3: cont'd

a) Suppose $\hat{\rho}_I$ is a pure state; then it must be possible to write it as $\hat{\rho}_I = |\phi_I\rangle\langle\phi_I|$, with $|\phi_I\rangle = \sum_n a_n |n\rangle$. In this assumption:

$$|\phi_I\rangle\langle\phi_I| = \sum_{nm} a_n a_m^* |n\rangle\langle m| \tag{24}$$

We can immediately set to 0 the coefficients a_i with $i > 1$ since the corresponding states do not appears in $\hat{\rho}_I$. We have therefore:

$$|\phi_I\rangle\langle\phi_I| = |a_0|^2 |0\rangle\langle 0| + |a_1|^2 |1\rangle\langle 1| + a_0 a_1^* |0\rangle\langle 1| + a_1 a_0^* |1\rangle\langle 0| \tag{25}$$

We can see that there exists no choice of a_0, a_1 such that $|\phi_I\rangle\langle\phi_I| = \hat{\rho}_I$.

b) The second way relies on the fact that for pure states $\hat{\rho}^2 = \hat{\rho}$.

$$\hat{\rho}_I^2 = \frac{1}{2}\hat{\rho}_I \neq \hat{\rho}_I \tag{26}$$

and therefore it is not a pure state.

Task 3: cont'd

(iv) Since the density matrix is an hermitian operator it can be diagonalized in a certain orthonormal basis $\{|\rho_k\rangle\}_{k\in\mathbb{N}}$; each element of the basis is related to an eigenvalue p_k. Applying the cyclicity of the trace and the definition of logarithm of an operator, the result holds:

$$S = -\sum_k p_k \log(p_k) \tag{27}$$

The maximum entropy in a Hilbert space of dimension d is reached when $p_k = p = \frac{1}{d} \quad \forall k$. The entropy is then:

$$S = -\sum_{k=0}^{d-1} p_k \log(p_k) = -\sum_{k=0}^{d-1} p \log(p) = -d\frac{1}{d}\log\left(\frac{1}{d}\right) = \log(d) \tag{28}$$

For a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$; it is a diagonal operator with one eigenvalue 1, its eigenvector $|\psi\rangle$ and all the others eigenvalues are 0. It follows that the entropy for a pure state is always $S_{\text{pure}} = 0$.

Given the density matrices of the subsystem I and II of the previous point we immediately see that they are both diagonal, with $p_0, p_1 = \frac{1}{2}$ and $p_1, p_2 = \frac{1}{2}$, respectively; the entropy is therefore

$$S_I = S_{II} = \log(2) \tag{29}$$

Task 4: Schmidt Decomposition

Exercise 2.78: Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

Exercise 2.79: Consider a composite system consisting of two qubits. Find the Schmidt decompositions of the states

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}; \quad \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}; \text{ and } \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}. \qquad (2.212)$$