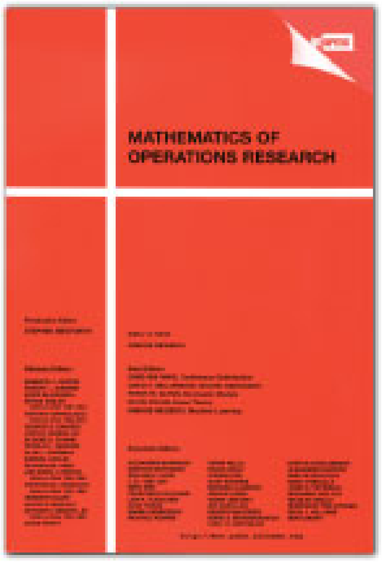

Mathematics of Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 25, No. 4, November 2000, pp. 645–656
Printed in U.S.A.

A NEAR-OPTIMAL SOLUTION TO A TWO-DIMENSIONAL CUTTING STOCK PROBLEM

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We present an asymptotic fully polynomial approximation scheme for strip-packing, or packing rectangles into a rectangle of fixed width and minimum height, a classical *NP*-hard cutting-stock problem. The algorithm, based on a new linear-programming relaxation, finds a packing of n rectangles whose total height is within a factor of $(1 + \varepsilon)$ of optimal (up to an additive term), and has running time polynomial both in n and in $1/\varepsilon$.

1. Introduction.

1.1. Results. We consider the *strip-packing problem*, which is the following version of a two-dimensional cutting stock problem: Given a supply of material consisting of one rectangular strip of fixed width 1 and large height, given a demand of n rectangles with widths and heights in the interval [0,1], the problem is to cut the strip into the demand rectangles while minimizing the waste, i.e., minimizing the total height used.

This is a natural generalization of bin-packing to two dimensions. We do not allow the demand rectangles to be rotated (in many applications, rotations are not allowed because of constraints such as the patterns of the cloth or of the grain of the wood). In computer science, strip-packing models the scheduling of independent tasks, each requiring a certain number of contiguous processors or memory locations for a certain length of time; the width of the strip represents the total number of processors or memory locations available, and the height represents the completion time.

Strip-packing is *NP*-hard because it includes bin-packing as a special case (when all heights are equal). Thus, unless $P = NP$, one cannot find an efficient algorithm for constructing the optimal packing. One then seeks to design approximate heuristics A with good performance guarantees.

DEFINITION 1. Let $A(L)$ denote the height used by A on input L, and let $Opt(L)$ denote the height used by the optimal algorithm on input L. The absolute performance ratio of A is $\sup_L A(L)/\mathrm{Opt}(L)$. The asymptotic performance ratio of A is $\limsup_{\mathrm{Opt}(L)\to\infty} A(L)/\mathrm{Opt}(L)$.

In this paper, we focus on the asymptotic performance ratio. Our main result is the following:

THEOREM 1. *There is an algorithm A which, given a list L of n rectangles whose side lengths are at most* 1, *and a positive number* ε, *produces a packing of L in a strip of width* 1 *and height $A(L)$ such that*:

$$A(L) \le (1+\varepsilon)Opt(L) + O(1/\varepsilon^2).$$

The time complexity of A is polynomial in n and $1/\varepsilon$.

Received January 6, 1997; revised October 22, 1998, and July 12, 1999.
MSC 1991 subject classification. Primary: 67Q25; secondary: 90B35.
OR/MS subject classification. Primary: Analysis of algorithms/suboptimal algorithms; secondary: production scheduling/cutting stock/trim.
Key words. Cutting-stock, strip-packing, fully polynomial approximation scheme.

0364-765X/00/2504/645/$05.00
1526-5471 electronic ISSN, © 2000, INFORMS

In other words, the paper presents an asymptotic fully polynomial time approximation scheme for strip-packing.

The running time is $O(n(\log n)\varepsilon^{-8} \operatorname{polylog}(\varepsilon) + (\log^2 n)\varepsilon^{-16} \operatorname{polylog}(\varepsilon))$ by appealing to the result from Karmarkar et al. (1982) Theorem 1. (It may be possible to improve on this by using ideas from Plotkin et al. 1995 and Young 1995).

Two-dimensional stock-cutting with stages goes as far back as 1965 with work by Gilmore and Gomory (1965). In computer science, many ideas for strip-packing originally arose from bin-packing studies. In 1980, Baker et al. showed that the "Bottom-Left" heuristic has asymptotic performance ratio equal to 3 when the rectangles are sorted by decreasing widths (Baker et al. 1980). Coffman et al. (1980) studied algorithms where the rectangles are placed on "shelves" using one-dimensional bin-packing heuristics, and showed that the First-Fit shelf algorithm has asymptotic performance ratio of 2.7 when the rectangles are sorted by decreasing height (this defines the First-Fit-Decreasing-Height algorithm) (Coffman et al. 1980). The asymptotic performance ratio of the best heuristic was further reduced to 2.5 (Sleater 1980), then to 4/3 (Golan 1981) and finally to 5/4 (Baker et al. 1981). The absolute performance ratio has also been the object of much research, with the best current algorithm having a performance of ratio 2 (Steinberg 1997 and Schiermeyer 1994).

In 1991, Fernandez de la Vega and Zissimopoulos used a very different approach, based on a reduction to integer linear programming, to design a $(1+\varepsilon)$ asymptotic approximation scheme for strip packing, in the case when all rectangle widths and heights are bounded below and above by constants (Fernandez de la Vega and Zissimopoulos 1991). In other words, they solve the strip-packing problem as long as the rectangles are neither too flat nor too narrow. Their work was inspired by approximation schemes developed for one-dimensional bin-packing, based on linear programming. This direction was explored by Fernandez de la Vega and Lueker in 1981 (with a reduction of bin-packing to constant-size integer linear programming) (Fernandez de la Vega and Lueker 1981) and later by Karp and Karmarkar, to yield an asymptotic fully polynomial time approximation scheme for bin-packing (Karmarkar and Karp 1982). To compare our algorithm with the one developed in Fernandez de la Vega and Zissimopoulos (1991), one must note that the algorithm in that paper, though linear-time in terms of the number of rectangles, is worse than exponential in terms of ε, thus inherently impractical.

In this paper (which is an extended version of Kenyon and Remila 1996), we use many ideas from Fernandez de la Vega and Zissimopoulos (1991), Fernandez de la Vega and Lueker (1981), and Karmarkar and Karp (1982).

1.2. Methods. Bin-packing and strip-packing are closely related, and many ideas that originally arose from bin-packing can also be applied to strip-packing. It is thus natural to try to extend the linear programming approach from bin-packing (Fernandez de la Vega and Lueker 1981, Karmarkar and Karp 1982) to strip-packing.

One obstacle to such an extension comes from the small input items (rectangles of small width or height), since both approximation schemes developed in Fernandez de la Vega and Lueker (1981) and Karmarkar and Karp (1982) for bin-packing first set small input values aside, then construct an efficient packing of the other values, and finally add the small values in a greedy way so as to form a packing which is still efficient.

In the case of strip-packing, however, there is no efficient way in general to complete a packing of a strip (which may have many little gaps of odd shapes), when adding the rectangles of small width or of small height.

However, one should note that rectangles of small width are not inherently difficult: In the extreme case when all input rectangles have small width, the First-Fit-Decreasing-Height shelf heuristic (FFDH) is very efficient.

THEOREM 2 (COFFMAN ET AL. 1980). *If all rectangles of L have width less than or equal to* $1/k$, *then the height* $FFDH(L)$ *achieved by the FFDH heuristic on list L satisfies*

$$FFDH(L) \leq s(L)(1 + 1/k) + 1,$$

where $s(L)$ *is the total area of the rectangles in L.*

Our original hope was that the method of Fernandez de la Vega and Zissimopoulos (1991) and the FFDH heuristics might be combined to get an efficient approximation scheme for general strip-packing. The integer linear program devised in Fernandez de la Vega and Zissimopoulos (1991) unfortunately cannot be combined with FFDH, but another linear programming approach, which *only constructs packings with few odd-shaped gaps*, works.

The algorithm is structured as follows: First, the rectangles are divided into two sublists: the "narrow" rectangles, i.e., whose widths are less than a positive constant ε', and the "wide" rectangles (i.e., which are not narrow). We take the wide rectangles and change their widths by using a variation of Karmarkar and Karp's linear rounding, so as to build another list L_{sup} of wide rectangles with only a *bounded number of distinct widths*. Relaxing the constraints to allow horizontal cuts of the rectangles (in the scheduling setting this corresponds to allowing preemption), we obtain a *fractional bin-packing problem*, defined in the next section, from which (either by Karmarkar and Karp 1982 or by Plotkin et al. 1995) we deduce a strip-packing for L_{sup} that is close to optimal. Finally, the packing produced has a specific "nice" shape, which makes the insertion of narrow rectangles possible while still keeping the packing close to optimal.

For stock-cutting applications, where machines can only perform edge-to-edge cuts parallel to the strip's length or width, called "guillotine cuts," it is worthwhile to remark that our algorithm is also applicable to guillotine cuts. In fact, it can be realized by five stages of consecutive parallel (and, consequently, permutable) guillotine cuts. This will be explained further at the time of the presentation of the algorithm.

2. The algorithm.

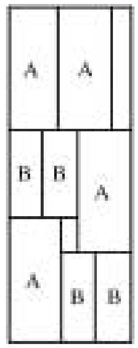
2.1. Definitions. A rectangle is given by its width w_i and height h_i, with $0 < w_i$, $h_i \leq 1$. The area (resp. height) of a list $L = ((w_1, h_1), (w_2, h_2), \ldots, (w_n, h_n))$ of rectangles is the sum of the areas (resp. heights) of the rectangles of L. We assume that the list is ordered by nonincreasing widths: $w_1 \geq w_2 \geq \cdots \geq w_n$.

REMARK. The assumption that the heights are less than 1 is also made in several other papers such as Coffman et al. (1980). Without it, one could scale the items arbitrarily and thus the absolute and the asymptotic performance ratios would coincide, so that there would be no hope of getting a fully polynomial asymptotic scheme.

A strip-packing of a list L of rectangles is a positioning of the rectangles of L within the vertical strip $[0,1] \times [0, +\infty)$, so that all rectangles have disjoint interiors. If rectangle (w_i, h_i) is positioned at $[x, x + w_i] \times [y, y + h_i]$, then y is called the lower boundary and $(y + h_i)$ the upper boundary of the rectangle. The height of a strip-packing is the uppermost boundary of any rectangle. Let $\text{Opt}(L)$ denote the minimum height of a strip-packing of L: $\text{Opt}(L) = \inf\{\text{height of } f \text{ such that } f \text{ is a packing of } L\}$.

A fractional strip-packing of L is a packing of any list L' obtained from L by subdividing some of its rectangles by horizontal cuts: Each rectangle (w_i, h_i) is replaced by a sequence $(w_i, h_{i_1}), (w_i, h_{i_2}), \ldots, (w_i, h_{i_{k_i}})$ of rectangles, such that $h_i = \sum_j h_{i_j}$.

First we present the algorithm when the number of distinct widths of the rectangles is bounded by some value m, and all widths are larger than some constant ε'. This special case is called the "few and wide" case.



	configuration	α_{1j} = number of As	α_{2j} = number of Bs
C_1	$3/7, 2/7, 2/7$	1	2
C_2	$3/7, 3/7$	2	0
C_3	$2/7, 2/7, 2/7$	0	3
C_4	$3/7, 2/7$	1	1
C_5	$2/7, 2/7$	0	2
C_6	$3/7$	1	0
C_7	$2/7$	0	1

FIGURE 1. A strip-packing of L.

2.2. Solving the "few and wide" case.

2.2.1. From the "few and wide" case to fractional strip-packing. Throughout this subsection, we assume that the n rectangles of L only have m distinct widths, $w'_1 > w'_2 > \cdots > w'_m > \varepsilon'$.

This section contains one main new idea: a reduction from this special case of strip-packing to fractional strip-packing.

To the input L, we associate a set of *configurations*. A configuration is defined as a nonempty multiset of widths (chosen among the m widths) that sum to less than 1 (i.e., capable of occurring at the same level). Their sum is called the *width* of the configuration. Without loss of generality, the configurations can be assumed to be ordered by nonincreasing widths.

Let q be the number of distinct configurations, and let α_{ij} denote the number of occurrences of width w'_i in configuration C_j.

To each (possibly fractional) strip-packing of L of height h, we associate a vector $(x_1, \ldots, x_q)$, $x_i \geq 0$, in the following manner. Scan the packing bottom-up with a horizontal sweep line $y = a$, $0 \leq a \leq h$. Each such line is canonically associated to a configuration $(\alpha_1, \ldots, \alpha_m)$, where α_i is the number of rectangles of width w'_i whose interior is intersected by the sweep line. Let $x_j, 1 \leq j \leq q$, denote the measure of the as such that the sweep line $y = a$ is associated with configuration C_j. For example, let A denote the rectangle $3/7 \times 1$ and B denote the rectangle $2/7 \times 3/4$, and assume that the input L consists of three rectangles of type A and four rectangles of type B. There are seven configurations, listed in Figure 1.

The vector corresponding to the strip-packing in Figure 1 is $(3/2, 5/4, 0, 0, 0, 0, 0)$.

The fractional strip-packing problem is canonically defined as follows: Given a list L of rectangles, construct a fractional strip packing of minimal height.

LEMMA 1. *Consider the linear program*:

$$\textit{minimize } (1.x) \textit{ subject to } x \geq 0 \textit{ and } Ax \geq B,$$

where 1 *is the all-ones vector, A is the $m \times q$ matrix $(\alpha_{ij})_{1\le i\le m, 1\le j\le q}$, and $B = (\beta_1, \ldots, \beta_m)$, β_i denoting the sum of the heights of all rectangles of width w_i'. Then any fractional strip-packing naturally corresponds to a feasible vector x, and conversely to any feasible vector x we can associate a fractional strip-packing of height $(1.x)$ and in which the number of configurations actually occurring is at most m plus the number of nonzero variables x_i.*

PROOF. Consider the vector $(x_1, \ldots, x_q)$ associated to a fractional strip-packing of L of height h. We clearly have: $x_j \ge 0$, $\sum_j x_j = h$, and for every i, $\sum_j \alpha_{ij} x_j = \beta_i$.

Conversely, take a vector $(x_1, \ldots, x_q)$ such that $x_i \ge 0$ and $\sum_j \alpha_{ij} x_j \ge \beta_i$; we can associate a fractional strip-packing of L of height $\sum_j x_j$ in the following manner.

Partition the strip of width 1 and height $\sum_j x_j$ into j pieces of width 1 and heights x_j $(1 \le j \le q)$. In the jth piece, for each i such that $\alpha_{ij} = 0$, draw α_{ij} columns of width w_i' and height x_j. Finally, for each i, fill up the columns of width w_i' with the input rectangles of width w_i' in a greedy manner, cutting the rectangles as you go so as to fill each column exactly up to height x_j (note that each column possibly contains the top part of a rectangle which had been started in a different column, followed by a few rectangles, possibly followed by the bottom part of a rectangle which is too tall to fit in the column and has to be cut). Since $\sum_j \alpha_{ij} x_j$ is greater than or equal to β_i, the total height of the rectangles of width w_i', we will have enough or more than enough rectangles of each width w_i'. If we have too much, we can just erase the extraneous part. We have constructed a fractional strip-packing of L of height $\sum_j x_j$. Moreover the number of configurations which are actually present in the fractional strip-packing thus constructed is at most m plus the number of nonzero variables x_j. □

We now recall the fractional bin-packing problem studied by Karmarkar and Karp (1982). In this problem, the input is a set of n items of m different types; i.e., they take only m distinct sizes in $(\varepsilon, 1]$. A configuration is a multi-set of types which sum to at most 1 (i.e., capable of being packed within a bin). If q denotes the number of configurations, then a feasible solution to the fractional bin-packing problem is a vector $(x_1, \ldots, x_q)$ of nonnegative numbers such that if α_{ij} is the number of pieces of type i occurring in configuration j, then for every i, $\sum_j \alpha_{ij} x_j$ is at least equal to the number n_i of input pieces of type i. The goal is to minimize $\sum_j x_j$.

Notice that fractional bin-packing and fractional strip-packing give rise to the same linear program. The only difference is that vector $B = (\beta_1, \ldots, \beta_m)$ of the strip-packing is replaced by the vector $B' = (n_1, \ldots, n_m)$ with integer coordinates.

Let OPT be the minimum possible value of $\sum_j x_j$. The fractional bin-packing problem with tolerance t has for its goal to find a basic feasible solution such that $\sum_j x_j \le \mathrm{OPT} + t$, and was solved in Karmarkar and Karp (1982) in polynomial time. More precisely, we have the following theorem:

THEOREM 3 (KARMARKAR AND KARP (1982, THEOREM 1).) *There exists a polynomial-time algorithm for fractional bin-packing with additive tolerance t, such that if n is the number of items, m the number of distinct items, and a the size of the smallest item, then the running time is*

$$O\left(m^8 \log m \log^2\left(\frac{mn}{at}\right) + \frac{m^4 n \log m}{t} \log\left(\frac{mn}{at}\right)\right).$$

The proof of this theorem uses linear programming techniques but does not use the fact that vector B' is integer. It can obviously be extended to strip-packing: with the notations of Lemma 1, there exists an algorithm with positive tolerance t whose running time is polynomial in m, $\sum_i \beta_i$ (which is less than the number n of rectangles) and t, which gives a solution with at most $2m$ nonzero coordinates.

In our setting a, m and t will all be polynomials in $1/\varepsilon$, and so the running time will be $O_\varepsilon(n \log n)$. Note that using a Lagrangian relaxation technique, in Plotkin et al. (1995, Theorem 5.11), an alternative approach is proposed.

2.2.2. From fractional strip-packing to strip-packing. We now relate fractional strip-packing to strip-packing.

Lemma 2. *If L has a fractional strip-packing $(x_1, \ldots, x_q)$ of height h and with at most $2m$ nonzero x_js, then L has an (integral) strip-packing of height at most $h + 2m$.*

Proof. Consider a fractional strip-packing $(x_1, \ldots, x_q)$ of L, of height $\sum_i x_i = h$, and with at most $2m$ nonzero coordinates x_is. Up to renaming, we assume that the nonzero coordinates are $x_1, \ldots, x_{m'}$, with $m' \leq 2m$. Let $h_{\max}$ be the maximum height of any rectangle of L. We construct a strip packing of L of height $h + 2mh_{\max}$ in the following way.

We fill in the strip bottom up, taking each configuration in turn. Let $x_j > 0$ denote the variable corresponding to the current configuration. Configuration j will be used between level $l_j = (x_1 + h_{\max}) + \cdots + (x_{j-1} + h_{\max})$ and level $l_{j+1} = l_j + x_j + h_{\max}$ (initially $l_1 = 0$). For each i such that $\alpha_{ij} \neq 0$, we draw α_{ij} columns of width w'_i going from level l_j to level l_{j+1}.

In this way, each column C of the fractional strip-packing of width w'_i and height x_j can be associated to a column C_+ width w'_i and height $x_j + h_{\max}$. In C_+, we place the rectangles which are completely in C, and the rectangle whose bottom is in C and whose top is in another column. There is at most one rectangle of this type from the proof of Lemma 1. Obviously, C_+ is sufficiently large to contain those rectangles. This proves that the construction yields a valid strip-packing of L. Its height is $(x_1 + h_{\max}) + \cdots + (x_{m'} + h_{\max}) = h + m'h_{\max} \leq h + 2m$, hence the lemma. □

This gives a straightforward algorithm for strip-packing in the special case studied in this section.

(1) Solve fractional strip-packing on L with tolerance 1 (the solution has at most $2m$ nonzero coordinates).

(2) From the fractional strip-packing, construct a strip-packing of L as in the proof of the lemma above.

Moreover, a crucial point for the sequel (i.e., for the addition of narrow rectangles) is that this strip packing leaves some well-structured free space. Note that in the proof of Lemma 2, column C_+ is almost fully used: the unused part of the column has height at most 2, one for the bottom rectangle of C which may have been placed in another column, and one for the extra space on top.

Important remark: Structure of a layer (See Figure 2). Let $c_1 \geq c_2 \geq \cdots \geq c_{m'}$ denote the widths of the m' configurations used above. The layer $[0,1] \times [l_i, l_{i+1}]$ can be divided into three rectangles:

(i) the rectangle $R_i = [c_i, 1] \times [l_i, l_{i+1}]$, which is completely free and will later be used to place the narrow rectangles;

(ii) the rectangle $R'_i = [0, c_i] \times [l_i, l_{i+1} - 2]$, which is completely filled by wide rectangles; and

(iii) the rectangle $R''_i = [0, c_i] \times [l_{i+1} - 2, l_{i+1}]$, which is partially filled in some complicated way by wide rectangles overlapping from R_i, and whose free space is now considered as wasted space, and will not be used in the remainder of the construction.

2.3. From general strip-packing to the "few and wide" case. In the general case, we have a list L_{general} with many distinct widths, some of which may be arbitrarily small.

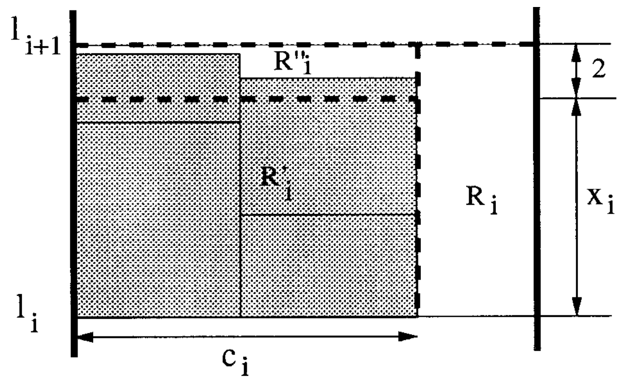


FIGURE 2. Structure of a layer.

We use appropriate extensions of two ideas of Fernandez de la Vega and Lueker (1981): elimination of small pieces, and grouping. The purpose of elimination is to insure all rectangles are wider than some ε'. The purpose of grouping is to insure that the number of distinct widths of the wide rectangles is bounded.

2.3.1. Elimination of narrow rectangles. During the elimination phase, we partition the list L_{general} into two sublists: L_{narrow}, containing all the rectangles of width at most ε', and L, containing all the rectangles of width larger than ε'. During the next stage, we will focus on L.

2.3.2. Grouping. This is one of the main ideas of the paper.

We define a partial order on lists of rectangles by saying that $L \leq L'$ if there is an injection from L to L' such that each rectangle of L has smaller width and height than the associated rectangle of L'.

Given a list L of rectangles whose widths are larger than ε', we will now approximate L by a list L_{sup} such that $L \leq L_{\text{sup}}$, and such that the rectangles of L_{sup} only have m distinct widths.

To define L_{sup}, we first stack up all the rectangles of L by order of nonincreasing widths, to obtain a left-justified stack of total height $h(L)$. We define $(m-1)$ threshold rectangles, where a rectangle is a threshold rectangle if its interior or lower boundary intersects some line $y = ih(L)/m$, for some i between 1 and $m-1$ (see, for example, Figure 3). The threshold rectangles separate the remaining rectangles into m groups. The widths of the rectangles in the first group are then rounded up to 1, and the widths of the rectangles in each subsequent group are then rounded up to the width of the threshold rectangle below their group. This defines L_{sup}. Note that if all rectangle heights are equal, this is exactly the linear grouping defined in Fernandez de la Vega and Lueker (1981), and thus this can be seen as an extension of that paper. Also note that L_{sup} consists of rectangles which have only m distinct widths, all greater than ε'.

We construct a strip-packing of L_{sup} using the ideas of §2.2. A packing of L is trivially deduced by using the relation $L \leq L_{\text{sup}}$ and placing each rectangle of L inside the position of the associated rectangle of L_{sup}.

To get a packing of L_{general}, the narrow rectangles must now be added.

2.3.3. Adding the narrow rectangles. Order the rectangles of L_{narrow} by decreasing heights. We add the rectangles of L_{narrow} to the current strip-packing, trying to use the m' free rectangular areas $R_1, R_2, \ldots, R_{m'}$ as much as possible, according to a Modified-Next-Fit-Decreasing-Height algorithm as follows. Use the Next-Fit-Decreasing-Height (NFDH)

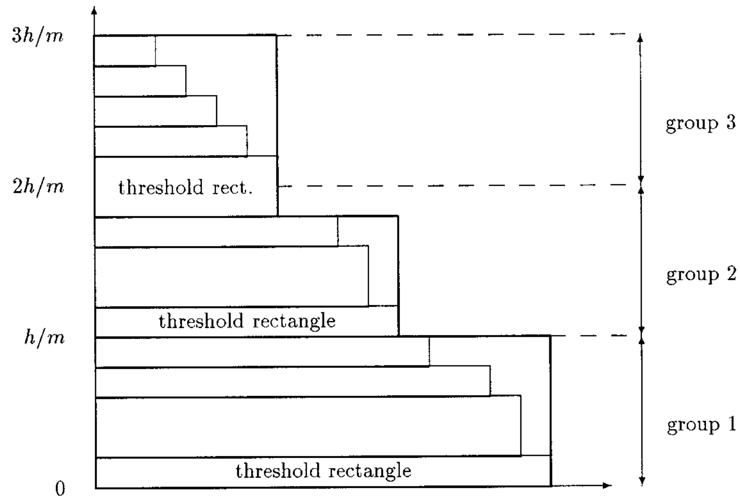


FIGURE 3. Grouping the rectangles, example when $m = 3$. The thick lines show how to extend the rectangles to construct L_{sup}.

heuristic to pack rectangles in R_1: In this heuristic, the rectangles are packed so as to form a sequence of sublevels. The first sublevel is simply the bottom line. Each subsequent sublevel is defined by a horizontal line drawn through the top of the first (and hence highest) rectangle placed on the previous sublevel. Rectangles are packed in a left-justified greedy manner, until there is insufficient space to the right to accommodate the next rectangle; at that point, the current sublevel is discontinued, the next sublevel is defined and packing proceeds on the new sublevel.

When a new sublevel cannot be started in R_1, start the next sublevel at the bottom left corner of R_2 using NFDH again, and so on until $R_{m'}$. When a rectangle cannot be packed in $R_1, \dots,$ or $R_{m'}$, use NFDH to pack the remaining rectangles in the strip of width 1 starting above $R_{m'}$, at level $l_{m'+1}$. This gives a packing of $L_{general}$.

We are now ready to summarize the overall algorithm.

2.4. The overall algorithm.

Parameters: ε' (the threshold narrow/wide) and m (the number of groups). We set $\varepsilon' = \varepsilon/(2+\varepsilon)$ and $m = (1/\varepsilon')^2$.

Input: a list of rectangles $L_{general}$.

(1) Perform the partition $L_{general} = L_{narrow} \cup L$ to set aside the rectangles of width less than ε'.

(2) Sort the rectangles of L according to their widths; form m groups of rectangles of approximately equal cumulative heights; round up the widths in each group, to yield a list L_{sup} with $L \leq L_{sup}$.

(3) Solve fractional strip-packing on L_{sup} with tolerance 1.

(4) From the fractional strip-packing, construct an integral strip-packing of L_{sup} and hence a well-structured strip-packing of L.

(5) Sort L_{narrow} according to decreasing heights and add the rectangles of L_{narrow} to the strip-packing of L using the Modified-Next-Fit-Decreasing-Height heuristic.

REMARK ON GUILLOTINE CUTS. We remark that this algorithm can be performed in five stages of guillotine cuts. First, we perform the horizontal cuts which perform the layers. Second, in each layer j, we perform a vertical cut at C_j, thus separating the part reserved to the wide rectangles from the part R_j for the narrow rectangles, and also perform all the vertical cuts defining the columns of configuration used in layer j. Third, we cut the columns with horizontal cuts, and in R_j, we cut all the sublevels with horizontal cuts. Fourth, we finish cutting out the wide rectangles using vertical cuts to adjust the widths to their true values and in each sublevel, we perform vertical cuts corresponding to the narrow rectangles. Finally, in each sublevel, we perform horizontal cuts to finish cutting out the narrow rectangles. All in all, we have used five stages of guillotine cuts.

3. The analysis. The running time is clearly polynomial in n and $1/\varepsilon$. Its bottleneck lies in the resolution of the fractional bin-packing problem in Step (3), which can be done as in Karmarkar and Karp (1982) or Plotkin et al. (1995). Thus the main difficulty in the analysis consists in showing that the strip-packing is close to optimal. This is done through a series of lemmas.

LEMMA 3. *The list L_{sup} obtained after the grouping of Step* (2) *is such that*

$$\text{lin}(L_{sup}) \le \text{lin}(L)(1 + 1/(m\varepsilon'))$$

and

$$s(L_{sup}) \le s(L)(1 + 1/(m\varepsilon')),$$

where $s(L)$ is the area of L and lin(L) *is the height of the optimal fractional strip-packing of L.*

PROOF. We follow the same plan as Fernandez de la Vega and Lueker (1981). Define the following extension of our partial order on lists of rectangles: $L \le L'$ if the stack associated to L (used for the grouping), viewed as a region of the plane, is contained in the stack associated to L'. Thus, $L \le L'$ clearly implies $\text{lin}(L) \le \text{lin}(L')$ and $s(L) \le s(L')$. We now define lists L'_{inf} and L'_{sup} such that $L'_{\text{inf}} \le L \le L_{\text{sup}} \le L'_{\text{sup}}$. These lists are obtained by first cutting the threshold rectangles using the lines $y = ih(L)/m$ $(1 \le i \le m-1)$, then considering the m subsequent groups of rectangles in turn (where each group now has cumulative height exactly $h(L)/m$); to define L'_{sup}, we round the widths in each group up to the widest width of the group (up to 1 for the first group); to define L'_{inf}, we round the widths within each group down to the widest width of the next group (down to 0 for the last group). It is easy to see that $L'_{\text{inf}} \le L \le L_{\text{sup}} \le L'_{\text{sup}}$. Moreover, the fractional bin-packing problem for L'_{inf} is almost the same as for L'_{sup}: the stack associated to L'_{sup} is the union of a bottom part of width 1 and height $h(L)/m$, and of a translated copy of the stack of L'_{inf}. This implies

$$\text{lin}(L'_{\text{sup}}) = \text{lin}(L'_{\text{inf}}) + h(L)/m$$

and

$$s(L'_{\text{sup}}) = s(L'_{\text{inf}}) + h(L)/m.$$

Finally, we note that since all rectangles have width at least ε', we have $h(L)\varepsilon' \le s(L) \le \text{lin}(L)$. This implies the statement of the lemma. □

LEMMA 4. *Let L_{aux} be the list formed from the union $L_{sup} \cup L_{narrow}$. If the height h_{final} at the end of Step* (5) *is larger than the height h' of the packing of the wide rectangles, then*: $h_{final} \le s(L_{aux})/(1-\varepsilon') + 4m + 1$.

PROOF. As in Coffman et al. (1980), we will charge the surface of a sublevel to the rectangles in the sublevel immediately below it. The core of the proof is in Figure 4.

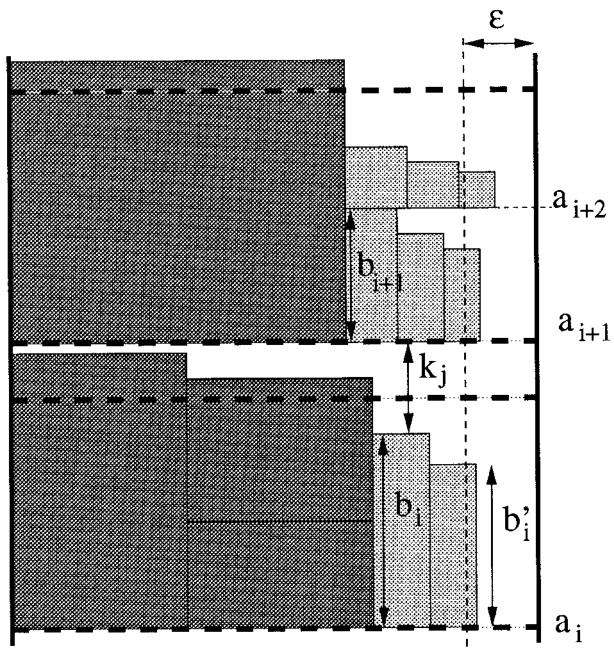


FIGURE 4. Addition of narrow rectangles: notations. Notice that the rectangle $(1 - c_j - \varepsilon') \times b_i'$ is completely covered. Moreover this rectangle has surface area larger than the rectangle $(1 - c_j - \varepsilon') \times b_{i+1}$. This is the main argument of the proof of Lemma 4.

Assume that the height h_{final} at the end of Step (5) is larger than the height h' of the packing of the wide rectangles.

Let $(a_1, a_2, \ldots, a_r)$ be the ordered sequence of the lower boundaries of the successive levels constructed by Modified NFDH when inserting the narrow rectangles (hence $a_1 < a_2 < \cdots < a_r$) and let b_i (respectively b_i') be the height of the first (respectively last) narrow rectangle placed on the ith sublevel (see Figure 4). We focus on the sublevels between l_j and l_{j+1}. By definition of NFDH, sublevel i is closed only when the next narrow rectangle is too wide to fit in the current level, which must thus have remaining unused width less than ε'. The total surface occupied in R_j on sublevel i is at least $b_i'(1-c_j\varepsilon')$. Since $b_i' \geq b_{i+1}$, the total surface occupied in R_j is at least

$$\sum_{i \text{ s.t. sublevel } i \text{ is in } [l_j, l_{j+1}]} b_{i+1}(1-c_j - \varepsilon') \geq (l_{j+1} - l_j - 2)(1 - c_j - \varepsilon'),$$

since this accounts for all sublevels between l_j and l_{j+1}, except the first one. Adding this to R_j', we get that the total surface occupied between l_j and l_{j+1} is at least $(l_{j+1} - l_j - 2)(1 - \varepsilon')$.

In the part between h' and h_{final}, an analysis similar to the analysis of R_j shows that the surface occupied is at least $(1 - \varepsilon')(h_{\text{final}} - h' - 1)$. Overall, since there are at most $2m$ layers $[l_j, l_{j+1}]$, the total surface occupied by L_{aux} is

$$s(L_{\text{aux}}) \geq (h' - 4m)(1 - \varepsilon') + (h_{\text{final}} - h' - 1)(1 - \varepsilon'),$$

from which we obtain

$$h_{\text{final}} \leq \frac{s(L_{\text{aux}})}{1 - \varepsilon'} + 4m + 1. \quad \square$$

Notice that Lemma 3 obviously implies that $s(L_{\text{aux}}) \leq s(L_{\text{general}})(1+1/m\varepsilon')$, which yields the following corollary:

COROLLARY 1. *If the height h_{final} at the end of Step* (5) *is larger than the height h' of the packing of the wide rectangles, then*:

$$h_{final} \leq s(L_{general})(1 + 1/(m\varepsilon'))/(1 - \varepsilon') + 4m + 1.$$

PROOF OF THEOREM 1. Let h_{final} be the height of the packing of L_{general} constructed by our algorithm. Then

$$h_{\text{final}} \leq \max\{h', s(L_{\text{general}})(1 + 1/(m\varepsilon'))/(1 - \varepsilon') + 4m + 1\},$$

where h' is the height of the strip-packing of the wide rectangles constructed in Step (4). But $h' \leq h + 2m$, where h is the height of the fractional strip-packing constructed in Step (3). By Karp and Karmarkar's theorem (with tolerance 1), h is at most $\text{lin}(L_{\text{sup}}) + 1$. Noticing that $\text{lin}(L) \leq \text{Opt}(L)$, we obtain:

$$\begin{aligned} h' &\leq h + 2m \quad \text{from Lemma 2,} \\ &\leq \text{lin}(L_{\text{sup}}) + 1 + 2m \\ &\leq \text{lin}(L)(1 + 1/m\varepsilon') + 1 + 2m \quad \text{from Lemma 3,} \\ &\leq \text{Opt}(L)(1 + 1/m\varepsilon') + 1 + 2m, \\ &\leq \text{Opt}(L_{\text{general}})(1 + 1/m\varepsilon') + 1 + 2m. \end{aligned}$$

Noticing that $s(L_{\text{general}}) \leq \text{Opt}(L_{\text{general}})$, we obtain:

$$h_{\text{final}} \leq \text{Opt}(L_{\text{general}})(1 + 1/(m\varepsilon'))/(1 - \varepsilon') + 4m + 1.$$

Replacing m and ε' by their values $\varepsilon' = \varepsilon/(2 + \varepsilon)$ and $m = (1/\varepsilon')^2$, we get

$$h_{\text{final}} \leq \text{Opt}(L_{\text{general}})(1 + \varepsilon) + 4(2 + \varepsilon)^2/(\varepsilon^2) + 1,$$

hence the theorem.

4. Remarks. In this paper, we proposed a fully polynomial time approximation scheme for strip-packing when the rectangle widths and heights are in $(0, 1]$. In many applications, it makes sense to allow rotations of the rectangles (in the case of cutting window panes out of glass or shapes out of leather, for example). We conjecture that our approach can be extended to solve the strip-packing problem when rotations of 90 degrees are allowed. It should however be noted that sometimes the optimal packing may use rotations of angles other than 90 degrees, even in the simple situation when one wants to pack squares in a strip (El Moumni 1997, Erdös and Graham 1975).

We also note that since the preliminary version of this work (Kenyon and Remila 1996), subsequent papers have used similar linear programming relaxations for various scheduling applications (Amoura et al. 1997, Jansen and Porkolab 1999). Finally, since we find the solution developed here relatively simple, we hope that it may help for attacking the three-dimensional version of the problem, as well as other variants of multidimensional cutting-stock problems.

Acknowledgments. The authors would like to thank Neal Young for a fruitful discussion that simplified Stage(3) of the algorithm. This simplification later enabled them to appeal to the fractional packing results from Karmarkar and Karp (1982), thus easily rendering the scheme *fully* polynomial. The authors would also like to thank Klaus Jansen for valuable comments.

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