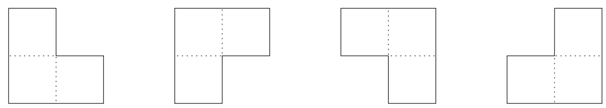
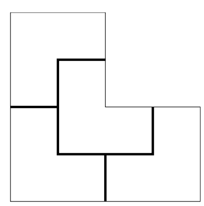
4. Consider a $2^n \times 2^n$ board, with one of its four quadrants missing. That is, the board consists of only three quadrants, each of size $2^{n-1} \times 2^{n-1}$. Let's call such a board a quad-deficient board. For $n = 1$, such a board becomes an L-shape 3-cell piece called a **tromino**, as shown below.



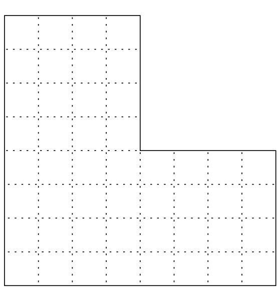
   (a) (12 pts.) Use a **divide-and-conquer** technique to prove by induction that a quad-deficient board of size $2^n \times 2^n$, $n \geq 1$ can always be **covered** using some number of trominoes. (By covering we mean that every cell of the board must be covered by a tromino piece, and the pieces must not overlap or extend outside of the board.) Use a diagram to help describing your algorithm and proof.

   Solution: For the base, $n = 1$, the borad is in the shape of a single tromino, thus it can be covered by a single tromino.

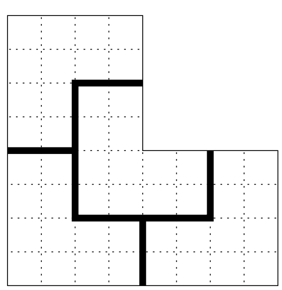
   For $n \geq 2$, we will prove that if a $2^{n-1} \times 2^{n-1}$ quad-deficient board can be covered, then a $2^n \times 2^n$ quad-deficient board can also be covered. Consider a $2^n \times 2^n$ quad-deficient board. The board can be divided into four $2^{n-1} \times 2^{n-1}$ quad-deficient boards, as shown below. By the hypothesis, each of these smaller boards can be covered. Therefore the $2^n \times 2^n$ board can be covered. $\square$



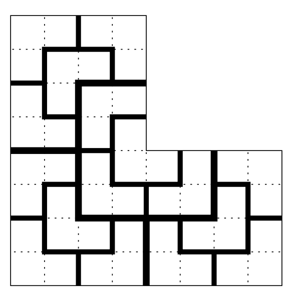
   (b) (8 pts.) Illustrate the covering produced by the algorithm for $n = 3$ (that is, $2^3 \times 2^3$ board).



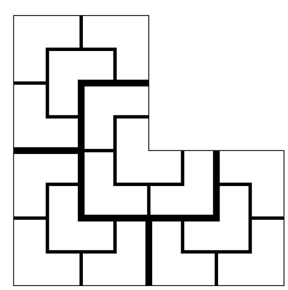
Initial Board



Top–Level Divide

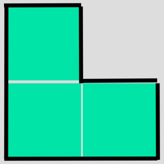


Next–Level Divide

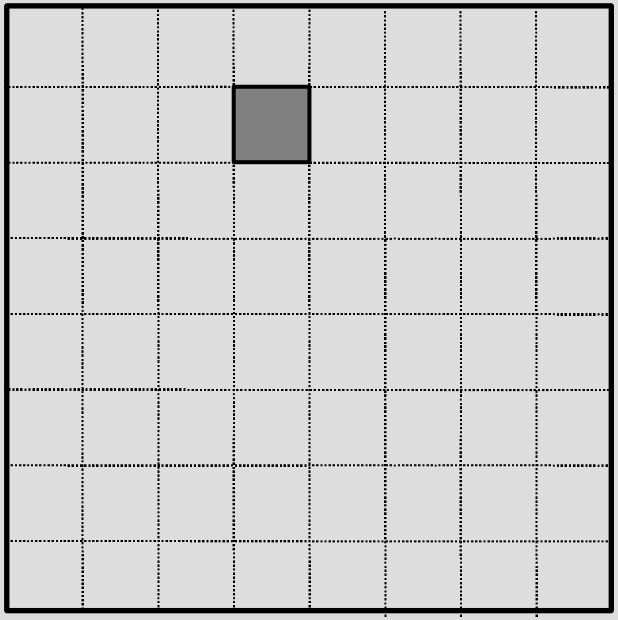


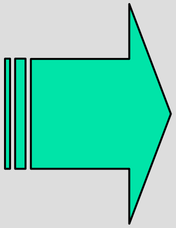
Polished Final Tiling

Tiling

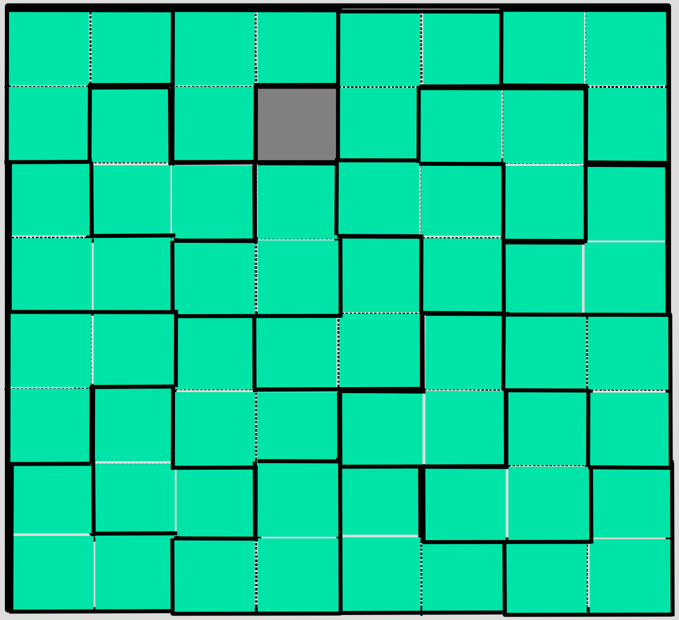
A tromino tile: 

A 2^nx2^n board with a hole:



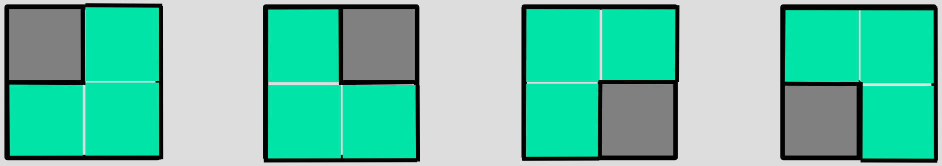


A tiling of the board with trominos:



Tiling: Trivial Case (n = 1)

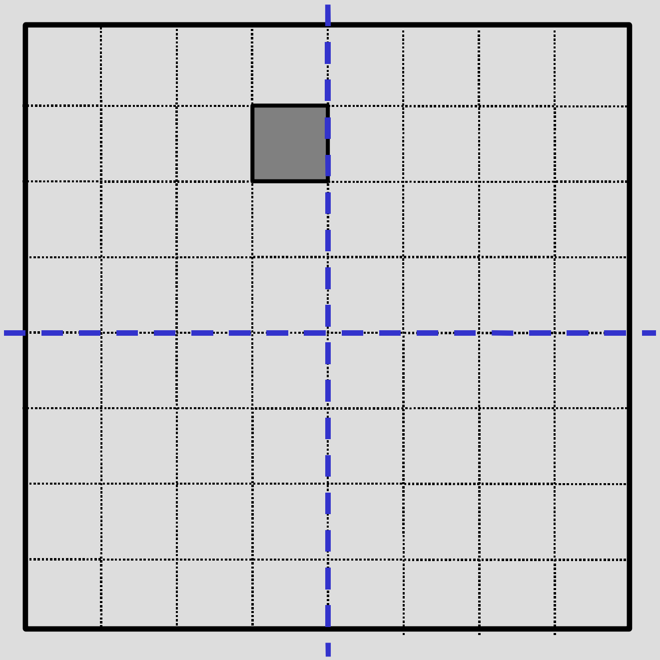
- Trivial case (n = 1): tiling a 2x2 board with a hole:



- Idea: reduce the size of the original problem, so that we eventually get to the 2x2 boards, which we know how to solve.

Tiling: Dividing the Problem

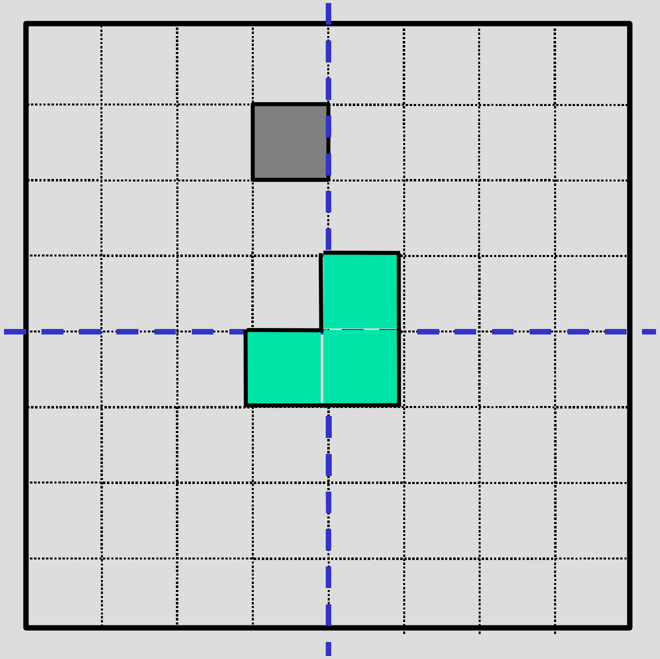
- To get smaller square boards let's divide the original board into four boards.



- Good: We have a problem of size $2^{n-1}x2^{n-1}$!
- Bad: The other three problems are not similar to the original problem – they do not have holes!

Tiling: Dividing the Problem/2

- Idea: insert one tromino at the center to "cover" three holes in each of the three smaller boards



- Now we have four boards with holes of the size $2^{n-1} \times 2^{n-1}$.
- Keep doing this division, until we get the 2x2 boards with holes – we know how to tile those.

Tiling: Algorithm

```
INPUT:  n – the board size (2^n x 2^n board),
        L – location of the hole.
OUTPUT: tiling of the board

Tile(n, L)
  if n = 1 then //Trivial case
    Tile with one tromino
    return
  Divide the board into four equal-sized boards
  Place one tromino at the center to cover 3 additional
      holes
  Let L1, L2, L3, L4 be the positions of the 4 holes
  Tile(n-1, L1)
  Tile(n-1, L2)
  Tile(n-1, L3)
  Tile(n-1, L4)
```