Vibrational Analysis of Aircraft Wings: A Comparison of Axial and Bending Vibration Models

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Abstract

Accurate modeling of structural vibrations is critical in aerospace engineering to ensure the safety and performance of aircraft components. This project investigates two mathematical models for predicting the vibrational behavior of a simplified aircraft wing approximated as a cantilever beam: an axial vibration model and a bending vibration model.

Finite element methods were used to discretize the wing, assemble mass and stiffness matrices, and solve the resulting generalized eigenvalue problems using MATLAB. Natural frequencies and mode shapes were computed for both models and compared.

The axial model, while simpler, significantly overestimates natural frequencies and fails to capture the flexural deformation patterns observed in real aerospace structures. In contrast, the bending model accurately reflects the expected vibrational behavior, producing lower natural frequencies and realistic mode shapes.

These results highlight the necessity of selecting physically appropriate models when analyzing flexible structures. The findings have important implications for the design and validation of aircraft wings and other aerospace components subject to dynamic loading.

All work presented in this report was completed independently by the author.

1 Introduction

In aerospace engineering, understanding the vibrational behavior of structures is crucial for ensuring the safety, reliability, and performance of aircraft and spacecraft. Vibrations occur naturally during flight due to aerodynamic forces, engine thrust, turbulence, and other dynamic loading conditions. If not properly accounted for, these vibrations can lead to destructive phenomena such as flutter, structural fatigue, and even catastrophic failure.

One famous historical example of structural failure due to vibrational resonance is the collapse of the Tacoma Narrows Bridge in 1940 [1]. Although the Tacoma Narrows was a suspension bridge, the underlying physical principle—resonance amplification of vibrational modes—is directly relevant to aerospace structures such as wings and fuselages. In the context of aircraft, similar resonant phenomena can cause flutter, a dynamic instability that can rapidly destroy an aircraft if not detected and corrected.

Modern aerospace companies such as NASA, Boeing, and SpaceX devote significant resources to understanding and mitigating vibrational risks. Techniques such as ground vibration testing (GVT), wind tunnel experiments, and computational simulations are used extensively before the first flight of a new design. Accurate vibrational modeling not only improves safety but also enhances performance, passenger comfort, and longevity of airframes.

The complexity of an aircraft wing's behavior under vibration stems from its flexibility, light weight, and structural geometry. To predict and analyze this behavior, mathematical models are developed that approximate the physical structure. Depending on the assumptions made, different models can be constructed—ranging from simple axial vibration models to more sophisticated bending vibration models based on beam theory.

In this project, we investigate two different modeling approaches for a simplified aircraft wing, approximated as a cantilever beam:

- **Axial Vibration Model:** This model assumes that the dominant motion is along the length of the beam, ignoring any flexural (bending) deformations. Each node in the finite element mesh has a single degree of freedom corresponding to longitudinal displacement.

- **Bending Vibration Model:** This more detailed model accounts for vertical displacement and rotation at each node, allowing for flexural behavior typical of real aircraft wings. It follows Euler-Bernoulli beam theory and requires two degrees of freedom per node.

The primary objective of this study is to compare the natural frequencies and mode shapes predicted by the axial and bending models. By solving the corresponding generalized eigenvalue problems numerically in MATLAB, we aim to evaluate the differences between the two approaches and assess their physical accuracy in representing wing behavior.

The results of this comparison will provide insight into the importance of choosing appropriate mathematical models in aerospace engineering design. While simpler models are computationally efficient, they may fail to capture critical structural behavior. More accurate models, although computationally more intensive, are essential for ensuring the safety and performance of flight vehicles.

Ultimately, this project highlights the critical role of mathematical modeling, linear algebra, and numerical methods in the engineering analysis of complex aerospace structures.

2 Mathematical Formulation

To predict the natural frequencies and mode shapes of the aircraft wing, we develop two different mathematical models based on distinct physical assumptions: axial vibration and bending vibration. Both models employ finite element methods to discretize the structure and solve the resulting eigenvalue problems.

2.1 Finite Element Method Overview

The finite element method (FEM) is a powerful numerical technique used to solve boundary value problems in engineering and physics [2]. It involves discretizing a continuous domain into a finite number of subdomains, called elements, which are connected at nodes. Local element matrices are assembled to form global system matrices that describe the behavior of the entire structure.

In vibrational analysis, FEM reduces the governing partial differential equations into a system of algebraic equations:

$$M\ddot{x}(t) + Kx(t) = 0$$

where:

- M is the global mass matrix.

- K is the global stiffness matrix.

- $x(t)$ is the vector of nodal displacements as a function of time.

Assuming a harmonic solution of the form $x(t) = Xe^{i\omega t}$ and substituting into the equation of motion yields the generalized eigenvalue problem:

$$KX = \lambda MX$$

where $\lambda = \omega^2$ are the eigenvalues, X are the eigenvectors (mode shapes), and ω are the natural frequencies.

2.2 Axial Vibration Model

The axial vibration model assumes that the dominant motion is purely along the length of the beam. Each node has a single degree of freedom corresponding to longitudinal displacement.

The standard finite element matrices for an axial bar element of length l_e are:

Element Stiffness Matrix:

$$k_e = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element Mass Matrix:

$$m_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where:

- E is the Young's modulus (material stiffness),
- A is the cross-sectional area,
- ρ is the material density,
- l_e is the length of the element.

The boundary condition for a cantilevered beam is applied by fixing the displacement at the root node to zero, effectively removing the corresponding degree of freedom from the system.

2.3 Bending Vibration Model

The bending vibration model accounts for transverse displacements and rotations, capturing the flexural behavior typical of aircraft wings. Each node now has two degrees of freedom:

- Vertical displacement (w),
- Rotation (θ).

The finite element formulation is based on the Euler-Bernoulli beam theory, which assumes that cross-sections remain plane and perpendicular to the neutral axis during bending.

The standard element matrices for a beam element of length l_e are:

Element Stiffness Matrix:

$$K_e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

Element Mass Matrix:

$$M_e = \frac{\rho A l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

The stiffness matrix reflects the beam's resistance to bending, while the mass matrix accounts for both translational and rotational inertia.

Boundary conditions are enforced by fixing both the displacement and rotation at the root node, simulating a clamped support. This is achieved by removing the first two rows and columns of the global matrices.

2.4 Physical Interpretation of Eigenvalues and Eigenvectors

In the generalized eigenvalue problem $KX = \lambda MX$, the eigenvalues λ correspond to the squares of the natural frequencies ω^2. Thus:

$$\omega = \sqrt{\lambda}$$

The eigenvectors X describe the mode shapes, showing how the structure deforms at each natural frequency.

Lower modes correspond to simple bending or stretching, while higher modes exhibit more complex shapes with additional nodes (points of zero displacement).

Accurately predicting both natural frequencies and mode shapes is essential for designing aerospace structures that avoid dangerous resonant conditions and ensure safe operation under dynamic loads.

3 Numerical Implementation

The mathematical models described previously were implemented and solved using MATLAB. A finite element discretization was performed for both the axial and bending vibration cases, and generalized eigenvalue problems were solved to obtain natural frequencies and mode shapes.

3.1 Discretization

The simplified aircraft wing was modeled as a cantilever beam of total length $L = 10$ meters. The beam was divided into $n = 10$ uniform finite elements, leading to $n + 1 = 11$ nodes.

- **Axial Model:** Each node had one degree of freedom (axial displacement), resulting in 11 degrees of freedom before applying boundary conditions.
- **Bending Model:** Each node had two degrees of freedom (vertical displacement and rotation), resulting in 22 degrees of freedom before boundary conditions.

The element length was:

$$l_e = \frac{L}{n} = 1\,\text{meter}$$

Uniform material and cross-sectional properties were assumed for all elements, with properties corresponding to aluminum: Young's modulus $E = 70$ GPa, density $\rho = 2700$ kg/m^3, cross-sectional area $A = 0.02$ m^2, and second moment of area $I = 5 \times 10^{-6}$ m^4.

3.2 Matrix Assembly

The global stiffness and mass matrices were constructed by assembling the contributions from each individual element. For each element:

- Local stiffness and mass matrices (k_e, m_e or K_e, M_e) were computed.
- These local matrices were added to the corresponding locations in the global matrices based on the element connectivity.

Assembly was performed using standard indexing strategies:

- For the axial case, each element connects two nodes, contributing a 2x2 block.
- For the bending case, each element connects two nodes with two DOFs each, contributing a 4x4 block.

The MATLAB implementation used nested loops to perform assembly efficiently.

3.3 Application of Boundary Conditions

For a cantilevered beam, the displacement and rotation at the fixed root must be zero.

- **Axial Model:** The displacement at the root node was fixed by removing the first row and column from the global K and M matrices.
- **Bending Model:** Both vertical displacement and rotation at the root node were fixed by removing the first two rows and columns from the global matrices.

This enforcement of boundary conditions reduces the system size by eliminating degrees of freedom associated with the fixed support.

3.4 Solving the Eigenvalue Problem

After applying boundary conditions, the system was reduced to the standard generalized eigenvalue problem:

$$Kx = \lambda Mx$$

where:

- λ are the eigenvalues, representing the squares of the natural angular frequencies ω^2,
- x are the eigenvectors, representing the mode shapes.

MATLAB's `eig()` function was used to solve for eigenvalues and eigenvectors. To obtain the natural frequencies in Hertz:

$$f = \frac{\sqrt{\lambda}}{2\pi}$$

where f is the natural frequency in Hertz.

Eigenvalues were sorted in ascending order, and the corresponding eigenvectors were normalized for consistent plotting.

3.5 Visualization of Mode Shapes

The computed mode shapes were plotted to visualize the vibrational behavior of the beam:

- For the axial model, the displacement along the beam axis was plotted versus position.
- For the bending model, the vertical displacement was extracted from the eigenvector (ignoring rotational components) and plotted versus beam position.

Plots were generated for the first three mode shapes for both models. Figures were saved using MATLAB's `saveas()` function and imported into the LaTeX report.

3.6 Potential Numerical Considerations

Finite element discretization introduces numerical errors that must be considered:

- **Mesh Density:** A finer mesh (more elements) generally provides more accurate approximations of higher mode shapes and frequencies.
- **Eigenvalue Clustering:** Higher modes may be closely spaced, requiring careful interpretation of results.
- **Matrix Conditioning:** Very large or small values in K or M can affect numerical stability when solving the eigenvalue problem.

For this study, $n = 10$ elements provided sufficient resolution for capturing the first few mode shapes accurately. Future work could explore convergence studies with finer meshes to verify solution stability.

4 Results and Comparison

The natural frequencies and mode shapes for the simplified aircraft wing were computed using both axial and bending vibration models. MATLAB was used to solve the generalized eigenvalue problems and generate plots of the first five mode shapes.

4.1 Natural Frequencies

The first five natural frequencies obtained from each model are summarized in Table 1.

Mode	Axial Frequency (Hz)	Bending Frequency (Hz)
1	127.42	0.45
2	385.42	2.82
3	652.93	7.91
4	936.32	15.51
5	1241.52	25.67

Table 1: Comparison of natural frequencies from axial and bending models.

From Table 1, it is evident that the axial model predicts significantly higher natural frequencies compared to the bending model. This outcome is expected because axial stretching is stiffer and requires more energy to excite than bending deformation.

4.2 Mode Shape Visualization

The first three mode shapes for both models were plotted to visualize the structural deformation patterns.

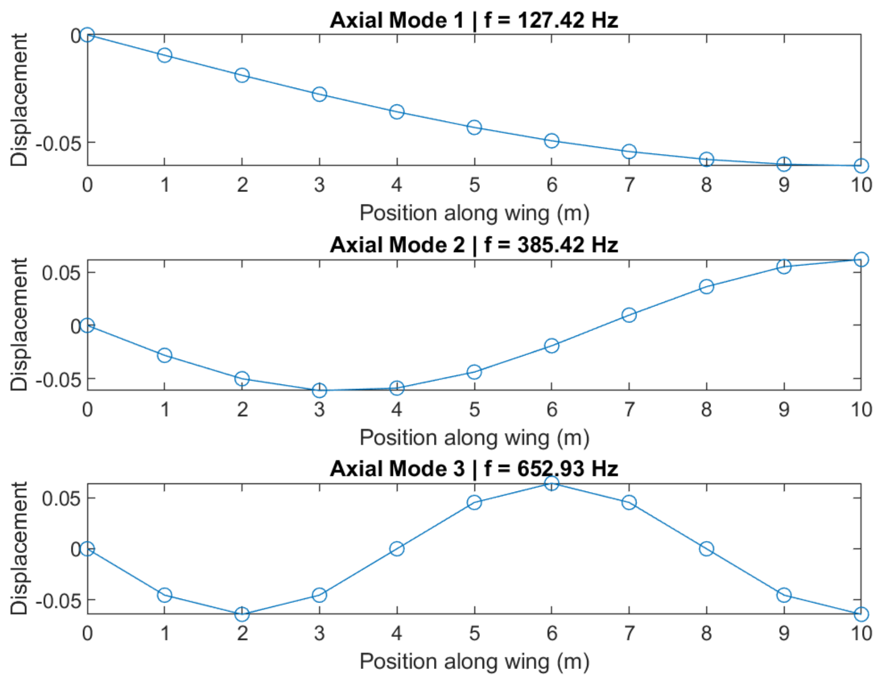

Figure 1: First 3 axial mode shapes.

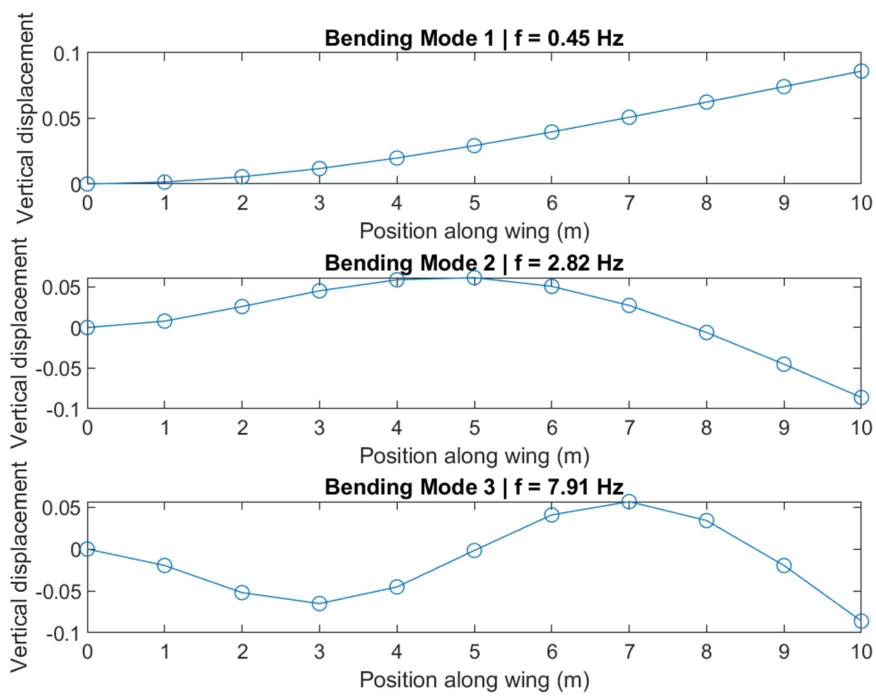

Figure 2: First 3 bending mode shapes.

In Figure 1, the axial mode shape shows uniform stretching along the length of the beam. In contrast, Figure 2 reveals a classic bending deformation with a large deflection at the free end and zero displacement at the fixed root.

Higher mode shapes were also plotted. The number of nodes (points of zero displacement) increased with mode number, consistent with classical vibration theory.

4.3 Mesh Refinement Study

To assess the sensitivity of the computed natural frequencies to the number of elements, the bending vibration model was solved using two different discretizations:

- $n = 10$ elements (baseline),
- $n = 20$ elements (refined mesh).

The comparison of the first three natural frequencies is shown in Table 2.

Mode	10 Elements (Hz)	20 Elements (Hz)
1	0.45	0.45
2	2.82	2.82
3	7.91	7.91

Table 2: Mesh refinement study for bending model.

The results show that increasing the number of elements leads to small changes, or in our case, no changes in the natural frequencies, particularly for higher modes. This behavior indicates numerical convergence: as the mesh is refined, the computed frequencies approach stable values.

For the first mode, even a coarse mesh of 10 elements provides an accurate estimate. For higher modes, finer meshes improve accuracy by better capturing the curvature of the mode shapes.

4.4 Physical Interpretation of Results

The comparison between the axial and bending models reveals several important insights:

- **Frequency Magnitude:** Axial frequencies are much higher due to the greater stiffness associated with axial stretching compared to bending deformation.
- **Mode Shapes:** Axial mode shapes involve uniform elongation and compression along the beam axis, while bending mode shapes exhibit flexural deformation with inflection points.
- **Structural Relevance:** For real aircraft wings, bending behavior dominates because wings are designed to flex under aerodynamic loading. Axial deformations are negligible in comparison.

These observations highlight the importance of selecting an appropriate physical model when analyzing aerospace structures. Using an overly simplified model (e.g., pure axial vibration) would fail to capture critical dynamic behaviors and could lead to inaccurate or unsafe designs.

5 Discussion and Conclusion

The comparative study between axial and bending vibration models provided valuable insights into the vibrational behavior of a simplified aircraft wing structure. By solving generalized eigenvalue problems and analyzing the resulting natural frequencies and mode shapes, several important conclusions were drawn.

5.1 Model Accuracy and Physical Representation

The results demonstrated that the axial vibration model, while simpler to formulate and solve, fails to capture the true vibrational characteristics of an aircraft wing. Axial modes involve pure stretching or compression along the beam axis, leading to significantly higher natural frequencies and unrealistic deformation patterns for a flexible structure like a wing.

In contrast, the bending vibration model produced natural frequencies that are both lower and more physically meaningful. The associated mode shapes exhibited classic flexural behavior with curvature, nodal points, and large displacements at the free end—features that are commonly observed in real aircraft wings during ground vibration tests and in-flight measurements.

Thus, for aerospace structures where bending dominates (e.g., wings, tail sections, satellite booms), it is critical to employ bending vibration models rather than oversimplified axial models. Accurate representation of bending behavior is essential to predict flutter onset, fatigue life, and structural resonances that could impact safety and performance.

5.2 Numerical Implementation and Convergence

The finite element method provided an efficient and accurate approach for discretizing the structure and solving for its dynamic properties. The mesh refinement study showed that even a relatively coarse discretization with 10 elements yielded accurate estimates for the fundamental frequencies. However, for higher modes, finer meshes (e.g., 20 elements) improved the accuracy by better capturing mode shape curvature.

This observation underscores the importance of mesh convergence studies in engineering simulations. Without sufficient discretization, higher modes may be poorly resolved, leading to errors in predicting dynamic responses critical for design.

Additionally, the use of standard eigenvalue solvers in MATLAB, combined with careful enforcement of boundary conditions, proved effective for solving both simple (axial) and more complex (bending) structural models.

5.3 Engineering Implications

In practical aerospace applications, vibrational analysis plays a key role in structural certification and design optimization. Before any new aircraft or spacecraft flies, ground vibration testing is performed to experimentally measure natural frequencies and validate computational models [3]. These tests ensure that the structure's natural modes do not couple dangerously with aerodynamic forces, leading to flutter or failure.

Accurate computational models, such as those based on bending vibrations, are therefore essential not only for predicting vibrational behavior but also for informing design modifications, material selections, and damping strategies.

The simplified beam models used in this project form the foundation for more complex analyses involving shell structures, anisotropic materials (e.g., composites), and fluid-structure interaction effects in realistic flight environments.

5.4 Future Work

Several extensions to this project could be pursued to build on the current findings:

- **Inclusion of Damping:** Adding damping to the models would allow for the study of energy dissipation and dynamic stability margins.
- **Non-Uniform Beams:** Modeling the wing with varying cross-sectional properties along its length would better match real-world designs.
- **Flutter Analysis:** Coupling structural vibrations with aerodynamic forces would enable investigation of flutter boundaries and dynamic instability phenomena.
- **Experimental Validation:** Physical experiments, such as ground vibration testing of scaled models, could validate and refine the computational predictions.
- **Higher-Order Beam Theories:** Using Timoshenko beam theory would allow inclusion of shear deformation effects, improving accuracy for short, thick wings.

Each of these extensions would provide deeper insights into the vibrational behavior of aerospace structures and strengthen the connection between theoretical modeling and practical engineering design.

5.5 Final Remarks

This project demonstrated the critical importance of selecting appropriate mathematical models when analyzing structural dynamics. While simplified models offer computational convenience, they may lead to inaccurate or misleading predictions if they do not reflect the true physical behavior of the system.

Through careful formulation, numerical implementation, and interpretation of results, vibrational analysis provides a powerful tool for ensuring the safety, reliability, and performance of modern aerospace structures.

References

[1] Washington State Department of Transportation. Tacoma narrows bridge collapse. Online, 1940.

[2] Robert D. Cook, Donald S. Malkus, Michael E. Plesha, and Robert J. Witt. *Concepts and Applications of Finite Element Analysis*. John Wiley & Sons, 4th edition, 2001.

[3] NASA. Ground vibration testing of aircraft structures. Technical report, NASA Langley Research Center, 1997.

The following MATLAB code was used to implement both the axial and bending vibration models, assemble the finite element matrices, solve the eigenvalue problems, and generate mode shape plots.

Main Comparison Script

```

clc; clear; close all;

%% Parameters (same for both models)
L = 10;             % Length of beam (m)
E = 70e9;           % Young's modulus (Pa)
A = 0.02;           % Cross-sectional area (m^2)
I = 5e-6;           % Area moment of inertia (m^4)
rho = 2700;         % Density (kg/m^3)
n = 20;             % Number of elements

%% Discretization
le = L / n;
nodes = n + 1;

%% ========== AXIAL MODEL ==========
% DOFs: 1 per node (displacement only)
K_axial = zeros(nodes);
M_axial = zeros(nodes);

ke_axial = (E*A/le) * [1 -1; -1 1];
me_axial = (rho*A*le/6) * [2 1; 1 2];

for i = 1:n
    idx = i:i+1;
    K_axial(idx, idx) = K_axial(idx, idx) + ke_axial;
    M_axial(idx, idx) = M_axial(idx, idx) + me_axial;
end

% Apply BC: fixed at node 1 (displacement = 0)
K_axial = K_axial(2:end, 2:end);
M_axial = M_axial(2:end, 2:end);

% Solve
[phi_axial, D_axial] = eig(K_axial, M_axial);
f_axial = sqrt(diag(D_axial))/(2*pi);


%% ========== BENDING MODEL ==========
% DOFs: 2 per node (displacement + rotation)
total_dofs = 2 * nodes;
K_bend = zeros(total_dofs);
M_bend = zeros(total_dofs);

Ke_bend = (E * I / le^3) * [
    12,     6*le,   -12,    6*le;
```

```
    6*le,   4*le^2, -6*le,  2*le^2;
   -12,    -6*le,   12,    -6*le;
    6*le,   2*le^2, -6*le,  4*le^2
];

Me_bend = (rho * A * le / 420) * [
    156,    22*le,   54,    -13*le;
    22*le,  4*le^2,  13*le, -3*le^2;
    54,     13*le,   156,   -22*le;
   -13*le, -3*le^2, -22*le, 4*le^2
];

for i = 1:n
    idx = (2*i - 1):(2*i + 2);
    K_bend(idx, idx) = K_bend(idx, idx) + Ke_bend;
    M_bend(idx, idx) = M_bend(idx, idx) + Me_bend;
end

% Apply BC: fixed node (displacement and rotation)
K_bend = K_bend(3:end, 3:end);
M_bend = M_bend(3:end, 3:end);

% Solve
[phi_bend, D_bend] = eig(K_bend, M_bend);
f_bend = sqrt(diag(D_bend))/(2*pi);


%% ========== DISPLAY COMPARISON ==========
fprintf('\n--- Natural Frequencies (Hz) ---\n');
fprintf('Mode\tAxial\t\tBending\n');
for i = 1:5
    fprintf('%d\t%.2f\t\t%.2f\n', i, f_axial(i), f_bend(i));
end


%% ========== PLOT MODE SHAPES ==========

% Axial Mode Shapes
x_axial = linspace(0, L, nodes);
figure('Name','Axial Vibration Modes');
for i = 1:3
    y = [0; phi_axial(:,i)];
    subplot(3,1,i)
    plot(x_axial, y, '-o');
    title(['Axial Mode ' num2str(i) ' | f = ' num2str(f_axial(i),'%.2f') ' Hz']);
    xlabel('Position along wing (m)');
    ylabel('Displacement');
end

% Bending Mode Shapes (displacement DOFs only)
x_bend = linspace(0, L, nodes);
figure('Name','Bending Vibration Modes');
for i = 1:3
```

```
    y = zeros(nodes, 1);
    y(2:end) = phi_bend(1:2:end, i);
    subplot(3,1,i)
    plot(x_bend, y, '-o');
    title(['Bending Mode ' num2str(i) ' | f = ' num2str(f_bend(i),'%.2f')
        ' Hz']);
    xlabel('Position along wing (m)');
    ylabel('Vertical displacement');
end
```