MATHEMATICAL APPENDIX

Rounding at the Geometric Mean

In this chapter the question was asked, why rounding at the geometric mean should be related to minimizing relative differences in the number of people needed for a congressional seat. Here we present a proof of the astonishing statement that both procedures give identical results.

Let $p_1, p_2, \ldots p_n$ be the populations of the states, let d be the divisor, and let $a_1, a_2, \ldots a_n$ be the apportionment that results from rounding the ratios p_i/d at the geometric mean.

Then for every state i we have

$$\sqrt{(a_i[a_i - 1])} \le p_i/d \le \sqrt{(a_i[a_i + 1])}.$$

This is equivalent to

$$a_i(a_i - 1)/p_i^2 \le 1/d^2 \le a_i(a_i + 1)/p_i^2.$$

Since this holds for every i, it follows that for every i and j,

$$a_i(a_i - 1)/p_i^2 \le a_j(a_j + 1)/p_j^2.$$

Now assume, by way of contradiction, that the apportionment $a_1, a_2, \ldots a_n$ does not minimize relative differences in the number of people needed for a congressional seat. In that case there exists a pair of states i, j such that state i is better represented than state j, and a transfer of one seat from i to j would lessen the relative inequality between them.

This means that

$$([a_j + 1]/p_j)/([a_i - 1]/p_i) < (a_i/p_i)/(a_j/p_j).$$

But this implies that

$$a_i(a_i - 1)/p_i^2 > a_j(a_j + 1)/p_j^2,$$

which contradicts the earlier inequality. Hence, the apportionment that results from rounding the ratios p_i/d at the geometric mean must minimize relative differences in the number of people needed for a congressional seat.

After: Peyton H. Young, *Equity in Theory and Practice*, Princeton University Press, 1995.

I would like to thank Daniel Barbiero, Manager of Archives and Records of the National Academy of Sciences, for making available to me the content of the documents quoted here. The documents are contained in the folder: NAS-NRC Archives, Central File: ADM: ORG: NAS: Committee on Mathematical Aspects of Reapportionment: 1928–29.

CHAPTER ELEVEN

THE PESSIMISTS

We now leave the matter of apportionment for a while and return to the troublesome problem of electing a leader. Remember Condorcet and his paradox? And how Lewis Carroll wrestled with it? Well the problem did not go away. Nor did it mellow with age. If anything it became more vexing. Enter Kenneth Arrow, Nobel Prize winner of economics in 1972 and one of the most important economists of the twentieth century.

An outstanding graduate student at Columbia University in the late 1940s, Arrow was thinking about his doctoral thesis. It was an exciting time for budding economists, observing and shaping subjects in the making. Arrow was caught up in these "heady days of emerging game theory and mathematical programming," as he would put it later. In the meantime he neglected his PhD thesis. He had high aspirations and his teachers and colleagues also expected a lot from him. But it was as if he was spellbound. No topic that he considered seemed sufficiently challenging. Even though his coursework had been completed at Columbia back in 1942, he was still short a thesis six years later. While everybody knew he was brilliant, the years passed without his putting pen to paper.

There was hope, however. Just a few years earlier, at the Institute for Advanced Study in Princeton, John von Neumann, together with Oskar Morgenstern, a refugee from the Nazis in Austria, had finished a thick primer that would become one of the most influential scientific works of the twentieth century. *Theory of Games and Economic Behavior*, published in 1944, was to have a profound influence on the further development of economics and political science. Based on only a handful of axioms, the theory contained in their book, henceforth called "game theory," ushered in the age of mathematical economics. What Euclid did for geometry, von Neumann and Morgenstern did for economic behavior.

One of the fundamental assumptions of their new theory was that each participant in a game has a so-called utility function. As we shall see presently, utility functions are a fundamental concept not only for the understanding of economic behavior but also of the Condorcet Paradox.

Attempts to explain economic behavior had started more than two centuries earlier, with work by the famous Swiss mathematician Daniel Bernoulli. In 1713 Daniel's cousin Nikolaus posed the following question: imagine a game in which you flip a coin and if it comes up heads, you get two dollars. If it comes up tails you flip again and if it now comes up heads you get four dollars. If it does not, you continue flipping until it comes up heads, the prize money doubling at every flip. How much would you be willing to pay in order to participate in this game? Most people would be willing to wager somewhere between two dollars and ten dollars.

But why so little? After all, the prize money could be enormous. If the coin comes up heads only after the tenth flip, the payoff would be 1,024 dollars, after the twentieth flip it would be more than a million and after thirty flips it would be a cool billion. Admittedly, the probability of getting nineteen or twenty-nine tails in a row, and heads only on the twentieth or thirtieth flip is very small. But the huge prize compensates for the small probability. In fact, Nikolaus Bernoulli found that the expected prize is infinite! (The expected prize is calculated by multiplying all the possible prizes by their probabilities, and adding the resulting numbers: $(1/2 \times 2) + (1/4 \times 4) + (1/8 \times 8) + \ldots = 1 + 1 + 1 + \ldots$, an unbounded sum.) So, once more, we have a paradox: if the expected prize is infinite, why is nobody willing to pay a thousand dollars to enter this game?

After thinking about the question for a while, Daniel came to a surprising conclusion: a dollar is not always worth a dollar. At first blush this statement may sound like a contradiction in terms, but upon further inspection it is not at all unreasonable. After all, a beggar who owns only a single dollar puts a high value on a second dollar, whereas a millionaire would hardly notice the receipt of an additional dollar. Hence the "utility" of money differs according to how much wealth one possesses. The utility of an additional dollar declines the more dollars one already has. Therefore, Daniel argues, one has to take into account not the expected prizes, but the expected utilities of the prizes.

Now the search was on for a suitable utility function. The requirements were that it grows—more is better than less, and even a rich person prefers more dollars to fewer dollars—but that the utility of an additional dollar decline with increasing wealth—the millionth dollar is valued less than the first. Hence, the two requirements on the shape of the utility function are that it always be increasing but to an ever lesser degree. One

function that fulfills both requirements hand in glove is the logarithmic function: it always rises but at a decreasing rate. Hence Daniel posited it as a suitable utility function. Accordingly, the utility of a second dollar after the first is 0.3, but the expected utility of the millionth dollars is only 0.0000004. Subjecting the coin-flipping game to the computations, it turns out that the expected utility of the game's prize is four dollars. This amount, which accords with our intuition, is the amount that Daniel Bernoulli thought an average person should be willing to pay in order to participate in the game.

The exact form of the utility function is open to discussion, and different people have different utilities for wealth. Daniel used the logarithmic only as an example. But the principle has become clear. In 1738, twelve years after Nikolaus's death, Daniel's solution to the problem was published in the *Commentaries of the Imperial Academy of Sciences* of St. Petersburg. Henceforth the problem became known as the St. Petersburg Paradox. (To see how utility function and the requirements on their shape imply the existence of the insurance industry, see chapter 36 in my book *The Secret Life of Numbers*.)

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Starting in 1948, Arrow spent the summers at the RAND Corporation in Santa Monica, the original nonprofit global policy research institute that would set the standard for all think tanks that followed. (Altogether more than half a dozen men who would eventually win economics Nobel Prizes worked at the RAND Corporation at various times. Apart from Arrow, there were Herbert Simon [1978], Harry Markowitz [1990], John Nash [1994], Thomas Schelling [2005], Edmund Phelps [2006], and Leonid Hurwicz [2007].) Game theory and operations research were hot topics at the corporation's headquarters. The Cold War was just gaining momentum and it was not surprising that the think tank was commissioned to study how game theory could be used to analyze international conflicts and strategy.

It turned out not be a straightforward assignment. If the Cold War were to be considered a game between the United States and the Soviet Union—albeit a serious one—what are the utility functions of the two players? Do collectives such as nations possess utility functions at all? Individuals do, but how can their preferences be aggregated into something to which game theory can be applied? All of a sudden, Arrow had found his thesis topic.

His epiphany at RAND had occurred in the summer of 1948; now he had to sit down and write everything up. In October, as soon as he got back to Chicago from his summer job, he started working seriously, continuing to write for the next nine months. By June 1949 a paper emerged and was published a year later under the title "A Difficulty in the Concept of Social Welfare" in the *Journal of Political Economy*. At first his advisors were perplexed. Nobody was quite sure what this was, and—whatever it was—whether it belonged to economics. But when Arrow finally presented his PhD thesis it was a bombshell.

Published as a booklet with the title "Social Choice and Individual Values" by the Cowles Foundation two years later, the thesis was hailed a "critical evaluation of democratic theory in general, as well as of economic policy and welfare economics in particular." Containing barely ninety printed pages, it ushered in the theory of social choice. Its importance can be ascertained already on page 1 of the preface; no less than five future Nobel Prize winners in economics are listed in the acknowledgments: Tjalling C. Koopmans (1975), Milton Friedman (1976), Herbert Simon (1978), Theodore W. Schultz (1979), and Franco Modigliano (1985).

Arrow points out that the simplest way for a collective to make decisions is either to have a single person or a small group of people dictate the choices for the community as a whole, or to let choices be imposed by a set of traditional rules, for example by a religious code. The former method is a dictatorship, the latter is a method guided solely by convention. Both are undesirable because they disenfranchise the individual. In contrast, a democracy knows of two methods by which individuals can participate in the collective decision-making process: they may vote to decide on political issues and they may use the market mechanism—work for wages, buy and sell goods and services—to make economic decisions.

But as we already know, in particular from the writings of the Marquis de Condorcet (chapter 6), the voting process may lead to a problem. Allowing the majority to decide may result in cycles. Let me remind the reader of the paradoxical situation by way of an example: Tom's ranking in the presidential elections in 2000 was Bush before Gore and Gore before Nader; Dick ranked the candidates Gore, Nader, Bush; and Harry preferred Nader to Bush and Bush to Gore.

Tom:	Bush > Gore > Nader
Dick:	Gore > Nader > Bush
Harry:	Nader > Bush > Gore

A majority (Tom and Harry) prefers Bush to Gore, another majority (Tom and Dick) prefers Gore to Nader, and another majority (Harry and Dick) prefers Nader to Bush. The society, composed here of Tom, Dick, and Harry, prefers Bush to Gore, Gore to Nader, Nader to Bush, Bush to Gore . . . lo and behold, we have a cycle!

One may find reason to argue with Tom, Dick, and Harry's political preferences, but as orderings go, each individual ranking is perfectly reasonable. Nevertheless, when aggregating the preferences into a communal ranking by majority vote, something quite unreasonable happens. Arrow concludes that the simple majority vote is not acceptable as a method of arriving at a social preference from individual preferences. He also showed by example that less simple methods that involve, say, putting weights on top-ranked choices and then summing the weights of the individuals' choices do not work either. So how can the preferences of individuals be amalgamated into a social preference schedule that would be capable of ranking the many alternatives facing society?

The short answer is, they can't. But let us do the long answer. At the outset of his thesis, on page 2, Arrow states the objective of his endeavor. He poses the question whether "it is formally possible to construct a procedure for passing from a set of known individual tastes to a pattern of social decision-making." In other words, if every individual has a utility function, how can these utility functions be amalgamated into a social utility function?

In order to be a reasonable method of aggregation, the procedure in question would have to satisfy certain natural conditions of rationality. The problem is that the utilities of two or more people cannot simply be added. They cannot even be compared. Let me illustrate with an example. When deciding what to have at the bar, Dwaine orders a glass of wine and Dwight orders a can of beer. Pressed about his preferences, Dwayne may say that according to his utility schedule, he attaches 5 "utiles" to wine and 3 "utiles" to beer. In Dwight's worldview, beer is worth 12 of some other units, say "yotiles," and wine is worth 8 "yotiles." Would it make sense to say that Dwight likes beer four times as much as Dwaine? Or to say two beers are worth 15 "somethings" to Dwaine and Dwight? No and no again. One cannot add, subtract, or even compare utiles and yotiles. The utilities that different people attach to drinks, or to any other goods, cannot be compared.

It is like measuring temperatures. Let us say that on a particular day the temperature is 26 degrees Celsius in Paris and 78 degrees Fahrenheit in San Francisco. Obviously, it would be quite wrong to conclude that it is three times as hot in San Francisco as it is in Paris. When measured to different scales, temperatures cannot be compared simply by comparing the numerical readings. In a similar manner, each person has a specific, individual utility scale that cannot be compared to any other person's utility scale.

Arrow was nothing if not formal and rigorous. To set things in motion he postulated two axioms about the way people make choices, and five attributes that reasonable social utility functions should possess. For such a broad and all-encompassing theory as the theory of social choice this is quite parsimonious. That is as it should be. One of the hallmarks of a good theory is that it requires few axioms. They are, after all, unproven assertions on which everything else depends. The fewer such prerequisites a theory needs, the more powerful it is since it explains more with less. And the fewer axioms there are, the lower the chances are that they contradict each other, which would make the theory inconsistent.

As an example of an axiomatic system, let me cite Euclidean geometry as a case in point. In the third century BCE, the Greek mathematician had postulated a set of five axioms from which all of plane geometry could be derived. He had also claimed that this is the smallest possible set; each of the five axioms is required. Nevertheless, a feeling persisted among mathematicians that one of them, the parallel axiom, which states that exactly one line parallel to another line passes through any specific point on the plane, may be superfluous. They believed that this statement—and therefore all of plane (not plain!) geometry—could be derived from the other four axioms alone. For centuries mathematicians attempted to reduce the set to just four axioms, only to realize in the early nineteenth century that the parallel axiom was, indeed, indispensable for plane geometry. Without

it, something completely different appears on the scene: non-Euclidean geometry. For example, on a sphere like planet Earth "parallel" lines must cross somewhere. Take a point on the equator and draw a line through it, pointing exactly north. Now take a point 100 kilometers to the west of that line, also pointing exactly north. The two lines, seemingly parallel, will cross at the north and south poles. Hence, Euclid's parallel axiom is violated on a sphere. Thus, one obtains Euclidean geometry with the parallel axiom, other geometries without it.

If five axioms are required for all of Euclidean geometry and four axioms for all of non-Euclidean geometry, then postulating no more than two axioms to describe rational decision-making is certainly not excessive. Obviously, Arrow includes no unnecessary dead weight. So what are the two axioms, the two unproven assertions that are both indispensable and sufficient to derive all of social choice theory? The first axiom says that when presented with two alternatives a decision maker is always able to make a comparison between them. Either he prefers one alternative to the other, or the other to the one, or he is indifferent between the two of them. For once, apples and oranges *can* be compared, at least with respect to the utilities they provide any individual. Recall in this context the legend about Buridan's donkey that starved to death standing halfway between two identical haystacks. Obviously, the animal took Arrow's axiom to its extreme. By dying, rather than turning to one of the haystacks, the poor animal showed that it was quite indifferent between the two alternatives.

The second axiom concerns the matter that has been dogging us since we discussed the work of the Marquis de Condorcet: cyclical preferences. It is conceivable that someone prefers juice to milk, and milk to water, but then decides that he prefers water to juice after all. Well in a perfect world this must not be, and Arrow postulates that a rational person's preferences must be transitive. This is a fancy way of saying that preferences carry through; if juice is preferred to milk, and milk to water, then juice must be preferred to water. Thus, while the majorities of a group of people may exhibit cycles, in Arrow's scheme of things individuals are not allowed to do so.

The only requirements that Arrow imposes on decision makers are that they can always make a choice, and that these choices must not result in cycles. The two axioms are very plausible; it could not get much simpler than that. (However, see the appendix "The Axiom of Choice" for a subtle problem with the first of these axioms.) We may now move on to the social utility function. Arrow felt that the requirements he put on individuals' choices also make sense for group choice. How can one get from individual preferences to the collective choice of a group? Since utilities cannot be added, something more sophisticated is needed. A mechanism that aggregates individual preferences into a "social welfare function" must satisfy certain conditions in order to be acceptable. As we shall see, they are also quite reasonable. The conditions accord with common sense and with our intuition about fairness and the democratic process.

The first condition goes under the technical-sounding name "unrestricted domain." It says that there must not be any restrictions on the personal utility functions that are to be aggregated except for the two mentioned axioms. Hence, the mechanism should work for all possible combinations of individual preference schedules. As long as they satisfy the two axioms, no orderings should be excluded. As reasonable as this requirement sounds, it is not always satisfied. Some orderings may be excluded for cultural or religious reasons, or a constitution guarantees certain rights, even if everybody would prefer otherwise. Limiting the choices to two candidates, as in runoff elections, also violates the unrestricted domain condition. The more serious problem occurs, however, when the condition of unrestricted domain *is* fulfilled. If electors are allowed to rank alternatives in any way they choose—as long as the individual rankings are transitive–cycles may result. We saw this in numerous examples throughout this book. Charles Lutwidge Dodgson a.k.a. Lewis Carroll made the suggestion to break cycles by disregarding some voters' preference orderings. (See chapter 8.) This would be a violation of Arrow's first condition.

Arrow's next condition is called the monotonicity requirement. It says that if one individual raises the ranking of an option, while everybody else keeps it constant, society as a whole cannot react by reducing this option's rank. To illustrate, if society decided that lemonade is preferable to orange juice, and one individual now changes his preference from orange juice to lemonade, it cannot happen that lemonade will suddenly be ranked lower than orange juice by society.

The third requirement that Arrow lists is that the social welfare function not be influenced by extraneous factors. If A is preferred to B, then the sudden appearance of C should not influence one's choice between A and B. The situation can be illustrated by the following scene in a restaurant. (The anecdote is ascribed to the philosopher Sidney Morgenbesser from Columbia University.) "Today we feature apple pie and brownies," the waiter informs the customer who, after commenting on the limited choice, decides on apple pie. A few minutes later the flustered waiter returns to inform the patron that he had forgotten to mention that the restaurant also features ice cream. "In that case I will have a brownie," the customer announces after short reflection. The poor waiter is completely thrown off and rightly so. Obviously, the diner did not care one way or another about ice cream, since he did not choose it even when it was offered. But its sudden availability did reverse his choice between the two other alternatives, apple pie and brownies.

Our intuition tells us that this simply should not happen. And that was exactly the opinion that Kenneth Arrow advanced. Stipulating that a social ordering should not be influenced by unimportant options, he formulated this requirement as an axiom: the "independence of irrelevant alternatives." Neither individuals nor a group of people should, if they are rational, ever reverse their choice simply because a lower-ranked, and hence irrelevant, alternative becomes available.

As reasonable as the axiom of the independence of irrelevant alternatives sounds, it is quite a strict requirement and is not always fulfilled. In the context of voting, the axiom is often violated. Let's see how. In an election, one may prefer the ecologically minded candidate to the socially oriented candidate and would definitely vote for her if only the two of them ran for office. But when a capitalist contender suddenly decides to enter the race, one may decide to throw one's weight behind the socialist after all just so the capitalist does not come out on top. That is why Ralph Nader, presidential candidate for the Green Party, always scored so low in presidential races, even though many people supported his values. Realizing that there is no chance of his winning, perfectly rational electors vote for their next best candidate. (Not enough people were "rational," however in 2000—and I say this without any political implication. The diehard "green" voters who stayed with Nader took sufficiently many votes away from Al Gore, to hand the victory to George Bush. His record on global warming speaks for itself.) This is also why the appearance on the scene of a serious third candidate like, say, Ross Perot, can create havoc. Anyway, Arrow, like the waiter, advocates the independence of irrelevant alternatives.

The fourth condition that Arrow requires from good aggregation mechanisms is "citizens' sovereignty." This means that choices must not be imposed on the electorate. Never should a situation occur in which X is preferred by society to Y, no matter what the individuals' preferences between the two alternatives. The condition of nonimposition, as it is often called, implies that no outcome is precluded. For every possible outcome there exist individual rankings such that, when aggregated, they result in this outcome. This condition is often violated in real life. Even if all individuals prefer one alternative over another, some preferences are taboo. For example, there are taxes that are imposed on citizens, whether they like it or not, and in most states a red traffic light unequivocally means "stop," even if all drivers are of the opinion that right-hand turns entail no danger.

Finally, and most importantly, "the social welfare function must not be dictatorial." With this crucial postulate Arrow requires the aggregation mechanism to satisfy a democracy's most basic principle. On the one hand, a social welfare function is said to be dictatorial if the aggregation mechanism always parallels one specific person's preferences, no matter what the other individuals prefer. Nobody would accept that; we do not allow our choices to be dictated by anyone. On the other hand, the head of a government, once elected, calls the shots whether we like it or not. But in a democracy such a situation cannot persist forever. If the president's or prime minister's decisions do not express the will of the people—aggregated in whatever manner—he will not be reelected.

The proof starts with the definition of a set of individuals as being "decisive" for the choice between x and y, if society prefers x to y, whenever all members of the decisive set do. (This is regardless of what the other members of society prefer, and regardless of the preferences anybody has concerning the remaining alternatives.) Arrow then uses the five conditions the aggregation mechanism must fulfill to derive five consequences for decisive sets. By way of example, one of them says that "society as a whole is a decisive set." The proof is easy: if every individual in the society prefers Grappa to Amaretto, society as a whole also prefers Grappa to Amaretto. The other consequences are more difficult to explain, which is

why I won't list them here. But—take Arrow's word for it—they are no more than direct consequences of the five requirements that an aggregation mechanism must fulfill.

Once the five consequences are stated, Arrow juggles around the alternatives among which choices must be made and the sets of individuals—all the time assuming that the five conditions of a rational aggregation mechanism hold. After a while he suddenly arrives at a contradiction; under certain conditions a set of individuals is simultaneously decisive and not decisive. Obviously this cannot be. The implication is that the five conditions that are required of an aggregation mechanism cannot hold simultaneously.

In a later version of his proof, Arrow replaced the monotonicity and nonimposition requirements by the somewhat weaker "Pareto condition." Its name goes back to the nineteenth-century Italian sociologist and economist Vilfredo Pareto who formulated various versions of this condition. If everybody ranks a certain alternative higher than another one, then this alternative must not be ranked lower in the social ordering. To illustrate: if everybody prefers coffee to tea, then it cannot be that society as a whole prefers tea to coffee. There is a stronger version: if every person save one is indifferent between coffee and tea, but one person prefers tea, then, for the community as a whole tea should be preferable.

The Pareto condition is one of the ingredients of democracy, as is the monotonicity requirement, in the sense that collective choice should be responsive to the preferences of the individuals. Specifically, Pareto asserted that if moving from A to B makes a single individual better off without hurting anybody else, then the community as a whole must prefer B to A. After all, the now better-off individual could compensate everybody else with his gains. The Pareto condition is closely related to the notion of Pareto efficiency, which describes an economic state in which nobody's situation can be improved without making at least one other person worse off.

Now that these five perfectly reasonable conditions have been stated, all that remains is to devise an aggregation mechanism that satisfies them. But here the efforts hit a brick wall. In chapter 5 of his booklet, Arrow gives a rigorous mathematical proof that it is not possible to devise a social welfare function whenever there are more than two alternatives from which to choose. Without fail, any attempt to aggregate the preferences of a group of people into a collective choice violates at least one of the five axioms.

The news struck like a thunderbolt. Since the times of Plato and Pliny, Llull and Kues, Borda and Condorcet, there had been hope that a mechanism to aggregate voters' preferences would eventually be discovered. Arrow's thesis put an end to such expectations. An appropriate method just cannot be found. No aggregation mechanism exists that simultaneously fulfills all five requirements.

Looking first at the bright side of things, Arrow started out by formulating a proposition that salvages whatever is salvageable. The proposition said that if there are only two options from which to choose, the method of majority decisions fulfills the requirements of an aggregation mechanism. In a possible attempt to sound upbeat in spite of the pessimistic message, Arrow called the proposition "the possibility theorem," indicating that under these very restrictive circumstances—only two alternatives from among which to choose—majority voting is an acceptable social choice mechanism. The theorem could be seen as an affirmation of the Anglo-American two-party system. It satisfies all conditions except the first one: by limiting the choice to just two alternatives, the domain is not unrestricted.

But soon Arrow had to face up to the inevitable. In Theorem 2 he formulated the central assertion of his PhD thesis, a statement that would turn the general reliance in the democratic process topsy-turvy. It said the following: if there are at least three alternatives, any aggregation method that satisfies reasonable conditions of rationality is either imposed or dictatorial. The democratic world would never be the same again. Only dictators could breathe a sigh of relief, no problem with their style of government.

Arrow required just eight pages to rigorously prove the theorem that would put a question mark to the theories of social choice, welfare economics, and political science. It should properly have been named "the impossibility theorem" and subsequently the literature usually referred to it thus, although Arrow stuck to his original term also for Theorem 2.

Of course, the majority vote of which we have grown so fond over the centuries is just one of the unacceptable aggregation mechanisms. Actually, this was nothing new. Condorcet was already aware that perfectly reasonable preference schedules among three voters or more may lead to

a cycle. Hence plurality voting violates the condition of "unrestricted domain": only if certain combinations of individuals' preferences are excluded can cycles be avoided. Then Arrow really rubs it in: no other scheme of proportional representation, no matter how complicated, can remove the paradox of voting. Voters' sovereignty is simply incompatible with collective rationality.

There could be a way out of the dilemma, Arrow reminds us. By comparing and manipulating the utilities of the various electors, a communal preference ranking could be constructed arithmetically. But that path was excluded because utilities of different people cannot be added or compared. Something has to give; at least one of the five conditions must be dropped. Exclude Condition 1 (unrestricted domain) and the majority vote can serve as an aggregation mechanism that satisfies the other four conditions. But that would mean accepting the possibility of cycles. Drop Condition 4 (nonimposition) and be governed, Soviet-style, by predefined laws, regulations, taboos, and customs. Not a very enticing option. Or skip Condition 5 (nondictatorship). But who wants a dictator?

Two possibilities are left. For one, the monotonicity requirement could be dropped. But that would mean allowing a choice to be ranked lower by society after its ranking is raised by one or more individuals. This would be very counterintuitive indeed. (It would be reminiscent of the Alabama Paradox: raise the size of the House by one and get fewer representatives.) So we won't throw out that one. Finally, there is the Independence of Irrelevant Alternatives. It is the most controversial of the requirements, and one could envisage dropping it in order to salvage democracy. After all, some people, individually, do violate it. It would not be a very enticing prospect, however, since it implies irrational acts like ordering the wrong dessert or shifting one's support from Laurel to Hardy as soon as Goofy appears on the scene.

Others had recognized the dilemma and were looking for ways out of the quagmire. Let me give two examples. The Scottish economist Duncan Black considered restricting the citizens' preference schedules. Let us say the alternatives among which the people have to choose can be arranged along a line according to a parameter. For example, the range of political parties may be ordered from the extreme left, to the left, to the center, to the moderately conservative, to the very conservative. For such cases, Black proved that if each individual's preference schedule has a single peak—say he prefers the center party to the parties both on the right and on the left—then the majority vote fulfills all of Arrow's conditions, provided the number of electors is odd. The price of this is that Black restricted the domain from which citizens are allowed to choose, or the way in which they rank their choices. The voters thus violate Arrow's first condition of unrestricted domain. Furthermore, if a new party appears on the political scene that does not fit into the left-right scheme, like the greens or a gay lib party, or if some voter ranks both the extreme left and the center higher than the moderate left, Black's method of aggregating the preferences by majority vote would again fall victim to cycles.

Then, in the late 1960s and early 1970s, the Indian-born economist and philosopher Amartya Sen, winner of the 1998 Nobel Prize for economics, showed that there exist aggregation mechanisms that fulfill all of Arrow's requirements, except transitivity. (Recall that transitivity, implied by the axiom of "unrestricted domain," means that if, say, a committee for public buildings prefers an opera house to a football stadium and a football stadium to a skating rink, then the committee must prefer the opera house to the skating rink.) He investigated the implications of relaxing the transitivity requirement to quasi-transitivity (if the committee is *indifferent* between an opera house and a football stadium, and *indifferent* between a football stadium and a skating rink, it might still prefer the opera house to the skating rink). There were other heroic efforts to relax this or that requirement just a teeny little bit. But in the final analysis, all attempts to amend, append, or adjust the conditions required of a good aggregation mechanism are cop-outs.

Arrow's finding was extremely troubling. No democratic constitution exists that produces a coherent method of social choice; only a dictatorship can fulfill a handful of innocuous sounding conditions. We are caught between five rocks and a hard place. Either we accept cycles, or dictatorship, or imposed choices, or one of two kinds of irrational behavior, or we throw democracy out the door. We can't have it all ways.

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In his pathbreaking book, Arrow showed that the preference schedules of a population of voters cannot be aggregated into a social preference ordering. If that were not bad enough already, it is only part of the story. Worse was yet to come. Arrow had taken the preference schedules of the

voters for granted. But what if the electors' answers are not truthful? What if, realizing that their first choice has no chance of winning, they pretend to support a different alternative or candidate, thus pushing their second or third choice to the fore? When someone pointed out to Jean-Charles de Borda that his method could easily be manipulated by a group of electors who decide to deprive the front-runner from victory, he was indignant. "My scheme is only intended for honest men," the navy officer snapped. But what if, in the interest of the second best, the electors are not honest?

It is a problem with which Pliny the Younger had already wrestled in the first century AD and voters are still wrestling with it in the twenty-first. During the American election in 2000, for example, numerous supporters of Ralph Nader preferred to cast their ballots for Al Gore rather than voting for their true first choice, hoping at least to beat George Bush. It was to avoid such misrepresentation of their true feelings that electors in ancient times and in the Middle Ages were often obliged to take oaths that they would vote honestly. The problem of strategic voting was not addressed by Arrow. But in the early 1970s, two graduate students, the philosopher Allan Gibbard and the economist Mark Satterthwaite, independently decided to investigate the question. Specifically, they asked themselves how susceptible voting systems are to manipulation by electors. Can voters influence the outcome by misrepresenting their true intentions? They considered a somewhat simpler setting than the one Arrow investigated. While Arrow was concerned with a complete ranking of all candidates, or alternatives, from best to worst, Gibbard and Satterthwaite showed that problems arise even if only a single winner is sought.

In 1969, Kenneth Arrow, Amartya Sen, and the philosopher John Rawls announced a seminar series, run jointly by the departments of economics and philosophy of Harvard University, on "Decision Making in Organizations." It was a grandiose event with the foremost economists and philosophers from MIT and Harvard sitting in the audience week after week. Only two graduate students were present at the first meeting, one of them being Gibbard. When Arrow announced that they too would be expected to present papers during the seminar series, the other student, a personal acquaintance of this author, rushed to the registrar's office to drop the course. Gibbard stayed and when his time came to give a lecture he talked about his doctoral thesis, the manipulation of elections. Everyone present, including the other graduate student who continued to audit the series, was greatly impressed, and with that Gibbard's academic career was all but made. Four years later, in 1973, he published the landmark paper "Manipulation of Voting Schemes: A General Result," in *Econometrica*, one of the foremost journals in the field of economics. Below, I will tell more about what the paper is about.

Unbeknownst to Gibbard, Mark Satterthwaite, a doctoral student in the department of economics of the University of Wisconsin was working on a PhD thesis in the early 1970s that dealt with the exact same subject matter. Satterthwaite did not know of Gibbard's article. After all, it would only be published in 1973, which is when Satterthwaite's own thesis was accepted by the faculty of the University of Wisconsin. By the time an edited version of his thesis appeared in the *Journal of Economic Theory* it was already 1975. In fact, the first Satterthwaite heard of Gibbard's previously published work was when a referee who was checking his submission to the *Journal of Economic Theory* pointed out its existence. Nevertheless, the theorem is today known as the Gibbard-Satterthwaite Theorem, and rightly so, since both Gibbard and Satterthwaite had thought of it and worked out the proof simultaneously, albeit using different techniques. One of the differences is that Gibbard described the misrepresentation of one's preferences as a manipulation while Satterthwaite called it a strategy.

What is the sad news about the theorem? Gibbard and Satterthwaite proved that any democratic election method that purports to elect a winner from among at least three candidates can be manipulated. By misrepresenting his true preferences and pretending to prefer a candidate that he actually does not, a voter can influence the electoral outcome. No matter which election method is used–plurality, absolute majority, Borda count, two-by-two elimination, whatever—the Gibbard-Satterthwaite Theorem says that a liar may help get a candidate elected who would not have stood a chance if all electors had been truthful about their preferences. (It may take a whole coalition of liars, but if the results are very close, a single liar may suffice.) Hence, no election method exists that is both democratic and strategy-proof!

There is only one method that cannot be manipulated. By now it may come as no surprise that it is the dictatorship. Obviously, it makes no difference whether you vote honestly or dishonestly in a totalitarian regime

since the dictator has his say in any case. And the dictator has no need to lie since his preference becomes law automatically.

Now, is it immoral to hide one's true preferences, thus falsifying the supposedly honest outcome? To live in society, one often has to make compromises. This happens with all aspects of daily life: which job to take, what house to buy, where to go on vacation. Many more couples would separate than actually do, if spouses would not stand back and settle for an agreeable, if not preferred, decision. Thus, a family may end up going neither to the boxing match, nor to the ballet, neither to the picnic, nor to the fancy restaurant. They may settle for a visit to the movie theater followed by a snack at the local diner, and nobody will be the worse for it.

Why should it be different in elections? Abandoning one's first choice to vote for the second is just such a compromise. Yet, setting the agenda in a committee meeting and being dishonest in order to push one's preferred alternative or candidate forward, is unfair. Say a committee made up of eleven members is to elect a new Director of Social Affairs by a knockout procedure. You and four allies prefer Alice over Bruce. Five other committee members prefer Bruce over Alice. And nobody wants Dofus, except for Mrs. Dofus. Now you do two things. First you set the agenda such that Bruce has to compete against Dofus in the first round. When the time comes to fill in the ballots, you and your friends lie about their true preference and vote for Dofus. Together with Mrs. Dofus's vote, Mr. Dofus wins the round six against five. The next round, between Alice and Dofus, will be a walkover for Alice. Now that was no compromise; it most certainly was a manipulation of the decision process.

The two professors whose names the theorem carries did not put any labels on the electors' activities. Nevertheless, Gibbard's use of the word "manipulation" has a negative connotation while their portrayal as "strategy" by Satterthwaite glosses over some of their disconcerting aspects. As so often, it all depends on the context.

* * *

First Arrow, then Gibbard and Satterthwaite. . . . The state of affairs was truly pessimistic and, unfortunately, there is no happy ending to this chapter. Furthermore, the outlook is bleak with more bad news to come. In the next chapter, where we return to the problem of apportionment of seats, we will see what else is impossible.

BIOGRAPHICAL APPENDIX

Kenneth Joseph Arrow

Arrow was born in New York in 1921 and spent his youth and student days in the Big Apple. The family lived in comfortable circumstances until the Great Depression wiped out most of their wealth; for the next ten years the family lived in poverty. Arrow passed the entrance exam to Townsend Harris High School in Queens, a school that was known for its high academic standard, and attended it from 1933 to 1936. The teachers, some of them with doctorates and hoping to become professors at a university, were of a very high caliber. It is thus not surprising that the school produced no less than three Nobel Prize winners—Arrow himself, his classmate Julian Schwinger (Nobel Prize in Physics 1965), and Herbert Hauptman (graduated high school 1933, Nobel Prize in Chemistry 1985). But when the time came for Arrow to go to college, his parents could barely afford the cost. Fortunately, City College of New York offered higher education without tuition fees, and Arrow was forever grateful for the opportunity accorded him. Even in the autobiography he would write for the Nobel Foundation more than thirty years later, he did not forget to mention "that excellent free institution." At City College he majored in mathematics with minors in history, economics, and education, and had the intention of becoming a math teacher. But when he graduated, winning the Gold Pell medal for the highest grades, there were no positions available in the New York City school system. So he entered Columbia University to continue studying mathematics. Arrow received his master's degree in 1941 and then was not sure of what he wanted to do next.

To his great fortune, he had taken a course at Columbia in mathematical economics with Harold Hotelling, a statistician who held an appointment at the department of economics. This experience proved to be propitious: Arrow decided that mathematical economics was the subject to which he would henceforth devote his life. A fellowship to the department of economics ensued but then life was interrupted by the Second World War. He entered the U.S. Air Force in 1942 as a weather officer, rising to the rank of captain in the Long Range Forecasting Group. One day, making use of their academic training, Arrow and his colleagues decided to submit their work to a statistical test. They investigated whether the group's aim—forecasting the number of rainy days one month in advance—was being attained. Not surprisingly, the conclusion was that it was not. They sent a letter to the General of the Air Force, advising the dissolution of the Long Range Forecasting Group. The response came half a year later: "The general is well aware that your forecasts are no good. However, they are required for planning purposes." So the group continued to prognosticate sunny days and rainy days using techniques that were about as good as drawing lots from a hat. Arrow left

the Air Force in 1946. Something positive did nevertheless originate from Arrow's work in the Air Force; his first scientific paper, "On the Optimal Use of Winds for Flight Planning," was published in the *Journal of Meteorology* in 1949.

After the war, Arrow continued graduate work at Columbia. Mindful of the hardships that his family had suffered during the Depression, he was on the lookout for a solid, down-to-earth profession. For a while, he toyed with the idea of becoming a life insurance actuary and actually passed a series of actuarial exams. While actively searching for a job in the insurance industry an older colleague dissuaded him, and Arrow decided to embark on a career in research. In 1947, he joined the Cowles Foundation for Research in Economics at the University of Chicago. There he encountered "a brilliant intellectual atmosphere . . . with eager young econometricians and mathematically inclined economists." It was also there that he met Selma Schweitzer, a young graduate student, whom he subsequently married. She was at the Cowles Foundation on a fellowship designed for women pursuing quantitative work in the social sciences. Originally the fellowship indicated preference for "women of the Episcopal Church," but the religious affiliation was subsequently dropped, which was fortunate because Selma, like Ken, was Jewish.

After earning his PhD degree Arrow was hired by the economics and statistics department at Stanford University in 1949 and then promoted through the ranks, eventually becoming Professor of Economics and Professor of Operations Research. Except for an eleven-year interlude at Harvard University, and visits to Cambridge, Oxford, Siena, and Vienna, Arrow spent his whole career at Stanford until retirement in 1991. He won numerous prizes, among them the John Bates Clark Medal in 1957, awarded every year to an outstanding economist under forty, and of course the Nobel Prize in 1972. He was elected to the National Academy of Sciences and to the American Philosophical Society and has received more than twenty honorary degrees. Even the Vatican honored him by making him a member of the Pontifical Academy of Social Sciences. During his rich career, Arrow also served on the staff of the United States Council of Economic Advisors, as president of the Econometric Society, and as fellow and member of numerous learned societies. He did not slow down after his retirement and has remained very active as an emeritus professor. During many summers, for example, he leads an advanced workshop in economic theory at the Hebrew University in Jerusalem.

In 1986 the Institute of Management Science and the Operations Research Society of America awarded Arrow the John von Neumann Theory Prize. The laudation stated that "his lightning-quick mind, his awesome wealth of knowledge, his extraordinary breadth of interests, his elegant prose and language, and his great personal warmth have inspired and charmed countless students, colleagues, and associates."

Allan Gibbard

Born in Providence, Rhode Island in 1942, Gibbard grew up in West Virginia and studied mathematics at Swarthmore College. There he received his BA degree, with minors in physics and philosophy, and then joined the Peace Corps in Africa. For two years he taught math and physics at the Achimota Secondary School, an elite high school in Accra, Ghana. Back in the United States, he returned to his college minor, philosophy, and obtained his PhD from Harvard University in 1971. As a professor of philosophy, first at the University of Chicago, then at Pittsburgh and Michigan, Gibbard tried to characterize the nature of moral judgment and define the meaning of moral statements. Gibbard has contributed important advancements to ethics, metaphysics, philosophy of language, and the theory of identity.

Philosophers are wont to rise to lofty heights of discourse, often asking questions about questions rather than answering them, thus leaving the realm of nitty-gritty, real-time decisions far behind. Gibbard fits this mold but only to a degree. Even though his books, *Wise Choices, Apt Feelings* (1990) and *Thinking How to Live* (2003), seem to promise practical advice of the how-to variety, they are nothing of the sort. And with papers like "Norms for Guilt and Moral Concepts," "Preference and Preferability," "Truth and Correct Belief," one would not expect Gibbard to stoop to down-to-earth subjects like how elections can be manipulated. But this is exactly what he did and, in the process, the mathematics he had learned as an undergraduate stood him in good stead.

Mark Satterthwaite

Satterthwaite received his BA in economics at the California Institute of Technology and then went to Wisconsin for his MA and PhD. Submitted and accepted in 1973, his thesis carried the title "The Existence of Strategy-Proof Voting Procedures." After earning his doctoral degree, he joined the faculty at the Kellogg School of Management at Northwestern University, never to leave it again, except for a semester as visiting professor at Caltech. He worked his way through the ranks, starting as an assistant professor even before he was formally awarded his PhD, all the way up to department head and to a named chair in Hospital and Health Care Management. His interests were, and still are, microeconomic theory, the economics of industrial organization, and health economics.

MATHEMATICAL APPENDIX

The Axiom of Choice

In mathematical logic there exists a very famous—and controversial—postulate called the axiom of choice, which has a subtle connection to Arrow's first axiom. Formulated in 1904 by the German mathematician Ernst Zermelo, the axiom of choice says that if one is presented with an infinite collection of sets, each of which contains a number of elements, one is always able to pick a representative member of each set. If presented with infinitely many pairs of gloves, there could be a rule that would say, "from each pair, pick the left glove." But when presented with pairs of socks, for example, there is a problem since left and right socks are indistinguishable. Thus there exists no picking rule and one finds oneself in the position of Buridan's donkey. The axiom of choice is a way out of this predicament. Without specifying a procedure or a choice rule, it simply postulates that one is able to pick an element from each set.

In mathematical proofs it is often assumed—sometimes without the prover or the reader actually being aware of it—that one can always pick a member from a group of objects. For a finite number of sets, one may simply move from set to set and physically point to one of the objects at random. But for infinitely many sets one requires a rule. "Pick the tallest child from each class" or "pick the lowest-calorie drink from each bar" are acceptable rules. But "pick one match from each box" is questionable because it implicitly assumes that such a pick can always be made. Hence it requires the axiom of choice.

Now on to Arrow's first axiom. It says that when a decision maker is presented with two alternatives he can always make a comparison between them. Even when faced with pairs of socks, he is able to pick one sock as his preferred choice. Thus Arrow's axiom treats socks and gloves in the same manner, just as the axiom of choice does. However, there is a difference between Arrow's axiom and the axiom of choice. In addition to preferring one alternative to the other, Arrow allows indifference between them. Indifference is an acceptable choice and the decision maker won't die like Buridan's donkey.

Gödel and the Impossibility Theorem

The way in which the Impossibility Theorem threw democratic principles into turmoil reminds us of a similar event twenty years earlier. A young man in Vienna, Austria, had done to mathematics what Arrow did to social science and political theory. In 1931 the logician Kurt Gödel published a paper in the German journal *Monatshefte für Mathematik und*

Physik. It was titled *"Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme"* (On formally undecidable theorems in Principia Mathematica and related systems) and showed that statements exist that are true in a mathematical system but cannot be proven within its axioms. (Gödel's paper referred to Alfred North Whitehead and Bertrand Russell's monumental work *Principia Mathematica*.) Gödel's Incompleteness Theorem, as it was henceforth called, proved that under certain circumstances an axiomatic system cannot be both consistent and complete, thus putting to rest all attempts to set mathematics on an axiomatic basis. Thereby it did to mathematics what Arrow's Impossibility Theorem did to social choice theory. An aggregation mechanism, based on a handful of axioms, cannot fulfill a few reasonable requirements and be democratic. (To the general malaise one may add that Werner Heisenberg had done something similar to physics four years earlier, in 1927, with his Uncertainty Principle.)

There is a well-known anecdote about the eminent logician that may have some bearing on what we just discussed. Gödel left Austria during the Second World War and found a new home at the Institute for Advanced Study in Princeton. In 1948, when he decided that he would not return to his fatherland, he applied for American citizenship. His colleagues Albert Einstein and Oskar Morgenstern, naturalized Americans themselves, accompanied the otherworldly mathematician to the immigration office for the crucial interview. On the way they coached him on the American Constitution. Gödel, who had studied it the night before, spent the drive to the bureau arguing that the venerated document left the door open to a dictatorship. Knowing that this would not be the kind of argument the immigration official would like, Einstein and Morgenstern persuaded him not to insist on that point when questioned. Luckily Gödel heeded the warning and was duly awarded citizenship. It is not documented what the loophole was, that Gödel thought he had discovered. Was it maybe, just maybe, the idea that the democratic, one-man-one-vote majority election system that all Americans hold dear could lead to cycles, thence to a revolution and finally to a dictatorship?

CHAPTER TWELVE
THE QUOTARIANS

We return to the frustrating subject of apportionment. In the preceding chapter I recounted that Kenneth Arrow proved that any election method that satisfies reasonable conditions of rationality—like avoiding cycles—is either imposed or dictatorial, and that Allan Gibbard and Mark Satterthwaite showed that any democratic election method can be manipulated. This chapter will, unfortunately, be the bearer of further bad tidings: a fair and true allocation of seats in Congress is also a mathematical impossibility.

With the size of the House fixed at 435 in 1912, the Alabama Paradox no longer loomed. And after the inclusion of Alaska and Hawaii in 1959 no new states were likely to join the Union, so the New State Paradox also no longer posed a problem. But the population keeps growing; hence, the Population Paradox is here to stay. And of course, all the inequities that occur when rounding seats up or down remained.

Even though Congress did its best to plaster over the differences whenever possible, the problem never really went away. According to the 1950 census, California would have gained a seat, to the detriment of Kansas, if the Webster-Willcox method, a.k.a. W-W method, or Cornell method, had been used. In 1960, North Dakota would have lost one of its two seats to Massachusetts and ten years later, in 1970, Kentucky and Colorado would have gained one seat each, to the detriment of South Dakota and Montana, had the Cornell method been in force. From time to time challenges were raised. Following the 1980 census, Indiana, which stood to gain an eleventh seat under W-W, raised a ruckus. The House actually considered changing the apportionment method—New Mexico would have been the loser—but the proposal never got off the ground.

After the 1990 census, tempers flared especially high. This time the Huntington-Hill method, a.k.a H-H method, or Harvard method, caused the state of Montana to lose out by having to forfeit one of its two seats in the House. Montana did not like this at all and sued the government, specifically the Department of Commerce, which administers the apportion-