force coherent parties with stable programs is to foreclose this artistry. It is the nature of politics that issues are in continuous creation. No institutional device to force binary choice can stop that creation and, furthermore, no device should stop it. Fairness to losers and an interest in what losers create forbid the imposition of binary choice.

* * *

So I conclude: Simple majority decision on binary alternatives satisfies three fundamental criteria of fairness and in that sense seems superior to other methods. But it cannot be fair in a democratic sense because the imposition of binary alternatives is itself unfair. Hence the relativistic conclusion of Chapter 2 still stands.

4

Voting Methods with Three or More Alternatives

Simple majority decision on binary alternatives requires some social embodiment of Procrustes, who chopped off the legs of his guests to fit them into the bed in his inn. The number of alternatives *must* be reduced to exactly two, and this means that some alternatives worthy of consideration must be excised. Furthermore, there must be some Procrustean leader or elite to excise them. Even the apparently unbiased method of reducing alternatives by a series of binary elections requires that someone decide on the *order* of elections—and control of the order is often enough to control the outcome, as we shall see. Thus, however democratic simple majority decision initially appears to be, it cannot in fact be so. Indeed, it is democratic only in the very narrow sense of satisfying certain formal conditions. In any larger sense, it is not democratic because its surrounding institutions must be unfair.

If a voting system is to be really fair, more than two alternatives must be allowed to enter the decision process; a decision method must be able to operate on three or more alternatives. But here is a snag. Many decision methods can deal with several alternatives, but no one method satisfies all the conditions of fairness that have been proposed as reasonable and just. Every method satisfies some and violates others. Unfortunately, there are, so far as I know, no deeper ethical systems nor any deeper axioms for decision that would allow us to judge and choose among these conditions of fairness. Hence there is no generally convincing way to show that one decision method is truly better than another.

So we are faced with a dilemma. Simple majority decision between two alternatives, while narrowly fair, is unattractive because it requires unfair institutions to operate it. On the other hand, no particular decision methods for three or more alternatives can be unequivocally demonstrated to be fair or reasonable. The problem is that we cannot prove that any method truly and fairly amalgamates the judgments of citizens simply because we do not know what "truly and fairly amalgamates" means.

Chapter 4 is devoted to a demonstration of that proposition. I have no deep preference for any method of decision, and I am not trying to prove that one is better or worse than another. I want merely to show that the notion of true and fair amalgamation is neither obvious nor simple, which is why we cannot easily choose among the voting methods discussed in Chapter 2.

4.A. Some Preliminaries

Because no method for decision among three or more alternatives is entirely satisfactory, many methods have been invented. Despite the variety, they can be classified into three families, the members of which are enough alike that their merits and deficiencies can be discussed together. I call the three families (1) majoritarian methods, (2) positional methods, and (3) utilitarian methods. To begin the discussion I will define the categories and offer some examples.

As a preliminary, however, the vocabulary must be revised and expanded. The set, X, of alternatives now must have $m > 2$ members, $X = (1, 2, \ldots, m)$, which will be referred to with lower-case italic letters, x, y, a, b, and so on. The relations, P, I, and R, remain binary in that they represent individual judgments and votes between a pair of alternatives. But now it must be decided whether they are also transitive—namely, that, if $x\, R_i\, y$ and $y\, R_i\, z$, then $x\, R_i\, z$. (For simplicity, I will write $x\, y\, z$ to mean "$x\, R_i\, y$, $y\, R_i\, z$, and $x\, R_i\, z$," and $x(yz)$ to mean "$x\, P_i\, y$, $y\, I_i\, z$, and $x\, P_i\, z$.") Many arguments are offered for and against individual transitivity.[1] I will, however, conventionally assume that P, R, and I are transitive, except when it is stated otherwise for I and R.

Furthermore, I will also assume that, for all i in the set, N, of decision-makers, some one of these relations connects every pair, x and y, in X. This means that D_i, which is the individual judgment on members of X, is an *ordering*—that is, a transitive and complete arrangement of X. By ordering is meant that the position of each alternative in a particular arrangement is unambiguous. Since X is connected, this means that, for each x and y in X, either xy, yx, or (xy) must be true for each i in N; that is, D_i is one of $\{xy, yx, (xy)\}$. This establishes an unambiguous order in every pair of alternatives. Since D_i is transitive, if it contains $x\, y$ and $y\, z$, then it must contain $x\, z$ (rather than $z\, x$ or $(x\, z)$), which establishes an unambiguous order in every triad of alternatives. And since triads can

overlap (e.g., $w\, x\, y$ and $x\, y\, z$), transitivity establishes an order over an X of any finite size.

The profile, D, will be a set of (not necessarily different) orderings, D_i, of X, for each of n participants; and $\boldsymbol{D}$ will be the set of all possible profiles. Finally, the social choice, F, will now be the result of a decision rule, g, operating on a specified set, X, and a particular profile, D, of judgments to produce a choice: $F_g(X, D)$. Notice that now the set, X, from which the choice is taken will customarily be specified because sometimes the discussion will concern choices from two different sets, X and X'. Typically, if the decision rule, g, is clear from the context, it will not be indicated in the identification of F.

For the time being, except when expressly stated otherwise, I will also assume that all voting is in accordance with preferences. Hence, if $x\, P_i\, y$, person i votes for x over y; and, if $x\, I_i\, y$, person i does not vote for either x or y.

4.B. Majoritarian Methods of Voting

Majoritarian systems are those in which the principles of simple majority voting are extended to three or more alternatives. The rationale is that majority decision is fair and reasonable in its logical structure, and it is assumed that the defect of limiting alternatives can be offset by admitting all desired alternatives. This assumption, however, is profoundly dubious, as we shall see in the discussion of positional voting in section 4.E.

The way of extending the method of majority decision from two to more alternatives is the so-called Condorcet method, in which a winner is defined as that alternative which can beat *all* others in X in a simple majority vote. When X has two members, this is merely simple majority decision. But when X has more than two, this is the requirement that the winner beat $m - 1$ others in $m - 1$ pairwise decisions.

Unfortunately, there are many situations in which the Condorcet winner is undefined simply because no alternative can beat $m - 1$ others. One extreme example, in Display 4-1, is the so-called paradox of voting (see section 1.H). In Display 4-1 each alternative beats one other and loses to another. Which, if any, ought to win?

A more confusing example is set forth in Display 4-2. There, since w ties or beats x, y, and z, presumably w ought to be among the winners. But what of x and y? They tie and so might reasonably win along with w; but although x ties w, y does not, so just as reasonably only w and x

Display 4-1

The Paradox of Voting

D_1:	$x\ y\ z$
D_2:	$y\ z\ x$
D_3:	$z\ x\ y$

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	x	y	z
x	—	2	1
y	1	—	2
z	2	1	—

A Condorcet winner would have a majority (here at least 2) in every cell (except the blank one) in a row, signifying that the alternative in the row beats all others. In the paradox, no alternative can beat all others, and so each row has at least one cell with less than a majority (here, at most, 1).

might win. In the absence of a Condorcet winner, problems like these inevitably arise.

Different resolutions of such problems result in different decision methods. Those called *majoritarian* or *Condorcet extensions* have the common property that they select the Condorcet winner, if one exists; if one does not exist, they provide for some further resolution. All such rules depend on knowing whether one alternative beats another in a simple majority vote. Thus I define a *social* relation of majority, M, so that $x\,My$ means "more people prefer x to y than prefer y to x."[2] In any event, to use the Condorcet criterion and its extended rules, it is necessary, as a practical matter, either for each voter to report his or her entire preference structure, D_i, or for the group to hold a number of ballotings, perhaps as many as a round robin, $m(m-1)/2$.

Display 4-2

A Profile for Which the Choice of a Winner Is Not Clear

D_1:	$w\ y\ x\ z$
D_2:	$x\ w\ z\ y$
D_3:	$y\ z\ w\ x$
D_4:	$x\ w\ y\ z$

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	w	x	y	z
w	—	2	3	3
x	2	—	2	3
y	1	2	—	3
z	1	1	1	—

While w beats or ties all others, x and y tie. Ought x to win? It is true that x ties with w, the alternative that otherwise beats all others. But x ties y, which otherwise loses. Should y win? Its only claim is a tie with x, which in turn ties w, which beats y.

4.C. Examples of Majoritarian Methods

There are many rules that utilize paired comparisons of alternatives to discover a Condorcet winner. If a Condorcet winner exists, then all these methods come out the same way. If a Condorcet winner does not exist, however, these rules typically produce different results, no one of which, in my opinion, seems more defensible than another.

The Amendment Procedure

The only widely used Condorcet extension, the goal of the amendment procedure is to select the Condorcet winner, if there is one, or, if there is not, to select the status quo. (Hence the procedure is not neutral, since the status quo is favored.) It works as intended on three alternatives—(1) motion, (2) amendment, (3) neither; but it does not ensure its objective for more than three.

The actual rule is: Alternatives (motions) are put forward to form X, which always includes the implicit motion to sustain the status quo. The winner of an identified pair is chosen by simple majority decision. That winner is placed against another identified motion and a second winner is chosen, and so forth until a just-previous winner is placed against the status quo, and a final winner is chosen by simple majority. In the typical American procedure, X can actually have six members:

- t. Original motion
- w. Amendment
- x. Amendment to the amendment
- y. Substitute amendment
- z. Amendment to substitute
- s. Status quo

The six are voted on by this rule:

Step 1: z vs. y

Step 2: x vs. w

Step 3: Survivor of step 1 vs. survivor of step 2

Step 4: Survivor of step 3 vs. t

Step 5: Survivor of step 4 vs. s[3]

With just three alternatives—motion (t), amendment (w), and status quo (s)—this procedure actually works as intended, for it discovers the Condorcet winner, if one exists, or, if none exists, it chooses the status quo. With three alternatives, this procedure reduces to two steps:

Step 1: w vs. t

Step 2: Survivor of step 1 vs. s

If a substantive motion (w or t) wins, it must beat both other alternatives and is thus the Condorcet winner (see Figure 4-1). Thus outcome 1 means

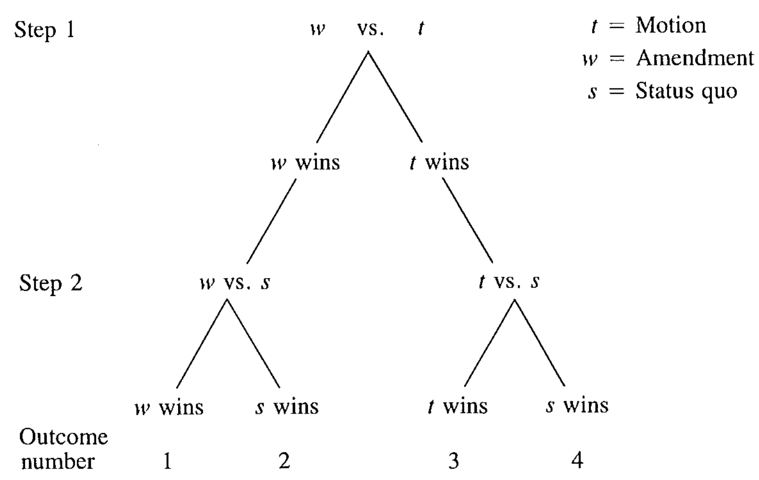

Figure 4-1 Parliamentary voting on three alternatives.

w has beaten both t and s, and outcome 3 means t has beaten w and s. When outcomes 2 or 4 occur, it is not clear whether s is the Condorcet winner or simply the tie-breaker. Outcome 4, for example, could be reached from either profile D or profile D' of Display 4-3.

In profile D, s beats both substantive alternatives and is the Condorcet winner. In profile D', no alternative can beat both others; so, by the non-neutral rule, s is chosen. The failure in this case to reveal whether D or D' exists is irrelevant because, by the procedure, s is intended to win in either case. It might be argued that s ought not win in D' because, if w had not been eliminated by t, then w would have beaten s and *should* in fact do so. Still, the assumption in the procedure is that, since w loses to t, it is better to do nothing than to adopt w. For that reason, the procedure is designed to make s win.

The fact that the amendment procedure embodies a reasonable extension of the Condorcet criterion to three alternatives is what, in my opinion, has made it generally acceptable. Nevertheless, when X has four or more members, this procedure is quite arbitrary and may select a *substantive* motion that fails the Condorcet test, even one that is unanimously believed inferior to a rejected motion. Thus, for motion (t), amendment (w), amendment to the amendment (x), substitute amendment (y), and status quo (s), a profile like that in Display 4-4 might occur.[4]

In this profile, x (the initially rejected amendment) is *unanimously* preferred to t (the original and winning motion). Clearly here the proce-

Display 4-3

Alternate Profiles Leading to a Victory for the Status Quo Under the Amendment Procedure with Only One Amendment

D
(with a Condorcet winner)

D_1, D_2:	*s t w*
D_3, D_4:	*t s w*
D_5:	*w s t*

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	s	*t*	*w*	
s	—	3	4	(Condorcet winner)
t	2	—	4	
w	1	1	—	

Alternative *s* is the Condorcet winner in profile *D*.

D′
(without a Condorcet winner)

D'_1, D'_2:	*t w s*
D'_3:	*s t w*
D'_4:	*s w t*
D'_5:	*w s t*

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	s	*t*	*w*
s	—	3	2
t	2	—	3
w	3	2	—

There is no Condorcet winner, and *s* wins by default in profile *D′*.

dure fails to achieve the intended goal of identifying and selecting a clear-cut winner.

The Successive Procedure

The successive procedure is the analogue for electing candidates of the amendment procedure for selecting motions, but it is neutral since all candidates may lose. It guarantees the choice of the candidate with an absolute majority, if one exists; but it does not choose at all when there is no Condorcet winner, and it may even reject a Condorcet winner.

The set X is composed of nominated candidates, who are voted upon individually in some specified order until one wins or the list is exhausted. If at the end no winner has been selected, the procedure is abandoned or repeated. At each stage the procedure in effect divides the alternative candidates into two sets, the singleton $\{x\}$ and the remainder $(X - \{x\})$, and places them against each other. Each voter then decides how to vote by this rule: If i prefers x to all other members of X, then i votes yes on $\{x\}$; but if i prefers some y in X, where $y \neq x$, then i votes no on $\{x\}$—in effect choosing $(X - \{x\})$ over $\{x\}$. If $\{x\}$ wins, x must, by definition, be able to beat *all* of the others simultaneously. Even a Condorcet winner can lose, as in the example in Display 4-5, where the alternative x is the Condorcet winner but the singleton set $\{x\}$ will lose to $(X - \{x\})$. If there is no winner, this rule has no method of resolution among nominees, although in practice it probably often influences voters to reorder their preferences to obtain a winner.

Display 4-4

A Profile in Which a Losing Alternative Is Unanimously Preferred to the Winning Alternative

	D
D_1:	$w\ x\ t\ y\ s$
D_2:	$y\ w\ x\ t\ s$
D_3:	$s\ x\ t\ y\ w$

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	s	t	w	x	y
s	—	1	1	1	1
t	2	—	2	0	2
w	2	1	—	2	1
x	2	3	1	—	2
y	2	1	2	1	—

Amendment Procedure by Which t Wins Although All Voters Prefer x to t

Step 1: x vs. w; w wins

Step 2: w vs. y; y wins

Step 3: y vs. t; t wins

Step 4: t vs. s; t wins

Runoff Elections

Runoff elections combine plurality decision (a positional method to be discussed in section 4.E) with simple majority decision in a two-stage process. Although the intent is probably to ensure a Condorcet winner,

Display 4-5

A Profile in Which the Condorcet Winner Loses Under the Successive Procedure

D_1:	$w\ x\ y\ z$
D_2:	$y\ x\ w\ z$
D_3:	$z\ x\ y\ w$

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	w	x	y	z	
w	—	1	1	2	
x	2	—	2	2	(Condorcet winner)
y	2	1	—	2	
z	1	1	1	—	

Although x is the Condorcet winner, in a contest between the singleton set $\{x\}$ and the set $(X - \{x\})$, $\{x\}$ loses, which means, of course, that x loses.

	Contest		
	$\{x\} = x$	vs.	$(X - \{x\}) = w, y, z$
Voters	None		1, 2, 3

runoff elections may fail to do so. Typically, positional methods do not satisfy the Condorcet criterion, and in this combination the plurality feature introduces violations of it. The rule is: When X contains three or more members, choose that alternative which receives more than half of the total vote. If none do, then hold a runoff election between the two alternatives with the largest numbers of votes, deciding between them by simple majority.

That this method can fail to satisfy the Condorcet criterion is easily seen in Display 4-6. Although x wins, z has a simple majority over x and y, and z is the Condorcet winner. Furthermore, as Smith has proved (see section 3.C and Chapter 3, note 10), this method is not necessarily monotonic because alternatives are eliminated. An example of the failure of runoff elections to satisfy monotonicity is seen in Display 4-7, where an increase from D' to D in support for alternative x results in its defeat in D even though it had won in D'. The reason is that the weakening of w changes the runoff from "x vs. w" in D' to "x vs. y" in D and, in both D' and D, x loses to y.

There are other Condorcet extensions, a few of which I will discuss in the following paragraphs. They have not been widely used, if at all, and are of interest because they show the difficulty of defining winners when no Condorcet winner exists. There is no good reason, in my opinion, for preferring one to another, yet they can lead to different outcomes (see note 8).

The Copeland Rule

The Copeland rule is based on the notion that those alternatives ought to be selected that win relatively most frequently in simple majority contests with all other alternatives. The rule is: Select that alternative, x, with the largest Copeland index, $c(x)$, which is the number of times x beats other alternatives less the number of times x loses to other alternatives.[5] Of course, if there is a Condorcet winner, $c(x) = m - 1$; if not, there may be ties.[6]

The Schwartz Rule

If no alternative beats all others, then some alternatives are in a cycle: $x\, M\, y\, M\, z\, M\, x$. The "top" cycle is one that either extends over the whole set X, or, if not, each alternative in the top cycle beats all alternatives not in it. The goal of the Schwartz rule is to select the top cycle, which is a unique alternative if a Condorcet winner exists, but may be most or all of the alternatives otherwise.[7]

The Kemeny Rule

The goal of the Kemeny rule is to find a permutation, P, of the members of X that is "closest" to all the D_i in D. A permutation of X is a transitive ordering of X and consists of ordered pairs, so that if xy is in P, then yx cannot be. Thus, for $X = (x, y, z)$, one permutation is zxy or the

Display 4-6

A Profile in Which the Condorcet Winner Loses Under the Runoff Procedure

D_1, D_2:	$x\,z\,y$
D_3, D_4:	$y\,z\,x$
D_5:	$z\,x\,y$

Plurality Stage

First-place votes:

x: 2

y: 2

z: 1

The alternative with the fewest first-place votes, z, is eliminated.

Majority Stage

	Contest		
	x	vs.	y
Votes	3		2

Alternative x, supported by voters 1, 2, and 5, wins. Nevertheless, z is the Condorcet winner.

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	x	y	z	
x	—	3	2	
y	2	—	2	
z	3	3	—	(Condorcet winner)

Display 4-7

Why Runoff Elections Do Not Satisfy the Criterion of Monotonicity

D'		D	
D'_1, D'_2:	$w\,x\,y\,z$	D_1, D_2:	$x\,w\,y\,z$
D'_3, D'_4:	$w\,y\,x\,z$	D_3, D_4:	$w\,y\,x\,z$
D'_5, D'_6, D'_7, D'_8:	$x\,w\,y\,z$	D_5, D_6, D_7, D_8:	$x\,w\,y\,z$
D'_9, D'_{10}, D'_{11}:	$y\,z\,x\,w$	D_9, D_{10}, D_{11}:	$y\,z\,x\,w$
D'_{12}, D'_{13}:	$z\,w\,y\,x$	D_{12}, D_{13}:	$z\,w\,y\,x$

Profile D'

Plurality stage, first-place votes:

w: 4

x: 4

y: 3

z: 2

The alternatives with the fewest first-place votes, y and z, are eliminated.

Majority stage:

	Contest		
	w	vs.	x
Votes	6		7

Alternative x, supported by voters 5, 6, 7, 8, 9, 10, and 11, wins.

Profile D

Profile D differs from Profile D' only in that voters 1 and 2 have raised x from second place to first place and lowered w to second place.

Plurality stage, first-place votes:

w: 2

x: 6

y: 3

z: 2

The alternatives with the fewest first-place votes, w and z, are eliminated.

Majority stage:

	Contest		
	x	vs.	y
Votes	6		7

Alternative y, supported by voters 3, 4, 9, 10, 11, 12, and 13, wins. Thus an increase in support of x in D as against D' brought about the defeat of x in D even though x had won in D'.

---

pairs zx, zy, xy. To find the "closest" permutation to D, the rule is: For each P, sum the number of ordered pairs in all the D_i of D that are the same as pairs in P. There is a set P^* of those P with the largest sum of ordered pairs in D and the winner(s) are those alternative(s) standing first in the permutations in P^*.

Although the detail of this extremely complicated calculation is not in itself important, an example (illustrating, perhaps, the desperation of theorists to discover an adequate Condorcet extension) is set forth in Display 4-8, where the ordered pairs in the six possible permutations of the members of X are checked against the ordered pairs in the D_i of D. The pairs in P_6 are found more frequently in D than are the pairs of any other permutation. Hence, $P^* = P_6$; and z, which is first in P_6, is the Kemeny winner. Notice that, if a Concorcet winner exists, it will be chosen; but if one does not exist, there is still likely to be a unique winner. Although the Kemeny rule is based on a clever and defensible idea, the

Display 4-8

Calculation of the Kemeny Winner

D	
D_1:	$z\,y\,x$
D_2:	$y\,z\,x$
D_3:	$x\,z\,y$

P			
P_1:	$x\,y\,z$	P_4:	$y\,z\,x$
P_2:	$x\,z\,y$	P_5:	$z\,x\,y$
P_3:	$y\,x\,z$	P_6:	$z\,y\,x$

Comparison of Ordered Pairs in D_i and P_1 to P_6

P_1: $x\,y\,z$

Pairs in P_1	Pairs in D		
	D_1	D_2	D_3
xy	0	0	1
xz	0	0	1
yz	0	1	0
Total	$0 + 1 + 2 = 3$		

P_2: $x\,z\,y$

Pairs in P_2	Pairs in D		
	D_1	D_2	D_3
xz	0	0	1
xy	0	0	1
zy	1	0	1
Total	$1 + 0 + 3 = 4$		

P_3: $y\,x\,z$

Pairs in P_3	Pairs in D		
	D_1	D_2	D_3
yx	1	1	0
yz	0	1	0
xz	0	0	1
Total	$1 + 2 + 1 = 4$		

P_4: $y\,z\,x$

Pairs in P_4	Pairs in D		
	D_1	D_2	D_3
yz	0	1	0
yx	1	1	0
zx	1	1	0
Total	$2 + 3 + 0 = 5$		

P_5: $z\,x\,y$

Pairs in P_5	Pairs in D		
	D_1	D_2	D_3
zx	1	1	0
zy	1	0	1
xy	0	0	1
Total	$2 + 1 + 2 = 5$		

P_6: $z\,y\,x$

Pairs in P_6	Pairs in D		
	D_1	D_2	D_3
zy	1	0	1
zx	1	1	0
yx	1	1	0
Total	$3 + 2 + 1 = 6$		

Since the pairs in P_6 are found more frequently in D than the pairs of any other permutation, the Kemeny winner is z, which is the alternative ordered first in P_6.

Kemeny winner may not be the same one chosen by Copeland's rule or by Schwartz's rule, which are also based on clever and defensible ideas.[8]

Conclusions

Two features stand out in this survey of majoritarian rules. First, majority choice over three or more alternatives, though not easy to operate, usually has a clear meaning when a Condorcet winner exists. Second, when a Condorcet winner does not exist, the rules actually used to pick a winner may in fact pick what many would regard as a loser, and there is no consensus on what ought to be regarded as the winner.

One can therefore say that, if the Condorcet criterion is not satisfied, majoritarian decision is only partially adequate from a practical point of view. It can pick winners, but they need not be unique. From a theoretical point of view, majoritarian decision is even worse because it is hard to define and, furthermore, the several definitions do not necessarily lead to the same winner from the same profile.

4.D. Positional Methods of Voting

The fact that majoritarian methods are clearly defined when a Condorcet winner exists does not win universal approval for them. As I mentioned at the beginning of section 4.B, majoritarian methods assume that the defect in simple majority decision of limiting alternatives can be cured by the easy solution of admitting more of them. But the remedy creates a new disease. One is the difficulty, just discussed, of stating an adequate rule when a Condorcet winner is undefined. This is, however, only the first and simplest difficulty. A more profound one is that majoritarian methods use only information about binary relations in the social profile, D, even though, once X is expanded to more than two members, much more information exists—namely, the position of each alternative in the individual orderings, D_i, of D.

Positional methods are intended to resolve this more profound difficulty by taking some or all of the information about whole individual orderings into account. These methods include (1) plurality voting (which uses only information about first places), (2) approval voting (which uses information about a variable number of places), and (3) scoring functions such as the Borda count, which uses information about all places. The Borda count—the most systematic of the three positional methods—has each individual assign $m - 1$ points to her or his first-place alternative, $m - 2$ points to the second-place alternative, and so on, to zero points to the m^{th}-place (that is, last) alternative. The Borda score for an alternative is the sum of points given it by the n voters. The Borda winner, then, is the alternative with the highest Borda score.[9]

The main argument for the positional approach is that it uses some or all of the information added by expanding X and thus expanding the D_i from paired comparisons to orderings of three or more alternatives. This added information makes a great difference. In an especially interesting example, Peter Fishburn points to the following pair of situations for $X = (x, y, a, b, c)$ and five voters:[10]

D: By binary majority decision: $x\,M\,y\,M\,a\,M\,b\,M\,c$

D': x has:	2 first-place votes	y has 2 first-place votes
	1 second-place vote	2 second-place votes
	1 fourth-place vote	1 third-place vote
	1 fifth-place vote	

As Fishburn remarks, although one might reasonably choose x to win in D and y to win in D' (since, in D', y has the same number of first-place votes and more second- and third-place votes), the interesting fact is that it is possible that $D = D'$, as shown in Display 4-9.

Clearly the kind of information used makes a difference. Paired comparisons lead to the choice of x, and positional comparisons lead to the choice of y. Furthermore, as Condorcet himself pointed out, the conflict is profound and does not depend at all on the particular method of numbering positions.[11] One can generalize the Borda method into a scoring system to count relative positions without using points at all—thereby implicitly revealing the justification for positional methods: For each alternative, x, determine a score $s(x)$, by counting the number of times x precedes $y, z, \ldots$ in all of the D_i of D and then subtracting the number of times x succeeds $y, z, \ldots$. The alternative with the highest score wins, which means the winner most frequently stands ahead of others in the whole profile.[12] For the example just given, $s(y) = 12$, $s(x) = 4$, $s(a) = -4$, $s(b) = -6$, $s(c) = -6$.[13] Since this is a count of positions (not points), it follows that y will beat x for *any* system that uses data on the entire profile.

Dodgson invented an excellent example, set forth in Display 4-10, to point up the ethical conflict between majoritarian and positional methods.[14] There are 11 voters. Although b is clearly the Condorcet winner, in some sense a has an excellent claim because

Display 4-9

A Profile in Which the Borda Winner Seems Appropriate Even Though a Different Condorcet Winner Exists

$D = D'$	
D_1:	$x\,y\,a\,b\,c$
D_2:	$y\,a\,c\,b\,x$
D_3:	$c\,x\,y\,a\,b$
D_4:	$x\,y\,b\,c\,a$
D_5:	$y\,b\,a\,x\,c$

Majoritarian Decision: Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	x	y	a	b	c	
x	—	3	3	3	3	(Condorcet winner)
y	2	—	5	4	4	
a	2	0	—	3	3	
b	2	1	2	—	3	
c	2	1	2	2	—	

Borda Count: Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	D_4	D_5	Total	
x	4	0	3	4	1	12	
y	3	4	2	3	4	16	(Borda winner)
a	2	3	1	0	2	8	
b	1	1	0	2	3	7	
c	0	2	4	1	0	7	

Display 4-10

A Profile in Which the Condorcet Winner Has a Majority of First-Place Votes While a Different Borda Winner Has a Much Higher Score Than the Condorcet Winner

D	
D_1, D_2, D_3:	$b\ a\ c\ d$
D_4, D_5, D_6:	$b\ a\ d\ c$
D_7, D_8, D_9:	$a\ c\ d\ b$
D_{10}, D_{11}:	$a\ d\ c\ b$

Majoritarian Decision: Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	a	*b*	*c*	*d*	
a	—	5	11	11	
b	6	—	6	6	(Condorcet winner)
c	0	5	—	6	
d	0	5	5	—	

Borda Count: Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	Total	
a	2	2	2	2	2	2	3	3	3	3	3	27	(Borda winner)
b	3	3	3	3	3	3	0	0	0	0	0	18	
c	1	1	1	0	0	0	2	2	2	1	1	11	
d	0	0	0	1	1	1	1	1	1	2	2	10	

a has 5 first-place votes
6 second-place votes

b has 6 first-place votes
5 fourth- (last-) place votes

Everybody who likes *b* better than *a* nevertheless likes *a* second-best, while everybody who likes *a* better than *b* puts *b* at the bottom of the list. Accordingly, *a* wins by a dramatic margin under the Borda count.

4.E. Examples of Positional Methods

I do not intend at this point to render a moral judgment on the relative merits of majoritarian and positional methods. I wish now merely to observe that the use of one rather than the other makes a great difference in outcomes. Furthermore, because the kind of positional voting one uses also makes a great difference, I will describe three kinds briefly.

Plurality Voting

In constitutional structure, the plurality method is close to simple majority decision. Both require single-member districts or unique outcomes, and which one occurs is determined by the number of candidates or alternatives put forward. If there are only two, then simple majority decision occurs; if more than two, plurality occurs. *Plurality* means selecting the alternative with the most votes (which may or may not be a majority of the votes cast). This method of voting thus in effect selects the alternative most frequently placed first in voters' preference orders. Naturally, this need not be a Condorcet winner and probably often is not. For example, in Display 4-11, *x* is the plurality winner, *y* is the Condorcet winner, and *x* loses to *all* other candidates in a majority decision. Elections in which something like this may have occurred are the presidential election of 1912 and the New York senatorial election of 1970, for which I have guessed at voters' orderings, as depicted in Displays 4-12 and 4-13.

This feature of plurality voting appears illogical and inadequate to many social choice theorists. In terms of a particular election, it may lead to an undesirable choice. In terms of a series of elections, however, it has the effect of encouraging politicians to offer exactly two candidates, thus achieving simple majority decision without the unfairness involved in the notion of monolithic "responsible" parties. The motive is, as in the foregoing cases, that supporters of the losers (*y* and *z* in Display 4-11, Taft and

Display 4-11

A Profile in Which the Plurality Winner Loses to All Other Alternatives Under Majority Decision

D	
D_1, D_2, D_3, D_4:	$x\ y\ z$
D_5, D_6, D_7:	$y\ z\ x$
D_8, D_9:	$z\ y\ x$

Plurality Winner

Number of First-Place Votes		
x	4	(Plurality winner)
y	3	
z	2	

Majoritarian Decision: Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	x	y	z	
x	—	4	4	
y	5	—	7	(Condorcet winner)
z	5	2	—	

Notice that x loses to both y and z.

Display 4-12

Possible Profiles of the 1912 Presidential Election

First-place votes	Possible orderings by voters with given first-place votes
42% Wilson	Wilson, Roosevelt, Taft
27% Roosevelt	Roosevelt, Taft, Wilson
24% Taft	Taft, Roosevelt, Wilson
7% Others	

The plurality winner is Wilson.

Majoritarian Decision: Percent of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	Wilson	Roosevelt	Taft	
Wilson	—	42%	42%	
Roosevelt	51%	—	69%	(Condorcet winner)
Taft	51%	24%	—	

Display 4-13

Possible Profiles of the 1970 New York Senatorial Election

First-place votes	Possible ordering by voters with given first-place votes
39% Buckley (Con.)	Buckley, Ottinger, Goodell
37% Ottinger (Dem.)	Ottinger, Goodell, Buckley
24% Goodell (Rep.)	Goodell, Ottinger, Buckley

The plurality winner is Buckley.

Display 4-13 *(Continued)*

Majoritarian Decision: Percent of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	Buckley	Ottinger	Goodell	
Buckley	—	39%	39%	
Ottinger	61%	—	76%	(Condorcet winner)
Goodell	61%	24%	—	

Roosevelt in Display 4-12, and Ottinger and Goodell in Display 4-13) like the winner least of all, *and* they know that their failure to compromise is the proximate cause of the winner's success. In the next election, therefore, they *want* to compromise, and this is what produces a two-candidate election.

In the instances cited, this is precisely what happened. After 1912, Progressives returned to the Republican fold. After the 1970 election, where many Republicans could not vote for their second choice, Ottinger, because they believed him quite incompetent, the Democrats compromised by offering an obviously intelligent candidate, Moynihan, and obtained thereby much moderate Republican support against Buckley, who ran in 1976 as a Republican. Thus plurality voting, while often defective in single elections, is probably the main force maintaining over time the simple majority decision that most people regard as desirable.[15]

Approval Voting

Approval voting permits the voter to cast one vote for as many candidates as she or he chooses, and the candidate with the most votes wins. This method is used, so far as I know, only for honorary societies, but it has recently been proposed for national primaries and elections.[16]

It is an essentially positional or point-count system, where one point is given for each of the j places, $j < m$, in the preference ordering, D_i, for which i chooses to vote. So approval voting is somewhere between plurality voting and the Borda count. As a positional system, approval voting can lead to the defeat of the Condorcet winner. Indeed, it can lead to the defeat of a candidate with a large absolute majority. If all voters decide to cast two approval votes—one for each of the first two positions, as in Display 4-14—then y wins, though x is the Condorcet winner.

Display 4-14

A Comparison of Approval Voting and Majoritarian Voting

	D
D_1–D_{61}:	$x\ y\ z$
D_{62}–D_{81}:	$y\ x\ z$
D_{82}–D_{101}:	$z\ y\ x$

Approval Voting, Assuming Each Voter Votes for Two Alternatives

	First-place votes	Second-place votes	Total	
x	61	20	81	
y	20	81	101	(Approval winner)
z	20	0	20	

Majoritarian Decision: Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	x	y	z	
x	—	61	81	(Condorcet winner)
y	40	—	81	
z	20	20	—	

The reason approval voting is urged as a replacement for the plurality method is that it would probably select the Condorcet winner in cases like those of 1912 and 1970, where supporters of the losers rate the winners last. Furthermore, in presidential primaries it would probably encourage the selection of moderates, because party members of all beliefs would be likely to vote for the moderates. Still, its dynamic effect would probably be to increase the number of candidates offered (especially in primaries), so that simple majority decision would almost never occur.

The motive would be that supporters of a minority candidate might believe that they could collect enough second-, third-, . . . , j^{th} place votes to win. In primaries, this dynamic fractionalization would simply split up the parties. But in elections the effect would probably be worse: The force in the plurality system toward simple majority decision might be dissipated. It does seem myopic to abandon a system dynamically moving toward two-candidate choice and to accept one kind of failure to choose a Condorcet winner (namely, permanent multipartyism or multifactionalism), simply to avoid another kind of failure (occasional plurality winners).

The Borda Count

The Borda count is, as noted in section 4.D, a simple version of scoring methods—that is, of ascertaining the order of alternatives based on the net number of positions of precedence. It is important not to think of the Borda score as some kind of utility function.[17] Just because a voter places x first among five (4 points) does not mean he or she likes x twice as much as y in third place (2 points). The points are simply a device to count "aheadness." This definition of winning in terms of aheadness is what renders the Borda count especially attractive for many people. It is also, however, what renders it especially vulnerable to strange and paradoxical results when the number of alternatives (and hence of positions) is varied. In such circumstances, there are problems with all voting systems, but the Borda count seems particularly prone to such distortions. For example, Fishburn has shown the possibility of these bizarre results:[18]

1. If the ordering induced on X by the Borda count is $y\ a\ b\ c$, and if y is then removed from X, the ordering on $(X - \{y\})$ can then be reversed to $c\ b\ a$, as in Display 4-15.
2. The inverse of this paradox is: If the ordering on X is $a\ b\ c\ y$, then the ordering on $(X - \{y\})$ can be $c\ b\ a$.
3. Suppose y is the Borda winner in X. It may be that, for all the possible proper subsets of X containing y and at least one other alternative, y is the Borda winner in only *one* subset. (A proper subset is a subset that does not include all members of the set.) So, if X has, say, $m = 6$ alternatives, there are $(2^{m-1} - 2) = 30$ proper subsets containing y and at least one other alternative, and it is possible that y loses in 29 of them and still wins in X.

The implicit justification of the Borda count is that positional information should be considered. These perversities, however, suggest that the method is quite sensitive to the number of positions, and that fact casts some doubt on the justification.

Finally, there is a practical difficulty with the Borda method, one it shares with Condorcet extensions such as Kemeny's. That difficulty is the onerous task of collecting complete preference orders, D_i, from all i in N. For more than three alternatives, many voters find ordering difficult.

Conclusions

Positional methods are sensitive to the positions used. Indeed, it is possible that with exactly the same profile, D, plurality voting will produce one winner, approval voting another, and the Borda count yet another. This is perhaps as it should be, because each involves adding some positional information; but it also suggests that the outcome is as much dependent on the method of counting votes as it is on the voters' judgments. In Display 4-16 is an example where D is constant, $X = (a, b, c, d)$, and $n = 13$; the winners are

Plurality: a

Approval: b, where each voter votes for two candidates

Approval: c, where each voter votes for three candidates

Borda: d

Incidentally, there is no Condorcet winner; but the Copeland winners are a and d; the Schwartz winners are a, b, c, d; and the Kemeny winners are b and d.

In Display 4-16, with just one profile of preferences, there are four different social choices from four methods of aggregation. Furthermore, these methods are not wildly different. They are all based on the notion of aheadness, which is implicit in positional methods. The inescapable inference is that social choice depends as much on methods of aggregation (that is, on social institutions) as it does on individual values and tastes.

Display 4-15

A Paradox with the Borda Count

Let $X = (y, a, b, c)$.

D on X	
D_1, D_2:	$y\ c\ b\ a$
D_3, D_4:	$a\ y\ c\ b$
D_5, D_6:	$b\ a\ y\ c$
D_7:	$y\ c\ b\ a$

Borda Count for D on X: Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	Total
y	3	3	2	2	1	1	3	15
a	0	0	3	3	2	2	0	10
b	1	1	0	0	3	3	1	9
c	2	2	1	1	0	0	2	8

The ordering induced on X by the Borda count is $y\ a\ b\ c$.

Let $(X - \{y\}) = \{a, b, c\}$.

D on $(X - \{y\})$	
D_1, D_2:	$c\ b\ a$
D_3, D_4:	$a\ c\ b$
D_5, D_6:	$b\ a\ c$
D_7:	$c\ b\ a$

Borda Count for D on $(X - \{y\})$: Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	Total
a	0	0	2	2	1	1	0	6
b	1	1	0	0	2	2	1	7
c	2	2	1	1	0	0	2	8

The ordering induced on $(X - \{y\})$ by the Borda count is $c\ b\ a$.

Display 4-16

A Profile of Four Alternatives in Which Four Positional Methods Each Lead to a Different Winner

D	
D_1, D_2, D_3, D_4:	$a\ d\ c\ b$
D_5:	$b\ a\ d\ c$
D_6, D_7:	$b\ a\ c\ d$
D_8, D_9, D_{10}:	$c\ b\ d\ a$
D_{11}, D_{12}:	$d\ b\ c\ a$
D_{13}:	$d\ c\ b\ a$

Plurality Method

	Number of First-Place Votes
a	4 (Plurality winner)
b	3
c	3
d	3

Display 4-16 *(Continued)*

Approval Voting Assuming Each Voter Casts Two Votes

	First-place votes	Second-place votes	Total votes	
a	4	3	7	
b	3	5	8	(Winner)
c	3	1	4	
d	3	4	7	

Approval Voting Assuming Each Voter Casts Three Votes

	First-place votes	Second-place votes	Third-place votes	Total votes	
a	4	3	0	7	
b	3	5	1	9	
c	3	1	8	12	(Winner)
d	3	4	4	11	

Borda Count: Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	Total	
a	3	3	3	3	2	2	2	0	0	0	0	0	0	18	
b	0	0	0	0	3	3	3	2	2	2	2	2	1	20	
c	1	1	1	1	0	1	1	3	3	3	1	1	2	19	
d	2	2	2	2	1	0	0	1	1	1	3	3	3	21	(Borda winner)

4.F. Utilitarian Methods of Voting

Majoritarian methods base decision on how many times x is ahead of one other alternative. Positional methods base decision on how many times x is ahead of all other alternatives. Neither bases decision on voters' direct valuation of alternatives, although positional methods are sometimes mistakenly so interpreted. Many writers, however, have argued that the intensity of judgment or direct valuation *ought* to be incorporated into the decision.[19] The voting method should reveal not only that x is ahead of y, but by *how much*. Methods that attempt this are here called *utilitarian* because they involve the construction and summation of utility numbers.

Utility is a measure on preference. Given two objects, x and y (such as yea and nay votes), and an individual choice of x over y (which is a judgment of preference), the act of choice reveals to an observer that the chooser likes x better than y. But nothing in the act itself reveals *how much* better. The task of measurement is to state "how much" on some scale or another. Distances on the scale are utilities and as such are a measure on preference.

Until recently, most social scientists were reluctant to use notions of cardinal utility because all methods of measurement seemed arbitrary. In the last generation, however, the Von Neumann–Morgenstern method has come to be accepted, and proposals for utilitarian voting depend, at least theoretically and conceptually, on the *existence* of this method.[20]

The theme of the Von Neumann–Morgenstern method is that the outside observer can, by conducting an experiment, arrive at an exact, external, cardinal measure of the chooser's ordinal preference. The protocol of the experiment is:

1. Offer the subject three or more alternatives and obtain a transitive and complete ordering (e.g., $x\, R_i\, w\, R_i \ldots R_i\, z$).
2. For three alternatives, the most and least preferred and the one in between (say, x, y, and z), give the most preferred, x, a utility, $u(x)$ of some maximum, $u_{\max}$ (e.g., let $u(x) = 1$); and similarly for the least preferred, z, give some minimum, $u_{\min}$ (e.g., let $u(z) = 0$). This generates a scale from zero to 1 on which y must be located.
3. Offer the subject a choice between two alternatives.
   a. The lottery alternative: Using some chance mechanism, the subject receives x with probability p and z with probability $(1 - p)$. Here p is an ordinary probability number, $0 \leq p \leq 1$ and $p + (1 - p) = 1$. The expected utility of this gamble is: $p\, u(x) + (1 - p)\, u(z)$.

b. The certain alternative: The subject receives y for certain.
Of course, the subject's choice will be affected by the value of p. If p is large (i.e., close to 1), then the expected value of the lottery is large, which means that, when the chance mechanism is operated, the subject is likely to receive the most-preferred alternative. Hence he or she is likely to choose alternative a (the lottery alternative). If, however, p is small (i.e., close to zero), then the expected value of the lottery is low, which means that the subject is likely to receive the least-preferred alternative. Hence he or she is likely to choose b, (the certain alternative), which assures the receipt of the medium alternative.

4. Offer the subject these choices repeatedly, beginning with p sufficiently high that he or she chooses the lottery. Repeat, lowering p systematically, until the subject cannot decide between alternatives a and b. This indecision is interpreted as meaning that the expected utility of the lottery is exactly equal to the utility of the certain alternative, which thus generates a value for $u(y)$. That is

$$p\, u(x) + (1 - p)\, u(z) = u(y)$$

Recall that $u(z) = 0$, so the equation reduces to

$$p\, u(x) = u(y)$$

And since $u(x) = 1$,

$$u(y) = p$$

5. For other alternatives besides y, repeat using $u(x) = 1$ and $u(z) = 0$.

This experimental procedure is an externalized analogue of what people probably go through—in a casual, private, unformalized way—when they decide whether to take a risk, for example, whether to buy insurance at a particular price. Hence it assumes just about what is assumed in the notion of risks. Specifically, to produce cardinal utilities by this experiment, the following axioms must be assumed to be true:

1. *Transitivity and completeness.* The relations of preference and indifference are transitive and complete over the set of alternatives, X, and $x\, P_i\, y$ for at least one pair, x and y.

2. *Axiom of Archimedes.* There is a probability number, p, and a lottery, $L(p)$, based on p, such that the chooser is indifferent between the utility of the lottery and the utility of an alternative y that is neither best nor worst.

Let the chooser order the alternatives from best to worst, say, $x, \ldots, z$, so that the maximum utility, $u_{\max}$, is the utility of x, $u(x)$, and the minimum utility, $u_{\min}$, is the utility of z, $u(z)$. Then

$$u\, L\,(p) = p\, u_{\max} + (1 - p)\, u_{\min}$$

Then the axiom is that

$$u\,(y)\, I_i\, u\, L\,(p)$$

3. *Continuity.* As p varies from zero to 1, $u\, L\,(p)$ varies from $u_{\min}$ to $u_{\max}$.

4. *Substitutability.* An alternative y can be replaced by its equivalent lottery, $L\,(p)$, as identified by the axiom of Archimedes (axiom 2).

It is apparent that axioms 1, 3, and 4 are simply technical conveniences for the experiment, though axiom 1 does impose the substantive limitation that X consist of comparable things. Axiom 1 is necessary to justify the subject's initial ordering in the experiment. Axiom 3 is a conventional continuity axiom to justify the use of p. Axiom 4 is necessary to justify the lottery procedure. The main substantive assumption is, therefore, axiom 2, which is often expressed bluntly as "Everything has a price."

Does everything, or rather every risk, have a price? People do indeed price trading risks, and the experiment externalizes the internal risk-pricing mechanism. Consequently, cardinal utility is quite usable in economic theory. But what about political theory? It seems to me that many alternatives outside the economic sphere are unpriceable or, more accurately, incommensurate in value with other alternatives. Doubtless it is easy to measure some political values such as how much someone would prefer more police officers to more trash collectors or how much one would prefer lower taxes to either or both. But it is probably impossible to measure how much one prefers more national sovereignty to less national wealth, or vice versa. Many persons might find such tradeoffs incalculable simply because in their judgments independence (or wealth) must first be achieved before the other can be considered at all. If so, the goals are incommensurate, and there is no tradeoff and no price. So one can make ordinal, but not cardinal, comparisons.

Nevertheless, several kinds of voting with cardinal utilities have been proposed.

4.G. Examples of Utilitarian Methods

The common theme of the following methods is that the supposed intensity of each voter's preference is included in the social summation. The differences among them are mainly to remedy technical defects, no one of which is as serious as the misguided attempt to measure incommensurables.

Summation of Cardinal Utility

Apparently, Bentham considered the summation of cardinal utility as a decision process, even evidently using money as a measure.[21] It is now well understood, as it was not in Bentham's day, that the relation between money and the utility of money is not linear (that is, one may want a second dollar over a first far more than one wants a millionth over a 999,999th). Hence, today Bentham might opt for Von Neumann–Morgenstern utility rather than for money; but the notion of summation of some kind of cardinal utility is fundamental to his idea of maximizing social happiness by choosing the alternative with the largest sum.[22]

Demand-Revealing Methods

These methods of decision, described in section 3.D, can be extended easily to three or more alternatives.[23] They are a summation method with the added feature of (partially) encouraging a true revelation of judgment.

Multiplication of Utilities

This method consists of measuring all persons' utilities, using as the origin some status quo point, and, for each alternative, multiplying voters' utilities for it, selecting the alternative with the largest product.[24] The rationale for using the maximum product rather than the maximum sum is that the former satisfies an axiom of consistency in choosing from a set and its subsets. (See section 5.F.)

Conclusions

Utilitarian methods are neither majoritarian nor positional. Because they base decision on intensity, not on numbers, they can easily select alternatives not favored by a majority, even when a Condorcet winner exists. Simultaneously, they can fail any one of the positional tests by selecting alternatives not more frequently standing ahead of others in D. One reason some people reject utilitarian methods is that the emphasis on intensity might lead people to exaggerate their desires simply to make them count for more. Demand-revealing methods tend to squelch that maneuver, but they do not otherwise change the nonmajoritarian, nonpositional character of the utilitarian decision process.

4.H. Criteria for Judging Voting Methods

It is clear from the foregoing survey that majoritarian, positional, and utilitarian methods of voting lead to different social results, often strikingly different, from the same underlying individual judgments. It seems natural to wonder, therefore, if there is any way to discriminate among this plethora of devices and to discover a way to amalgamate judgments fairly and truly. There are many criteria for discrimination, and I will discuss some here. Whether they lead to the discovery of a fair and true method I leave to the reader's judgment—though my own belief is that they do not.

I will begin by showing how the properties of simple majority decision discussed in Chapter 3—undifferentiatedness (anonymity), neutrality, and monotonicity—can be generalized to accommodate three or more alternatives. Those three criteria are, in some sense, elementary requirements of fairness and consistency, and most of the voting methods discussed in this chapter satisfy most of them. I will then introduce three deeper requirements of fairness, which embody some of the moral requirements raised in this chapter.

Undifferentiatedness

If the choice from D is x, and if the individual orderings are reassigned among voters to form D', the social choice in D' remains x. Since permuting the preference orders does not change the outcome, it follows that the identity of voters has no effect on the social choice, which is exactly what undifferentiatedness should mean.[25]

Neutrality

If a social profile yields a choice x, and if the elements of X are permuted, thereby creating a new profile, then the choice from the new profile is the permutation of x. Since a rearrangement of alternatives leads to a corresponding rearrangement of outcomes, it follows that no alternative has a favored position in the voting system, which is exactly what neutrality should mean.[26]

Monotonicity

If a profile changes because some person raises the valuation of x relative to other alternatives, then, if x was originally the social choice, it remains so. Conversely, if a person lowers the valuation of an x that was not originally the social choice, it does not become so. This means that a higher judgment on a winning alternative cannot make it lose; nor can a lower judgment on a loser make it win.[27]

A similar generalization applies to unanimity, as an implication from monotonicity. For unanimity, corresponding to the weak unanimity of equation (N3.6), if, for some alternative, y, all persons prefer some other alternative(s) to it, then y cannot be the social choice. This means that an alternative unanimously beaten by one or more others cannot win.[28]

The Condorcet Criterion

According to the first "deeper" requirement of fairness and consistency, the Condorcet criterion, if an alternative beats (or ties) all others in pairwise contests, then it ought to win.[29] This notion is closely related to the notion of equality and "one man, one vote," in the sense that, when an alternative opposed by a majority wins, quite clearly the votes of some people are not being counted the same as other people's votes.

Consistency

The consistency requirement concerns the way votes are taken from the electorate. If the electorate is divided into two parts for election purposes and if one alternative is chosen in both parts, then it ought to be chosen in the whole.[30]

That this requirement contains a fundamental kind of fairness seems obvious. If a winner is a true winner, subdividing the electorate ought not to make it a loser. If consistency fails to hold, therefore, manipulation is rendered easy. Suppose x wins in the whole electorate and the voting method fails to satisfy consistency. Then opponents of x have merely to set up two appropriately chosen subelectorates, define the winner as that alternative which wins in both, and thereby make y win.

Independence from Irrelevant Alternatives

The independence criterion requires that a method of decision give the same result every time from the same profile of ordinal preferences.[31] This too seems a fundamental requirement of consistency and fairness to prevent the rigging of elections and the unequal treatment of voters; but it has nevertheless been seriously disputed.[32]

4.I. Judgments on Voting Methods

Do the methods described in this chapter satisfy the criteria just listed? The answer is unequivocally negative. I will dispose first of what seem obvious (and, in some cases, minor) violations, so that I can then discuss the main (and mostly serious) violations by each method of voting.

Minor Violations of Fairness

Monotonicity is satisfied by all the voting methods except runoff elections. (*All* elimination procedures can violate monotonicity and are thus highly suspect.) All methods here discussed except the Schwartz rule satisfy unanimity. The Schwartz rule fails simply because, when there is an all-inclusive cycle, it includes all alternatives, even those unanimously dominated.[33]

The neutrality criterion is satisfied by all but the amendment procedure, and it is precisely this violation that allows the procedure to work reasonably well on three alternatives—although the violation does not help on more alternatives.

Except for the violation of neutrality, the foregoing are indeed serious; but there are even more significant violations, to which I now turn.

No one of the three categories of voting methods satisfies all criteria:

Majoritarian methods violate the consistency criterion.

Positional methods violate the Condorcet criterion; moreover, the Borda count violates independence, and approval voting violates undifferentiatedness (anonymity).

Utilitarian methods violate the independence criterion and, as pointed out in section 3.D, demand-revealing methods violate undifferentiatedness.

Violation of Consistency by Majoritarian Methods

That majoritarian methods violate the consistency criterion has been shown by H. P. Young, who also proved in general that "scoring functions" (which are positional methods) are the only ordinal methods that satisfy consistency.[34] Utilitarian methods, involving summations of cardinal numbers, also satisfy this criterion. The reason majoritarian methods fail is that, since they count only simple majorities on pairs, an alternative that wins or comes close in N^1 but fails in N^2 may pick up enough votes in N^2 to win in N. Positional methods, which keep track of the pairwise relations, do not admit this anomaly.

For example, in Display 4-17, using some majoritarian methods (say, the Copeland rule or the Schwartz rule), if 75 voters are divided into district 1 ($N^1 = 30$ voters) and district 2 ($N^2 = 45$ voters), with D^1 and D^2 on three alternatives, then x wins in N^1 and ties in N^2 but loses to y in $N^1 + N^2$—in clear violation of consistency, which requires that a winner in both N^1 and N^2 also win in $N^1 + N^2$. As Gideon Doron remarks, the failure of consistency is an invitation to gerrymander.[35]

The failure of majoritarian methods to satisfy consistency is, practically, a very serious defect. This is illustrated in the history of senatorial elections in the United States. Initially, state legislatures used several voting methods. From 1866 to 1913, however, the uniform majoritarian rule was: If a candidate obtains a simple majority in both houses sitting separately, he wins; otherwise, the houses sitting together elect by simple majority. This is exactly the situation of the consistency criterion. As might be expected, there were many deadlocks—45 out of about 130 from 1891 to 1905, of which 14 resulted in no election at all.[36] Almost certainly some of these deadlocks resulted from the failure to satisfy the consistency criterion.

To see how deadlock might occur, assume some candidate, y, can win on the joint ballot, as in Display 4-18, while some other candidate, w, can win or tie in both houses separately. Some of the supporters of w—namely, the six members of group IV in the Display—are crucial for y to beat x on the joint ballot. Since the members of group IV have a strong motive not to allow y to win, one can expect deadlock in the houses sitting together, thus:

41 votes for x

43 votes for y

6 votes for w

Display 4-17

Violation of Consistency by Majoritarian Methods

D^1 (for N^1)	
D_1–D_{17}:	$x\ y\ z$
D_{18}–D_{25}:	$y\ z\ x$
D_{26}–D_{30}:	$z\ x\ y$

D^2 (for N^2)	
D_{31}–D_{44}:	$x\ z\ y$
D_{45}–D_{60}:	$y\ x\ z$
D_{61}–D_{75}:	$z\ y\ x$

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

D^1

	x	y	z
x	—	22	17
y	8	—	25
z	13	5	—

D^2

	x	y	z
x	—	14	30
y	31	—	16
z	15	29	—

In N^1, x is the Condorcet winner. In N^2, x, y, and z tie in the cycle $x\ z\ y\ x$.

Using the Copeland or Schwartz methods, x is chosen in N^1, and x, y, and z are chosen in N^2.

$D^1 + D^2$ (for $N^1 + N^2$)	
D_1–D_{17}:	$x\ y\ z$ (17 voters)
D_{31}–D_{44}:	$x\ z\ y$ (14 voters)
D_{18}–D_{25}:	$y\ z\ x$ (8 voters)
D_{45}–D_{60}:	$y\ x\ z$ (16 voters)
D_{26}–D_{30}:	$z\ x\ y$ (5 voters)
D_{61}–D_{75}:	$z\ y\ x$ (15 voters)

Display 4-17 *(Continued)*

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

$D^1 + D^2$

	x	y	z	
x	—	36	47	
y	39	—	41	(Condorcet winner)
z	28	34	—	

Notice that, while x is chosen in both N^1 and N^2 (that is, x is the Condorcet winner in N^1 and is in the cycle of tied alternatives in N^2), still y wins under any majoritarian method in $N^1 + N^2$. Since consistency requires that, if an alternative wins in N^1 and in N^2, then it must win in $N^1 + N^2$, and since x wins in N^1 and in N^2 but loses in $N^1 + N^2$, the majoritarian methods fail to satisfy consistency. If, however, the Borda count, as a typical positional method, is used, then consistency is satisfied.

Number of Points for Alternatives in Rows, Preference Orders in Columns

D^1 (for N^1)

	D_1–D_{17}	D_{18}–D_{25}	D_{26}–D_{30}	Total	
x	34	0	5	39	(Borda winner)
y	17	16	0	33	
z	0	8	10	18	

D^2 (for N^2)

	D_{31}–D_{44}	D_{45}–D_{60}	D_{61}–D_{75}	Total	
x	28	16	0	44	
y	0	32	15	47	(Borda winner)
z	14	0	30	44	

$D^1 + D^2$ (for $N^1 + N^2$)

	D_1–D_{17}	D_{18}–D_{25}	D_{26}–D_{30}	D_{31}–D_{44}	D_{45}–D_{60}	D_{61}–D_{75}	Total	
x	34	0	5	28	16	0	83	(Borda
y	17	16	0	0	32	15	80	winner)
z	0	8	10	14	0	30	62	

Notice that, since no alternative wins by the Borda method in both N^1 and N^2, consistency imposes no requirement on the winner of $N^1 + N^2$.

Failing consistency, then, majoritarian methods are at least impractical as well, perhaps, as unfair.

Violation of Independence by the Borda Count

The failure of the Borda count to satisfy independence can be seen in Display 4-19. Suppose a committee (members 1, 2, 3) is to award a scholarship to one of three applicants (a, b, c), using the Borda count. Initially, in profile D', the vote results in a tie between a and c. Suppose member 2 revises his ordering to $c\ b\ a$, producing a new profile, D. Then c wins. The independence criterion requires that identical profiles on a subset, S, of a set, X, produce identical outcomes. Letting $S = (a, c)$ and $X = (a, b, c)$, then D' and D are identical on S because member 2 changed only the relation of a and b in D, not of a and c. In D' applicants a and c tie, but in D applicant c wins in direct violation of the independence criterion. It is true that in both D and D' the Condorcet winner is c and member 2 can be said to have produced a "fairer" outcome by his switch; but the switch nevertheless reveals a violation of independence. The criterion of independence from irrelevant alternatives, by requiring that the same order on S in profiles D' and D produce the same result, reveals a serious defect in the Borda rules themselves.

Violation of Undifferentiatedness by Approval Voting

Undifferentiatedness requires that the choice remain the same when preference orders are permuted among voters.[37] Since, however, different

Display 4-18

Deadlock in a Senatorial Election

D^1 (for N^1 with 30 voters)

Group	Alternatives
I. D_1–D_{16}:	$w\,x\,y\,z$ (16 voters)
II. D_{17}–D_{26}:	$y\,w\,x\,z$ (10 voters)
III. D_{27}–D_{30}:	$z\,y\,x\,w$ (4 voters)

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

D^1

	w	x	y	z	
w	—	26	16	26	(Condorcet winner)
x	4	—	16	26	
y	14	14	—	26	
z	4	4	4	—	

D^2 (for N^2 with 60 voters)

Group	Alternatives
IV. D_{31}–D_{36}:	$w\,y\,x\,z$ (6 voters)
V. D_{37}–D_{61}:	$x\,z\,y\,w$ (25 voters)
VI. D_{62}–D_{84}:	$y\,z\,w\,x$ (23 voters)
VII. D_{85}–D_{90}:	$z\,y\,w\,x$ (6 voters)

D^2

	w	x	y	z
w	—	35	6	6
x	25	—	25	31
y	54	35	—	29
z	54	29	31	—

Thus D^2 results in a cycle: $w\,x\,z\,y\,w$.

$D^1 + D^2$ (for $N^1 + N^2$ with 90 voters)

Group	Alternatives
I. D_1–D_{16}:	$w\,x\,y\,z$ (16 voters)
IV. D_{31}–D_{36}:	$w\,y\,x\,z$ (6 voters)
V. D_{37}–D_{61}:	$x\,z\,y\,w$ (25 voters)
II. D_{17}–D_{26}:	$y\,w\,x\,z$ (10 voters)
VI. D_{62}–D_{84}:	$y\,z\,w\,x$ (23 voters)
III. D_{27}–D_{30}:	$z\,y\,x\,w$ (4 voters)
VII. D_{85}–D_{90}:	$z\,y\,w\,x$ (6 voters)

Number of Votes for the Alternative in the Row When Placed in Contest Against the Alternative in the Column

	w	x	y	z	
w	—	61	22	32	
x	29	—	41	57	
y	68	49	—	55	(Condorcet winner)
z	58	33	35	—	

Notice that the six voters in group IV prefer w, who is the absolute winner in N^1 and is tied in N^2, and that they are crucial for the success of y in $N^1 + N^2$—all of which gives them a strong motive to promote deadlock in the joint ballot.

voters can have different intentions about how many votes to cast, and since a permutation of D_i does not necessarily permute i's intentions about the number of votes to be cast, the choice from D will not necessarily remain constant under permutations of D_i. For example, in Display 4-20, for $X = (a, b, c, d)$ and $n = 4$, voter 1 casts three votes in both D and D^σ and voters 2, 3, and 4 cast two votes in both profiles. Then candidate c wins in D and candidate d wins in D^σ in direct violation of the condition of undifferentiatedness.

Display 4-19

The Failure of the Borda Count to Satisfy Independence from Irrelevant Alternatives

D'	
D'_1:	$a\ b\ c$
D'_2:	$c\ a\ b$
D'_3:	$c\ a\ b$

Number of Points for Alternatives in Rows, Preference Orders in Columns

	D'_1	D'_2	D'_3	Total
a	2	1	1	4 (Tied with c)
b	1	0	0	1
c	0	2	2	4 (Tied with a)

D	
D_1:	$a\ b\ c$
D_2:	$c\ b\ a$
D_3:	$c\ a\ b$

Number of Points for Alternatives in Rows, Preference Orders in Columns

	D_1	D_2	D_3	Total
a	2	0	1	3
b	1	1	0	2
c	0	2	2	4 (Borda winner)

In D as against D' member 2 has revised the relation of a and b but has not changed the relation of a and c; yet the social choice between a and c has changed in D from D'.

Display 4-20

The Failure of Approval Voting to Satisfy Undifferentiatedness

D	
D_1:	$a\ b\ c\ d$
D_2:	$b\ c\ d\ a$
D_3:	$c\ d\ a\ b$
D_4:	$d\ a\ b\ c$

Assume voter 1 casts three votes and voters 2, 3, and 4 each cast two votes:

Number of Votes for Candidates

Candidates	First place	Second place	Third place	Total
a	1	1	0	2
b	1	1	0	2
c	1	1	1	3 (Approval winner)
d	1	1	0	2

Assume now that persons 1 and 2 trade preference orders in the following permutation, where $D_{\sigma(i)}$ means "the preference order that replaces D_i":

$$D_{\sigma(1)} = D_2$$
$$D_{\sigma(2)} = D_1$$
$$D_{\sigma(3)} = D_3$$
$$D_{\sigma(4)} = D_4$$

This permutation produces a new profile, D^{σ}.

Display 4-20 *(Continued)*

D^{σ}	
$D_{\sigma(1)}$:	$b\ c\ d\ a$
$D_{\sigma(2)}$:	$a\ b\ c\ d$
$D_{\sigma(3)}$:	$c\ d\ a\ b$
$D_{\sigma(4)}$:	$d\ a\ b\ c$

Assume, as in D, that voter 1 casts three votes and voters 2, 3, and 4 each cast two votes:

Number of Votes for Candidates

Candidates	First place	Second place	Third place	Total
a	1	1	0	2
b	1	1	0	2
c	1	1	0	2
d	1	1	1	3 (Approval winner)

Hence $F_{\text{approval}}(X, D) = c$; and $F_{\text{approval}}(X, D^{\sigma}) = d$.

The practical effect of this violation is that the indifferent voters or the voters with a broad tolerance for positions dominate in the choice. As is well known, simple majority decision—and even plurality voting insofar as it contributes to simple majority—places a premium on the median voter in a distribution. One can be at or near the median either (1) because of reasoned judgment or (2) because one does not care. Whether one wants to exaggerate the influence of type 2 voters in social decision seems to me doubtful.

Violation of Independence by Utilitarian Methods

That utilitarian voting violates the independence criterion is immediately apparent from the example in Display 4-21 of two voters and three alternatives. Although D and D' are ordinally identical, still by a utilitarian F x wins in D and y in D'. The moral question derived from this fact is whether a change in cardinal numbers without a change in ordinal relations should make a difference in choice. The utilitarian argument is that the change between D and D' involves a true change in the intensity of valuation. From D to D', voter 1 upgraded y and voter 2 downgraded x. Surely, the argument runs, y should now replace x as the winner.

The counterargument is that, since the cardinal sums are not soundly based and ought not therefore to affect the outcome, x and y should tie in both D and D'. The doubt about the soundness of the cardinal sums arises from the way they were initially constructed. A scale was created for an individual person, and alternatives were located on it by individual comparison. Hence, the numbers created are uniquely personal. They are useful to study behavior by the subject and even useful, perhaps, for the subject to guide his or her own behavior. But these personal numbers cannot be compared between people because they do not mean the same thing to each person. The sums are therefore meaningless. If one adds oranges and apples to get oranapps, one knows nothing more because one doesn't know what an oranapp is. Neither does one know what the sum of utilities is.

Of course, one may insist that the cardinal relations be fixed so long as the ordinal relations are, but then one loses the main advantage of cardinal utility—namely, the reflection of intensity of preference. If, however, one allows cardinal relations to reflect intensity, then the independence criterion may be violated, whenever, as in Display 4-21, ordinal relations remain constant.[38]

4.J. The Absence of True and Fair Amalgamations

In the beginning of this chapter I promised to show that we cannot prove that any method of voting truly and fairly amalgamates the judgments of voters. I think I have done so. I have shown that all members of each of the three main categories of methods violate at least one reasonable criterion of fairness or consistency. Furthermore, I have shown that many methods violate several criteria. Had space permitted, I would have introduced many additional criteria, some of which are violated by every category of voting methods. To assert, therefore, that one method truly and fairly amalgamates, one must show that the criterion or criteria it violates is or are unreasonable or trivial. But there is a good rationale in terms of fairness or consistency or both for every one of the criteria I have discussed as well as for many I have omitted. So it seems unlikely that

Display 4-21

The Failure of Utilitarian Methods to Satisfy Independence from Irrelevant Alternatives

D	
D_1:	$x\ y\ z$
D_2:	$y\ x\ z$

For Profile *D*:
Utility of the Alternative in the Row; Utility to the Voter in the Column

	U_1	U_2	Total
x	1.0	0.6	1.6 (Utilitarian winner)
y	0.5	1.0	1.5
z	0	0	0

D′	
D'_1:	$x\ y\ z$
D'_2:	$y\ x\ z$

For Profile *D′*:
Utility of the Alternative in the Row; Utility to the Voter in the Column

D′

	U_1	U_2	Total
x	1.0	0.5	1.5
y	0.6	1.0	1.6 (Utilitarian winner)
z	0	0	0

Although $D = D'$, the winner in D is x, and the winner in D' is y.

these criteria can be rejected as morally or logically irrelevant. If not, then we are inexorably forced to the conclusion that every method of voting is in some applications unfair or inadequate.

For my own taste, I would use different methods in different circumstances. In legislatures, I would use the amendment procedure on three alternatives and, somewhat hesitantly, the Borda count or Kemeny's method for more than three alternatives. In elections of executives or legislators, I would use the plurality method, more for its dynamic effects of maintaining the two-party system than for its effectiveness of choice in particular elections. In primary elections, I would use approval voting—though I would avoid it absolutely in general elections, where its effect would be to destroy the two-party system. In planning an economy, if that activity were forced upon me, I would use a kind of utilitarianism—namely, demand-revealing procedures—recognizing, of course, that however "honest" they are supposed to be, it is easy enough to manipulate them.

There is some niche, it seems to me, for majoritarian, positional, and utilitarian methods of voting. Nevertheless, we have now learned, I think, that we should never take the results of any method always to be a fair and true amalgamation of voters' judgments. Doubtless the results often are fair and true; but, unfortunately, we almost never know whether they are or are not. Consequently, we should not generally assume that the methods produce fair and true amalgamations. We should think of the methods, I believe, simply as convenient ways of doing business, useful but flawed. This gives them all a place in the world, but it makes none of them sacrosanct.

5

The Meaning of Social Choices

In Chapter 4 I showed that no method of voting could be said to amalgamate individual judgments truly and fairly because every method violates some reasonable canon of fairness and accuracy. All voting methods are therefore in some sense morally imperfect. Furthermore, these imperfect methods can produce different outcomes from the same profile of individual judgments. Hence it follows that sometimes—and usually we never know for sure just when—the social choice is as much an artifact of morally imperfect methods as it is of what people truly want. It is hard to have unbounded confidence in the *justice* of such results.

It is equally hard, as I will show in this chapter, to have unbounded confidence in the *meaning* of such results. Individual persons presumably can, if they think about it deeply enough, order their personal judgments transitively. Hence their valuations mean something, for they clearly indicate a hierarchy of preference that can guide action and choice in a sensible way. But the results of voting do not necessarily have this quality. It is instead the case that *no* method of voting can simultaneously satisfy several elementary conditions of fairness and also produce results that always satisfy elementary conditions of logical arrangement. Hence, not only may the results of voting fail to be fair, they may also fail to make sense. It is the latter possibility that will be analyzed in this chapter.

5.A. Arrow's Theorem

Kenneth Arrow published *Social Choice and Individual Values* in 1951. Although his theorem initially provoked some controversy among economists, its profound political significance was not immediately recognized by political scientists.[1] In the late 1960s, however, a wide variety of philosophers, economists, and political scientists began to appreciate how profoundly unsettling the theorem was and how deeply it called into question some conventionally accepted notions—not only about voting, the subject of this work, but also about the ontological validity of the concept of social welfare, a subject that, fortunately, we can leave to metaphysicians.

The essence of Arrow's theorem is that no method of amalgamating individual judgments can simultaneously satisfy some reasonable conditions of fairness on the method and a condition of logicality on the result. In a sense this theorem is a generalization of the paradox of voting (see section 1.H), for the theorem is the proposition that something like the paradox is possible in *any* fair system of amalgamating values. Thus the theorem is called the *General Possibility Theorem.*

To make the full meaning of Arrow's theorem clear, I will outline the situation and the conditions of fairness and of logicality that cannot simultaneously be satisfied.[2] The situation for amalgamation is:

1. There are n persons, $n \geq 2$, and n is finite. Difficulties comparable to the paradox of voting can arise in individuals who use several standards of judgment for choice. Our concern is, however, *social* choice, so we can ignore the Robinson Crusoe case.

2. There are three or more alternatives—that is, for the set $X = (x_1, \ldots, x_m)$, $m \geq 3$. Since transitivity or other conditions for logical choice are meaningless for fewer than three alternatives and since, indeed, simple majority decision produces a logical result on two alternatives, the conflict between fairness and logicality can only arise when $m \geq 3$.

3. Individuals are able to order the alternatives transitively: If $x\, R_i\, y$ and $y\, R_i\, z$, then $x\, R_i\, z$. If it is not assumed that individuals are able to be logical, then surely it is pointless to expect a group to produce logical results.

The conditions of fairness are:

1. *Universal admissibility of individual orderings (Condition U).* This is the requirement that the set, **D**, includes all possible profiles, D, of individual orders, D_i. If each D_i is some permutation of possible orderings of X by preference and indifference, then this requirement is that individuals can choose any of the possible permutations. For example, if $X = (x, y, z)$, the individual may choose any of the following 13 orderings:

1. $x\,y\,z$	7. $x\,(y\,z)$	10. $(x\,y)\,z$	13. $(x\,y\,z)$
2. $y\,z\,x$	8. $y\,(z\,x)$	11. $(y\,z)\,x$	
3. $z\,x\,y$	9. $z\,(x\,y)$	12. $(z\,x)\,y$	
4. $x\,z\,y$			
5. $z\,y\,x$			
6. $y\,x\,z$			(5-1)

The justification for this requirement is straightforward. If social outcomes are to be based exclusively on individual judgments—as seems implicit in any interpretation of democratic methods—then to restrict individual persons' judgments in any way means that the social outcome is based as much on the restriction as it is on individual judgments. Any rule or command that prohibits a person from choosing some preference order is morally unacceptable (or at least unfair) from the point of view of democracy.

2. *Monotonicity.* According to this condition, if a person raises the valuation of a winning alternative, it cannot become a loser; or, if a person lowers the valuation of a losing alternative, it cannot become a winner. The justification for monotonicity was discussed in section 3.B. Given the democratic intention that outcomes be based in some way on participation, it would be the utmost in perversity if the method of choice were to count individual judgments *negatively,* although, as I have shown, some real-world methods actually do so.

3. *Citizens' sovereignty or nonimposition.* Define a social choice as imposed if some alternative, x, is a winner for any set, D, of individual preferences. If x is always chosen, then what individuals want does not have anything to do with social choice. It might, for example, happen that x was everyone's least-liked alternative, yet an imposed choice of x would still select x. In such a situation, voters' judgments have nothing to do with the outcome and democratic participation is meaningless.

4. *Unanimity or Pareto optimality (Condition P).* This is the requirement that, if everyone prefers x to y, then the social choice function, F, does not choose y. (See Chapter 3, note 8, and Chapter 4, note 28.) This is the form in which monotonicity and citizens' sovereignty enter all proofs of Arrow's theorem. There are only two ways that a result contrary to unanimity could occur. One is that the system of amalgamation is not monotonic. Suppose in D' everybody but i prefers x to y

and $y P_i' x$. Then in D, i changes to $x P_i y$ so everybody has x preferred to y; but, if F is not monotonic, it may be that x does not belong to $F(\{x, y\}, D)$. The other way a violation of unanimity could occur is for F to impose y even though everybody prefers x to y. Thus the juncture of monotonicity and citizens' sovereignty implies Pareto optimality.

Many writers have interpreted the unanimity condition as purely technical—as, for example, in the discussion of the Schwartz method of completing the Condorcet rule (see section 4.C). But Pareto optimality takes on more force when it is recognized as the carrier of monotonicity and nonimposition, both of which have deep and obvious qualities of fairness.

5. *Independence from irrelevant alternatives (Condition I).* According to this requirement (defined in section 4.H), a method of amalgamation, F, picks the same alternative as the social choice every time F is applied to the same profile, D. Although some writers have regarded this condition simply as a requirement of technical efficiency, it actually has as much moral content as the other fairness conditions (see section 4.H). From the democratic point of view, one wants to base the outcome on the voters' judgments, but doing so is clearly impossible if the method of amalgamation gives different results from identical profiles. This might occur, for example, if choices among alternatives were made by some chance device. Then it is the device, not voters' judgments in D, that determines outcomes. Even if one constructs the device so that the chance of selecting an alternative is proportional in some way to the number of people desiring it (if, for example, two-thirds of the voters prefer x to y, then the device selects x with $p = 2/3$), still the expectation is that, of several chance selections, the device will choose x on p selections and y on $1 - p$ selections from the same profile, in clear violation of Condition I. In ancient Greece, election by lot was a useful method for anonymity; today it would be simply a way to by-pass voters' preferences. Another kind of arbitrariness prohibited by the independence condition is utilitarian voting. Based on interpersonal comparisons of distances on scales of unknown length, utilitarian voting gives advantages to persons with finer perception and broader horizons. Furthermore, independence prohibits the arbitrariness of the Borda count (see section 5.F).
6. *Nondictatorship (Condition D).* This is the requirement that there be no person, i, such that, whenever $x P_i y$, the social choice is x, regardless of the opinions of other persons. Since the whole idea of democracy is to avoid such situations, the moral significance of this condition is obvious.

Finally, the condition of logicality is that the social choice is a weak order, by which is meant that the set, X, is connected and its members can be *socially* ordered by the relation, R, which is the transitive social analogue of preference and indifference combined. (This relation, as in $x R y$, means that x is chosen over or at least tied with y.) In contrast to the previous discussion, in which the method of amalgamation or choice, F, simply selected an element from X, it is now assumed that F selects repeatedly from pairs in X to produce, by means of successive selections, a social order analogous to the individual orders, D_i. And it is the failure to produce such an order that constitutes a violation of the condition of logicality.[2]

Since an individual weak order or the relation R_i is often spoken of as individual rationality, social transitivity, or R, is sometimes spoken of as collective rationality—Arrow himself so described it. And failure to produce social transitivity can also be regarded as a kind of social irrationality.

Arrow's theorem, then, is that every possible method of amalgamation or choice that satisfies the fairness conditions fails to ensure a social ordering. And if society cannot, with fair methods, be certain to order its outcome, then it is not clear that we can know what the outcomes of a fair method mean. This conclusion appears to be devastating, for it consigns democratic outcomes—and hence the democratic method—to the world of arbitrary nonsense, at least some of the time.

Naturally there has been a variety of attempts to interpret and sidestep this conclusion. One line of inquiry is to raise doubts about its practical importance; another is to look for some theoretical adjustment that deprives the theorem of its force. The rest of this chapter is devoted to a survey of both branches of this huge and important literature, so that in Chapter 6 it will be possible to assess fully the political significance of Arrow's theorem.

I will begin with inquiries about the practical importance of the theorem. One such inquiry is an estimate of the expected frequency of profiles, D, that do not lead to a transitive order.

5.B. The Practical Relevance of Arrow's Theorem: The Frequency of Cycles

One meaning of Arrow's theorem is that, under any system of voting or amalgamation, instances of intransitive or cyclical outcomes can occur.

Since, by definition, no one of the alternatives in a cycle can beat all the others, there is no Condorcet winner among cycled alternatives. All cycled alternatives tie with respect to their position in a social arrangement in the sense that $x\,y\,z\,x$, $y\,z\,x\,y$, and $z\,x\,y\,z$ have equal claims to being the social arrangement. Borda voting similarly produces a direct tie among cycled alternatives. Hence a social arrangement is indeterminate when a cycle exists. When the arrangement is indeterminate, the actual choice is arbitrarily made. The selection is not determined by the preference of the voters. Rather it is determined by the power of some chooser to dominate the choice or to manipulate the process to his or her advantage. Every cycle thus represents the failure of the voting process. One way to inquire into the practical significance of Arrow's theorem is, therefore, to estimate how often cycles can occur.

For this estimate, a number of simplifying assumptions are necessary. For one thing, majority voting (rather than positional voting or any other kind of amalgamation) is always assumed. This assumption of course limits the interpretation severely. For another thing, only cycles that preclude a Condorcet winner are of interest. Voting may fail to produce a weak order in several ways:

1. With all three alternatives, there may be a cycle: $x\,R\,y\,R\,z\,R\,x$ or simply $x\,y\,z\,x$.
2. With four or more alternatives, there may be
   a. A Condorcet winner followed by a cycle: $w\,x\,y\,z\,x$
   b. A cycle among all alternatives: $w\,x\,y\,z\,w$; or intersecting cycles: $s\,t\,w\,x\,y\,z\,w\,v\,s$
   c. A cycle in which all members beat some other alternative: $x\,y\,z\,x\,w$

If one is interested in social welfare judgments involving an ordering of all alternatives, then all cycles are significant no matter where they occur. But if one is interested in picking out a social choice, as in the voting mechanisms discussed here, then the significant cases are only 1, 2(b), and 2(c), where there is no unique social choice. (These are often called *top cycles.*) Attempts to estimate the significance of Arrow's theorem by some sort of calculation have all been made from the point of view of social choice rather than welfare judgments and have therefore concerned the frequency of top cycles.

For Arrow's theorem, Condition U allows individuals to have any weak ordering, R_i, of preference and indifference, as in (5.1). Calculation is simpler, however, based on strong orders—that is, individual preference orders, P_i, with indifference not allowed.

With m alternatives, there are $m!$ (i.e., $1 \cdot 2 \cdot \ldots \cdot m$) such linear orders possible; and, when $m = 3$, these are:

$$x\,y\,z, \quad x\,z\,y, \quad y\,x\,z, \quad y\,z\,x, \quad z\,x\,y, \quad z\,y\,x$$

Each such order is a potential D_i. When each of n voters picks some (not necessarily different) D_i, a profile, D, is created. Since the first voter picks from $m!$ orders, the second from $m!, \ldots,$ and the last from $m!$, the number of possible different profiles, D, is $(m!)^n$, which is the number of members of the set, $\mathbf{D}$, of all profiles, when voters have only strong orders.

A calculation that yields some estimate of the significance of cycles is the fraction, $p(n, m)$, of D in $\mathbf{D}$ without a Condorcet winner:

$$p(n, m) = \frac{\text{Number of } D \text{ without a Condorcet winner}}{(m!)^n}$$

If one assumes that each D is equally likely to occur (which implies also that, for each voter, the chance of picking some order is $1/m!$), then $p(n, m)$ is an a priori estimate of the probability of the occurrence of a top cycle. Several calculations have been made, as set forth in Display 5-1.[3] As is apparent from the Display, as the number of voters and alternatives increases, so do the number of profiles without a Condorcet winner. The calculation thereby implies that instances of the paradox of voting are very common. Most social choices are made from many alternatives (though often we do not realize this fact because the number has been winnowed down by various devices such as primary elections and committees that select alternatives for agendas) and by many people, so the calculations imply that Condorcet winners do not exist in almost all decisions.

But, of course, there are a number of reasons to believe that such calculations are meaningless. People do not choose an ordering with probability $1/m!$. Rather, at any particular moment, some orders are more likely to be chosen than others. The six strong orders over triples generate two cycles:

"Forward Cycle"	"Backward Cycle"	
1. $x\,y\,z$	4. $x\,z\,y$	
2. $y\,z\,x$	5. $z\,y\,x$	
3. $z\,x\,y$	6. $y\,x\,z$	(5-2)

Display 5-1

Values of $p(n, m)$: Proportion of Possible Profiles Without a Condorcet Winner

m = Number of Alternatives	n = Number of Voters						
	3	5	7	9	11	...	Limit
3	.056	.069	.075	.078	.080		.088
4	.111	.139	.150	.156	.160		.176
5	.160	.200	.215				.251
6	.202						.315
Limit	1.000	1.000	1.000	1.000	1.000		1.000

The entry in the row for four alternatives and in the column for seven voters—namely, .150—is the ratio of the number of profiles without a Condorcet winner to the number of profiles possible when seven voters order four alternatives.

Cycles occur when voters concentrate on one or the other of these sets of three orders. But suppose voters are induced by, for example, political parties, to concentrate heavily on, say, (1), (2), and (5). Then there is no cycle. Furthermore, there is good reason to believe that debate and discussion do lead to such fundamental similarities of judgment. Calculations based on equiprobable choices very likely seriously overestimate the frequency of cycles in the natural world.

On the other hand, it is clear that one way to manipulate outcomes is to generate a cycle. Suppose that in Display 5-2 profile D exists and that person 2 realizes that his or her first choice, y, will lose to the Condorcet winner, x. Person 2 can at least prevent that outcome by generating a cycle (or a tie) by voting as if his or her preference were $y\ z\ x$ as in D'.

The tendency toward similarity may thus reduce the number $p(n, m)$, while the possibility of manipulation may increase the number. It seems to me that similarity probably reduces the number of profiles without Condorcet winners on issues that are not very important and that no one has a motive to manipulate, while the possibility of manipulation

Display 5-2

The Generation of a Cycle

D		D'	
D_1:	$x\ y\ z$	D'_1:	$x\ y\ z$
D_2:	$y\ x\ z$	D'_2:	$y\ z\ x$
D_3:	$z\ x\ y$	D'_3:	$z\ x\ y$

Note. Majoritarian ordering of D: $x\ P\ y\ P\ z$.

Note. Cycle in D' under majoritarian voting: $x\ P\ y\ P\ z\ P\ x$.

In D' person 2 has reversed z and x from D, thereby generating a cycle.

increases the number of such profiles on important issues, where the outcome is worth the time and effort of prospective losers to generate a top cycle. Neither of these influences appears in the calculations and thus renders them suspect from two opposite points of view.

5.C. The Practical Relevance of Arrow's Theorem: Conditions for Condorcet Winners

Another approach to estimating the practical significance of Arrow's theorem is to inquire into what kinds of profiles are certain to produce a Condorcet winner. As in the previous approach, only majoritarian voting is considered, which limits the relevance of the inquiry to the theorem but does say something about its practical effect on this kind of decision process. For example, as can be seen in Display 5-1, for $m = n = 3$, the number of elements of $\boldsymbol{D} = (m!)^n = 216$ and $p(n, m) = 12/216 = .056$. It is natural to look for the features that guarantee a Condorcet winner for 204 of the profiles in $\boldsymbol{D}$. If one can generalize about the sets of preference orders that produce these results, then it may be possible to estimate the practical significance of the theorem for majoritarian voting.

To give a simple example: If each voter chooses the same preference order, D_i, then under majoritarian rules the social order for the profile D will be identical with the chosen D_i, and the unique social choice will be the first alternative in that social order.

The goal of this approach is to identify kinds of preference orders, D_i, such that when the whole profile, D, is composed of such orders, then D will lead by majoritarian methods to a weak order and a Condorcet winner as a social outcome.

Even before Arrow's theorem was uttered, Duncan Black observed one such pattern of orders in D—namely, that the profile can be expressible as a set of single-peaked curves.[4] A preference order can be graphed as in Figure 5-1. On the vertical axis is measured the degree of preference from lowest at the origin to highest at the top. On the horizontal axis is placed some ordering of the alternatives in X, an ordering appropriately chosen to depict one particular D_i as a single-peaked curve. This is always possible if D_i is a strong order (with indifference not allowed). The general definition of single-peaked curves (with indifference permitted at the top) is, as displayed in Figure 5-2, reading from left to right: (1) always downward sloping, (2) always upward sloping, (3) sloping upward to a particular point and then sloping downward, (4) sloping upward to a plateau and then sloping downward, (5) horizontal and then downward sloping, (6) upward sloping and then horizontal. Curves that are *not* single-peaked are shown in Figure 5-3.

A profile, D, is single-peaked if some ordering of alternatives on the horizontal axis allows every D_i in D to be drawn as a single-peaked curve. As already observed, for a single D_i, it is always possible to find such an ordering. But with three or more D_i, an ordering that renders D_j single-peaked may preclude that D_k be single-peaked. Indeed, it is exactly when cycles exist that single-peakedness cannot be attained for D. In Figure 5-4 assume there are three persons who have chosen different preference orders in the forward cycle (5-2). Then *all* possible orderings of $X = (x, y, z)$ on the horizontal axis result in a set of curves that fail to be single-peaked, as in Figure 5-4a–4f, where the axes are *all* the $m!$ permutations of $\{x, y, z\}$. The same is true of the backward cycle. So to say a profile, D, is single-peaked is to say it does not admit of cycles. In general, if D is single-peaked, then:

1. If all D_i are strong orders and n is odd, the social ordering is strong.
2. If all D_i are weak orders, n is odd, and no D_i involves complete indifference over a triple, the social ordering is a weak order.[5]

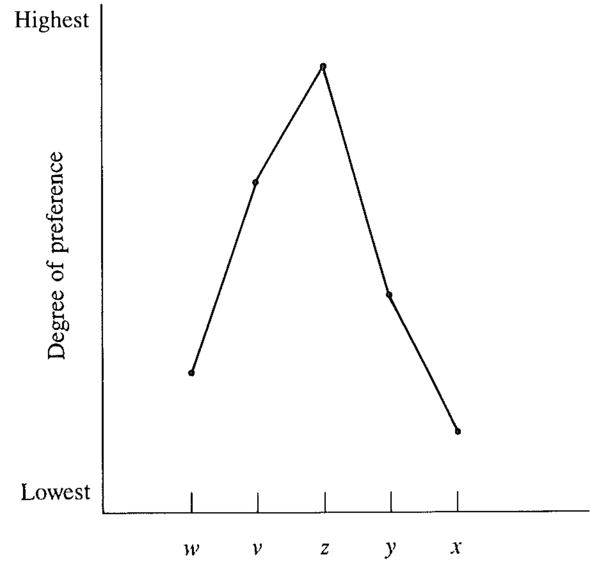

Figure 5-1 A single-peaked curve with the linear order $z\ v\ y\ w\ x$.

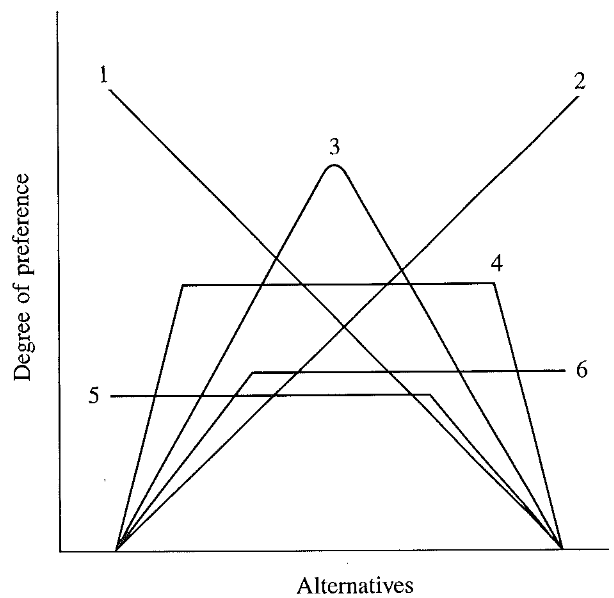

Figure 5-2 Single-peaked curves.

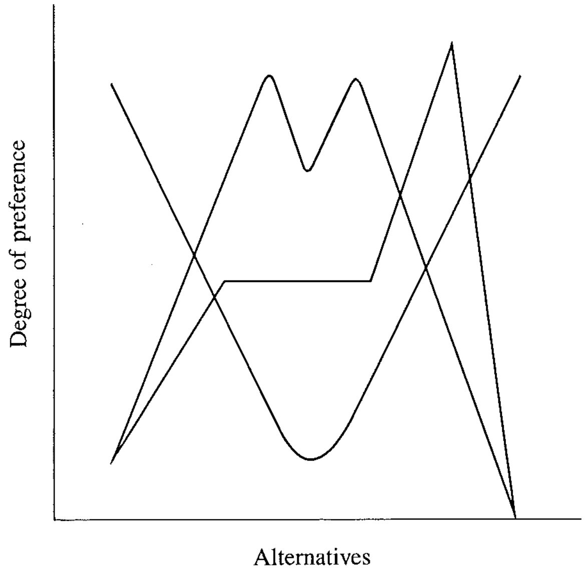

Figure 5-3 Non-single-peaked curves.

So single-peakedness implies transitivity and hence ensures the existence of a Condorcet winner.

It is furthermore a remarkable fact that, if D is single-peaked and n is odd, the Condorcet winner is immediately identifiable as the alternative on the horizontal axis beneath the median peak.[6] (If n is even, the winner is some alternative between the $n/2^{\text{th}}$ peak and the $(n/2) + 1^{\text{th}}$ peak, if such an alternative exists. And, if none exists, the alternatives at these peaks tie.) In Figure 5-5, with five peaks, the alternative beneath the median peak (3) is identified as x_{med}. If x_{med} is put against some alternative to its left, say x_1, then x_{med} wins because a majority consisting of voters 3, 4, and 5 prefer x_{med} to x_1 (that is, their curves are upward sloping from x_1 to x_{med}). Similarly, x_{med} can beat any alternative to its right, say x_4, with a majority consisting of voters 1, 2, and 3, whose curves are downward sloping from x_{med} to x_4, which means they prefer x_{med} to x_4. Hence x_{med} can beat anything to its right or left and is a Condorcet winner.

Single-peakedness is important because it has an obvious political interpretation. Assuming a single political dimension, the fact that a profile, D, is single-peaked means the voters have a common view of the political situation, although they may differ widely on their judgments. Person i may choose $D_i = x\,y\,z$, and person j may choose $D_j = z\,y\,x$; yet

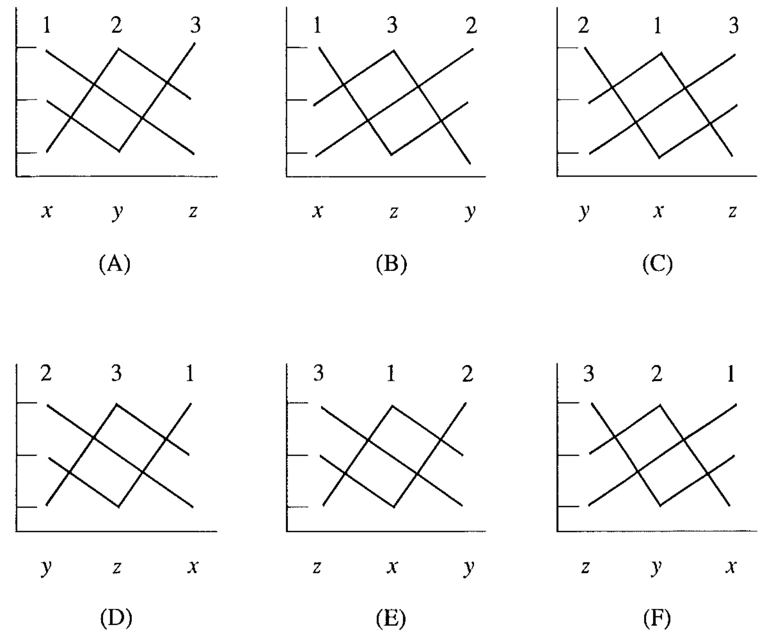

Figure 5-4 Non-single-peakedness for the forward cycle.

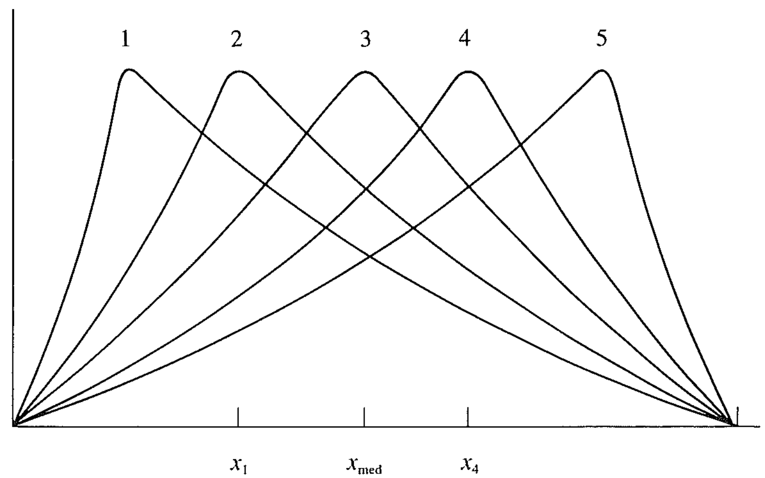

Figure 5-5 Single-peaked curves with Condorcet winner.

they agree that x is at one end of the scale, z at the other, and y in the middle, which means they agree entirely on how the political spectrum is arranged. This kind of agreement is precisely what is lacking in a cycle, where voters disagree not only about the merits of alternatives but even about where alternatives are on the political dimension.

If, by reason of discussion, debate, civic education, and political socialization, voters have a common view of the political dimension (as evidenced by single-peakedness), then a transitive outcome is guaranteed. So if a society is homogeneous in this sense, there will typically be Condorcet winners, at least on issues of *minor* importance. This fact will not prevent civil war, but it will at least ensure that the civil war makes sense.

A number of other kinds of restrictions on preference orders, D_i, that guarantee that D will produce a transitive outcome have been identified. Like single-peakedness they minimize disagreement over the dimensions of judgment. Consider "value-restrictedness," which is an obvious development from the forward and backward cycles of (5-2). One property of those cycles (observable by inspection) is that each alternative in X appears in first place in some D_i, in second place in another, and in third place in a third. So, if, for strong orders in D_i, some alternative is never first in a D_i, or never second, or never last—if, in short, an alternative is "value-restricted"—then no cycle can occur and transitivity is guaranteed.

A number of other such provisions for transitivity have been identified. They have been exhaustively analyzed by Peter Fishburn.[7] They are important because they indicate that quite a wide variety of rather mild agreement about the issue dimension guarantees a Condorcet winner. Furthermore, not all voters need display the agreement to obtain the guarantee. Richard Niemi has shown that the probabilities of the occurrence of top cycles, by calculations similar to those set forth in Display 5-1, reduce to tiny proportions (e.g., .02 to .04) when as few as three-fourths of 45 or 95 voters agree on the issue dimension while disagreeing on orders.[8] This result implies that agreement about dimensions probably renders uncontrived cyclical outcomes quite rare. So I conclude that, because of agreement on an issue dimension, intransitivities only occasionally render decisions by majoritarian methods meaningless, at least for somewhat homogeneous groups and at least when the subjects for decision are *not* politically important. When, on the other hand, subjects are politically important enough to justify the energy and expense of contriving cycles, Arrow's result is of great practical significance. It suggests that, on the very most important subjects, cycles may render social outcomes meaningless.

5.D. The Theoretical Invulnerability of Arrow's Theorem: Independence

Assuming that the practical significance of Arrow's theorem increases with the political importance of the subject for decision, it is then reasonable to inquire whether the theorem is too demanding. Does it overstate the case by stressing the possibility of intransitivity and its consequent incoherence when perhaps this is too extreme an interpretation?

To weaken the force of Arrow's theorem, it is necessary to question the conditions of either fairness or logicality. Most of the fairness conditions seem intuitively reasonable—at least to people in Western culture—so most of the attack has been focused on logicality. One fairness condition, independence, has, however, often been regarded as too strong.

The independence condition has at least three consequences:

1. It prohibits utilitarian methods of choice (for reasons discussed in section 4.I).
2. It prohibits arbitrariness in vote-counting, such as lotteries or methods that work in different ways at different times.
3. It prohibits, when choosing among alternatives in a set S, which is included in X, reference to judgments on alternatives in $X - S$.

It seems to me that one can defend the independence condition for each of these consequences. As for consequence 1, since interpersonal comparisons of utility have no clear meaning, the prohibition of utilitarian methods seems quite defensible, although a weaker form of Condition I might accomplish the same result. With respect to consequence 2, earlier in this chapter it was shown that arbitrary counting is just as unfair as violations of Conditions U, P, and D. It is difficult to imagine that any weaker form of Condition I would accomplish what I does, because the arbitrariness must be prohibited for any set.

Most attention has been given to consequence 3, because many people believe that judgments on alternatives in $X - S$ are germane to judgments on S itself.[9] In a presidential preference primary, for example, choice among several candidates may depend on judgments of still other candidates. For example, in the 1976 Democratic primaries, in thinking about a decision between Carter and Udall as if they covered the whole spectrum of party ideology, a mildly left-of-center voter might prefer Udall. But if the voter thought about Jackson also, so that Udall appeared as an extremist, that same voter might have preferred Carter to Udall. So "irrelevant" alternatives (here, Jackson) may really be "relevant."

The question is whether there should be some formal way to allow judgments on the "irrelevant" alternatives to enter into the choice. And the difficulty in answering is: How can one decide which nonentered candidates are relevant? Why not allow consideration of still other, even a hundred, irrelevant alternatives? But if no irrelevant alternatives are considered, then y might beat x; but with such consideration, if there is no Condorcet winner, x might beat y. Thus meaning and coherence depend on variability in the voting situations (on the size, that is, of X and S) as much as on voters' judgment.

There seems, unfortunately, no wholly defensible method to decide on degrees of irrelevance.[10] In the absence of such a method, Condition I seems at least moderately defensible. Furthermore, while some might argue about the desirability of consequence 3, Condition I seems necessary because consequence 2 is indispensable for fair decision.

5.E. The Theoretical Invulnerability of Arrow's Theorem: Transitivity

If the fairness conditions survive, then the only condition left to attack is transitivity. The sharpest attack is to assert that transitivity is a property of humans, not of groups. Hence the individual relation, R_i, should be transitive, but it is simple anthropomorphism to ask that the social relation, R, be transitive also.[11] Still, there is some reason to seek transitivity for outcomes.

Without transitivity, there is no order; and without order, there is no coherence. Social outcomes may in fact be meaningless, but one would like to obtain as much meaning as possible from social decisions. So the obvious question is: Can one, by modifying the definition of coherence, obtain some lesser coherence compatible with fairness? Unfortunately, the answer is mainly negative.

The social relation, R, which generates a weak order in Arrow's logicality condition, combines social preference, P, and social indifference, I. And R is useful for the purpose Arrow had in mind—namely, social judgments involving comparisons and ordering of all feasible social policies, such as distributions of income. Suppose, however, that one does not require quite so general a result. For purposes of making a social choice, which is the interest in this book, one does not need to impose a complete order on the whole set X merely to find a best alternative in X.

We can think of a best element in X as one that is chosen over or tied with every other alternative.[12] The best alternative is then the choice from X or $C(X)$.[13]

A requirement, weaker than transitivity, that nevertheless ensures the existence of one best alternative is *quasi-transitivity*—that is, the transitivity of P, but not of R or I. This means that, if $x\,P\,y$ and $y\,P\,z$, then $x\,P\,z$; but if the antecedent does not hold (e.g., if $x\,I\,y$), then the consequent need not hold either. For example, quasi-transitivity allows (as in note 12) $y\,P\,z$, $z\,I\,x$, and $x\,I\,y$, which is clearly intransitive in both R and I, although it is enough to establish that the choice from $X = (x, y, z)$ is $C(X) = (x, y)$.

Another, even weaker requirement for a choice, is *acyclicity,* which is the requirement that alternatives in X can be arranged so that there is no cycle.[14] It turns out that, by using the logical requirement of acyclicity rather than transitivity, it is possible to find social choice that satisfies all of Arrow's fairness conditions as well as the revised condition of logicality. A. K. Sen offers an example of such a method: For a set $X = (a, b, \ldots,)$, let a be chosen for $C(X)$ over b if everybody prefers a to b and let a and b both be chosen if not everybody prefers a to b or b to a.[15] This rule satisfies Condition U because all individual orders are allowed. It satisfies Condition P because it is based on the principle of unanimity. It satisfies Condition I because the choice between any pair depends only on individual preferences on that pair, and it satisfies Condition D because the only way a can be better socially than b is for everyone to prefer a to b. Finally, it is always acyclic. So even if one cannot guarantee an order with fair procedure, it appears that one can at least guarantee a best choice.

Unfortunately, however, something very much like dictatorship is required to guarantee quasi-transitivity or acyclicity. Quasi-transitive social outcomes can be guaranteed only if there is an oligarchy.[16] (An *oligarchy* is a subset of choosers who, if the members agree, can impose a choice, or, if they do not agree, enables all members individually to veto the choice.) If one modifies Condition D from no dictator to no vetoer, then even a quasi-transitive social outcome cannot be guaranteed.[17] As for acyclicity, Donald Brown has shown that acyclic choice requires a "collegium" such that alternative a is chosen over b if and only if the whole collegium and some other persons prefer a to b. Thus, although a collegium cannot unilaterally impose a choice, unlike an oligarchy it can always at least veto.[18]

Furthermore, if one strengthens Arrow's conditions just a little bit by requiring not just the monotonicity that enters into Condition P, but a condition of positive responsiveness (Condition PR), then quasi-transitivity again involves dictatorship. (Monotonicity requires merely that, if a voter raises her or his valuation of an alternative, the social valuation does not go down. In contrast, positive responsiveness requires that, if a voter raises her or his valuation, society does so as well, if that is possible.) It is then the case that any quasi-transitive social result that satisfies Conditions U, P, I, and PR must violate Condition D if there are three alternatives; and, furthermore, someone must have a veto if there are four or more alternatives.[19]

Weakening transitivity into some logical condition that requires only a social choice but not a full ordering does not gain very much. This brief survey indicates there is a *family* of possibility theorems of which Arrow's theorem is a special case. And in the whole family there is still some kind of serious conflict between conditions of fairness and a condition of logicality. In general, the only effective way to guarantee consistency in social outcomes is to require some kind of concentration of power in society—a dictator, an oligarchy, or a collegium. So fairness and social rationality seem jointly impossible, which implies that fairness and meaning in the content of social decisions are sometimes incompatible.

5.F. The Theoretical Invulnerability of Arrow's Theorem: Conditions on Social Choice

Of course, one can abandon entirely the effort to guarantee some kind of ordering for social "rationality," whether it be transitivity or merely acyclicity. One can simply provide that a social choice is made and impose no kind of ordering condition. The reason, however, that transitivity or even less restrictive ordering conditions are attractive is that they often forestall manipulation by some participants either of agenda or of sets of alternatives to obtain outcomes advantageous to the manipulator. As Arrow remarked at the conclusion of the revised edition of *Social Choice and Individual Values,* "the importance of the transitivity condition" involves "the independence of the final choice from the path to it."[20] "Transitivity," he said, "will ensure this independence," thereby ensuring also that the preferences of the participants (rather than the form of or manipulation of the social choice mechanism) determine the outcome. He went on to point out that both Robert Dahl and I had described ways in which intransitive social mechanisms had produced "unsatisfactory" results. So Arrow concluded that "collective rationality" was not merely an "illegitimate" anthropomorphism, "but an important attribute of a genu-

ine democratic system." Consequently, if one gives up on social transitivity or some weaker form of ordering, one is in effect abandoning the effort to ensure socially satisfactory outcomes.

To ensure satisfactory outcomes without imposing an anthropomorphic collective rationality, one might impose consistency conditions on the social choice mechanism—conditions that could have the same effect of forestalling manipulation that transitivity does, but that would not attribute to society the ability to order possessed only by persons. Hopefully, one would thereby avoid all the problems of the possibility theorems put forth by Arrow and his successors. Unfortunately, however, it turns out that these consistency conditions also cannot be satisfied by social choice mechanisms that satisfy the fairness conditions. Consequently, although the problem can be elegantly restated in terms of choice rather than ordering, the main defect of the methods of amalgamation is unaffected by the new language. Just to say, for example, that $x\,P_1\,y$ and $x\,P_2\,y$ lead to $C(x, y) = x$ rather than to say that they lead to $x\,P\,y$ does not solve the problem of amalgamation. Some kind of inconsistency is ineradicable.

Consistency requirements on choice have been discussed in two quite different ways, which, however, turn out to be substantially equivalent in this context. I will discuss both ways here, despite their equivalence, because their verbal rationales are complementary.

A. K. Sen and subsequently many others have imposed on social choice conditions of logicality that were originally devised as standards for individual choice behavior. This procedure has the advantage of relating consistency in groups to consistency in persons, but it is subject to the same charge of anthropomorphism that was leveled against the use of ordering conditions. Charles Plott, however, has devised a consistency condition for social choice itself, one that could not easily be applied to persons but captures the spirit of Arrow's insistence that the final choice ought to be independent of the path to it. It is interesting and remarkable that Sen's and Plott's conditions turn out to be closely related and almost equivalent.[21]

Looking first at Sen's conditions, let S and T be sets of alternatives in $X = (x_1, x_2, \ldots, x_m)$ and let S be a subset of T. Sen's conditions are restrictions on the choice sets from these two sets of alternatives, $C(S)$ and $C(T)$:

1. *Property* α: For sets S and T, with S a subset of T, if x is in both $C(T)$ and S, then x is in $C(S)$.
2. *Property* $\beta+$: For sets S and T, with S a subset of T, if x is in $C(S)$ and y is in S, then, if y is in $C(T)$, so also is x in $C(T)$.

The meaning of these conditions is easily explained: Property α requires that, if the choice from the larger set is in the smaller set, then it is in the choice from the smaller set as well.

To see the rationale of α, consider a violation of it: A diner chooses among three items on a menu, beef (B), chicken (C), and fish (F), which are the set $\{B, C, F\}$. The diner chooses beef (B); then the restaurant runs out of fish (F). The new menu is the set $\{B, C\}$, whereupon the diner chooses chicken (C) in violation both of property α and of apparent good sense.[22]

Property α guarantees consistency in choices as the number of alternatives is *contracted* because in going from T to S the choice does not change if it is in both sets. Property $\beta+$, on the other hand, guarantees consistency in choices as the number of alternatives is *expanded.* It requires that, if any element in the smaller set is the choice from the larger set, then all choices from the smaller set are choices from the larger set. Thus, in going from S to T, if any choices from S continue to be chosen from the larger set, all such choices continue to be chosen.

The rationale of $\beta+$ can be appreciated from a violation of it: For a seminar with students $S = (a, b, c, d)$, a teacher ranks d best. Then another student enrolls making $T = (a, b, c, d, e)$, whereupon the teacher ranks c best. Doubtless student d discerns an inconsistency and believes that if he is the best or among the best in S and if some other member of S is best in T, then he (d) ought to be among the best in T also.

As I have already noted, property α and property $\beta+$ apply as well to individuals as to society. Plott, however, attempted to embody Arrow's notion of "independence of the final result from the path to it" directly in a condition on *social* choice. Plott justified his condition, which, appropriately, he called "path independence," thus:

the process of choosing, from a dynamic point of view, frequently proceeds in a type of "divide and conquer" manner. The alternatives are "split up" into smaller sets, a choice is made over each of these sets, the chosen elements are collected, and then a choice is made from them. Path independence, in this case, would mean that the final result would be independent of the way the alternatives were initially divided up for consideration.[23]

The definition of *path independence* is that, for any pair of sets S and T, the choice from the union of the sets is the same as the choice from the union of the separate choices from each set.[24] Manifestly, if S and T are any ways of breaking up the set of alternatives, X, then to equate the choices from their union with the choice from the union of their choice sets is to say that it makes no difference to the final outcome how X is divided up for choosing.

Path independence (*PI*) can be broken up into two parts—*PI** and **PI*:

1. *PI** is the condition that the choice from the union of S and T be included in or equivalent to the choice from the union of their choice sets.
2. **PI* is the converse of *PI**. Specifically, **PI* is the condition that the choice from the union of the choice sets of S and T be included in or equivalent to the choice from the union of S and T.[25]

It is a remarkable and important fact that *PI** is exactly equivalent to property α.[26] Furthermore, a choice function satisfying property $\beta+$ satisfies **PI*, so that, though not equivalent, **PI* is implied by $\beta+$.[27]

These standards of consistency in choice turn out to be quite similar in effect to ordering principles.[28] Although property α does not guarantee transitivity, it does guarantee acyclicity in choices from X. So also, therefore, do *PI* and *PI**. Consequently, social choice methods satisfying these conditions are dictatorial or oligarchic, just as are those satisfying ordering principles.

On the other hand, property $\beta+$ does not guarantee even acyclicity when choices from X are made in a series of pairwise comparisons. Consequently, methods satisfying $\beta+$ and **PI* do not imply dictatorship or oligarchy or any other kind of concentration of power. If one is willing to give up consistency in contracting alternatives—and this is quite a bit to give up—then reliance on simple consistency in expanding alternatives might be a way around all the difficulties discovered by Arrow. Unfortunately, however, methods of choice satisfying $\beta+$ and **PI* violate another fairness condition—namely, unanimity or Pareto optimality.[29]

Suppose a choice is to be made by three people with these preference orders: (1) $x\,y\,z\,w$, (2) $y\,z\,w\,x$, (3) $z\,w\,x\,y$. This leads to a cycle in simple majority rule, $x\,P\,y\,P\,z\,P\,w\,P\,x$, so that the choice set is all the alternatives: $C(w, x, y, z) = (w, x, y, z)$. But everyone prefers z to w, although there is a path by which w can be chosen. Let $S_1 = (y, z)$ and $C(S_1) = y$; $S_2 = (x, y)$ and $C(S_2) = x$; $S_3 = (x, w)$ and $C(S_3) = w$. Using S_1 at step 1, S_2 at step 2, and S_3 at step 3, w is selected even though z, eliminated at step 1, is unanimously preferred to w. This result is generalized by Ferejohn and Grether.[30] It tells us that, even if we rely solely on an expansion consistency condition and thus avoid concentrations of power, we still do not achieve fairness. So, in a quite different way, we are back where we began. Nothing has been gained except an elegant formalism that avoids anthropomorphizing society.

5.G. The Absence of Meaning

The main thrust of Arrow's theorem and all the associated literature is that there is an unresolvable tension between logicality and fairness. To guarantee an ordering or a consistent path, independent choice requires that there be some sort of concentration of power (dictators, oligarchies, or collegia of vetoers) in sharp conflict with democratic ideals. Even the weakest sort of consistency ($\beta+$ or $*PI$) involves a conflict with unanimity, which is also an elementary condition of fairness.

These conflicts have been investigated in great detail, especially in the last decade; but no adequate resolution of the tension has been discovered, and it appears quite unlikely that any will be. The unavoidable inference is, therefore, that, so long as a society preserves democratic institutions, its members can expect that some of their social choices will be unordered or inconsistent. And when this is true, no meaningful choice can be made. If y is in fact chosen—given the mechanism of choice and the profile of individual valuations—then to say that x is best or right or more desired is probably false. But it would also be equally false to say that y is best or right or most desired. And in that sense, the choice lacks meaning. The consequence of this defect will be explored in the ensuing chapters.