Chapter 6: Discrete Random Variables

STAT 2601 – Business Statistics

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What is a Discrete Random Variable?

Definition

A **discrete random variable** is a variable whose values are countable and typically result from counting.

Examples

- Number of customers arriving in an hour
- Number of defective items in a batch
- Number of emails received per day
- Number of successes in n trials

Random Variables

Discrete: Possible values can be counted or listed with discrete gaps in between.
Continuous: MMay assume any numerical value in one or more intervals. We cannot count or list the numbers in such an interval or continuum because they are infinitesimally close together.

Probability Distribution Function (PDF)

Definition

The **probability distribution** of a discrete random variable X lists all possible values of X and their corresponding probabilities (in a table, graph, or formula).

x	**P(x)**
x_1	$P(x_1)$
x_2	$P(x_2)$
$\vdots$	$\vdots$
x_k	$P(x_k)$

Properties

For a valid probability distribution:

1. $0 \leq p(x_i) \leq 1$ for all x_i
2. $\sum p(x_i) = 1$

Example

Let X = number of heads in 2 coin tosses:

x	0	1	2
$P(X = x)$	0.25	0.50	0.25

Expected Value (Mean)

Definition

The **expected value** $E(X)$ or μ is the weighted average of all possible values of X, weighted by their probabilities.

$$E(X) = \mu = \sum_i x_i \cdot p(x_i)$$

Example

Let X = daily sales (in units):

x	10	20	30
$p(x)$	0.3	0.5	0.2

$E(X) = (10 \times 0.3) + (20 \times 0.5) + (30 \times 0.2) = 3 + 10 + 6 = 19$ units

Variance and Standard Deviation

Definition

Variance measures the spread or dispersion of a random variable around its mean.

$$\text{Variance: } V(X) = \sigma^2 = \sum (x_i - \mu)^2 \cdot p(x_i) = \sum_{\text{all } x} x^2 p(x) - \mu^2$$

$$\text{Standard Deviation: } SD(X) = \sigma = \sqrt{\sigma^2} = \sqrt{V(X)}$$

Example

Using previous sales example ($\mu = 19$):

$$\begin{aligned} \sigma^2 &= (10-19)^2(0.3) + (20-19)^2(0.5) + (30-19)^2(0.2) \\ &= 81(0.3) + 1(0.5) + 121(0.2) \\ &= 24.3 + 0.5 + 24.2 = 49 \\ \sigma &= \sqrt{49} = 7 \end{aligned}$$

Example: Calculating Variance of Sales

A coffee shop tracks the number of specialty drinks sold per hour (X). The probability distribution is:

Drinks (x)	10	15	20	25	30
P(X = x)	0.1	0.2	0.4	0.2	0.1

Calculate the expected value (μ) , the variance (σ^2), and the standard deviation (σ)

Option 1: Direct variance formula using $\sigma^2 = \sum(x_i - \mu)^2 \cdot p(x_i)$

x	$p(x)$	$x \cdot p(x)$	$x - \mu$	$(x-\mu)^2$	$(x-\mu)^2 \cdot p(x)$
10	0.1	$10 \times 0.1 = 1$	$(10-20) = -10$	100	$100 \times 0.1 = 10$
15	0.2	$15 \times 0.2 = 3$	$(15-20) = -5$	25	$25 \times 0.2 = 5$
20	0.4	$20 \times 0.4 = 8$	$(20-20) = 0$	0	$0 \times 0.4 = 0$
25	0.2	$25 \times 0.2 = 5$	$(25-20) = 5$	25	$25 \times 0.2 = 5$
30	0.1	$30 \times 0.1 = 3$	$(30-20) = 10$	100	$100 \times 0.1 = 10$
Sum		$\mu = \mathbf{20}$			$\sigma^2 = \mathbf{30}$

$$\mu = 20, \quad , \sigma^2 = 30 \quad \Rightarrow \quad \sigma = \sqrt{30} = \mathbf{5.477} \text{ drinks}$$

Option 2: Shortcut variance formula using $\sigma^2 = \sum_{\text{all } x} x^2 p(x) - \mu^2$

x	$p(x)$	x^2	$x^2 \cdot p(x)$
10	0.1	100	$100 \times 0.1 = 10$
15	0.2	225	$225 \times 0.2 = 45$
20	0.4	400	$400 \times 0.4 = 160$
25	0.2	625	$625 \times 0.2 = 125$
30	0.1	900	$900 \times 0.1 = 90$
Total			**430**

Note that the expected value is

$$\begin{aligned}\mu = E(X) &= \sum x \cdot p(x) \\ &= (10 \times 0.1) + (15 \times 0.2) + (20 \times 0.4) + (25 \times 0.2) + (30 \times 0.1) = \mathbf{20} \text{ drinks}\end{aligned}$$

$$\sum_{\text{all } x} x^2 p(x) = 430, \quad \mu^2 = (20)^2 = 400$$

Thus,

$$\sigma^2 = 430 - 400 = \mathbf{30} \quad \Rightarrow \quad \sigma = \sqrt{30} = \mathbf{5.477} \text{ drinks}$$

Characteristics of Binomial Experiments

A Binomial experiment must satisfy:

1. Fixed number of trials (n)
2. Each trial has only two outcomes: **success** or **failure**
3. Constant probability of success (p).
4. Trials are independent

Notation

- n: number of trials
- x: number of successes
- p: probability of success
- $q = 1 - p$: probability of failure

Binomial Probability Formula

$$P(X = x) = C_x^n p^x q^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

where

$$C_x^n = \frac{n!}{x!(n-x)!}, \quad n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Note, $1! = 0! = 1$

Mean and Variance of Binomial Distribution

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = npq$$

$$\sigma = SD(X) = \sqrt{npq}$$

Important

$q = 1 - p$, where p is the probability of success

Binomial Example 1: Quality Control

Example

A factory produces light bulbs with 5% defect rate. In a sample of 20 bulbs:

- $n = 20$, $p = 0.05$, $q = 0.95$

1. **Probability exactly 2 are defective:**

$$P(X = 2) = C_x^n p^x q^{n-x} = \frac{20!}{2!18!}(0.05)^2(0.95)^{18}$$
$$= 190 \times 0.0025 \times 0.3972 = 0.1887$$

2. **Expected number of defectives:**

$$\mu = np = 20 \times 0.05 = 1$$

3. **Standard deviation:**

$$\sigma = \sqrt{npq} = \sqrt{20 \times 0.05 \times 0.95} = \sqrt{0.95} = 0.9747$$

Example: Marksman Target Practice

A marksman hits a target 80% of the time. He fires 6 shots at the target. Let X represent the number of shots that the marksman hits the target.

1. State the name of the probability distribution of X. List the values of all relevant parameters.
2. What is the probability that exactly 4 shots hit the target?
3. What is the probability that more than 4 shots hit the target?
4. What is the probability that at least 2 shots hit the target?
5. What is the probability that at most 5 shots hit the target?
6. Would it be unusual that none of the shots hits the target?

Binomial Distribution Parameters

- $n = 6$ (number of trials)
- $p = 0.8$ (probability of success - hitting target)
- $q = 1 - p = 0.2$ (probability of failure - missing target)
- X = number of shots that hit the target

X follows a Binomial distribution with parameters $n = 6,$ and $p = 0.8$

or $\quad X \sim B(n = 6, p = 0.8)$

Example of Marksman Target Practice (cont'd):

Binomial Probability Formula

$$P(X = x) = C_x^n p^x q^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

1. Probability of Exactly 4 Hits:

$$P(X = 4) = \frac{6!}{4!(6-4)!}(0.8)^4(0.2)^2 = 15 \times 0.4096 \times 0.04 = \mathbf{0.24576}$$

2. Probability of More Than 4 Hits:

$$P(X > 4) = P(X = 5) + P(X = 6)$$

$P(X = 5)$

$$\begin{aligned} P(X = 5) &= \frac{6!}{5!1!}(0.8)^5(0.2)^1 \\ &= 6 \times 0.32768 \times 0.2 \\ &= 0.393216 \end{aligned}$$

$P(X = 6)$

$$\begin{aligned} P(X = 6) &= \frac{6!}{6!0!}(0.8)^6(0.2)^0 \\ &= 1 \times 0.262144 \times 1 \\ &= 0.262144 \end{aligned}$$

$$P(X > 4) = 0.393216 + 0.262144 \approx \mathbf{0.655}$$

Example of Marksman Target Practice (cont'd):

3. Probability of At Least 2 Hits:

What We Need to Calculate

At least 2 hits means: $X \geq 2$ or $X = 2, 3, 4, 5, 6$
Easier to use complement rule:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$$

$P(X = 0)$

$$\begin{aligned} P(X = 0) &= \frac{6!}{0!6!}(0.8)^0(0.2)^6 \\ &= 1 \times 1 \times 0.000064 \\ &= 0.000064 \end{aligned}$$

$P(X = 1)$

$$\begin{aligned} P(X = 1) &= \frac{6!}{1!5!}(0.8)^1(0.2)^5 \\ &= 6 \times 0.80 \times 0.00032 \\ &= 0.001536 \end{aligned}$$

$$P(X \geq 2) = 1 - (0.000064 + 0.001536) = 1 - 0.0016 = \mathbf{0.9984}$$

Example of Marksman Target Practice (cont'd):

4. Probability of At Most 5 Hits:

What We Need to Calculate

At most 5 hits means: $X \leq 5$ or $X = 0, 1, 2, 3, 4, 5$
Easier to use complement rule:

$$P(X \leq 5) = 1 - P(X = 6)$$

Recall $P(X = 6)$ from Part 2

$$P(X = 6) = 0.262144$$

Thus,

$$P(X \leq 5) = 1 - 0.262144 = \mathbf{0.737856}$$

Alternative Method (not recommended!)

$$\begin{aligned} P(X \leq 5) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ &= 0.000064 + 0.001536 + 0.01536 + 0.08192 + 0.24576 + 0.393216 \\ &= 0.737856 \end{aligned}$$

Example of Marksman Target Practice (cont'd):

Method 1: Use the z-score

$$z = \frac{X - \mu_x}{\sigma_x},$$

- Expected value: $\mu = np = 6 \times 0.8 = 4.8$ hits
- Standard deviation: $\sigma = \sqrt{npq} = \sqrt{6 \times 0.8 \times 0.2} = \sqrt{0.96} = 0.9798$

z-score for 0 hits:

$$z = \frac{0 - 4.8}{0.9798} = -4.9$$

Since $z = -4.9$ is beyond the interval $(-3, 3)$, it would be unusual that no shot hits the target.

Method 2:

$$P(X = 0) = 0.000064 = 0.0064\%$$

Since the probability 0.000064 is very tiny (i.e., chance is almost zero), it would be extremely unusual that no shot hits the target.

Poisson Distribution

Characteristics of Poisson Distribution: Poisson distributions are used to model the events occurring over time or space when:

- Events occur independently
- Average rate (μ) is constant
- Two events cannot occur at exactly the same instant/position

Common Applications

- Number of text messages a person receives on their cell phone per 30 minutes
- Number of patients arrive at a hospital emergency room per hour
- Number of insurance claims per day
- Number of cars served at the gas station in 24-hour period
- Number of defects per square meter
- Number of typographical errors in a given book per 10 pages

Poisson Probability Formula

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

Mean, Variance, and Standard Deviation

$$E(X) = \mu, \qquad V(X) = \mu, \qquad \sigma = \sqrt{\mu}$$

Poisson Example: Customer Arrivals

Example

A bank observes an average of 3 customers arrive every 15 minutes during lunch hour.

Let X represent the number of customers that arrive in the next 15 minutes. Thus, X follows a Poisson distribution with a mean of $\mu = 3$, or $X \sim \text{Poisson}(\mu = 3)$

1. **Probability exactly 2 customers in 15 minutes:**

$$P(X = 2) = \frac{e^{-3}3^2}{2!} = \frac{0.0498 \times 9}{2} = 0.2241$$

2. **Probability at most 2 customers:**

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} \\ &= 0.0498 + 0.1494 + 0.2241 = 0.4233 \end{aligned}$$

3. **Standard deviation:**

$$\sigma = \sqrt{\mu} = \sqrt{3} = 1.732$$

Poisson Example: Traffic Accidents

The average number of traffic accidents on a certain section of highway is 2 per week.

1. Find the probability of exactly one accident during a one-week period (Same Unit of Time)
2. Find the probability of at most three accidents during a two-week (Different Unit of Time)
3. Would it be unusual that 8 or more accidents happen per week? (Same Unit of Time)

Part 1:

$$\mu = 2/\text{week}$$

$$P(X = 1) = \frac{e^{-2}2^1}{1!} = 0.2707$$

Part 2:

$$\mu = 2/\text{week, adjusted } \mu = 4/\text{2-week} \quad \Rightarrow X \sim \text{Poisson}(\mu = 4)$$

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X \leq 3) = \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} = 0.4335$$

Part 3: **Method 1:**

$$z = \frac{X - \mu_x}{\sigma_x} = \frac{8 - 2}{\sqrt{2}} = 4.24$$

Since $z = 4.24$ is beyond the interval $(-3, 3)$, it would be unusual that 8 or more accidents happen per week

Method 2:

$$P(X \geq 8) = 1 - P(X < 8) = 1 - P(X \leq 7) \approx 0.001$$

Since the probability 0.001 is very tiny (i.e., chance is very thin), it would be unusual that 8 or more accidents happen per week.

Example

A statistics instructor observes that the number of typographical errors is Poisson distributed with a mean of 1.5 per **100 pages**.

(i) What is the probability that there are **no typos** in a new book of **100 pages**? (Same Unit of Space)

$$\mu = 1.5/100\text{-page}$$

$$P(X=0) = \frac{e^{-1.5}1.5^0}{0!} = 0.2231$$

(ii) Suppose that the instructor has just received a copy of a new statistics book of **400 pages**. Find the probability that there are five or fewer typos. (Different Unit of Space)

$$\mu = 1.5/100\text{-page, adjusted } \mu = 6/400\text{-page} \quad \Rightarrow X \sim \text{Poisson}(\mu = 6)$$

$$P(X \leq 5) = P(X=0) + P(X=1) + \cdots + P(X=5)$$

$$P(X \leq 5) = \frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} + \cdots + \frac{e^{-6}6^5}{5!} = 0.45$$

Comparison of Distributions

Feature	Binomial	Poisson
Type of variable	Count of successes	Count of occurrences
Number of trials	Fixed (n)	Not fixed
Probability of success	Constant (p)	μ = average rate
Possible values	$0, 1, 2, \ldots, n$	$0, 1, 2, \ldots$
Mean	$\mu = np$	μ
Variance	$\sigma^2 = npq$	$\sigma^2 = \mu$

Key Formulas

General Discrete Random Variable:

$$\mu = E(X) = \sum x \cdot p(x) \quad \sigma^2 = \sum (x-\mu)^2 p(x)$$

Binomial Distribution:

$$P(X = x) = C_x^n p^x q^{n-x} \quad \mu = np, \ \sigma = \sqrt{npq}$$

where $C_x^n = \dfrac{n!}{x!(n-x)!}$

Poisson Distribution:

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!} \quad \mu = \mu, \ \sigma = \sqrt{\mu}$$