Chapter 7: Continuous Random Variables
STAT 2601 – Business Statistics

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Continuous vs. Discrete Random Variables

- **Discrete Random Variable:** Countable outcomes (e.g., number of customers)
- **Continuous Random Variable:** Uncountable number of values (e.g., height, weight, time)

Probability Density Function $f(x)$

- Function where area under curve = probability
- $f(x) \geq 0$ for all $x \in (a, b)$
- Total area $= P(a \leq X \leq b) = \int_a^b f(x)dx = 1$

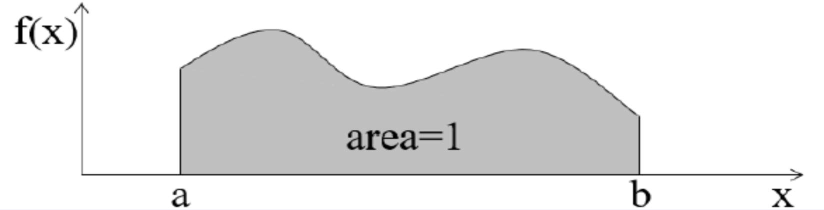


Figure: Area Represents Probability

Key Difference Between Probability Distribution of Discrete and Continuous Random Variables

- For continuous random variables:
  1. $P(X = x) = 0$ for any specific value x.
  2. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$.
- For discrete random variables:
  1. $P(X = x)$ is not necessarily $= 0$.
  2. $P(a \leq X \leq b) \neq P(a < X \leq b) \neq P(a \leq X < b) \neq P(a < X < b)$.

Note that the distribution of a discrete random variable is referred to as the **Probability Mass Function (PMF)**.

Normal Distribution: The Bell Curve

Mathematical Formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ = mean, σ = standard deviation.

Characteristics:

- Bell-shaped, symmetric
- Mean = median = mode
- Defined by μ and σ^2 (or σ)
- Range: $(-\infty, \infty)$

We say X follows a random distribution with μ = mean, σ^2 = variance (or standard deviation σ) denoted as $X \sim N(\mu, \sigma^2)$ (or $X \sim N(\mu, \sigma)$).

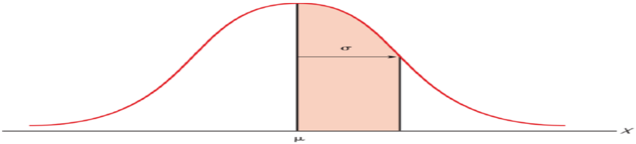
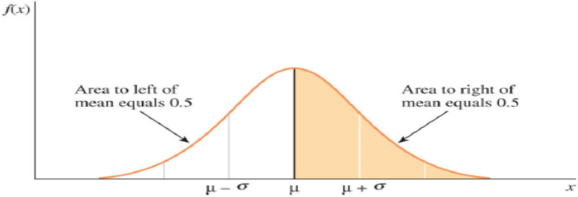


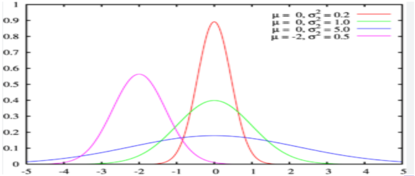
Figure: Normal Distribution

Characteristics of Normal Distribution

- The mean μ locates the centre of the distribution and the shape of the distribution is determined by σ, the population standard deviation.
- Total area under the curve is 1. As the distribution is symmetric, the left side is a mirror image of the right side i.e., the left area is 0.5 and the right area is 0.5.



- The shape and location of the normal curve changes as the mean and standard deviation change. Increasing the mean shifts the curve to the right, while increasing the standard deviation flattens the curve.



Why Standardize?

- Infinite combinations of μ and σ^2
- Need one table for all normal distributions
- Standardization converts any normal to standard normal

Z-score Formula

$$\text{If } X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X - \mu}{\sigma}$$

where $Z \sim N(0, 1)$

Characteristics of $Z \sim N(0, 1)$

- Mean = 0, Standard Deviation = 1
- Symmetric about 0
- $P(Z < 0) = P(Z > 0) = 0.5$

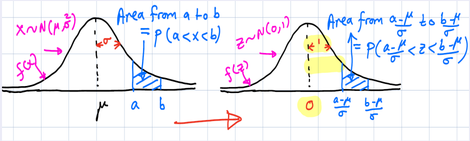
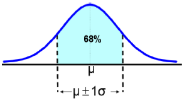


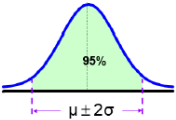
Figure: Transform Normal to Standard Normal

Empirical Rule (68 – 95 – 99.7 Rule)

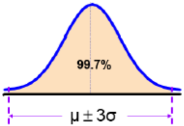
- If $X \sim N(\mu, \sigma^2)$, then the area within one standard deviation of the mean ($\mu \pm 1\sigma$) is approximately 68% of the total probability.



- If $X \sim N(\mu, \sigma^2)$, then the area within two standard deviation of the mean ($\mu \pm 2\sigma$) is approximately 95% of the total probability.



- If $X \sim N(\mu, \sigma^2)$, then the area within three standard deviation of the mean ($\mu \pm 3\sigma$) is approximately 99.7% of the total probability.



Using the Standard Normal Table

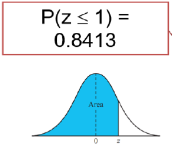
Cumulative Standard Normal Table

Gives $P(Z \leq z)$ for z values (see the table next page)

Example

Find $P(Z \leq 1.00)$

1. Find row 1.0 and column 0.00
2. Intersection gives 0.8413
3. $P(Z \leq 1.00) = 0.8413$



z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8328	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278

Figure: Standard Normal Table

Important Properties

- $P(Z \geq z) = P(Z \leq -z)$ (Using fact that the distribution curve is symmetric)
- $P(Z \geq z) = 1 - P(Z \leq z)$ (Using the compliment rule: $P(\bar{A}) = 1 - P(A)$)
- $P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a)$

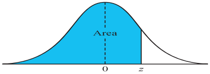


TABLE 3 Areas under the Normal Curve

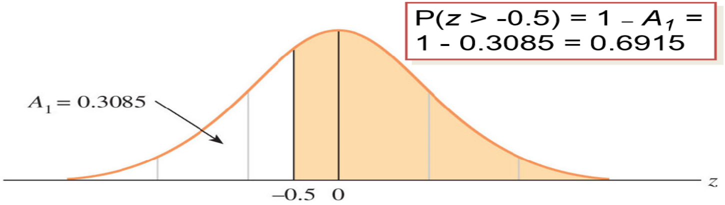
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

TABLE 3 *(continued)*

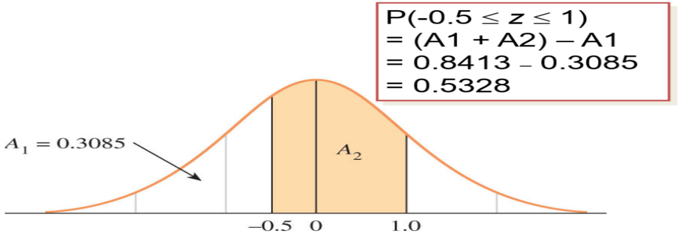
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Examples

- Example of Standard Normal Table: $P(z > -0.5)$



- Example of Standard Normal Table: $P(-0.5 \leq z \leq 1)$



Example

Find $P(-1.5 \leq Z \leq 2.0)$

Solution:

$$\begin{aligned} P(-1.5 \leq Z \leq 2.0) &= P(Z \leq 2.0) - P(Z \leq -1.5) \\ &= 0.9772 - 0.0668 \\ &= 0.9104 \end{aligned}$$

Example

Find $P(Z \geq 1.75)$

Solution:

- **Option 1**

$$\begin{aligned} P(Z \geq 1.75) &= 1 - P(Z \leq 1.75) \\ &= 1 - 0.9599 = 0.0401 \end{aligned}$$

- **Option 2**

$$\begin{aligned} P(Z \geq 1.75) &= P(Z \leq -1.75) \\ &= 0.0401 \end{aligned}$$

Example 2: Converting X to Z

Example

Let $X \sim N(100, 15^2)$. Find $P(X \leq 130)$

Solution:

1. Convert to $Z = \dfrac{X - \mu}{\sigma}$: $z = \frac{130-100}{15} = 2.0$
2. $P(X \leq 130) = P(\frac{X-100}{15} \leq \frac{130-100}{15}) = P(Z \leq 2.0)$
3. From the Z-table: $P(Z \leq 2.0) = 0.9772$

Example

Find $P(90 \leq X \leq 110)$ where $X \sim N(100, 15^2)$

Solution:

$$\begin{aligned}
P(90 \leq X \leq 110) &= P\left(\frac{90-100}{15} \leq \frac{X-100}{15} \leq \frac{110-100}{15}\right) \\
&= P(-0.67 \leq Z \leq 0.67) \\
&= P(Z \leq 0.67) - P(Z \leq -0.67) = 0.7486 - 0.2514 \\
&= 0.4972
\end{aligned}$$

Example: Weekly Sales Revenue

Suppose the weekly sales revenue (X) for a retail store follows a normal distribution with a mean of $\mu = 25{,}000$ dollars and a standard deviation of $\sigma = 4{,}000$ dollars, so that $X \sim N(25{,}000, 4{,}000^2)$.
Questions:

(a) What is the probability that the store's weekly revenue is less than \$20,000?
Solution: We want: $P(X < 20{,}000)$

$$P(X < 20{,}000) = P(Z < \frac{20{,}000 - 25{,}000}{4{,}000}) = P(Z < -1.25) = 0.1056$$

(b) What is the probability that the store's weekly revenue is greater than \$30,000?
Solution: We want: $P(X > 30{,}000)$

$$\begin{aligned} P(X > 30{,}000) &= P(Z > \frac{30{,}000 - 25{,}000}{4{,}000}) = P(Z > 1.25) \\ &= P(Z < -1.25) = 0.1056 \quad \text{(Option 1)} \\ &= 1 - P(Z \leq 1.25) = 1 - 0.8944 = 0.1056 \quad \text{(Option 2)} \end{aligned}$$

(c) What is the probability that the store's weekly revenue is between \$22,000 and \$28,000?
Solution: We want: $P(22{,}000 < X < 28{,}000)$

$$\begin{aligned} P(22{,}000 < X < 28{,}000) &= P(\frac{22{,}000 - 25{,}000}{4{,}000} < Z < \frac{28{,}000 - 25{,}000}{4{,}000}) \\ &= P(-0.75 < Z < 0.75) = P(Z < 0.75) - P(Z < -0.75) \\ &= 0.7734 - 0.2266 = 0.5468 \end{aligned}$$

Finding Percentiles: Working Backwards

Recall

The pth percentile is the value x_p such that $P(X \leq x_p) = p/100$

Example

Find the 90th percentile for $Z \sim N(0, 1)$

1. Find z-value where $P(Z \leq z_{0.90}) = 0.90$
2. From table: $z_{0.90} = 1.28$ (since $P(Z \leq 1.28) = 0.8997 \approx 0.90$)
3. 90th percentile = 1.28

Example

Find the 90th percentile for $X \sim N(100, 15^2)$

$$\begin{aligned} x_p &= \mu + z_p \cdot \sigma \\ x_{90} &= 100 + 1.28 \times 15 \\ &= 100 + 19.2 = 119.2 \end{aligned}$$

Example: Working Backwards

The distribution of commuting time for American workers follows Normal distribution with mean $\mu = 24.3$ and $\sigma = 10$. What is the score (i.e., time X) that separates the middle 90% of commuting time from the rest?

- Given: $X \sim N(\mu = 24.3, \sigma^2 = 10^2)$, Want: $P(? < X < ?) = 0.90$
- Transform X to Z:

$$P(\frac{? - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{? - \mu}{\sigma}) = P(\frac{? - 24.3}{10} < z < \frac{? - 24.3}{10}) = 0.90 \quad (1)$$

Using Standard Normal Table:

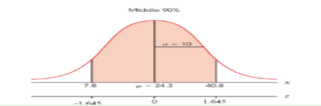
$$P(-1.645 < z < 1.645) = 0.90 \quad (2)$$

- Mapping (1) and (2):

$$\frac{? - 24.3}{10} = -1.645 \longrightarrow X(?) = 24.3 - 10 \times 1.645 = 7.85.$$

and

$$\frac{? - 24.3}{10} = 1.645 \longrightarrow X(?) = 24.3 + 10 \times 1.645 = 40.75.$$



Exponential Distribution

Used for:

- Time between events (e.g., time between transactions at an ATM Machine)
- Service times
- Product lifetimes

Key Formulas

$$X \sim \exp(\lambda) \Rightarrow$$

- PDF: $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$
- Mean: $E[X] = \frac{1}{\lambda}$
- Variance: $V(X) = \frac{1}{\lambda^2}$
- CDF: $P(X \leq b) = 1 - e^{-\lambda b}, \quad b > 0$
- $P(X \geq a) = e^{-\lambda a}, \quad a > 0$
- $P(a \leq X \leq b) = e^{-\lambda a} - e^{-\lambda b}$

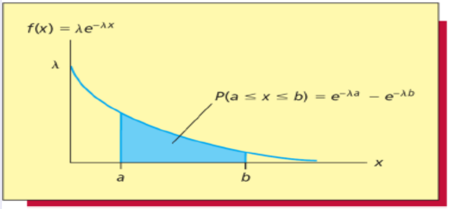


Figure: Exponential Distribution

Exponential Distribution Example

Example

Customer service calls arrive every 10 minutes on average ($\lambda = 0.1$/min).
What's the probability the next call arrives within 5 minutes?

Solution

$$\begin{aligned} E[X] &= \frac{1}{\lambda} = 10 \text{ minutes} \\ \lambda &= 0.1 \\ P(X \leq 5) &= 1 - e^{-0.1\times 5} \\ &= 1 - e^{-0.5} \\ &= 1 - 0.6065 = 0.3935 \end{aligned}$$

There's a 39.35% chance the next call arrives within 5 minutes.

Practice Problems

1. For $X \sim N(50, 4^2)$, find $P(X \geq 45)$, $P(X \leq 55)$, and $P(45 \leq X \leq 55)$.
2. For $X \sim N(200, 25^2)$, what value cuts off the top 15%?
3. The blood glucose test of elderly men, suffer from diabetes, in a certain clinic shows that the blood sugar level has a normal distribution with mean 250 mmol/L and variance 400 (mmol/L)^2.
    1. If an elderly man is chosen at random from the population, what is the probability that his blood glucose exceeds the level 245 mmol/L?
    2. What is the percentage of the elderly men, suffering from diabetes, with blood glucose level between 246 and 255 mmol/L?
    3. What blood glucose level represents the 95^{th} percentile?
4. Based on a report of Canada Mortgage Housing Corporation (CMHC), Canada had the highest household debt among G7 countries in 2023. The household debt of Canadian households excluding mortgages in 2023 followed an approximately normal distribution with a mean of $\mu = \$41,500$ and a standard deviation of $\sigma = \$3,900$. Let X represent the debt of Canadian household excluding mortgages in 2023.
    1. What percentage of Canadian households had burden of debts above $50,000 in 2023?
    2. What is the probability that a randomly selected Canadian household had household debt excluding mortgages between $35,000 and $45,000 in 2023?
    3. What was the household debt of the top 20% Canadian household in 2023?
    4. Suppose a Canadian household debt was at 65th percentile in 2023. How much was the debt?
5. If calls arrive every 15 minutes on average, what's the probability of waiting more than 20 minutes?