Chapter 11: Statistical Inferences Based on Two Samples

STAT 2601 – Business Statistics

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Learning Objectives

By the end of this chapter, you should be able to:

- Identify when to use independent vs. paired samples for comparing two population means
- State the assumptions and conditions required for each two-sample inference procedure
- Perform hypothesis tests comparing two means using:
  - Independent samples with known variances (z-test)
  - Understand the variance ratio rule for determining equal vs. unequal (unknown) population variances
  - Independent samples with unknown variances (t-test)
  - Paired/dependent samples (paired t-test)
- Construct and interpret confidence intervals for the difference between two population means
- Calculate test statistics, critical values, and p-values for two-sample tests
- Perform hypothesis tests comparing two population proportions using critical values and p-values methods
- Construct and interpret confidence intervals for the difference between two population proportions

Comparing Means: Independent Samples, Known Population Variances

Scenario

We want to compare two population means (μ_1 and μ_2) using two **independent** random samples. We **know** σ_1^2 and σ_2^2.

Assumptions/Conditions:

- Independent random samples from each population
- **Sample sizes are large:** $n_1 \geq 30$ and $n_2 \geq 30$ (by CLT)
- σ_1^2 and σ_2^2 are known

Examples

Right-tailed test example: A bank compares mean transaction times between a new software (Group 1) and old software (Group 2). Industry standards provide known variances. They test if the new software is **faster**.

$$H_0 : \mu_1 \geq \mu_2 \,(\text{ or } H_0 : \mu_1 - \mu_2 \geq 0) \quad vs. \quad H_a : \mu_1 < \mu_2 \,(\text{ or } H_a : \mu_1 - \mu_2 < 0)$$

(Note: $\mu_1 < \mu_2$ means "faster" if μ is time)

Left-tailed test example: A retailer tests if mean daily sales at downtown stores (Group 1) are **higher** than suburban stores (Group 2). Historical data provides known sales variability.

$$H_0 : \mu_1 \leq \mu_2 \,(\text{ or } H_0 : \mu_1 - \mu_2 \leq 0) \quad vs. \quad H_a : \mu_1 > \mu_2 \,(\text{ or } H_a : \mu_1 - \mu_2 > 0)$$

Two-tailed test example: A cereal company wants to test if two production lines have **different** mean fill weights. They know the machine precision (variance) from years of operation.

$$H_0 : \mu_1 = \mu_2 \,(\text{ or } H_0 : \mu_1 - \mu_2 = 0) \quad vs. \quad H_a : \mu_1 \neq \mu_2 \,(\text{ or } H_a : \mu_1 - \mu_2 \neq 0)$$

Test Statistic and Standard Error

We are interested in comparing two independent population means: $\mu_1 - \mu_2$.

$$H_0 : \mu_1 - \mu_2 = D_0 \quad \text{(Often the null value } D_0 = 0\text{)}$$

The point estimator of the parameter $\mu_1 - \mu_2$ is: $\bar{x}_1 - \bar{x}_2$.

Standard Error:

$$SE(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

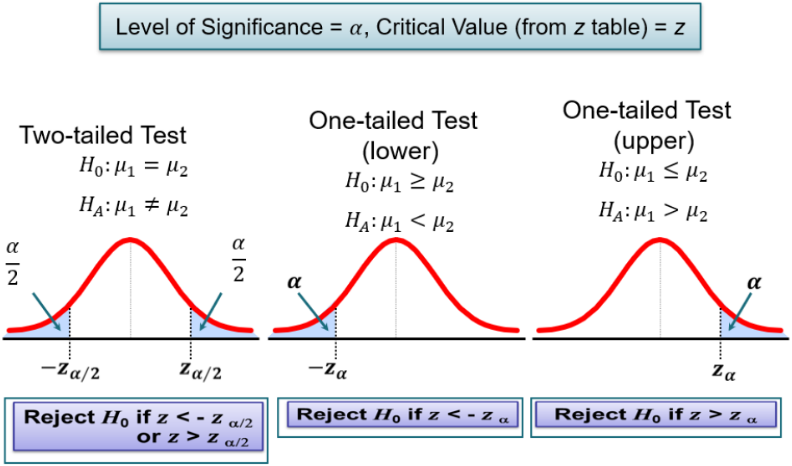
Test Statistic (z):

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{SE(\bar{x}_1 - \bar{x}_2)} = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Under H_0, $z \sim N(0, 1)$.

Hypothesis Testing: Critical Value Approach

At a level of significant α, compare the test statistic z to critical value z_α (one-sided) or $z_{\alpha/2}$ (two-sided) and reject H_0 if the test statistic z value is located inside the rejection region.

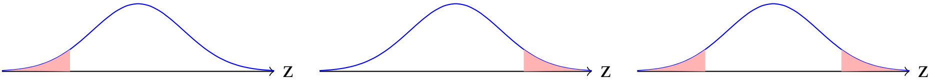


Hypothesis Testing: p-value Approach

p-value: is the probability of obtaining test statistic at least as extreme as the result actually observed, under the assumption that the null hypothesis is correct.

Calculation of p-value:

- p-value $= p(Z < -z)$ (Left-tailed Test)
- p-value $= p(Z > z)$ (Right-tailed Test)
- p-value $= 2p(Z > |z|)$ (or p-value $= p(Z < -z) + p(Z > z)$) (Two-tailed Test)



Left-tailed: $p = P(Z < z)$ Right-tailed: $p = P(Z > z)$ Two-tailed: $p = 2 \times P(Z > |z|)$

Decision Rule:

1. If p-value $< \alpha$, reject H_0.
2. If p-value $\geq \alpha$, do not reject H_0.

Confidence Interval around $\mu_1 - \mu_2$: Independent Samples, Known Variances

Two-sided $(1-\alpha)100\%$ **Confidence Interval:**

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Note: If 0 is not in the CI, we reject $H_0 : \mu_1 - \mu_2 = 0$ at significance level α.

Example: Two Production Line Performance Comparison (Known Variances)

A quality control manager at a snack food company needs to compare the performance of two production lines (Line A and Line B) that fill 25-gram bags of chips. From years of operational data, it is known that:

- Line A has a filling process standard deviation of $\sigma_A = 1.2$ grams
- Line B has a filling process standard deviation of $\sigma_B = 1.0$ grams

To conduct the comparison, random samples are taken:

- From Line A: $n_A = 50$ bags, with sample mean fill weight $\bar{x}_A = 24.6$ grams
- From Line B: $n_B = 40$ bags, with sample mean fill weight $\bar{x}_B = 23.9$ grams

Questions:

a. At the $\alpha = 0.05$ significance level, test whether there is a significant difference in the mean fill weights between the two production lines. Use both the critical value and p-value methods.

b. Construct and interpret a 95% confidence interval for the difference in mean fill weights ($\mu_A - \mu_B$).

c. Based on your results, what should the quality control manager conclude about the two production lines?

Solution: Two Production Line Performance Comparison

Step 1: Hypotheses.

$$H_0 : \mu_A - \mu_B = 0 \quad vs. \quad H_a : \mu_A - \mu_B \neq 0$$

Step 2: Test Statistic.

$$SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{1.2^2}{50} + \frac{1.0^2}{40}} = \sqrt{0.0288 + 0.025} \approx 0.232$$

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(24.6 - 23.9) - 0}{0.232} = \frac{0.7}{0.232} = 3.017241 \approx 3.02$$

Step 3: Make a Decision (Critical Value Method). $\alpha = 0.05$, two-tailed $\Rightarrow z_{\alpha/2} = z_{0.025} = 1.96$. Since $|z| = 3.017 > 1.96$, **Reject** H_0 and conclude that mean fill weight of Line A is **significantly different** from the mean fill weight of Line B.

Step 3: Make a Decision (p-value Method).

$$p\text{-value} = 2 \times P(Z \geq 3.02) \approx 2 \times 0.0013 = 0.0026 < 0.05 \Rightarrow \text{ Reject } H_0.$$

Step 4: 95% CI.

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \equiv 0.7 \pm 1.96 \times 0.232 = 0.7 \pm 0.455 = (0.245,\ 1.155) \text{ grams}$$

Interpretation: We are 95% confident that the mean weight of line A is between 0.245 and 1.155 grams higher than line B. Since 0 is not in the interval, we reject H_0.

Comparing Means: Independent Samples, Unknown Population Variances

Now, we need to compare the two independent population means where σ_1^2 and σ_2^2 are **unknown**. We use the sample variances s_1^2 and s_2^2.

Key Issue: Are the population variances **equal** ($\sigma_1^2 = \sigma_2^2$) or **unequal** ($\sigma_1^2 \neq \sigma_2^2$)?

Rule of Thumb (Check equal variances before proceeding)

$$\text{If } \frac{1}{3} \leq \frac{s^2_{\text{large}}}{s^2_{\text{small}}} \leq 3 \quad \Rightarrow \quad \text{Assume } \sigma_1^2 = \sigma_2^2 \text{ (Equal variances)}$$

$$\text{If } \frac{s^2_{\text{large}}}{s^2_{\text{small}}} > 3 \quad \Rightarrow \quad \text{Assume } \sigma_1^2 \neq \sigma_2^2 \text{ (Unequal variances)}$$

Case 1: Unequal Variances ($\sigma_1^2 \neq \sigma_2^2$)

Sample Statistics:

- $\bar{x}_1$ = sample mean from Population 1, based on n_1 observations
- $\bar{x}_2$ = sample mean from Population 2, based on n_2 observations
- s_1^2 = sample variance from Population 1: $s_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_{1i} - \bar{x}_1)^2$
- s_2^2 = sample variance from Population 2: $s_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(x_{2i} - \bar{x}_2)^2$

Standard Error for Unequal Variances (not pooled): $SE(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Test Statistic (t):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Degrees of Freedom (Conservative Approach):

$$df = \min(n_1 - 1,\ n_2 - 1)$$

$(1-\alpha)100\%$ **Confidence Interval:**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2,df} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Case 2: Equal Variances ($\sigma_1^2 = \sigma_2^2$)

When variances are equal, we **pool** the sample variances.

Pooled Variance Estimator:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Standard Error for Equal Variances (pooled): $SE(\bar{x}_1 - \bar{x}_2) = \sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Test Statistic (t):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Degrees of Freedom:

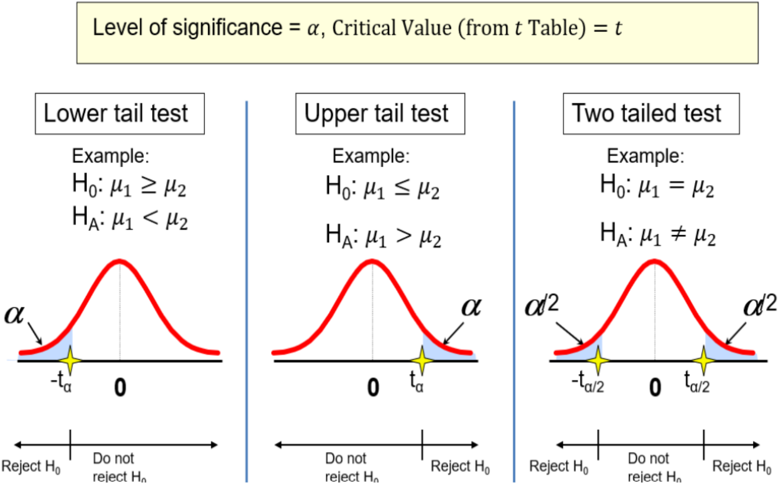
$$df = n_1 + n_2 - 2$$

$(1-\alpha)100\%$ **Confidence Interval:**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2, df} \cdot \sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Hypothesis Testing: Critical Value Approach

At a level of significant α, compare the test statistic t to critical value t_α (one-sided) or $t_{\alpha/2}$ (two-sided) and reject H_0 if the test statistic z value is located inside the rejection region.



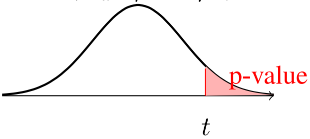
Hypothesis Testing: p-value Approach

p-value: is the probability of obtaining test statistic at least as extreme as the result actually observed, under the assumption that the null hypothesis is correct.

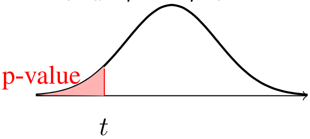
Calculation of p-value:

- p-value $= p(T > t)$ (Right-tailed Test)
- p-value $= p(T < -t)$ (Left-tailed Test)
- p-value $= 2p(T > |t|)$ (or p-value $= p(T < -t) + p(T > t)$) (Two-tailed Test)

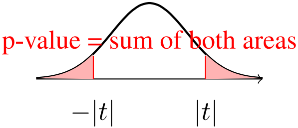
Right-tailed test
($H_a : \mu_1 > \mu_2$)



Left-tailed test
($H_a : \mu_1 < \mu_2$)



Two-tailed test
($H_a : \mu_1 \neq \mu_2$)



Decision Rule:

1. If p-value $< \alpha$, reject H_0.
2. If p-value $\geq \alpha$, do not reject H_0.

Example: Unequal Unknown Variances

Example (Retail Sales Comparison)

A regional retail manager is evaluating the performance of two store locations: Store 1 (downtown flagship) and Store 2 (suburban mall). The manager wants to determine if the mean weekly sales at Store 1 are **greater than** those at Store 2. Weekly sales data (in dollars) were collected over a random sample of weeks gives the following statistics:

$$\text{Store 1} : n_1 = 22, \quad \bar{x}_1 = 8500\$, \quad s_1 = 1100\$$$
$$\text{Store 2} : n_2 = 20, \quad \bar{x}_2 = 7800\$, \quad s_2 = 580\$.$$

Does Store 1 have significantly higher mean weekly sales than Store 2?

Retail Sales Comparison Example (Unequal Unknown Variances)

Required Analysis

1. Check the variance ratio to determine if equal or unequal variances should be assumed
2. Formulate appropriate null and alternative hypotheses
3. Calculate the test statistic
4. Make a decision using both critical value and p-value methods (significance level is not given, so use the default $\alpha = 0.05$). Note that this is a right tailed test
5. Construct and interpret a relevant confidence interval (optional; only required if explicitly requested)
6. Provide a practical conclusion for the retail manager

Step 1: Check Variance Ratio.

$$\frac{s^2_{\text{large}}}{s^2_{\text{small}}} = \frac{1100^2}{580^2} = \frac{1,210,000}{336,400} \approx 3.596$$

Since $3.596 > 3$, we assume $\sigma_1^2 \neq \sigma_2^2$ (unequal variances).

Solution: Retail Sales Comparison (cont.)

Step 2: Hypotheses.

$$H_0 : \mu_1 - \mu_2 \leq 0 \quad vs. \quad H_a : \mu_1 - \mu_2 > 0$$

where μ_1 = population mean weekly sales for Store 1, μ_2 = population mean weekly sales for Store 2.

Step 3: Test Statistic.

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1100^2}{22} + \frac{580^2}{20}} = \sqrt{55,000 + 16,820} = \sqrt{71,820} \approx 268$$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{8500 - 7800}{268} = \frac{700}{268} \approx 2.612$$

Degrees of Freedom: $df = \min(n_1 - 1,\ n_2 - 1) = \min(21, 19) = 19.$

Step 4: Make a Decision (Critical Value Method).
$t_{0.05,19} = 1.729$. Since $t = 2.212 > 1.729$, **we would reject** H_0. Thus, there is sufficient statistical evidence at the $\alpha = 0.05$ level to conclude that Store 1 (downtown) has **significantly higher** mean weekly sales than Store 2 (suburban).

Solution: Retail Sales Comparison (cont.)

Step 4: Make a Decision (p-value Method).
Referring to the t-table with degrees of freedom $df = 19$, the test statistic $t = 2.212$ lies between the critical values 2.093 and 2.539, which correspond to significance levels 0.025 and 0.01, respectively. Since this is a one-tailed test, the p-value for $t = 2.212$ and $df = 19$ falls between 0.01 and 0.025. Because this p-value is less than $\alpha = 0.05$, we reject H_0 and at the $\alpha = 0.05$ significance level, there is sufficient statistical evidence to conclude that Store 1 (downtown) has a significantly higher mean weekly sales than Store 2 (suburban).

95% Confidence Interval (Two-sided) For two-sided 95% CI with $df = 19$: $t_{0.025,19} = 2.093$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2,df} \cdot SE = 700 \pm 2.093 \times 268$$

$$700 \pm 560.9 = (139.1, 1260.9)$$

95% CI Interpretation:

- We are 95% confident that the true difference in population mean weekly sales between Store 1 (downtown) and Store 2 (suburban) lies between \$139.1 and \$1,260.9.
- The "95% confident" means: If we were to take many random samples and construct confidence intervals in the same way, 95% of those intervals would contain the true population difference $\mu_1 - \mu_2$.

Example: Equal Unknown Variances

Example (Manufacturing Process Comparison)

A manufacturing engineer is evaluating two different processing methods (Method A and Method B) for assembling an electronic component. The key performance metric is the mean processing time per unit (in minutes). Lower processing times are desirable as they increase production efficiency. The engineer conducted timed trials for each method using random samples:

$$\text{Method A: : } n_1 = 15, \quad \bar{x}_1 = 25.2, \quad s_1 = 4.1$$
$$\text{Method B: : } n_2 = 12, \quad \bar{x}_2 = 28.5, \quad s_2 = 3.8.$$

Is Method A **faster** (has lower mean processing time) than Method B?

Manufacturing Process Comparison (Equal Unknown Variances)

Required Analysis

1. Check the variance ratio to determine if equal or unequal variances should be assumed
2. Formulate appropriate null and alternative hypotheses
3. Calculate the test statistic
4. Make a decision using both critical value and p-value methods ($\alpha = 0.01$). Note that this is a left tailed test
5. Construct and interpret a relevant confidence interval (optional; only required if explicitly requested)
6. Provide a practical conclusion for the retail manager

Step 1: Check Variance Ratio.

$$\frac{s^2_{\text{large}}}{s^2_{\text{small}}} = \frac{4.1^2}{3.8^2} = \frac{16.81}{14.44} \approx 1.16$$

Since 1.16 is between $1/3$ and 3, we assume $\sigma_1^2 = \sigma_2^2$ (equal variances).

Solution: Manufacturing Process Comparison (cont.)

Step 2: Hypotheses.

$$H_0 : \mu_A - \mu_B \geq 0 \quad vs. \quad H_a : \mu_A - \mu_B < 0$$

Step 3: Pooled Variance.

$$s_p^2 = \frac{(14)(4.1^2) + (11)(3.8^2)}{15 + 12 - 2} = \frac{235.34 + 158.84}{25} = \frac{394.18}{25} = 15.767$$

Step 4: Test Statistic.

$$SE = \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{15.767\left(\frac{1}{15} + \frac{1}{12}\right)} \approx 1.538$$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(25.2 - 28.5) - 0}{1.538} = \frac{-3.3}{1.538} \approx -2.146$$

$df = 15 + 12 - 2 = 25.$

Solution: Manufacturing Process Comparison (cont.)

Step 5: Make a Decision (Critical Value Method).
Left-tailed test, $\alpha = 0.01$, $df = 25$: $t_{0.01,25} \approx -2.485$ (negative).
Since $t = -2.146 > -2.485$, **Fail to Reject** H_0 and conclude that there is **insufficient statistical evidence** to conclude that Method A has a lower mean processing time than Method B.
Step 5: Make a Decision (p-value Method).
$P(T_{25} \leq -2.146) = P(T_{25} \geq 2.146) =$? From table, $t_{0.025,25} = 2.060$, $t_{0.01,25} = 2.485$.
So $0.025 < p$-value (one-tail) < 0.05 which is greater than $\alpha = 0.01$. Thus **Fail to Reject** H_0 and we cannot conclude with statistical confidence that Method A is faster than Method B.
98% CI (Two-sided).
$t_{0.01,25} = 2.485$ for two-sided 98% CI.

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2,(n_1+n_2-2)} \cdot \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \equiv -3.3 \pm 3.821 = (-7.121,\, 0.521)$$

- The interval contains **zero**, which is consistent with failing to reject H_0.
- The interval also contains **positive values** (up to 0.521 minutes), meaning Method A could actually be *slower* than Method B in reality.
- The engineer can be 99% confident that the true difference could be anywhere from Method A being 7.121 minutes faster to Method A being 0.521 minutes slower.

Paired (Dependent) Difference Experiments

Used when observations are naturally **paired** or **matched**.

Examples

- Twins assigned to different treatments
- Same car tested with two tire types (winter vs. all-season)
- Patient blood pressure before and after medication
- Student test scores before and after a tutoring program

Idea: Instead of analyzing x_1 and x_2 separately, analyze the **differences** $d = x_1 - x_2$.

Notation and Procedure for Paired Samples

Let $d_i = x_{1i} - x_{2i}$ for pair i, $i = 1, \ldots, n$.

Sample mean difference:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n}$$

Sample standard deviation of differences:

$$s_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}}$$

Paired Data Table

Pair (i)	x_{1i}	x_{2i}	$d_i = x_{1i} - x_{2i}$
1	x_{11}	x_{21}	$d_1 = x_{11} - x_{21}$
2	x_{12}	x_{22}	$d_2 = x_{12} - x_{22}$
3	x_{13}	x_{23}	$d_3 = x_{13} - x_{23}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	x_{1n}	x_{2n}	$d_n = x_{1n} - x_{2n}$
		Means:	$\bar{d} = \frac{\sum d_i}{n}$
		Std. Dev.:	$s_d = \sqrt{\frac{\sum(d_i - \bar{d})^2}{n-1}}$

We will test hypotheses and construct a confidence interval for $\mu_d = \mu_1 - \mu_2$.

Paired Difference Experiments

Test Statistic (t):

$$t = \frac{\bar{d} - D_0}{s_d/\sqrt{n}}$$

Degrees of Freedom:

$$df = n - 1$$

$(1-\alpha)100\%$ **Confidence Interval:**

$$\bar{d} \pm t_{\alpha/2,n-1} \cdot \frac{s_d}{\sqrt{n}}$$

Example: Paired Samples (Blood Pressure Study)

Clinical Trial Data

A pharmaceutical company tests a new blood pressure medication. Systolic BP (in mmHg) is measured for 10 patients before and after 4 weeks of treatment. Test if the drug reduces BP at $\alpha = 0.05$.

Patient	**Before** (x_{1i})	**After** (x_{2i})	**Difference** $d_i = x_{1i} - x_{2i}$
1	150	142	8
2	148	143	5
3	152	145	7
4	155	150	5
5	162	155	7
6	158	152	6
7	145	138	7
8	160	155	5
9	153	148	5
10	160	155	5
Mean	$\bar{x}_1 = 154.3$	$\bar{x}_2 = 148.3$	$\bar{d} = 6.0$
Std. Dev.	$s_1 = 5.3$	$s_2 = 5.6$	$s_d = 1.247$
Sample Size	$n = 10$	$n = 10$	$n = 10$

Solution: Clinical Trial Data

Calculations:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{60}{10} = 6$$

$$s_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}} = \sqrt{\frac{(8-6)^2 + (5-6)^2 + \cdots + (5-6)^2}{9}} = \sqrt{\frac{14}{9}} \approx 1.247$$

Step 1: Hypotheses.

$$H_0 : \mu_d \le 0 \quad \text{(drug does not reduce BP)} \quad vs. \quad H_a : \mu_d > 0 \quad \text{(drug reduces BP)}$$

where $\mu_d = \mu_{\text{before}} - \mu_{\text{after}}$, so $\mu_d > 0$ means BP before > BP after (reduction).

Step 2: Test Statistic.

$$t = \frac{\bar{d} - D_0}{s_d/\sqrt{n}} = \frac{6.0}{1.247/\sqrt{10}} = \frac{6.0}{0.394} \approx 15.23$$

$df = 9$.

Step 3: Make a Decision (Critical Value Method).

The critical value is $t_{0.05,9} = 1.833$. Since $15.23 > 1.833$, **Reject** H_0.

Statistical Interpretation: There is **sufficient statistical evidence** to reject the null hypothesis at the 5% significance level and conclude that the new blood pressure drug is effective in reducing the systolic blood pressure.

Solution: Clinical Trial Data (Cont.)

Step 3: Make a Decision (p-value).
For $t = 15.23$ with $df = 9$ (right-tailed):

- From t-table: $t_{0.005,9} = 3.25$ (smallest value our table)
- Our $t = 15.23 \gg 3.25$, so p-value $\ll 0.005$

Since p-value extremely small and less than $\alpha = 0.05$: **Reject** H_0 and conclude that there is strong statistical evidence that the blood pressure medication significantly reduces systolic blood pressure.

95% CI for μ_d.
$t_{0.025,9} = 2.262$.

$$\bar{d} \pm t_{\alpha/2,n-1} \cdot \frac{s_d}{\sqrt{n}} \equiv 6.0 \pm 2.262 \times 0.394 = 6.0 \pm 0.891 = (5.109,\ 6.891) \text{ mmHg}$$

We are 95% confident the mean reduction in systolic BP is between 4.48 and 7.92 units.

Comparing Two Population Proportions (Large Samples)

Notations

- Population proportions of successes (parameter):

$$p_1 = \frac{X_1}{N_1}, \quad p_2 = \frac{X_2}{N_2} \quad (\text{Unknown})$$

- Population proportions of failures (parameter):

$$q_1 = 1 - p_1, \quad q_2 = 1 - p_2 \quad (\text{Unknown})$$

- Sample proportions of successes (statistic):

$$\hat{p}_1 = \frac{x_1}{n_1}, \quad \hat{p}_2 = \frac{x_2}{n_2}$$

- Sample proportions of failures (statistic):

$$\hat{q}_1 = 1 - \hat{p}_1, \quad \hat{q}_2 = 1 - \hat{p}_2$$

Comparing Two Population Proportions (Large Samples)

Sometimes we are interested in comparing the proportion of "successes" in two binomial populations. Below are three common testing scenarios:

Examples of Comparison

- **Right-tailed test ($p_1 > p_2$):** Comparing whether the proportion of customers who make a purchase after seeing Ad A is higher than after seeing Ad B.

$$H_0 : p_1 \leq p_2 \qquad \text{vs.} \qquad H_a : p_1 > p_2$$

- **Left-tailed test ($p_1 < p_2$):** Testing whether the proportion of smokers among urban residents is lower than among rural residents.

$$H_0 : p_1 \geq p_2 \qquad \text{vs.} \qquad H_a : p_1 < p_2$$

- **Two-sided test ($p_1 \neq p_2$):** Comparing whether the proportion of male voters favoring a candidate differs from the proportion of female voters favoring the same candidate.

$$H_0 : p_1 = p_2 \qquad \text{vs.} \qquad H_a : p_1 \neq p_2$$

Comparing Two Population Proportions (Large Samples)

Estimation of $p_1 - p_2$

- Point Estimate:

$$\hat{p}_1 - \hat{p}_2$$

- Interval Estimate:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

Test Statistic

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{SE(\hat{p}_1 - \hat{p}_2)} = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where we use the pooled proportion:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}, \quad \hat{q} = 1 - \hat{p}$$

Comparing Two Population Proportions (Large Samples)

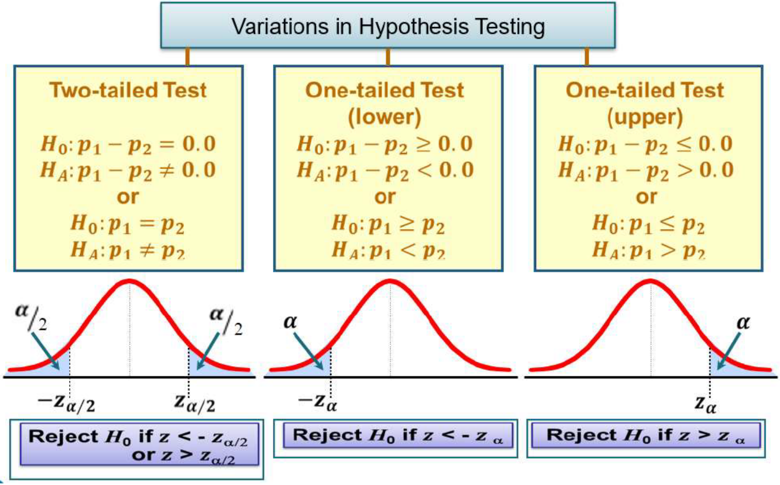
Assumptions and Conditions

- **Independence:** Sampled values are independent.
- **Randomization:** Data sampled randomly.
- **Success/Failure Condition:**

$$n_1\hat{p}_1 \geq 5, \quad n_1\hat{q}_1 \geq 5, \quad n_2\hat{p}_2 \geq 5, \quad n_2\hat{q}_2 \geq 5$$

Hypothesis Testing: Critical Value Approach

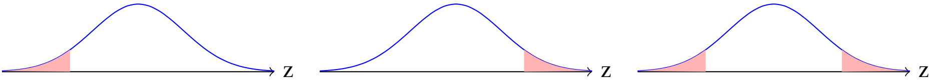
At a level of significant α, compare the test statistic z to critical value z_α (one-sided) or $z_{\alpha/2}$ (two-sided) and reject H_0 if the test statistic z value is located inside the rejection region.



Hypothesis Testing: p-value Approach

Calculation of p-value:

- p-value $= p(Z < -z)$ (Left-tailed Test)
- p-value $= p(Z > z)$ (Right-tailed Test)
- p-value $= 2p(Z > |z|)$ (or p-value $= p(Z < -z) + p(Z > z)$) (Two-tailed Test)



Left-tailed: $p = P(Z < z)$ Right-tailed: $p = P(Z > z)$ Two-tailed: $p = 2 \times P(Z > |z|)$

Decision Rule:

1. If p-value $< \alpha$, reject H_0.
2. If p-value $\geq \alpha$, do not reject H_0.

Example:

A researcher is studying whether support for Proposition A differs between men and women. In a random sample, 36 of 72 men and 31 of 50 women indicated that they would vote "Yes."

At the 0.05 level of significance:

1. Define the population parameters and state the null and alternative hypotheses to test whether the proportion of men who will vote "Yes" differs from the proportion of women.
2. Compute the appropriate test statistic for comparing the two proportions.
3. Use critical value method and state your decision regarding the null hypothesis.
4. Find the p-value and state your decision regarding the null hypothesis.
5. Interpret your conclusion in the context of the problem.
6. Construct a 95% confidence interval for the difference in proportions and interpret it.

Solution

Step 1: Define parameters and hypotheses

Let p_1 = proportion of men supporting Proposition A, p_2 = proportion of women supporting Proposition A.

Given: $n_1 = 72,\ x_1 = 36 \implies \hat{p}_1 = \dfrac{x_1}{n_1} = 0.50;$

$n_2 = 50,\ x_2 = 31 \implies \hat{p}_2 = \dfrac{x_2}{n_2} = 0.62.$

$$H_0 : p_1 = p_2, \quad H_a : p_1 \neq p_2$$

Step 2: Test statistic

$$\text{Pooled proportion: } \hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{36 + 31}{72 + 50} \approx 0.549, \quad \hat{q} \approx 0.451$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{SE(\hat{p}_1 - \hat{p}_2)} = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.50 - 0.62}{\sqrt{0.549 \times 0.451\left(\frac{1}{72} + \frac{1}{50}\right)}} \approx -1.31$$

Step 3: Make a Decision (Critical value method)

Two-tailed, $\alpha = 0.05$, critical value $z_{\alpha/2} = \pm 1.96$. Since $-1.310 \in (-1.96, 1.96)$, we fail to reject H_0.

Solution (Cont.)

Step 4: Make a Decision (p-value)

$$p\text{-value} \approx 2P(Z < -1.31) \approx 2 \times 0.0951 = 0.1902 > 0.05$$

Fail to reject H_0.
Step 5: Conclusion There is insufficient evidence at the 0.05 significance level to conclude that the proportion of men supporting Proposition A differs from the proportion of women supporting it.
Step 6: 95% confidence interval

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}} \equiv (0.5 - 0.62) \pm 1.96 \times \sqrt{\frac{0.5(0.5)}{72} + \frac{0.62(0.38)}{50}}$$
$$= (-0.30, 0.06)$$

Interpretation: We are 95% confident that the true difference $p_1 - p_2$ lies between -0.297 and 0.057. Zero being inside the interval supports no significant difference.

Summary of Two-Sample Inference

Case	Test Statistic	DF	Standard Error
Two Population Means (Independent)			
Known variances	$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{SE}$	N/A (z)	$SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
Unknown variances, unequal	$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{SE}$	$\min(n_1 - 1, n_2 - 1)$	$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
Unknown variances, equal	$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{SE}$	$n_1 + n_2 - 2$	$SE = \sqrt{s_p^2(\frac{1}{n_1} + \frac{1}{n_2})}$ $s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}$
Two Population Means (Paired/Dependent)			
Paired samples	$t = \frac{d - D_0}{SE}$	$n - 1$	$SE = \frac{s_d}{\sqrt{n}}$ $d_i = x_{1i} - x_{2i}$
Two Population Proportions (Independent)			
Large samples	$z = \frac{(\hat{p}_1 - \hat{p}_2) - D_0}{SE}$	N/A (z)	$SE = \sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$ $\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$ (pooled)
Confidence Interval	$\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2} \cdot SE$	N/A	$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

Key Steps for Two-Sample Means:

1. Check if samples are independent or paired
2. If independent, check if variances known or unknown
3. If unknown, use variance ratio rule: $\frac{s^2_{\text{large}}}{s^2_{\text{small}}} > 3 \rightarrow$ unequal; $\frac{1}{3} \leq$ ratio $\leq 3 \rightarrow$ equal
4. Choose correct formula and proceed

Key Steps for Two-Sample Proportions:

1. Check conditions: $n_1\hat{p}_1 \geq 5, n_1\hat{q}_1 \geq 5, n_2\hat{p}_2 \geq 5, n_2\hat{q}_2 \geq 5$
2. For hypothesis testing: Use pooled proportion $\hat{p}$ in standard error
3. For confidence intervals: Use separate proportions $\hat{p}_1$ and $\hat{p}_2$ in standard error