Chapter 10: Hypothesis Testing
STAT 2601 – Business Statistics

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Learning Objectives

By the end of this chapter, you will be able to:

- Formulate the null (H_0) and alternative (H_1) hypotheses.
- Understand the difference between Type I and Type II errors.
- Distinguish between the z-test and t-test and identify when each should be used.
- Conduct a hypothesis test for a population mean (μ) and population proportion (p) using:
    1. The Critical Value approach.
    2. The p-value approach.
- Apply the z-test and t-test for testing the hypotheses about a population mean.
- Interpret the results in a business context.

Formulating Hypotheses

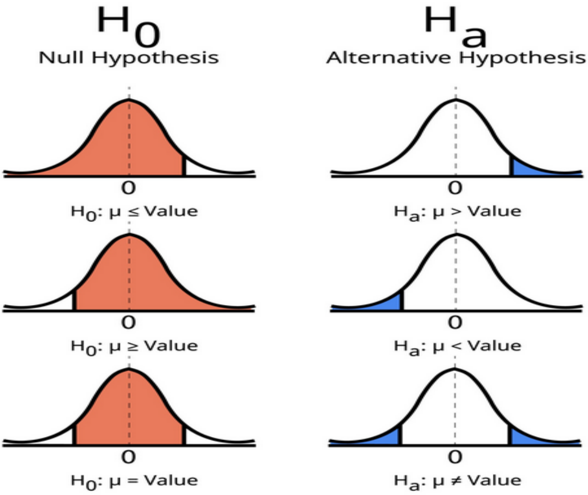
By reading the research problem, create two non-overlapping contrasting situations:

- **The Null Hypothesis (H_0):** The statement being tested. It usually represents the "status quo" or no change. It **always** contains the equality sign ($=$, or $\leq$, or $\geq$).
- **The Alternative Hypothesis (H_1 or H_a):** The statement we hope to find evidence for, which is the opposite of the null hypothesis. It never contains an equality sign ($\neq$, or $>$, or $<$).

Examples

1. A student claims that a specific study app helps them score higher than their current average of 85.
   - $H_0 : \mu \leq 85$ (85 denotes the null value $= \mu_0$)
   - $H_1 : \mu > 85$ (Right-tailed test)
2. A manager claims a new process reduces assembly time below 15 minutes.
   - $H_0 : \mu \geq 15$ minutes (15 denotes the null value $= \mu_0$)
   - $H_1 : \mu < 15$ minutes (Left-tailed test)
3. A tech firm checks if a new battery's life differs from the standard 8 hours.
   - $H_0 : \mu = 8$ hours (8 denotes the null value $= \mu_0$)
   - $H_1 : \mu \neq 8$ hours (two-tailed test)

Non-Overlapping Hypotheses



Errors in Hypothesis Testing

In statistics, we never "prove" anything; we only find evidence. This leads to two types of errors:

	H_0 **True**	H_0 **False**
Reject H_0	Type I Error (α = Significant Level)	Correct Decision = Power $(1-\beta)$
Accept H_0	Correct Decision = Confidence $(1-\alpha)$	Type II Error (β)

- **Type I Error:** Rejecting a true null hypothesis when it should not be rejected:

$$\alpha = P(\text{Reject } H_0 | H_0 \text{ true})$$

- **Type II Error:** Accepting the false null hypothesis when it should not be accepted:

$$\beta = P(\text{Accept } H_0 | H_0 \text{ false})$$

Hypothesis Testing: The Error Trade-off

Unfortunately we cannot minimize both errors (α and β) at the same time as decreasing one error necessarily increases the other. Thus, if we need to choose between bad (which is α) and worst (which is β), we have to choose α and try to avoid making β by never say "Accept H_0":

Correct terminology to avoid making type 2 error

- ✓ **Correct:** We fail to reject the null hypothesis
- ✓ **Correct:** There is insufficient evidence to reject H_0
- ✓ **Correct:** We do not find statistically significant evidence against H_0
- $\times$ **Incorrect:** We accept the null hypothesis
- $\times$ **Incorrect:** The null hypothesis is true

Note:

- Failing to reject H_0 doesn't prove it's true.
- In practice, we usually set $\alpha = 0.05$ increasing the sample size in order to increase the power.

Three Methods for Hypothesis Testing

There are three mathematically equivalent ways to make a decision regarding the Null Hypothesis (H_0). At a given significance level α:

1. **The Critical Value Method**
   - Compare the calculated test statistic to a critical value from the tables.
   - *Rule:* Reject H_0 if the test statistic falls in the "Rejection Region."
2. **The p-value Method**
   - Calculate the probability of obtaining the test statistic as extreme or more extreme than observed, assuming H_0 is true.
   - *Rule:* Reject H_0 if p-value $\leq \alpha$.
3. **The Confidence Interval (CI) Method**
   - Construct a $(1-\alpha)100\%$ confidence interval for the parameter.
   - *Rule:* Reject H_0 if the hypothesized value μ_0 falls **outside** the interval.

Note

All three methods will always lead to the same statistical conclusion.

z-Test Statistic for a Population Mean (σ Known)

Test Statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

where:

- $\bar{x}$: Sample mean
- μ_0: Null value (hypothesized population mean under H_0)
- σ: Known population standard deviation
- n: Sample size
- $\sigma/\sqrt{n}$: Standard error of the sample mean $\bar{x}$

Sampling Distribution

Under H_0, the test statistic (z-score) follows:

$$z \sim N(0, 1)$$

This is the standard normal distribution!

The Critical Value Method: Step-by-Step

1. **State the hypotheses:** H_0 and H_a

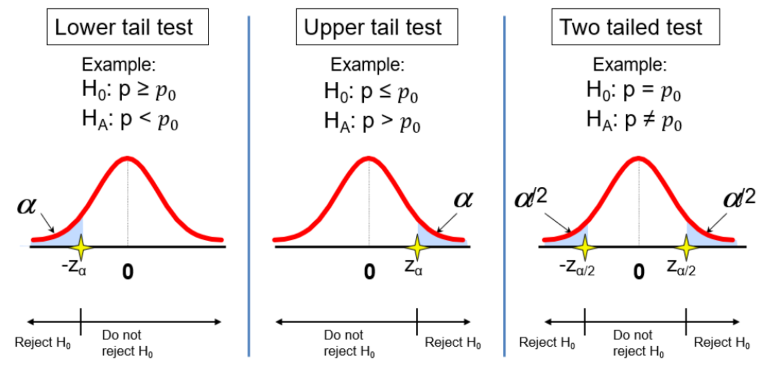
$$\begin{aligned}\text{Right tail } H_0: \mu \leq \mu_0 \text{ vs } H_a: \mu > \mu_0\\ \text{Left tail } H_0: \mu \geq \mu_0 \text{ vs } H_a: \mu < \mu_0\\ \text{Two tails } H_0: \mu = \mu_0 \text{ vs } H_a: \mu \neq \mu_0\end{aligned}$$

2. **Check assumptions:** Random sample, normality/large n, known σ
3. **Compute test statistic:**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

4. **Determine critical value(s) (tabular value(s))** based on α and H_a
5. **Make decision:**
   - Reject H_0 if test statistic falls in rejection region
   - Fail to reject H_0 otherwise
6. **State conclusion** in context

Rejection Regions



Rejection Region: Reject H_0 if:

- Left-tailed test: $z < -z_\alpha$
- Right-tailed test: $z > z_\alpha$
- Two-tailed test: $z > z_{\alpha/2}$ OR $z < -z_{\alpha/2}$ (or equivelently $|z| > z_{\alpha/2}$)

Critical Values for Common α Levels

Test Type	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.10$
Two-tailed	± 1.96	± 2.575	± 1.645
Right-tailed	1.645	2.33	1.28
Left-tailed	-1.645	-2.33	-1.28

Table: Critical z-values for common significance levels

Finding Critical Values

Use z-table:

- Two-tailed: $z_{\alpha/2}$ such that $P(Z > z_{\alpha/2}) = \alpha/2$
- Right-tailed: z_α such that $P(Z > z_\alpha) = \alpha$
- Left-tailed: $-z_\alpha$ (symmetry of normal distribution)

Example: Lightbulbs Claim (Critical Value Method)

A manufacturer claims their lightbulbs last 1000 hours on average. We test 50 bulbs and find $\bar{x} = 990$ hours. Assume $\sigma = 50$ hours. Test at $\alpha = 0.05$ if the mean lifetime is less than claimed.

1. **Hypotheses:**
   - $H_0 : \mu \geq 1000$
   - $H_a : \mu < 1000$ (left-tailed test)
2. **Assumptions:** Random sample, $n = 50 \geq 30$, σ known
3. **Test statistic:**
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{990 - 1000}{50/\sqrt{50}} = \frac{-10}{7.071} = -1.414$$
4. **Critical value:** For left-tailed test with $\alpha = 0.05$, the critical value is $z_{0.05} = -1.645$
5. **Decision:**
   - Compare: $z = -1.414$ vs $z_{0.05} = -1.645$
   - Since $-1.414 > -1.645$, test statistic is NOT in rejection region
   - **Fail to reject** H_0
6. **Conclusion:** At $\alpha = 0.05$, there is insufficient evidence to conclude that the mean lifetime is less than 1000 hours.

The p-Value Method: Step-by-Step

1. **State the hypotheses:** H_0 and H_a

$$\begin{aligned} \text{Right tail } H_0 &: \mu \leq \mu_0 \text{ vs } H_a : \mu > \mu_0 \\ \text{Left tail } H_0 &: \mu \geq \mu_0 \text{ vs } H_a : \mu < \mu_0 \\ \text{Two tails } H_0 &: \mu = \mu_0 \text{ vs } H_a : \mu \neq \mu_0 \end{aligned}$$

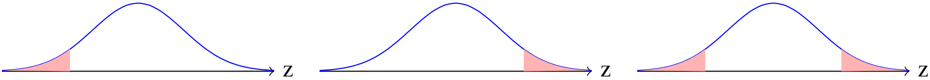
2. **Check assumptions:** Random sample, normality/large n, known σ
3. **Compute test statistic:**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

4. **Compute p-value:** Probability of obtaining test statistic as extreme or more extreme than observed, assuming H_0 is true
5. **Make decision:**
   - Reject H_0 if p-value $\leq \alpha$
   - Fail to reject H_0 if p-value $> \alpha$
6. **State conclusion** in context

Calculating p-Values

Alternative Hypothesis	**p-value Calculation**
$H_a : \mu > \mu_0$ (right-tailed)	$P(Z \geq z)$
$H_a : \mu < \mu_0$ (left-tailed)	$P(Z \leq z)$
$H_a : \mu \neq \mu_0$ (two-tailed)	$2 \times P(Z \geq \|z\|)$



Left-tailed: $p = P(Z < z)$ Right-tailed: $p = P(Z > z)$ Two-tailed: $p = 2 \times P(Z > |z|)$

Interpreting p-values

- **Small p-value**: Unlikely result if H_0 is true $\Rightarrow$ Evidence against H_0
- **Large p-value**: Likely result if H_0 is true $\Rightarrow$ Little evidence against H_0
- Common threshold: $\alpha = 0.05$

Example: Lightbulbs Claim (p-Value Method)

Problem Statement (Recall)

Lightbulbs: Claim $\mu = 1000$, Sample: $n = 50$, $\bar{x} = 990$, $\sigma = 50$, $\alpha = 0.05$, $H_a : \mu < 1000$

1. **Hypotheses:** $H_0 : \mu \geq 1000$, $H_a : \mu < 1000$
2. **Assumptions:** Satisfied
3. **Test statistic:** $z = -1.414 \approx -1.41$ (calculated earlier)
4. **p-value:** For left-tailed test:

$$\text{p-value} = P(Z \leq -1.41)$$

   From z-table: $P(Z \leq -1.411) = 0.0793$ So p-value = 0.0793
5. **Decision:**
   - Compare: p-value = 0.0793 vs $\alpha = 0.05$
   - Since 0.0793 > 0.05, p-value > α
   - **Fail to reject** H_0
6. **Conclusion:** At $\alpha = 0.05$, there is insufficient evidence to conclude that the mean lifetime is less than 1000 hours.

Practice Problem (Try This Yourself)

A cereal company claims each box contains 500g of cereal. A consumer group samples 40 boxes and finds $\bar{x} = 495g$. Assume $\sigma = 20g$. Test at $\alpha = 0.05$ if the mean weight is different from claimed.

Instructions:

1. State hypotheses
2. Check assumptions
3. Calculate test statistic
4. Use both critical value and p-value methods
5. State conclusion

Solution (Check Your Work)

- $H_0 : \mu = 500$ against $H_a : \mu \neq 500$
- $z = (495 - 500)/(20/\sqrt{40}) = -1.581 \approx -1.58$
- Critical values: ± 1.96 (fail to reject)
- p-value: $2 \times P(Z \geq 1.58) = 2 \times 0.0571 = 0.1142$ (fail to reject)
- Conclusion: Insufficient evidence that mean differs from 500g

From z-test to t-test

In most real-world situations:

- We **don't know** the population standard deviation σ
- We only have sample data: $\bar{x}$ and s

Question

What happens if we use s instead of σ in our test statistic?

$$\text{Incorrect: } z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

This doesn't follow $N(0, 1)$!

From Normal to t-Student Distribution

When we replace σ with s, the distribution changes from normal to t-distribution

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

t-Test Statistic for a Population Mean (σ Unknown)

The t-Test Statistic Formula

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where:

- $\bar{x}$: Sample mean
- μ_0: Hypothesized population mean under H_0
- s: Sample standard deviation
- n: Sample size
- $s/\sqrt{n}$: Standard error of the sample mean $\bar{x}$

Sampling Distribution

Under H_0, the test statistic follows:

$$t \sim t_{n-1}$$

The t-distribution with $n - 1$ degrees of freedom

Comparison: z-test vs t-test

z-test (σ known)	**t-test (σ unknown)**
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$	$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$
Distribution: $N(0, 1)$ **Critical values:** From z-table **Assumption:** σ known **Use when:** Rare in practice	**Distribution:** t_{n-1} **Critical values:** From t-table **Assumption:** σ unknown **Use when:** Most common case

Important

Always use t-test when σ is unknown, even for large samples!

- Some textbooks say "use z when $n \geq 30$"
- Modern practice: always use t when σ unknown
- t converges to z as n increases anyway

Critical Value Method: General Procedure

1. **State hypotheses** (H_0 and H_a)

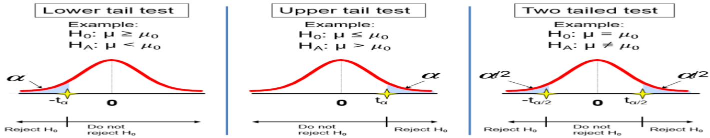
$$\begin{aligned} \text{Right tail } H_0 &: \mu \leq \mu_0 \text{ vs } H_a : \mu > \mu_0 \\ \text{Left tail } H_0 &: \mu \geq \mu_0 \text{ vs } H_a : \mu < \mu_0 \\ \text{Two tails } H_0 &: \mu = \mu_0 \text{ vs } H_a : \mu \neq \mu_0 \end{aligned}$$

2. **Check assumptions** (random sample, normality, independence)
3. **Compute test statistic**

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

4. **Determine critical value(s)** based on:
   - Significance level α
   - Type of test (one-tailed vs two-tailed)
   - Degrees of freedom $df = n - 1$
5. **Make decision**:
   - Reject H_0 if test statistic in rejection region
   - Fail to reject H_0 otherwise
6. **State conclusion** in context

Critical Values for Different Test Types



Test Type	**Rejection Region**	**Critical Value(s)**	**t-Table Lookup**
Right-tailed test	$t > t_{\alpha,df}$	$t_{\alpha,df}$	Use column for α
Left-tailed test	$t < -t_{\alpha,df}$	$-t_{\alpha,df}$	Use column for α
Two-tailed test	$\lvert t\rvert > t_{\alpha/2,df}$	$\pm t_{\alpha/2,df}$	Use column for $\alpha/2$

df	$t_{0.10}$	$t_{0.05}$	$t_{0.025}$
5	1.476	2.015	2.571
10	1.372	1.812	2.228
15	1.341	1.753	2.131
20	1.325	1.725	2.086
25	1.316	1.708	2.060
30	1.310	1.697	2.042
∞	1.282	1.645	1.960

Example: $\alpha = 0.05$, $df = 15$

- Right-tailed: $t_{0.05,15} = 1.753$
- Left-tailed: $-t_{0.05,15} = -1.753$
- Two-tailed: $\pm t_{0.025,15} = \pm 2.131$

Symmetry Property

For one-tailed tests:

$$t_{\alpha,df} = -t_{1-\alpha,df}, \text{ where } P(T > t_{\alpha,df}) = \alpha$$

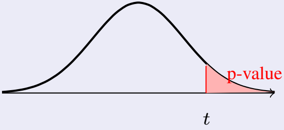
p-Value Method: General Procedure

1. **State hypotheses** (H_0 and H_a)
2. **Compute test statistic** $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$
3. **Calculate p-value**: Probability of obtaining results as extreme or more extreme than observed assuming H_0 is true
4. **Make decision**:
   - Reject H_0 if p-value $\leq \alpha$
   - Fail to reject H_0 if p-value $> \alpha$
5. **State conclusion** in context

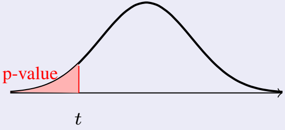
Calculating p-Values for Different Test Types

Test Type	**p-value Formula**	**Interpretation**
$H_a: \mu > \mu_0$ (right-tailed)	p-value $= P(T \geq t)$	Area in right tail
$H_a: \mu < \mu_0$ (left-tailed)	p-value $= P(T \leq t)$	Area in left tail
$H_a: \mu \neq \mu_0$ (two-tailed)	p-value $= 2 \times P(T \geq \|t\|)$	Area in both tails

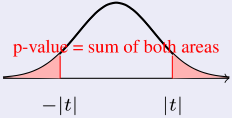
Right-tailed test ($H_a: \mu > \mu_0$)



Left-tailed test ($H_a: \mu < \mu_0$)



Two-tailed test ($H_a: \mu \neq \mu_0$)



Example

New Energy Drink Performance

A sports nutrition company introduces a new energy drink and claims that it leads to an increase in athletic performance relative to the current standard performance score of 75 points. To evaluate this claim at a significance level of $\alpha = 0.05$, a researcher takes a random sample of $n = 25$ athletes who consumed the new energy drink. The sample mean performance score is $\bar{x} = 82$ points, with a sample standard deviation of $z = 15$ points. Based on this information, is there sufficient evidence to conclude that the new energy drink increases athletic performance compared to the standard?

Solution:

Step 1: State the Hypotheses

The company claims an **increase** in performance, so we use a **right-tailed test**.

$$H_0 : \mu \leq 75$$
$$H_a : \mu > 75$$

where μ is the population mean performance score.

Step 2: Compute the Test Statistic

Since the population standard deviation σ is unknown, we use a **one-sample t-test**.

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{82 - 75}{15/\sqrt{25}} = \frac{7}{3} \approx 2.33$$

Degrees of freedom:

$$df = n - 1 = 25 - 1 = 24$$

Step 3: Critical Value Method

For a right-tailed test at $\alpha = 0.05$ with $df = 24$, the critical value from the t-table is:

$$t_{\alpha,df} = t_{0.05,\,24} = 1.711$$

$$\text{Reject } H_0 \text{ if } t > t_{\alpha,df}$$

Step 4: Decision (Critical Value Method)

Our Test Statistic: $t = 2.333 > t_{\alpha,df} = 1.711$, so we reject H_0.

Step 5: Conclusion (Critical Value Method)

At the 5% significance level, there is sufficient evidence to conclude that the new energy drink increases athletic performance compared to the standard.

Example: New Energy Drink Performance (Cont.)

p-Value Method

For right-tailed test:

$$p\text{-value} = P(T \geq t) = P(T \geq 2.333), \quad \text{with } df = 24$$

Using t-Table at $df = 24$:

t value	One-tailed p
$t_{0.025} = 2.064$	0.025
$t_{0.010} = 2.492$	0.010
2.333	between 0.01 and 0.025 which is less than $\alpha = 0.05$

Since 0.01p-value < 0.025 is less than $\alpha = 0.05$, we reject H_0 and conclude that at the 5% significance level, there is sufficient evidence to conclude that the new energy drink increases athletic performance compared to the standard.

Example: Coffee Temperature

A coffee shop claims their coffee is served at 75°C. A customer suspects it's different. They measure temperature of 12 cups:

$$73, \quad 76, \quad 74, \quad 72, \quad 77, \quad 75, \quad 74, \quad 73, \quad 76, \quad 75, \quad 74, \quad 75$$

Test at $\alpha = 0.05$ if the mean temperature differs from 75°C.

Solution

1. **Hypotheses**:
   - $H_0 : \mu = 75$
   - $H_a : \mu \neq 75$
2. **Sample Statistics**: From the data (calculate or given):
   - Sample size: $n = 12$
   - Sample mean: $\bar{x} = \sum_{i=1}^{n} = 74.5$
   - Sample standard deviation: $s = \sqrt{\frac{\sum x^2 - (\sum x)^2/n}{n-1}} = 1.5$
   - Degrees of freedom: $df = n - 1 = 11$

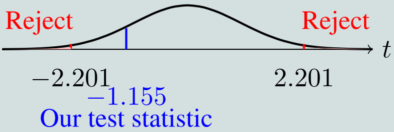
Example: Coffee Temperature (Cont.)

Test Statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{74.5 - 75}{1.5/\sqrt{12}} = \frac{-0.5}{1.5/3.464} = \frac{-0.5}{0.433} = -1.155$$

Example: Critical Value Method

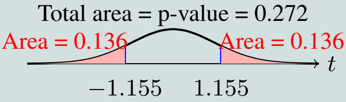
1. Significance level: $\alpha = 0.05$
2. Critical value: $t_{\alpha/2,df} = t_{0.025,11} = 2.201$
3. Decision rule: Reject if $|t| > 2.201$
4. Our test statistic: $|t| = 1.155$
5. Since $1.155 < 2.201$, we fail to reject H_0



Example: Coffee Temperature (Cont.)

p-value calculation

p-value $= 2 \times P(T \geq |1.155|)$. From t-table with $df = 11$, we find $t_{0.10,11} = 1.363$ ($p = 0.2$) where our test statistic: $t = -1.155$ is less than 1.363. Thus, we conclude that the test statistic has a corresponding p-value that is larger than $p = 0.2$ and hence larger than $\alpha = 0.05$. So we fail to reject H_0.



Conclusion for Both Methods

At $\alpha = 0.05$, we fail to reject H_0.

Interpretation: There is insufficient evidence to conclude that the mean coffee temperature differs from 75°C.

Notes

- Both methods gave the same conclusion (as they always should!)
- We say "fail to reject H_0" not "accept H_0"
- The coffee might actually be different, but we don't have enough evidence with this sample

Practice Problem: Textbook Weights (Try This Yourself)

A publisher claims their statistics textbook weighs 2.0 kg on average. You weigh 8 randomly selected textbooks:

$$1.9, \quad 2.1, \quad 2.0, \quad 1.8, \quad 2.2, \quad 1.9, \quad 2.0, \quad 2.1$$

Test at $\alpha = 0.05$ if the mean weight differs from 2.0 kg.

Guided Steps

1. State hypotheses
2. Calculate $\bar{x}$ and s
3. Compute test statistic
4. Find critical value
5. Make decision
6. State conclusion

Solution Check

- $\bar{x} = 2.0$, $s = 0.1414$
- $t = 0$ (exactly at null!)
- Critical value: $t_{0.025,7} = 2.365$
- $|0| < 2.365$: Fail to reject
- Conclusion: No evidence mean differs

z Tests about a Population Proportion

We use a z-test for a population proportion when:

- We want to test a claim about a **population proportion** (p)
- Our data are **categorical** (success/failure, yes/no)
- We have a **single sample** from the population

Examples:

- Testing if the proportion of voters supporting a candidate differs from 50%
- Testing if a drug's success rate is greater than 70%
- Testing if defect rate in a factory is less than 5%

Conditions for z-Test for Proportion

Three conditions must be satisfied:

Condition 1: Random Sample

The sample must be randomly selected from the population.

Condition 2: Success-Failure Condition

Both $np_0 \geq 5$ and $nq_0 \geq 5$, where:

- n = sample size
- p_0 = null value (hypothesized population proportion) and $q_0 = 1 - p_0$

Condition 3: Independence

The sampled values must be independent of each other.

The z-Test Statistic Formula

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Where:

- $\hat{p}$ = sample proportion = $\frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$
- p_0 = hypothesized population proportion

Note:

- Denominator uses p_0, not $\hat{p}$!
- The denominator, $\sqrt{\frac{p_0(1-p_0)}{n}}$, is the *standard error* of $\hat{p}$ under H_0

General Steps for Hypothesis Testing

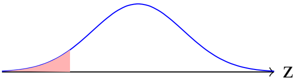
1. **State** the null and alternative hypotheses
2. **Check** the conditions
3. **Calculate** the test statistic
4. **Make a decision** using either:
   - Critical value method, OR
   - P-value method
5. **Interpret** the conclusion in context

Critical Value Method: Step-by-Step

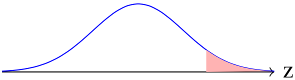
1. Determine significance level α (commonly 0.05)
2. Identify the critical value(s) from z-table:
   - Right-tailed test: z_α
   - Left-tailed test: $-z_\alpha$
   - Two-tailed test: $\pm z_{\alpha/2}$
3. Calculate test statistic $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0 q_0}{n}}}$, where $q_0 = 1 - p_0$
4. Compare z to critical value(s)
5. Decision rule:
   - If $z > z_\alpha$ (right-tailed): Reject H_0
   - If $z < -z_\alpha$ (left-tailed): Reject H_0
   - If $|z| > z_{\alpha/2}$ (two-tailed): Reject H_0

P-Value Method: Step-by-Step

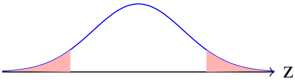
1. Calculate test statistic z
2. Find the p-value:
   - Right-tailed test: $p = P(Z > z)$
   - Left-tailed test: $p = P(Z < z)$
   - Two-tailed test: $p = 2 \times P(Z > |z|)$
3. Compare p-value to significance level α
4. Decision rule:
   - If $p \leq \alpha$: Reject H_0
   - If $p > \alpha$: Fail to reject H_0



Left-tailed: $p = P(Z < z)$



Right-tailed: $p = P(Z > z)$



Two-tailed: $p = 2 \times P(Z > |z|)$

Example: iPod Failure Rate

MacIn Touch reported that several versions of the iPod reported failure rates of 20% or more. From a customer survey, the colour iPod, first released in 2004, showed 64 failures out of 517. Is there any evidence that the failure rate for this model may be lower than the 20% rate of previous models? Assume $\alpha = 0.001$.

Step 1: Null and Alternative Hypotheses

$$H_0 : p \geq 0.20 \quad \text{vs} \quad H_a : p < 0.20 \quad \text{(Left-tailed Test)}$$

Step 2: Check Success/Failure Condition

$$np_0 = 517 \times 0.20 = 103.4 > 5 \checkmark$$
$$nq_0 = 517 \times 0.80 = 413.6 > 5 \checkmark$$
$$\Rightarrow \hat{p} \sim N\left(p_0, \sqrt{\frac{p_0 q_0}{n}}\right)$$

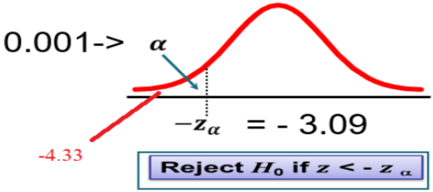
Step 3: Test Statistic Calculation

$$n = 517, \quad \hat{p} = \frac{x}{n} = \frac{64}{517} = 0.1238$$
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.1238 - 0.20}{\sqrt{\frac{(0.20)(0.80)}{517}}} = -4.33$$

iPod Failure Rate Example (Cont.)

Step 4: Make a Decision (Critical Value Method)

- **Using Critical Value Method**:
  - ★ Critical Value for $\alpha = 0.001$: $z_\alpha = -3.09$
  - ★ **Comparison:** Since $z = -4.33 < z_\alpha = -3.09$, we **reject the null hypothesis**.



- **Using P-value Method**:

$$P(Z < -4.33) \approx< 0.0001$$

Since p-value $< 0.0001 < \alpha = 0.001$, we **reject the null hypothesis**.

Conclusion

There is sufficient sample evidence to conclude that the failure rate for this iPod model is significantly lower than 20%.

Example: Indigenous population proportion test

According to census data, Indigenous peoples make up approximately 3.3% of the total Canadian population. However, the proportion of Indigenous peoples is much higher in the territories. For example, in Nunavut, about 85% of the population is Indigenous.
To investigate this claim, a random sample of 300 residents in Nunavut is selected, and 261 individuals identify as Indigenous.
At a significance level of $\alpha = 0.01$, does the sample provide sufficient statistical evidence to conclude that the proportion of Indigenous peoples in Nunavut differs from the proportion reported in the national census data?

Step 1: Null and Alternative Hypotheses

$$H_0 : p = 0.85 \quad \text{vs} \quad H_a : p \neq 0.85 \quad \text{(Two-tailed Test)}$$

Step 2: Check Success/Failure Condition

$$np_0 = 300 \times 0.85 = 255 > 5 \checkmark$$
$$nq_0 = 300 \times 0.15 = 45 > 5 \checkmark$$
$$\Rightarrow \hat{p} \sim N\left(p_0, \sqrt{\frac{p_0 q_0}{n}}\right)$$

Step 3: Test Statistic Calculation

$$n = 300, \hat{p} = \frac{x}{n} = \frac{261}{300} = 0.87$$
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.87 - 0.85}{\sqrt{\frac{0.85 \times 0.15}{300}}} = 0.97$$

Indigenous proportion Example (Cont.)

Step 4: Using Critical Value Method: Reject H_0 if $z < -z_{\alpha/2} = -2.575$ or $z > z_{\alpha/2} = 2.575$

Since $-2.575 < z = 0.97 < 2.575$, we fail to reject H_0.
Step 4: Using P-value Method:

$$\begin{aligned} 2P(Z > 0.97) &= P(Z < -0.97) + P(Z > 0.97) \\ &= 2(1 - 0.8340) = 0.332 \end{aligned}$$

Since p-value $= 0.332 > \alpha = 0.01$, we fail to reject H_0.
Step 5: Conclusion: We conclude that the proportion of indigenous people in Nunavut is NOT different from that reported in census data.

Practice Problem: Textbook Weights (Try This Yourself)

A pharmaceutical company claims their vaccine is 90% effective. In a clinical trial with 200 patients, 168 were successfully immunized. Is there evidence that the effectiveness is different from 90%? Use $\alpha = 0.05$.

Guided Steps

1. State hypotheses
2. Check the assumptions
3. Calculate $\hat{p}$
4. Compute test statistic
5. Find critical value
6. Make decision
7. State conclusion

Solution Check

- $H_0 : p = 0.90 \quad \text{vs.} \quad H_a : p \neq 0.90$
- $\hat{p} = \frac{168}{200} = 0.84$
- $z \approx -2.83$
- Critical values are ± 1.96
- Since $|z| = 2.83 > 1.96$, we reject H_0
- The p-value is approximately $0.0046 < 0.05$, so we reject H_0.
- At the 5% significance level, there is sufficient evidence to conclude that the vaccine effectiveness is different from 90%.

Summary of One-Sample Test Statistics

Population Mean μ

z-test (σ known):

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Conditions:

- σ is known
- $n \geq 30$ OR population normal

t-test (σ unknown):

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}, \quad df = n - 1$$

Conditions:

- σ is unknown

Population Proportion p

z-test:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

where $\hat{p} = \frac{x}{n}$ **Conditions:**

1. Random sample
2. Independent sample
3. $np_0 \geq 5$
4. $nq_0 \geq 5$, where $q_0 = 1 - p_0$

Two Decision Methods:

- **Critical value:** Compare test-statistic to critical value (tabular value) based on α and H_0
- **P-value:** Compare p-value to α and reject H_0 if p-value $< \alpha$.