root finding

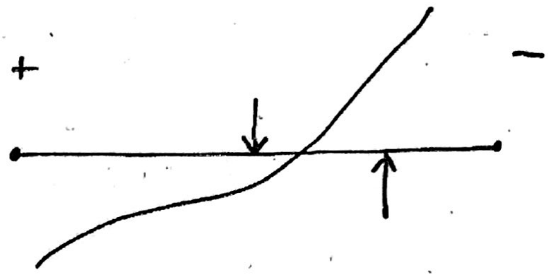
(1). Bracketing

a root is bracketed in (a, b) if $f(a)$ and $f(b)$ have opposite signs

* if $f(x)$ is continuous in (a, b), then there is a root in that interval.

* $f = \frac{1}{x-c}$ discontinuous, singularity.

(2). Bisection method.



* size of interval can be a measure of error.

We have $\epsilon_{n+1} = \epsilon_n / 2$ halved each time

We say this method converges linearly.

* if $\epsilon_{n+1} = c\,(\epsilon_n)^m$, $m > 1$, it converges superlinearly.

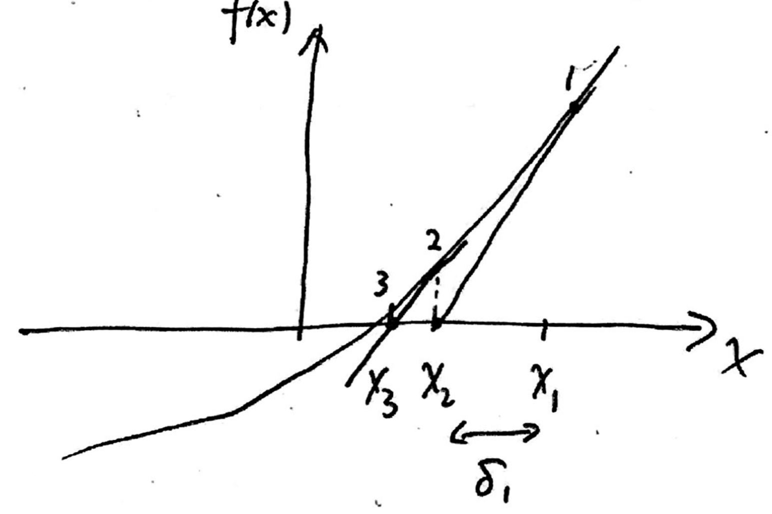
* numerically

a). if there are several roots, bisection will find one of them.

b). if there is no roots, it will find the singularity.

(3). Newton's method (Newton-Raphson)

* it is the best one in 1D root finding, requires information of $f(x)$ and $f'(x)$.



* observation $f(x+\delta) = f(x) + f'(x)\delta + \frac{1}{2} f''(x) \delta^2$

for small δ, take linear order, $f(x+\delta) = 0$

$$\Rightarrow \delta = -\frac{f(x)}{f'(x)} \quad \text{or} \quad \underbrace{x+\delta}_{\text{new point}} = x - \frac{f(x)}{f'(x)}$$

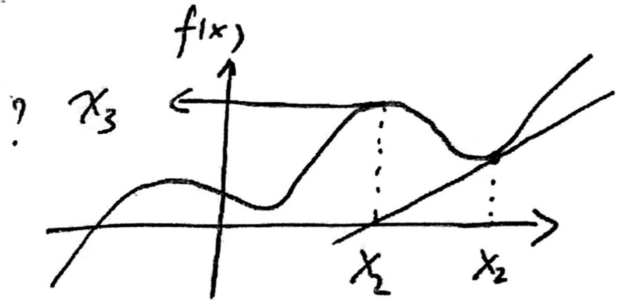
algorithm: given initial values of $f(x_1)$, $f'(x_1)$

$$\delta_1 = -\frac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = x_1 + \delta_1 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

evaluate $f(x_2)$, $f'(x_2)$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad \cdots\cdots$$

* this method can be generalized to multiple dimensions.
* initial guess of root should be close to the root.



Solution, combine bisection with Newton.

* convergence rate

in general we have $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ (A)

assume the true root is $\bar{x}$ and $x_{i+1} = \bar{x} + \epsilon_{i+1}$

$x_i = \bar{x} + \epsilon_i$

of course $f(\bar{x}) = 0$. (true root).

from (A) we have $\epsilon_{i+1} = \epsilon_i - \dfrac{f(x_i)}{f'(x_i)}$

$$f(x_i) = \underset{\substack{\parallel \\ 0}}{f(\bar{x})} + \epsilon_i f'(\bar{x}) + \frac{\epsilon_i^2}{2} f''(\bar{x})$$

$$f'(x_i) = f'(\bar{x}) + \epsilon_i f''(\bar{x})$$

$$\therefore \frac{f(x_i)}{f'(x_i)} = \frac{\epsilon_i f'(\bar{x}) + \frac{\epsilon_i^2}{2} f''(\bar{x})}{f'(\bar{x}) + \epsilon_i f''(\bar{x})} = \epsilon_i \frac{1 + \frac{\epsilon_i}{2}\frac{f''(\bar{x})}{f'(\bar{x})}}{1 + \epsilon_i \frac{f''(\bar{x})}{f'(\bar{x})}}$$

$$= \epsilon_i \left(1 + \frac{\epsilon_i}{2}\frac{f''}{f'}\right)\left(1 - \epsilon_i \frac{f''}{f'} + \cdots\right)$$

$$= \epsilon_i \left(1 - \frac{\epsilon_i}{2}\frac{f''}{f'}\right) = \epsilon_i - \frac{\epsilon_i^2}{2}\frac{f''}{f'}$$

$$\therefore \epsilon_{i+1} = \epsilon_i - \frac{f(x_i)}{f'(x_i)} = \frac{\epsilon_i^2}{2}\frac{f''}{f'} = \underline{\underline{c \ \epsilon_i^2}}$$

$\therefore$ Newton - Raphson converges quadratically.

roughly speaking the significant figures doubles each iteration.