1

Pade approximation

example $\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$

if $x \leq 1$, the series converges.

* an approximate form of the same function up to x^8 order can be written as a rational function

$$\cos x \simeq \frac{1 + p_1 x^2 + p_2 x^4}{1 + q_1 x^2 + q_2 x^4}$$ even function

Pade approximation

find p_1, p_2, q_1, q_2,

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots$$

$$\frac{1}{1+q_1x^2+q_2x^4} = 1 - (q_1 x^2 + q_2 x^4) + (q_1 x^2 + q_2 x^4)^2 + \cdots$$

$$= 1 - q_1 x^2 + (q_1^2 - q_2) x^4 + (2q_1 q_2 - q_1^3) x^6$$

$$+ (q_2^2 - 3 q_1^2 q_2 + q_1^4) x^8 + \cdots$$

$$\frac{1}{1+q_1x^2+q_2x^4} * (1 + p_1 x^2 + p_2 x^4) =$$

$\cos x$

x^2 term

$$p_1 - q_1 = -\frac{1}{2} \quad (A)$$

x^4

$$q_1^2 - p_1 q_1 + p_2 - q_2 = \frac{1}{24}$$

x^6 $$-(q_1^3 - p_1 q_1^2 + p_2 q_1 - 2q_1 q_2 + p_1 q_2) = -\frac{1}{720}$$

x^8 $$q_1^4 - p_1 q_1^3 + p_2 q_1^2 - 3q_1^2 q_2 + 2p_1 q_1 q_2 - p_2 q_2 + q_2^2 = \frac{1}{40,320}$$

$$* \quad q_1^2 - p_1 q_1 + p_2 - q_2 = q_1(q_1 - p_1) + p_2 - q_2$$
$$= \frac{1}{2} q_1 + p_2 - q_2 = \frac{1}{24} \quad (B)$$

x^6 term $\Rightarrow -\left(q_1^2 (q_1 - p_1) + p_2 q_1 - q_1 q_2 + q_2 (p_1 - q_1) \right)$ use (A)

$$= -\left(\frac{q_1^2}{2} + q_1 (p_2 - q_2) - \frac{1}{2} q_2 \right)$$ use (B)

$$= -\left(\frac{q_1}{24} - \frac{1}{2} q_2 \right) = -\frac{1}{720} \quad (C)$$

x^8 term $\Rightarrow$ $$\frac{1}{720} q_1 - \frac{1}{24} q_2 = \frac{1}{40,320} \quad (D)$$

solve linear equations to find

$$p_1 = -\frac{115}{252}, \quad p_2 = \frac{313}{15,120}, \quad q_1 = \frac{11}{252}, \quad q_2 = \frac{13}{15,120}$$

finally

$$\cos x \cong \frac{15,120 - 6900\, x^2 + 313\, x^4}{15,120 + 660\, x^2 + 13\, x^4}$$

test RHS $= 0$ at $x = -1.5708259$

LHS $= 0$ at $x = -\frac{\pi}{2} = -1.5707963$.

The general form of Padé approximation

$$f(x) = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$

$Q_m(x) = 1 + q_1 x + \cdots + q_m x^m$.

---

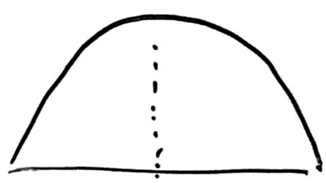
example, Fermi function $f(x) = \dfrac{1}{e^{\beta x} + 1}$

$$f(x) \simeq \frac{1}{2} - \sum_{j=1}^{N} \frac{2\alpha_j x}{x^2 + \beta_j^2}, \qquad \text{find } \alpha_j \text{ and } \beta_j$$

poles

$e^{\beta x} + 1 = 0 \Rightarrow x = ikT(2\pi n + \pi)$ infinite # of poles.

$\int_{-\infty}^{+\infty} f(x)\, g(x)\, dx$ using theorem of residue.



$$e^{-x} \approx \frac{1.00000\,00007 - 0.47593\,58618x + 0.08849\,21370x^2 - 0.00656\,58101x^3}{1 + 0.52406\,42207x + 0.11255\,48636x^2 + 0.01063\,37905x^3}$$

$$\frac{\tan^{-1} x}{x} \approx \frac{0.99999\,99992 + 1.13037\,54276x^2 + 0.28700\,44785x^4 + 0.00894\,72229x^6}{1 + 1.46370\,86496x^2 + 0.57490\,98994x^4 + 0.05067\,70959x^6}$$

$$\ln \tfrac{1}{2}(1+x) \approx \frac{-0.69314\,71773 + 0.06774\,12133x + 0.52975\,01385x^2 + 0.09565\,58162x^3}{1 + 1.34496\,44663x + 0.45477\,29177x^2 + 0.02868\,18192x^3} \tag{3-16}$$

The maximum errors for these three approximations are quoted in Fröberg (*Numerical Mathematics*, Benjamin/Cummings, Menlo Park, California, 1985) to be, respectively, 7.34×10^{-10}, 7.80×10^{-10}, and 3.29×10^{-9}. Note also that the constant term in the numerator in each expression is slightly different from the exact value of the function at $x = 0$. This is done so that the overall accuracy in the entire range may be improved.