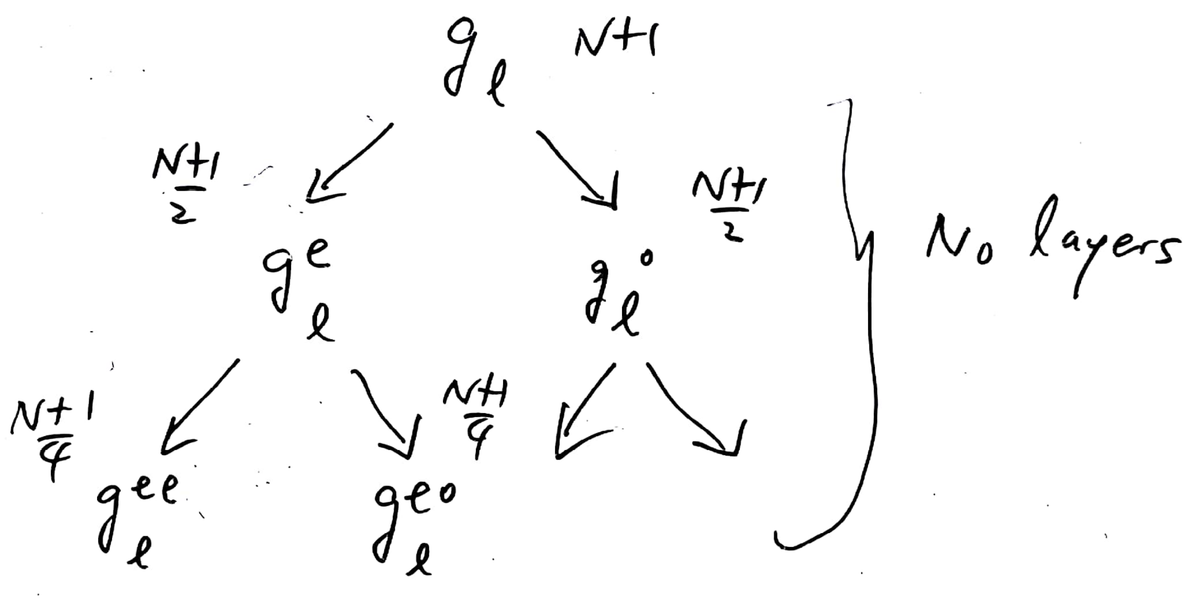
tree structure

$N+1 = 2^{N_0}$



for example $0, 1, 2, 3, 4, 5, 6, 7$

$e = \{0, 2, 4, 6\}$

$o = \{1, 3, 5, 7\}$

$ee = \{0, 4\}, \quad eo = \{2, 6\}, \quad oe = \{1, 5\}, \quad oo = \{3, 7\}$

\# of operations for FFT $(N+1)\log_2(N+1)$

$\Downarrow$

N_0

similarly $g_l^o = g_l^{oe} + e^{\frac{-2\pi i l}{\frac{N+1}{2}}} g_l^{oo}$

here we restrict l to be $(0, [\frac{N+1}{4}])$.

for $l = ([\frac{N+1}{2}], N+1)$, we have

$$g_{[\frac{N+1}{2}]+l} = \sum_{k=0}^{[\frac{N+1}{2}]} e^{\frac{-2\pi i(\frac{N+1}{2}+l)k}{\frac{N+1}{2}}} f_{2k} + \sum_{k=0}^{\frac{N+1}{2}} e^{-\frac{2\pi i}{N+1}(\frac{N+1}{2}+l)(2k+1)} f_{2k+1}$$

$$= \sum_{k=0}^{\frac{N+1}{2}} e^{\frac{-2\pi i k l}{\frac{N+1}{2}}} f_{2k} + e^{-\frac{2\pi i}{N+1}(\frac{N+1}{2}+l)} \sum_{k=0}^{\frac{N+1}{2}} e^{-\frac{2\pi i}{\frac{N+1}{2}}(\frac{N+1}{2}+l)k} f_{2k+1}$$

$$= g_l^e + e^{-\pi i} \alpha^{-l} g_l^o$$

$$\therefore \quad g_l = \begin{cases} g_l^e + \alpha^{-l} g_l^o & l = (0, [\frac{N}{2}]) \\ g_l^e - \alpha^{-l} g_l^o & l = ([\frac{N}{2}], N+1) \end{cases}$$

to find g_l, it is enough to find g_l^e, g_l^o for $l = (0, [\frac{N}{2}])$,

$$\begin{cases} g_l^o = \begin{cases} g_l^{oe} + \alpha^{\frac{l}{2}} g_l^{oo}, & l = 0, [\frac{N}{4}] \\ g_l^{oe} + \quad \text{"} & l = [\frac{N}{4}]+1, [\frac{N}{2}] \end{cases} \\ g_l^e = \dots \end{cases}$$

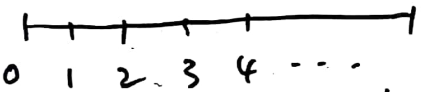
In general, assume $N+1 = 2^{N_0}$

$[\frac{N}{2}] = \frac{N-1}{2}$, if $N+1 = 8$, $[\frac{N}{2}] = 3$

$$g_l = \sum_{k=0}^{N} \alpha^{-lk} f_k$$

$$= \sum_{k=0}^{[\frac{N}{2}]} e^{-\frac{2\pi i}{N+1} l \cdot 2k} f_{2k} + \sum_{k=0}^{[\frac{N}{2}]} e^{-\frac{2\pi i}{N+1} l (2k+1)} f_{2k+1}$$

$$= \sum_{k=0}^{[\frac{N}{2}]} e^{-\frac{2\pi i l k}{\frac{N+1}{2}}} f_{2k} + e^{-\frac{2\pi i}{N+1} l} \sum_{k=0}^{[\frac{N}{2}]} e^{-\frac{2\pi i l k}{\frac{N+1}{2}}} f_{2k+1}$$



$$= g_l^e + e^{-\frac{2\pi i}{N+1} l} g_l^o$$

$(0, 2, 4) \Rightarrow$ even points

$(1, 3, 5) \Rightarrow$ odd points

$\Downarrow$ FT of $\frac{N+1}{2}$ points

where we have assumed that $l = 0, 1, \cdots, [\frac{N}{2}]$

and
$$g_l^e = \sum_{k=0}^{[\frac{N}{2}]} e^{-\frac{2\pi i l k}{\frac{N+1}{2}}} f_{2k}$$

$$= \sum_{k=0}^{[\frac{N}{4}]} e^{-\frac{2\pi i l \cdot 2k}{\frac{N+1}{2}}} f_{4k} + \sum_{k=0}^{[\frac{N}{4}]} e^{-\frac{2\pi i l (2k+1)}{\frac{N+1}{2}}} f_{4k+2}$$

$$= \sum_{k=0}^{[\frac{N}{4}]} e^{-\frac{2\pi i l k}{\frac{N+1}{4}}} f_{4k} + e^{-\frac{2\pi i l}{\frac{N+1}{2}}} \sum_{k=0}^{[\frac{N}{4}]} e^{-\frac{2\pi i l k}{\frac{N+1}{4}}} f_{4k+2}$$

$$= g_l^{ee} + e^{-\frac{2\pi i l}{\frac{N+1}{2}}} g_l^{eo}$$

$(0, 2, 4, 6, 8)/2$ $\quad$ $0, 4, 8, \cdots$ ee

$= (0, 1, 2, 3, 4, \cdots)$ $<$ $\quad$ $2, 6, 10 \cdots$ eo

Table 3-6 Reduction of g_ℓ to sums of two terms.

$8g_0 = f_{10} + f_{11}$ $8g_4 = f_{10} - f_{11}$	$f_{10} = f_0 + f_2 + f_4 + f_6$ $f_{11} = f_1 + f_3 + f_5 + f_7$
$8g_1 = f_{12} - i\alpha f_{13}$ $8g_5 = f_{12} + i\alpha f_{13}$	$f_{12} = f_0 - if_2 - f_4 + if_6$ $f_{13} = f_1 - if_3 - f_5 + if_7$
$8g_2 = f_{14} - i\, f_{15}$ $8g_6 = f_{14} + i\, f_{15}$	$f_{14} = f_0 - f_2 + f_4 - f_6$ $f_{15} = f_1 - f_3 + f_5 - f_7$
$8g_3 = f_{16} - \alpha\, f_{17}$ $8g_7 = f_{16} + \alpha\, f_{17}$	$f_{16} = f_0 + if_2 - f_4 - if_6$ $f_{17} = f_1 + if_3 - f_5 - if_7$

The intermediate quantities f_{1k}, for $k = 0, 1, \ldots, 7$, in the table again display a symmetry between the pairs of elements (0,4), (1,5), (2,6), and (3,7). In fact, a similar table to Table 3-6 can be constructed for f_{1k} (see Table 3-7).

Table 3-7 Linear combinations of f_k for calculating g_ℓ.

$f_{10} = f_{20} + f_{21}$ $f_{14} = f_{20} - f_{21}$	$f_{20} = f_0 + f_4$ $f_{21} = f_2 + f_6$
$f_{11} = f_{22} + f_{23}$ $f_{15} = f_{22} - f_{23}$	$f_{22} = f_1 + f_5$ $f_{23} = f_3 + f_7$
$f_{12} = f_{24} - i f_{25}$ $f_{16} = f_{24} + i f_{25}$	$f_{24} = f_0 - f_4$ $f_{25} = f_2 - f_6$
$f_{13} = f_{26} - i f_{27}$ $f_{17} = f_{26} + i f_{27}$	$f_{26} = f_1 - f_5$ $f_{27} = f_3 - f_7$

In an actual calculation, we start with the construction of the eight $f_{2\ell}$ from pairs of input f_k as given in Table 3-7. Next we calculate the eight $f_{1\ell}$ from pairs of f_{2k} just obtained. The final step involves the calculation of the eight g_ℓ from pairs of f_{1k}. The total number of operations is therefore $(N+1)\log_2(N+1) = 8 \times 3 = 24$

$f_{13} = f_1 - i f_3 - f_5 + i f_7$ odd points

then $8 g_1 = f_{12} - i\alpha f_{13}$

$8 g_5 = f_{12} + i\alpha f_{13}$

Similar treatment can be done for (g_2, g_6), (g_3, g_7).

See table 3-6.

$\therefore$ to find $g_0, g_1, \cdots, g_7$, all we need is to find $f_{10}, f_{11}, \cdots, f_{17}$

Now look at (f_{10}, f_{14}), (f_{11}, f_{15}), (f_{12}, f_{16}), (f_{13}, f_{17}).

define $f_{20}, f_{21}, \cdots, f_{27}$

then $f_{10} = f_{20} + f_{21}$ $\cdots$ see table 3-7.

$f_{14} = f_{20} - f_{21}$

---

actual calculation

FT : # of operation $(N+1)^2 = 64$

FFT : (1) f_{2i} 8 operations

(2) f_{1i} 8 operations

(3) g_i 8 $\Rightarrow 8 \times 3 = (N+1)\log_2(N+1)$

3 layers $(\log_2(N+1))$, each layer requires $(N+1)$ operations.

$\therefore$ A can be rewritten as

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & i & \cdots & i\alpha \\ 1 & i & -1 & \cdots & -i \\ \vdots \\ 1 & -i\alpha & & \cdots & \alpha \end{pmatrix}$$

$$B = \frac{A^{-1}}{8} = \frac{1}{8}\begin{pmatrix} & \cdots & \end{pmatrix},$$ A and B have the same symmetry.

$\underline{g = Bf}$

Fast Fourier transform, set $N+1 = 8$

We look at $(g_0, g_4), (g_1, g_5), (g_2, g_6), (g_3, g_7)$

$$8g_0 = f_0 + f_1 + f_2 + \cdots + f_7$$
$$8g_4 = f_0 - f_1 + f_2 - \cdots - f_7$$

define $f_{10} = f_0 + f_2 + f_4 + f_6$ even points

$f_{11} = f_1 + f_3 + f_5 + f_7$ odd points

then $8g_0 = f_{10} + f_{11}$

$8g_4 = f_{10} - f_{11}$

$$8g_1 = f_0 - i\alpha f_1 - if_2 - \alpha f_3 - f_4 + i\alpha f_5 + if_6 + \alpha f_7$$
$$8g_5 = f_0 + i\alpha f_1 - if_2 + \alpha f_3 - f_4 - i\alpha f_5 + if_6 - \alpha f_7$$

define $f_{12} = f_0 - if_2 - f_4 + if_6$ even points

As a result, the matrix $\boldsymbol{A}$ in (3-69) takes on a particularly simple form

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & i & i\alpha & -1 & -\alpha & -i & -i\alpha \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\alpha & -i & \alpha & -1 & -i\alpha & i & -\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\alpha & i & -i\alpha & -1 & \alpha & -i & i\alpha \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\alpha & -i & -\alpha & -1 & i\alpha & i & \alpha \end{pmatrix}$$

There are many "symmetries" in this matrix. For example, the elements in the first four rows are similar to those in the second four rows, except that the odd elements (start the counting of the elements in each row from zero) have the opposite signs.

These symmetries carry over to $\boldsymbol{B}$, the inverse of $\boldsymbol{A}$. For example, for the $(N+1) = 8$ case we are dealing with here,

$$\boldsymbol{B} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -i\alpha & -i & -\alpha & -1 & i\alpha & i & \alpha \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -\alpha & i & -i\alpha & -1 & \alpha & -i & i\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & i\alpha & -i & \alpha & -1 & -i\alpha & i & -\alpha \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \alpha & i & i\alpha & -1 & -\alpha & -i & -i\alpha \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array}$$

where we have made use of the fact that $\alpha^{-\ell} = \alpha^{N+1-\ell}$. By making use of the symmetries in $\boldsymbol{B}$, methods of FFT can reduce the number of operations required to obtain all $(N+1)$ Fourier coefficients by a large extent, usually from $(N+1)^2$ to $(N+1)\log_2(N+1)$. This is a very significant factor, especially when $(N+1)$ is large.

We can now make use of (3-60) to construct a set of equations relating the unknown Fourier coefficients $\{g_\ell\}$ with the input quantities $\{f_i\}$ for the function $f(x)$. For simplicity, we shall take $x_0 = 0$ and, as a result of the assumption of evenly spaced points, we have the relation $x_k = kh$. On applying (3-60) for $x = x_k$, we obtain the result

$$f_k = f(x_k) = \sum_{\ell=0}^{N} g_\ell e^{i\ell\pi x_k/L} \tag{3-67}$$

The summation goes from 0 to N here. This comes from the fact that we have only $(N+1)$ pieces of input information, $f_0, f_1, \ldots, f_N$, and, consequently, we can determine at most $(N+1)$ coefficients $g_0, g_1, \ldots, g_N$.

To simplify the notation, we shall write

$$\alpha \equiv e^{i\pi h/L} = e^{i2\pi/(N+1)} \tag{3-68}$$

since $2L = (N+1)h$. For each of the $(N+1)$ values of $f(x)$ given to us, we have an equation of the form of (3-67). In terms of α, these $(N+1)$ equations may be put into the form

$$\begin{array}{l}
g_0 + g_1 + g_2 + g_3 + \cdots + g_N = f_0 \\
g_0 + \alpha\, g_1 + \alpha^2 g_2 + \alpha^3 g_3 + \cdots + \alpha^N g_N = f_1 \\
g_0 + \alpha^2 g_1 + \alpha^4 g_2 + \alpha^6 g_3 + \cdots + \alpha^{2N} g_N = f_2 \\
g_0 + \alpha^3 g_1 + \alpha^6 g_2 + \alpha^9 g_3 + \cdots + \alpha^{3N} g_N = f_3 \\
\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
g_0 + \alpha^N g_1 + \alpha^{2N} g_2 + \alpha^{3N} g_3 + \cdots + \alpha^{NN} g_N = f_N
\end{array}$$

In matrix notation, they may be expressed as

$$Ag = f \tag{3-69}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^N \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \cdots & \alpha^{2N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^N & \alpha^{2N} & \alpha^{3N} & \cdots & \alpha^{NN} \end{pmatrix} \qquad g = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} \qquad f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

Our aim here is to solve this equation and obtain the values of the elements of g.

# of operation for usual FT

For a given l, g_l needs $(N+1)$ multiplication and addition.

for $l = 0, 1, 2, \cdots, N$, need $(N+1)^2$ operations.

* examine matrix A $\quad f = Ag$

$$f_k = \sum_{l=0}^{N} \alpha^{lk} g_l \qquad \therefore f_0 = \sum_{l=0}^{N} g_l$$

$$\therefore \left\{ \begin{array}{l} f_0 = g_0 + g_1 + \cdots + g_N \\ f_1 = g_0 + \alpha g_1 + \cdots + \alpha^N g_N \\ \vdots \end{array} \right.$$

For $N+1 = 8$, we have

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^7 \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{14} \\ \vdots & & & & \\ 1 & \alpha^7 & \cdots & \cdots & \alpha^{49} \end{pmatrix} \quad \text{where} \quad \alpha = e^{\frac{2\pi i}{N+1}}$$

if $N+1 = 2^{N_0}$ then $\alpha^{N+1} = 1$, $\alpha^{\frac{(N+1)}{2}} = e^{\pi i} = -1$

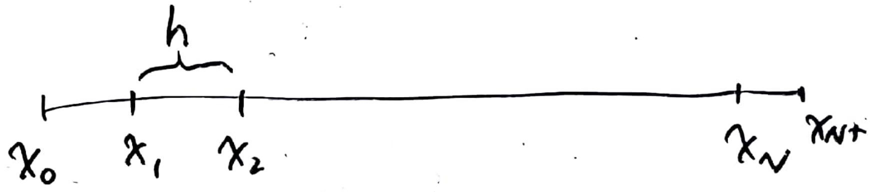
$\alpha^{\frac{N+1}{4}} = e^{\frac{\pi}{2} i} = i$, $\alpha^{\frac{N+1}{2} + l} = -\alpha^l$

for $N+1 = 8$ we have

$$\alpha^8 = 1,\ \alpha^4 = -1,\ \alpha^2 = i,\ \alpha^7 = \alpha^4 \cdot \alpha^2 \cdot \alpha = -i\alpha \text{ etc.}$$

$$\alpha^6 = -i,\ \alpha^5 = \alpha^4 \cdot \alpha = -\alpha,\ \alpha^3 = \alpha^2 \cdot \alpha = i\alpha,\ \alpha^9 = \alpha \cdots$$

Discrete Fourier transform



$$x_{N+1} - x_0 = 2L = (N+1)h$$

$$\therefore\ h = \frac{2L}{N+1}$$

set $x_0 = 0,\quad x_k = kh,\quad \alpha = e^{\frac{2\pi i h}{2L}} = e^{\frac{i\pi h}{L}} = e^{\frac{2\pi i}{N+1}}$

then $e^{\frac{i\pi l x_k}{L}} = e^{\frac{i\pi l k h}{L}} = \alpha^{lk}$

Fourier transform of $f(x_k) \equiv f_k$

$$g_l = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{i\pi l x}{L}}\, dx$$

$$= \frac{1}{2L}\sum_k f(x_k)\, e^{-\frac{i\pi l x_k}{L}}\, \Delta x, \quad \Delta x = h = \frac{2L}{N+1}$$

$$g_l = \frac{1}{N+1}\sum_{k=0}^{N} f_k\, \alpha^{-lk}$$

inverse Fourier transform

$$f_k = \sum_{l=0}^{N} \alpha^{lk} g_l$$

in matrix notation $\quad f = Ag,\quad g = A^{-1}f$