Legendre polynomials

wiki: https://en.wikipedia.org/wiki/Legendre_polynomials

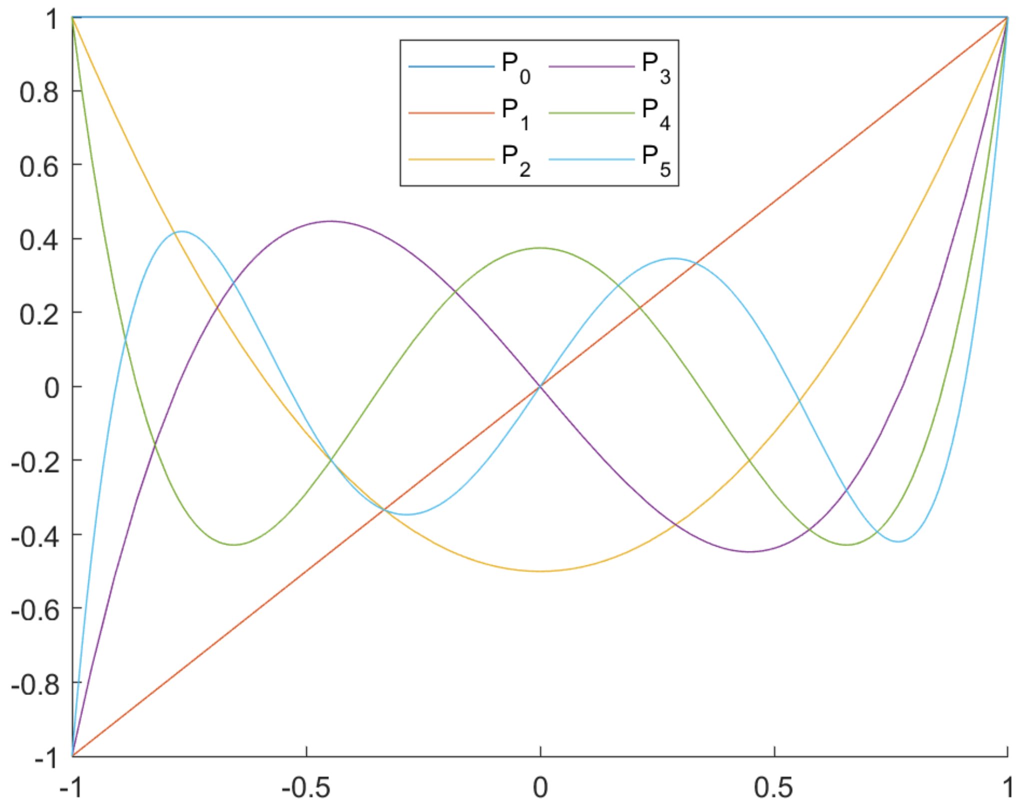
$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}$$

example:

1. $P_0(x) = 1$
2. $P_1(x) = x$
3. $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

Graph

```
fig = figure();
ax = axes(fig);
ax.NextPlot = 'add';
fp = gobjects(1,6);
for ind1 = 0:5
    [~,hf1] = my_legendre(ind1);
    fp(ind1+1) = fplot(hf1, [-1,1]);
end
legend(fp, {'P_0','P_1','P_2','P_3','P_4','P_5'},...
    'Location','north','NumColumns',2)
```



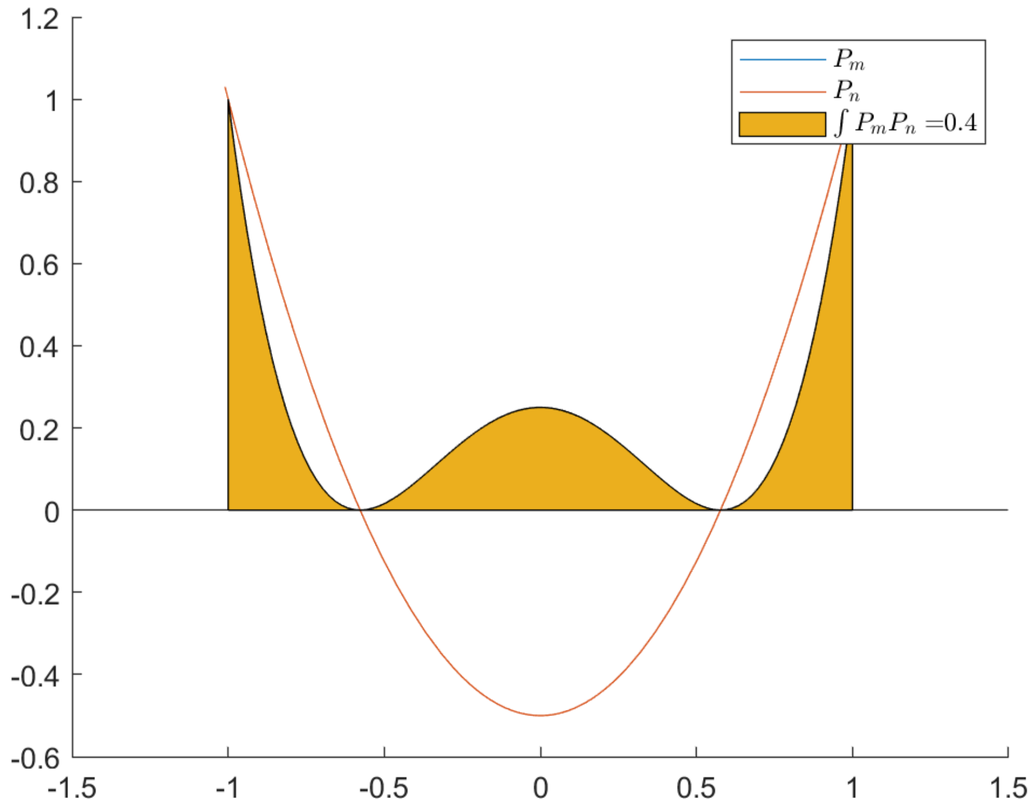
Orthognality

wiki: https://en.wikipedia.org/wiki/Legendre_polynomials#Orthogonality

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

```
M = 2;
N = 2;
[~,hf1] = my_legendre(M);
[~,hf2] = my_legendre(N);
hf3 = @(x) hf1(x).*hf2(x);
value = integral(hf3, -1, 1);

fig = figure();
ax = axes(fig);
ax.NextPlot = 'add';
fp1 = fplot(hf1, [-1.01,1.01]);
fp2 = fplot(hf2, [-1.01,1.01]);
x = linspace(-1,1,100);
harea = area(x, hf3(x));
tmp1 = {'$P_m$', '$P_n$', ['$\int{P_mP_n=}',num2str(value,4), '$']};
legend([fp1,fp2,harea], tmp1, 'Interpreter','latex')
```



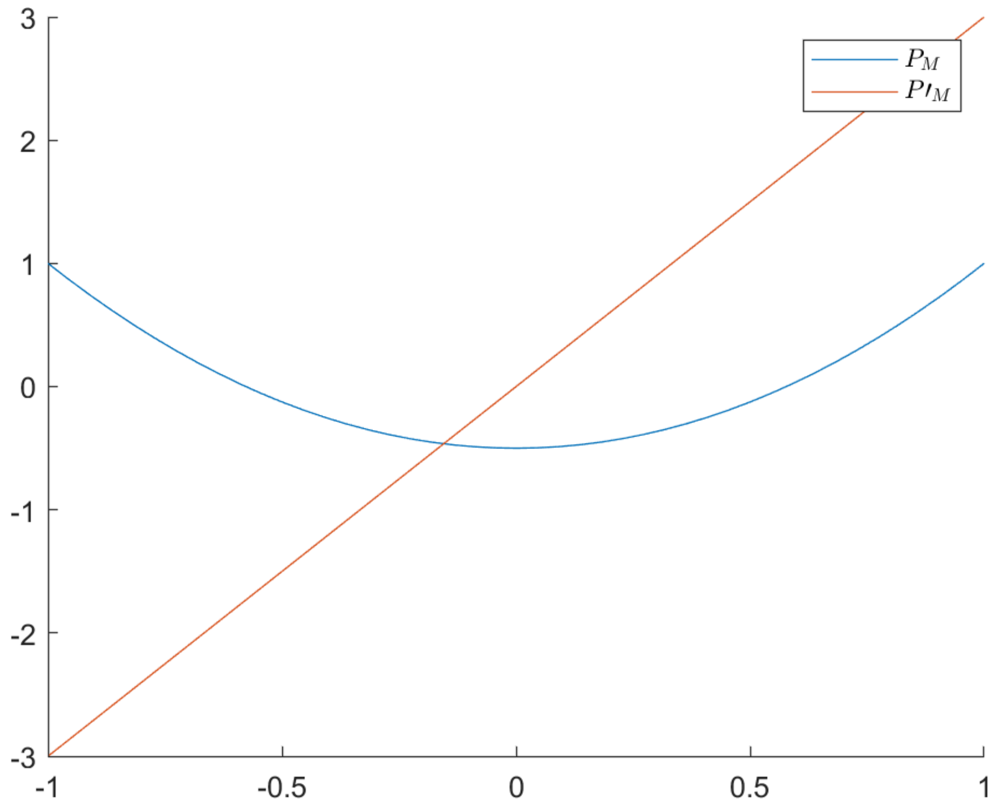
Recursion relations

wiki: https://en.wikipedia.org/wiki/Legendre_polynomials#Recursion_relations

$$\frac{x^2-1}{n}\frac{d}{dx}P_n(x) = xP_n(x) - P_{n-1}(x)$$

```
M = 2;
[~,hf1] = my_legendre(M);
hf2 = my_legendre_first_derivative(M);

fig = figure();
ax = axes(fig);
ax.NextPlot = 'add';
fp1 = fplot(hf1, [-1,1]);
fp2 = fplot(hf2, [-1,1]);
legend([fp1,fp2], {'$P_M$', '$P\prime_M$'}, 'Interpreter','latex')
```



Gaussian Quadrature Integral (Gaussian-Legendre)

wiki: https://en.wikipedia.org/wiki/Gaussian_quadrature

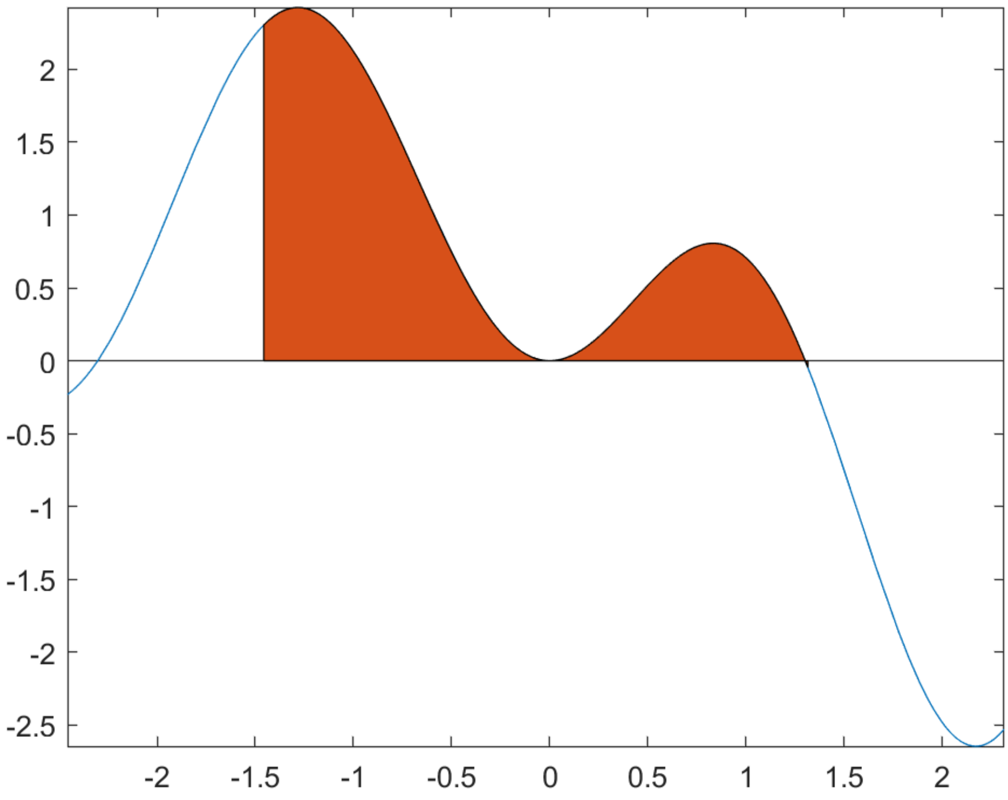
$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} g(x)dx; \quad g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)$$

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} w_i g(x_i)$$

$$w_i = \frac{2}{\left(1-x_i^2\right)\left[P_n'(x_i)\right]^2}$$

```
hf1 = @(x) (3-x-x.^2).*sin(x).^2; %integral function; hf short for function handle
a = -1-rand()/2; %lower limit
b = 1+rand()/2; %upper limit

figure()
fplot(hf1, [a-1,b+1])
hold on;
area(linspace(a,b), hf1(linspace(a,b)))
hold off
```




```
ret1 = integral(hf1, a, b); %doc('integral')
disp(['built-in integral: ', num2str(ret1,14)])
```

```
built-in integral: 2.5167613796195
```

```
ret1 = my_gaussian_quadrature(hf1, a, b, 5);
disp(['Gaussian-Quadrature integral: ', num2str(ret1,14)])
```

```
Gaussian-Quadrature integral: 2.5164503495063
```

```
function ret = my_gaussian_quadrature(hf1, a, b, num_order)
%reference: https://en.wikipedia.org/wiki/Gaussian_quadrature
% hf1(function handle)
% a(float)
% b(float)
% num_order(int)
% (ret)float
max_order = 15;
if num_order>max_order
    disp(['root of high order Legendre Polynomial is unstable, set num_order=',num2str(max_orde
    disp('detail see: https://math.stackexchange.com/questions/2636801/high-accuracy-root-finde
```

```
    num_order = max_order;
end

root_legendre = my_legendre_root(num_order);
hf2 = my_legendre_first_derivative(num_order);
weight = 2./((1-root_legendre.^2).*hf2(root_legendre).^2);

x = root_legendre*(b-a)/2 + (b+a)/2;
ret = (b-a)/2 * sum(weight.*hf1(x));
end


function ret = my_legendre_first_derivative(N)
% reference: https://en.wikipedia.org/wiki/Legendre_polynomials#Recursion_relations
% N(int)
% (ret)(function handle)
if N==0
    ret = @(x) zeros(size(x));
else
    [~,hf1] = my_legendre(N);
    [~,hf2] = my_legendre(N-1);
    ret = @(x) N*(x.*hf1(x) - hf2(x))./(x.^2-1);
end
end
```