Gaussian quadrature

* rectanglar rule

polynomial expansion.

$$f(x) = f_{i-\frac{1}{2}} + f'_{i-\frac{1}{2}} (x - x_{i-\frac{1}{2}}) + \frac{1}{2} f''_{i-\frac{1}{2}} (x - x_{i-\frac{1}{2}})^2 + \dots$$

$$\therefore \int_{x_{i-1}}^{x_i} f(x)dx = f_{i-\frac{1}{2}} \cdot h + 2^{nd} \text{ order term of } \Delta x$$

$$\downarrow$$

$$(\Delta x)^2$$

* Simpson's rule

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\, dx = f_i \cdot 2h + \frac{1}{3} f''_i h^3 + 3^{rd} \text{ order term of } \Delta x$$

$$\downarrow$$

$$(\Delta x)^3$$

* We could in general carry out polynomial expansion up to n^{th} order.

$$f(x) \simeq \sum_{l=0}^{n} \alpha_l P_l(x),$$

$\longrightarrow$ complete set of polynomials e.g. Legendre polynomials.

where $\int_{-1}^{1} P_k(x) P_l(x)\, dx = \frac{2}{2l+1} \delta_{kl}$

suppose $f(x)$ is a polynomial of order $2n-1$

$$f(x) = p_{2n-1}(x)$$

We wish to show that

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

i.e., the integration can be done using $2n$ parameters

* how to find the absissa and weight factors.

(1). polynomial division of $p_{2n-1}(x)$ by $P_n(x)$ gives
↳ Legendre po

$$p_{2n-1}(x) = p_{n-1}(x) P_n(x) + q_{n-1}(x)$$

where p_{n-1} and q_{n-1} are polynomials of order n

(2). we know that $p_{n-1}(x) = \sum_{k=0}^{n-1} a_k P_k(x)$

only involves up to $(n-1)^{th}$ Legendre poly.

$$\therefore \int_{-1}^{1} p_{n-1}(x) P_n(x)\,dx = 0 \quad \because \int P_k P_n \sim \delta_{nk}$$

or $$\int_{-1}^{1} p_{2n-1}(x)\,dx = \int_{-1}^{1} q_{n-1}(x)\,dx.$$

(3). Since $P_n(x)$ has n zeros in $[-1, 1]$, denoted as $x_1, x_2, \cdots, x_n$,

We have
$$p_{2n-1}(x_i) = p_{n-1}(x_i) P_n(x_i) + q_{n-1}(x_i)$$
$$= q_{n-1}(x_i)$$

(4) do Legendre expansion on $q_{n-1}(x)$, we find

$$q_{n-1}(x) = \sum_{k=0}^{n-1} \beta_k P_k(x)$$

$$\therefore \quad \underbrace{p_{2n-1}(x_i)}_{p_i} = \sum_{k=0}^{n-1} \beta_k \underbrace{P_k(x_i)}_{P_{ik}}, \quad \begin{array}{l} i = 1, 2, \cdots, n \\ k = 0, 1, \cdots, n-1 \end{array}$$

$$p_i = \sum_{k=0}^{n-1} P_{ik} \beta_k \quad \Rightarrow \text{ matrix notation}$$

$$p = P \beta$$

or $\beta = P^{-1} p$

$$\Rightarrow \beta_k = \sum_{i=1}^{n} (P^{-1})_{ik} \, p_i$$

note that $p_i = p_{2n-1}(x_i) = f(x_i)$

(5) now we are ready to calculate the integral

$$\int_{-1}^{1} f(x)\, dx$$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} p_{2n-1}(x)\,dx$$

$$= \sum_{k=0}^{n-1} \beta_k \int_{-1}^{1} P_k(x)\,dx \,, \quad \int_{-1}^{1} P_k(x)\,dx = \frac{2}{2k+1} \times \delta_{k0}$$

$$= 2\beta_0 = 2\sum_{i=1}^{n} p_i (P^{-1})_{i0}.$$

Since $p_i = f(x_i)$, $P_{ik} = P_k(x_i)$, define $w_i = 2(P^{-1})_{i0}$

We have $\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$.

* it can be shown that

$$w_i = \frac{2}{(1-x_i^2)\,[P_n'(x_i)]^2} \,, \quad P_n'(x_i) = \frac{d}{dx} P_n(x)\Big|_{x=x_i}$$

* For any $f(x)$, we approximate it by a polynomial of order $2n-1$, the larger the n, the better the approximation is.

then $\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{n} w_i f(x_i)$

* For $\int_a^b f(y)\,dy$, we do a transformation

$$y = \frac{b-a}{2}x + \frac{b+a}{2}$$

$$\int_a^b f(y)\,dy = \frac{b-a}{2}\int_{-1}^{1} f\left(\tfrac{1}{2}(b-a)x + \tfrac{1}{2}(b+a)\right) dx.$$