Chebyshev polynomial of the first kind

$T_n(x)$ (degree n)

1) definition $T_n(x)$ is solution of the following diff.-

$$(1-x^2)\,y'' - x y' + n^2 y = 0$$

Solution $T_n(x) = \cos(n \arccos x)$ $\therefore -1 \le T_n(x) \le 1$

set $x = \cos\theta$

then $T_n(\cos\theta) = \cos n\theta$

e.g.

$$\begin{cases} n=0 & T_0 = 1 \\ n=1 & T_1 = \cos\theta = x \\ n=2 & T_2 = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1 \\ & T_3 = 4\cos^3\theta - 3\cos\theta \Rightarrow 4x^3 - 3x \end{cases}$$

using $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$, $n \geq 1$

We have $T_{n+1}(x) + T_{n-1}(x) = 2x\,T_n(x)$

recursion relation.

(2) orthorgonal relation (continuous)

$$A = \int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = ?$$

$$= \int_{-1}^{1} \frac{\cos n\theta \cos m\theta \, d(\cos\theta)}{\sin\theta}$$

$$= - \int_{\pi}^{0} \cos n\theta \cos m\theta \, d\theta$$

a). $m = n = 0$, $A = \int_0^{\pi} d\theta = \pi$

b). $m = n \neq 0$, $A = \int_0^{\pi} \cos^2 n\theta \, d\theta = \frac{\pi}{2}$

c). $m \neq n$, $A = \int_0^{\pi} \frac{1}{2} (\cos (m+n)\theta + \cos(n-m)\theta) d\theta =$

$$\therefore A = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$$

* $T_n(x)$ is a polynomial of order n,
it has n zeros in $[-1, 1]$ at

$$x = \cos \frac{\pi(k-\frac{1}{2})}{n}, \quad k = 1, 2, \cdots, n$$

Proof

$$A = \sum_{k=1}^{m} \cos i\theta_k \cos j\theta_k = \frac{1}{2} \sum_k \left[\cos(i+j)\theta_k + \cos(i-j)\theta_k\right]$$

a). if $i=j=0$, $A = m$.

b). if $i=j\neq 0$, $A = \frac{1}{2} \sum_k \left(\cos 2i\theta_k + 1\right)$

We will show

$$B = \sum_{k=1}^{m} \cos 2n\theta_k = 0, \quad \theta_k = \frac{\pi}{m}\left(k-\frac{1}{2}\right)$$

$$B = Re\left[\sum_k e^{i2n\theta_k}\right]$$

$$= Re\left[e^{-\frac{2\pi n}{2m}i} \sum_k e^{\frac{2\pi n}{m}ik}\right]$$

$$= Re\left[e^{-\frac{\pi n}{m}i} e^{\frac{2\pi n}{m}i} \frac{1-e^{\frac{2\pi ni}{m}m}}{1-e^{\frac{2\pi iM}{m}}}\right]$$

$$\underset{0}{/\!/}$$

$$\therefore A = \frac{1}{2}\sum_k 1 = \frac{m}{2}.$$

c). $i\neq j$

We will show for $m\neq n$

$$A = \frac{1}{2}\sum_k \left[\cos(m+n)\theta_k + \cos(m-n)\theta_k\right] = 0, \quad \theta_k = \frac{\pi}{N}(k-$$

1^{st} term

$$\theta_k = \frac{\pi}{N}\left(k - \frac{1}{2}\right)$$

$$Re\left[\sum_k e^{i(m+n)\theta_k}\right] = Re\left[e^{-\frac{\pi}{2N}(m+n)i} \sum_k e^{\frac{\pi}{N}(m+n)ik}\right]$$

$$= Re\left[e^{-\frac{\pi}{2N}(m+n)i} e^{\frac{\pi}{N}(m+n)i} \frac{1 - e^{\frac{\pi}{N}(m+n)i \times N}}{1 - e^{\frac{\pi}{N}(m+n)i}}\right]$$

$$= Re\left[\frac{e^{\frac{\pi}{2N}(m+n)i}}{1 - e^{\frac{\pi}{N}(m+n)i}} \left(1 - (-1)^{m+n}\right)\right]$$

$\Downarrow$ real

$$= Re\left[\frac{\left(1 - e^{-\frac{\pi}{N}(m+n)i}\right) e^{\frac{\pi}{2N}(m+n)i}}{\left(1 - e^{\frac{\pi}{N}(m+n)i}\right)\left(1 - e^{-\frac{\pi}{N}(m+n)i}\right)}\right] \left(1 - (-1)^{m+n}\right)$$

denominator is real

$$= Re\left[e^{\frac{\pi}{2N}(m+n)i} - e^{-\frac{\pi}{2N}(m+n)i}\right] \frac{1 - (-1)^{m+n}}{\left|1 - e^{\frac{\pi}{N}(m+n)i}\right|^2}$$

$\Downarrow$

$2i \sin\frac{\pi}{2N}(m+n)$ no real part

$= 0.$

Similarly 2^{nd} term $= 0$.

Proof $\quad T_n(\cos\theta) = \cos n\theta = 0$

$$\therefore n\theta_k = k\pi - \frac{1}{2}\pi, \quad k = 1, 2, \cdots, n$$

$$\therefore \theta_k = \frac{\pi(k-\frac{1}{2})}{n}, \quad x_k = \cos\theta_k.$$

* $T_n(x)$ has $n+1$ extrema (maxima, minima)

at $x_k = \cos\frac{\pi k}{n}, \quad k = 0, 1, \cdots, n$

$$\because \cos n\theta = \pm 1 \Rightarrow n\theta_k = k\pi$$

$$\therefore \theta_k = \frac{k\pi}{n}$$

* *This is a very useful property of T_n.*

(3) discrete orthorgonal relation.

for $i, j < m$

$$\sum_{k=1}^{m} T_i(x_k)\, T_j(x_k) = \begin{cases} 0 & i \neq j \\ \frac{m}{2} & i = j \neq 0 \\ m & i = j = 0 \end{cases}$$

where x_k $(k = 1, 2, \cdots, m)$ are zeros of $T_m(x)$.

Chebyshev approximation.

$$f(x) \cong \sum_{k=1}^{N} C_k T_{k-1}(x) - \frac{1}{2} C_1 \quad \text{in } [-1, 1] \qquad (A)$$

* better approximation for large N.

* to find coefficient C_k, we use zeros of T_N

assume $T_N(x_k) = 0$, $k = 1, 2, \cdots, N$

then $f(x_k) = \sum_{j=1}^{N} c_j T_{j-1}(x_k) - \frac{1}{2} C_1$

$$B_i = \sum_{k=1}^{N} T_i(x_k) f(x_k) = \sum_k \sum_j C_j T_i(x_k) T_{j-1}(x_k) - \frac{1}{2} C_1 \sum_k T_i(x_k) \underbrace{T_0(x_k)}_{=1}$$

a). $i = 0$, $B_0 = N C_1 - \frac{1}{2} N C_1 = \frac{N}{2} C_1$

b). $i \neq 0$, $B_i = \frac{N}{2} C_{i+1}$

$\therefore$ $B_i = \frac{N}{2} C_{i+1}$ for general i

$$\therefore C_j = \frac{2}{N} B_{j-1} = \frac{2}{N} \sum_{k=1}^{N} T_{j-1}(x_k) f(x_k).$$

Note that eq. (A) is exact when $x = x_k$.

Why chebyshev approximation?

* minimax polynomial p_ℓ.

Among all polynomials g_ℓ of the same degree, p_ℓ has the smallest maximum deviation from the true function $f(x)$.

$$f(x) = \sum_\ell g_\ell a_\ell \qquad a_\ell \text{ coefficient.}$$

$$= \sum_\ell p_\ell a_\ell \quad \leftarrow \text{minimax.}$$

* although minimax polynomial is very difficult to find, chebyshev approx. polynomial is almost minimax.

* generating function of $T_n(x)$

$$\sum_{n=0}^{\infty} T_n(x)\, t^n = \frac{1-tx}{1-2tx+t^2}.$$