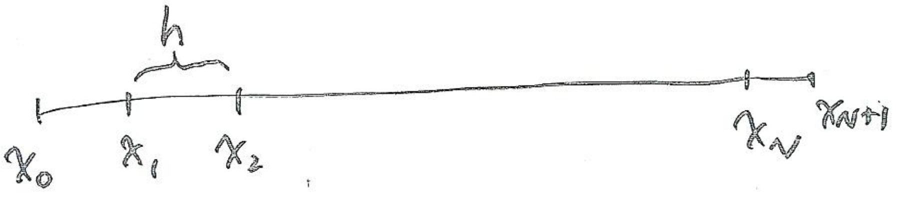
Discrete Fourier transform



$$x_{N+1} - x_0 = 2L = (N+1)h$$

$$\therefore h = \frac{2L}{N+1}$$

Set $x_0 = 0$, $x_k = kh$, $\alpha = e^{\frac{2\pi i h}{2L}} = e^{\frac{i\pi h}{L}} = e^{\frac{2\pi i}{N+1}}$

then $e^{\frac{i\pi l x_k}{L}} = e^{\frac{i\pi l k h}{L}} = \alpha^{lk}$

Fourier transform of $f(x_k) \equiv f_k$

$$g_l = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-\frac{i\pi l x}{L}}\, dx$$

$$= \frac{1}{2L} \sum_k f(x_k)\, e^{-\frac{i\pi l x_k}{L}}\, \Delta x \quad , \quad \Delta x = h = \frac{2L}{N+1}$$

$$\underline{g_l = \frac{1}{N+1} \sum_{k=0}^{N} f_k\, \alpha^{-lk}}$$

inverse Fourier transform

$$f_k = \sum_{l=0}^{N} \alpha^{lk}\, g_l$$

in matrix notation $\quad f = Ag \;,\; g = A^{-1} f$

# of operation for usual FT

For a given l, g_l needs $(N+1)$ multiplication and addition.

for $l = 0, 1, 2, \cdots, N$, need $(N+1)^2$ operations.

* examine matrix A $\quad f = Ag$

$$f_k = \sum_{l=0}^{N} \alpha^{lk} g_l \quad , \quad \therefore \; f_0 = \sum_{l=0}^{N} g_l$$

$$\therefore \quad \begin{cases} f_0 = g_0 + g_1 + \cdots + g_N \\ f_1 = g_0 + \alpha g_1 + \cdots + \alpha^N g_N \\ \quad \vdots \end{cases}$$

For $N+1 = 8$, we have

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^7 \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{14} \\ \vdots \\ 1 & \alpha^7 & \cdots & \cdots & \alpha^{49} \end{pmatrix} \quad \text{where } \alpha = e^{\frac{2\pi i}{N+1}}$$

if $N+1 = 2^{N_0}$ then $\alpha^{N+1} = 1$, $\alpha^{\frac{(N+1)}{2}} = e^{\pi i} = -1$

$\alpha^{\frac{N+1}{4}} = e^{\frac{\pi}{2} i} = i$, $\alpha^{\frac{N+1}{2} + l} = -\alpha^l$

for $N+1 = 8$ we have

$\alpha^8 = 1$, $\alpha^4 = -1$, $\alpha^2 = i$, $\alpha^7 = \alpha^4 \cdot \alpha^2 \cdot \alpha = -i\alpha$ etc.

$\alpha^6 = -i$, $\alpha^5 = \alpha^4 \cdot \alpha = -\alpha$, $\alpha^3 = \alpha^2 \cdot \alpha = i\alpha$, $\alpha^9 = \alpha \cdots$

We can now make use of (3-60) to construct a set of equations relating the unknown Fourier coefficients $\{g_\ell\}$ with the input quantities $\{f_i\}$ for the function $f(x)$. For simplicity, we shall take $x_0 = 0$ and, as a result of the assumption of evenly spaced points, we have the relation $x_k = kh$. On applying (3-60) for $x = x_k$, we obtain the result

$$f_k = f(x_k) = \sum_{\ell=0}^{N} g_\ell e^{i\ell\pi x_k/L} = \sum_{l=0}^{N} g_l \alpha^{lk} \tag{3-67}$$

The summation goes from 0 to N here. This comes from the fact that we have only $(N+1)$ pieces of input information, $f_0, f_1, \ldots, f_N$, and, consequently, we can determine at most $(N+1)$ coefficients $g_0, g_1, \ldots, g_N$.

To simplify the notation, we shall write

$$\alpha \equiv e^{i\pi h/L} = e^{i2\pi/(N+1)} \tag{3-68}$$

since $2L = (N+1)h$. For each of the $(N+1)$ values of $f(x)$ given to us, we have an equation of the form of (3-67). In terms of α, these $(N+1)$ equations may be put into the form

$$\begin{aligned}
g_0 + \quad g_1 + \quad g_2 + \quad g_3 + \cdots + \quad g_N &= f_0 \\
g_0 + \alpha\, g_1 + \alpha^2 g_2 + \alpha^3 g_3 + \cdots + \alpha^N g_N &= f_1 \\
g_0 + \alpha^2 g_1 + \alpha^4 g_2 + \alpha^6 g_3 + \cdots + \alpha^{2N} g_N &= f_2 \\
g_0 + \alpha^3 g_1 + \alpha^6 g_2 + \alpha^9 g_3 + \cdots + \alpha^{3N} g_N &= f_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\
g_0 + \alpha^N g_1 + \alpha^{2N} g_2 + \alpha^{3N} g_3 + \cdots + \alpha^{NN} g_N &= f_N
\end{aligned}$$

In matrix notation, they may be expressed as

$$Ag = f \qquad g = A^{-1} f \qquad \cancel{f = A^{-1} g} \tag{3-69}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^N \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \cdots & \alpha^{2N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^N & \alpha^{2N} & \alpha^{3N} & \cdots & \alpha^{NN} \end{pmatrix} \qquad g = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} \qquad f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

Our aim here is to solve this equation and obtain the values of the elements of g.

As a result, the matrix A in (3-69) takes on a particularly simple form

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & i & i\alpha & -1 & -\alpha & -i & -i\alpha \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\alpha & -i & \alpha & -1 & -i\alpha & i & -\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\alpha & i & -i\alpha & -1 & \alpha & -i & i\alpha \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\alpha & -i & -\alpha & -1 & i\alpha & i & \alpha \end{pmatrix}$$

There are many "symmetries" in this matrix. For example, the elements in the first four rows are similar to those in the second four rows, except that the odd elements (start the counting of the elements in each row from zero) have the opposite signs.

These symmetries carry over to B, the inverse of A. For example, for the $(N+1) = 8$ case we are dealing with here,

$$\begin{array}{cc} B = \dfrac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -i\alpha & -i & -\alpha & -1 & i\alpha & i & \alpha \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -\alpha & i & -i\alpha & -1 & \alpha & -i & i\alpha \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & i\alpha & -i & \alpha & -1 & -i\alpha & i & -\alpha \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \alpha & i & i\alpha & -1 & -\alpha & -i & -i\alpha \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \\ \end{matrix} \end{array}$$

where we have made use of the fact that $\alpha^{-\ell} = \alpha^{N+1-\ell}$. By making use of the symmetries in B, methods of FFT can reduce the number of operations required to obtain all $(N+1)$ Fourier coefficients by a large extent, usually from $(N+1)^2$ to $(N+1)\log_2(N+1)$. This is a very significant factor, especially when $(N+1)$ is large.

∴ A can be rewritten as

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & i & \cdots & i\alpha \\ 1 & i & -1 & \cdots & -i \\ \vdots & & & & \\ 1 & -i\alpha & & \cdots & \alpha \end{pmatrix}$$

$$B = \frac{A^{-1}}{8} = \frac{1}{8}\begin{pmatrix} & \cdots & \end{pmatrix},$$ A and B have the same symmetry.

$\underline{g = Bf}$

<u>Fast Fourier transform</u>, set $N+1 = 8$

We look at $(g_0, g_4), (g_1, g_5), (g_2, g_6), (g_3, g_7)$

$$8g_0 = f_0 + f_1 + f_2 + \cdots + f_7$$

$$8g_4 = f_0 - f_1 + f_2 - \cdots - f_7$$

define $f_{10} = f_0 + f_2 + f_4 + f_6$ even points

$f_{11} = f_1 + f_3 + f_5 + f_7$ odd points.

then $8g_0 = f_{10} + f_{11}$

$8g_4 = f_{10} - f_{11}$

$$8g_1 = f_0 - i\alpha f_1 - if_2 - \alpha f_3 - f_4 + i\alpha f_5 + if_6 + \alpha f_7$$

$$8g_5 = f_0 + i\alpha f_1 - if_2 + \alpha f_3 - f_4 - i\alpha f_5 + if_6 - \alpha f_7$$

define $f_{12} = f_0 - if_2 - f_4 + if_6$ even points