Pade Approximation

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Pade Approximation

Use Pade approximation to calculate e^{-x} up to 5 order. The first five order of Taylor series expansion is

$$a_0 = 1,\ a_1 = -1$$
$$a_2 = \frac{1}{2},\ a_3 = \frac{-1}{6}$$
$$a_4 = \frac{1}{24},\ a_5 = \frac{-1}{120}$$

Pade Approximation

We are going to use rational functions, r(x), of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{n} p_i x^i}{1 + \sum_{j=1}^{m} q_j x^j}$$

And say that the degree of such a function is $N = n + m$

Extension of **Taylor expansion** to rational functions; selecting the p_i's and q_i's so that $r^{(k)}(x_0) = f^{(k)}(x_0)\ \forall k = 0, 1, \ldots, N$.

$$f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)}$$

Now, use the Taylor expansion $f(x) \sim \sum_{i=0}^{\infty} a_i (x - x_0)^i$, for simplicity $x_0 = 0$:

Pade Approximation

For simplicity we (sometimes) define the "indexing-out-of-bounds" coefficients:

$$\begin{cases} p_{n+1} = p_{n+2} = \cdots = p_N = 0 \\ q_{m+1} = q_{m+2} = \cdots = q_N = 0, \end{cases}$$

$$f(x) - r(x) = \frac{f(x)q(x) - p(x)}{q(x)} = \frac{\sum_{k=0} a_k x^k \sum_{j=0}^{N} q_j x^j - \sum_{i=0}^{N} p_i x^i}{1 + \sum_{j=1}^{m} q_j x^j}$$

We can obtain

$$p_i = \sum_{k=0}^{i} a_k q_{i-k}$$

Pade Approximation

Find the Padé approximation of $f(x)$ of degree 5, where $f(x) \sim a_0 + a_1 x + \ldots a_5 x^5$ is the Taylor expansion of $f(x)$ about the point $x_0 = 0$.

The corresponding equations are:

$$
\begin{array}{c|lcccc}
x^0 & a_0 & - & p_0 & = & 0 \\
\hline
x^1 & a_0 q_1 + a_1 & - & p_1 & = & 0 \\
x^2 & a_0 q_2 + a_1 q_1 + a_2 & - & p_2 & = & 0 \\
x^3 & a_0 q_3 + a_1 q_2 + a_2 q_1 + a_3 & - & p_3 & = & 0 \\
x^4 & a_0 q_4 + a_1 q_3 + a_2 q_2 + a_3 q_1 + a_4 & - & p_4 & = & 0 \\
x^5 & a_0 q_5 + a_1 q_4 + a_2 q_3 + a_3 q_2 + a_4 q_1 + a_5 & - & p_5 & = & 0
\end{array}
$$

We get a linear system for $p_1, p_2, \ldots, p_N$ and $q_1, q_2, \ldots, q_N$:

$$
\begin{bmatrix}
a_0 & & & & \\
a_1 & a_0 & & & \\
a_2 & a_1 & a_0 & & \\
a_3 & a_2 & a_1 & a_0 & \\
a_4 & a_3 & a_2 & a_1 & a_0
\end{bmatrix}
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix}
-
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix}
= -
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.
$$

Pade Approximation

If we want n=3, m=2, we have

The corresponding equations are:

$$
\begin{array}{c|lcccc}
x^0 & a_0 & - & p_0 & = & 0 \\
\hline
x^1 & a_0 q_1 + a_1 & - & p_1 & = & 0 \\
x^2 & a_0 q_2 + a_1 q_1 + a_2 & - & p_2 & = & 0 \\
x^3 & \cancel{a_0 q_3} + a_1 q_2 + a_2 q_1 + a_3 & - & p_3 & = & 0 \\
x^4 & \cancel{a_0 q_4} + \cancel{a_1 q_3} + a_2 q_2 + a_3 q_1 + a_4 & - & \cancel{p_4} & = & 0 \\
x^5 & \cancel{a_0 q_5} + \cancel{a_1 q_4} + \cancel{a_2 q_3} + a_3 q_2 + a_4 q_1 + a_5 & - & \cancel{p_5} & = & 0
\end{array}
$$

$$
\begin{bmatrix}
a_0 & 0 & -1 & & \\
a_1 & a_0 & 0 & -1 & \\
a_2 & a_1 & 0 & 0 & -1 \\
a_3 & a_2 & 0 & 0 & 0 \\
a_4 & a_3 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3
\end{bmatrix}
= -
\begin{bmatrix}
a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5
\end{bmatrix}.
$$

Pade Approximation

If we want n=2, m=3, we have

The corresponding equations are:

$$
\begin{array}{c|lcccc}
x^0 & a_0 & - & p_0 & = & 0 \\
\hline
x^1 & a_0 q_1 + a_1 & - & p_1 & = & 0 \\
x^2 & a_0 q_2 + a_1 q_1 + a_2 & - & p_2 & = & 0 \\
x^3 & a_0 q_3 + a_1 q_2 + a_2 q_1 + a_3 & - & \cancel{p_3} & = & 0 \\
x^4 & \cancel{a_0 q_4} + a_1 q_3 + a_2 q_2 + a_3 q_1 + a_4 & - & \cancel{p_4} & = & 0 \\
x^5 & \cancel{a_0 q_5} + \cancel{a_1 q_4} + a_2 q_3 + a_3 q_2 + a_4 q_1 + a_5 & - & \cancel{p_5} & = & 0
\end{array}
$$

$$
\begin{pmatrix}
a_0 & 0 & 0 & -1 & 0 \\
a_1 & a_0 & 0 & 0 & -1 \\
a_2 & a_1 & a_0 & 0 & 0 \\
a_3 & a_2 & a_1 & 0 & 0 \\
a_4 & a_3 & a_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2
\end{pmatrix}
= -
\begin{bmatrix}
a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5
\end{bmatrix}.
$$

Pade Approximation

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$, hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

$$\begin{bmatrix} 1 & 0 & -1 & & \\ -1 & 1 & 0 & -1 & \\ 1/2 & -1 & 0 & 0 & -1 \\ -1/6 & 1/2 & 0 & 0 & 0 \\ 1/24 & -1/6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1/2 \\ -1/6 \\ 1/24 \\ -1/120 \end{bmatrix},$$

which gives $\{q_1, q_2, p_1, p_2, p_3\} = \{2/5,\ 1/20,\ -3/5,\ 3/20,\ -1/60\}$, *i.e.*

$$r_{3,2}(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}.$$

Pade Approximation

```
x=1;
N=5; n=3; m=N-n;
a=[1,-1,1/2,-1/6,1/24,-1/120];
% N=8;n=4; m=N-n;
% a=[1,0,-1/2,0,1/24,0,-1/720,0,1/40320];

A=zeros(N);
for ii=1:N
    for jj=1:N
        if ii>=jj
            A(ii,jj)=a(ii-jj+1);
        end
    end
end
for ii=1:n
    L=zeros(N,1);
    L(ii,1)=-1;
    A(:,m+ii)=L;
end
B=a(2:end).';
QP=-inv(A)*B;
```

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

$$\begin{bmatrix} a_0 & 0 & -1 & & \\ a_1 & a_0 & 0 & -1 & \\ a_2 & a_1 & 0 & 0 & -1 \\ a_3 & a_2 & 0 & 0 & 0 \\ a_4 & a_3 & 0 & 0 & 0 \end{bmatrix}$$

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$, hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

$$\begin{bmatrix} 1 & 0 & -1 & & \\ -1 & 1 & 0 & -1 & \\ 1/2 & -1 & 0 & 0 & -1 \\ -1/6 & 1/2 & 0 & 0 & 0 \\ 1/24 & -1/6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1/2 \\ -1/6 \\ 1/24 \\ -1/120 \end{bmatrix},$$

Pade Approximation

```
Px=a(1);
for ii=1:n
    Px=Px+QP(m+ii)*x^(ii);
end
Qx=1;
for ii=1:m
    Qx=Qx+QP(ii)*x^(ii);
end
Rx=Px/Qx
real=exp(-x)
% real=cos(x)
Err=real-Rx
```

```
Rx =

    0.9048


real =

    0.9048


Err =

   1.2790e-10
```

For x=0.1

The Taylor series expansion for e^{-x} about $x_0 = 0$ is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$, hence $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, -1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}\}$.

$$\begin{bmatrix} 1 & 0 & -1 & & \\ -1 & 1 & 0 & -1 & \\ 1/2 & -1 & 0 & 0 & -1 \\ -1/6 & 1/2 & 0 & 0 & 0 \\ 1/24 & -1/6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1/2 \\ -1/6 \\ 1/24 \\ -1/120 \end{bmatrix},$$

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$$r_{3,2}(x) = \frac{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}{1 + \frac{2}{5}x + \frac{1}{20}x^2}.$$

THANKS FOR ATTENTION !