Integration - Romberg Integration

$$I = \int_a^b f(x)dx$$

1. integral function: $f(x)$
2. upper limit: b
3. lower limit: a
4. number of splited intervals: $h, N; x_0, x_1, ..., x_N; f_0, f_1, ..., f_N$

(Personal view) these two methods are same

1. Richardson extrapolation wiki: https://en.wikipedia.org/wiki/Richardson_extrapolation
2. Romberg Integration wiki: https://en.wikipedia.org/wiki/Romberg%27s_method

power series expansion: $I(h) = I + a_1 h^2 + a_2 h^4 + ... + a_k h^{2k} + ...$

approximation: $I = \lim_{h \to 0} I(h)$

extrapolation method: given $I(h)$ for $h = b - a, \frac{b-a}{2}, \frac{b-a}{4}, ..., \frac{b-a}{2^N}$, we are required to estimate the value for $h = 0$, which is not in the range from $\frac{b-a}{2^N}$ to $b - a$

Naive version Romberg Integration - linear equations

$$\begin{aligned}
I(b-a) &= I + a_1(b-a)^2 + a_2(h-a)^4 + a_N(b-a)^{2N} \\
I\left(\frac{b-a}{2}\right) &= I + a_1\left(\frac{b-a}{2}\right)^2 + a_2\left(\frac{b-a}{2}\right)^4 + a_N\left(\frac{b-a}{2}\right)^{2N} \\
&... \\
I\left(\frac{b-a}{2^N}\right) &= I + a_1\left(\frac{b-a}{2^N}\right)^2 + a_2\left(\frac{b-a}{2^N}\right)^4 + a_N\left(\frac{b-a}{2^N}\right)^{2N}
\end{aligned}$$

In matrix form

$$\begin{pmatrix}
1 & (b-a)^2 & (b-a)^4 & ... & (b-a)^{2N} \\
1 & \left(\frac{b-a}{2}\right)^2 & \left(\frac{b-a}{2}\right)^4 & ... & \left(\frac{b-a}{2}\right)^{2N} \\
1 & \left(\frac{b-a}{4}\right)^2 & \left(\frac{b-a}{4}\right)^4 & ... & \left(\frac{b-a}{4}\right)^{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \left(\frac{b-a}{2^N}\right)^2 & \left(\frac{b-a}{2^N}\right)^4 & ... & \left(\frac{b-a}{2^N}\right)^{2N}
\end{pmatrix}
\begin{pmatrix} I \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}
=
\begin{pmatrix} I(b-a) \\ I\left(\frac{b-a}{2}\right) \\ I\left(\frac{b-a}{4}\right) \\ \vdots \\ I\left(\frac{b-a}{2^N}\right) \end{pmatrix}$$

Vandermonde matrix: https://en.wikipedia.org/wiki/Vandermonde_matrix

Romberg Integration

Recursive formula: $R_{n-1m} = \frac{4^m R_{nm-1} - R_{n-1m-1}}{4^m - 1}$

denote

1. $h = b - a$
2. $R_{00} = I(h), R_{10} = I(h/2), \dots, R_{N0} = I(h/2^N)$

$$\begin{pmatrix} 1 & h^2 & h^4 & \dots & h^{2N} \\ 1 & (h/2)^2 & (h/2)^4 & \dots & (h/2)^{2N} \\ 1 & (h/4)^2 & (h/4)^4 & \dots & (h/4)^{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (h/2^N)^2 & (h/2^N)^4 & \dots & (h/2^N)^{2N} \end{pmatrix} \begin{pmatrix} I \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} R_{00} \\ R_{10} \\ R_{20} \\ \vdots \\ R_{N0} \end{pmatrix}$$

First Iteration: $R_{01} = \frac{4R_{10} - R_{00}}{3}, R_{11} = \frac{4R_{20} - R_{10}}{3}, \dots, R_{N-11} = \frac{4R_{N0} - R_{N-10}}{3}$

$$\begin{pmatrix} 1 & h^2 & h^4 & \dots & h^{2N-2} \\ 1 & (h/2)^2 & (h/2)^4 & \dots & (h/2)^{2N-2} \\ 1 & (h/4)^2 & (h/4)^4 & \dots & (h/4)^{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (h/2^{N-1})^2 & (h/2^{N-1})^4 & \dots & (h/2^{N-1})^{2N-2} \end{pmatrix} \begin{pmatrix} I \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \end{pmatrix} = \begin{pmatrix} R_{01} \\ R_{11} \\ R_{21} \\ \vdots \\ R_{N-11} \end{pmatrix}$$

Repeat recursive formula

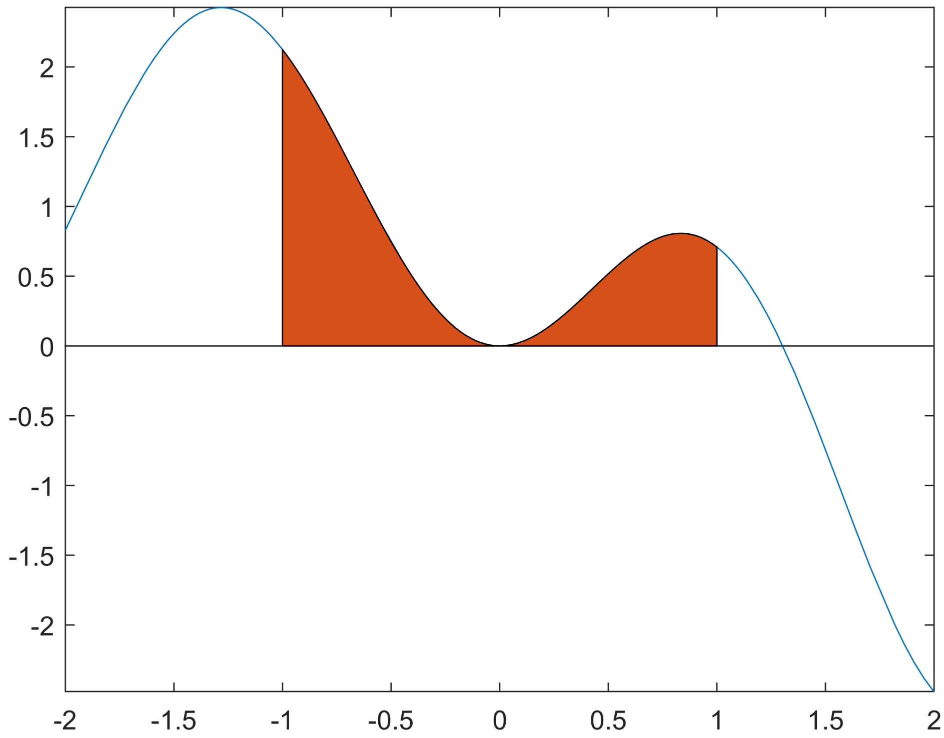
$$I = R_{0N} = \frac{4^N R_{1N-1} - R_{0N-1}}{4^N - 1}$$

paramter

```
hf1 = @(x) (3-x-x.^2).*sin(x).^2;
a = -1;
b = 1;
```

plot integral area

```
figure()
fplot(hf1, [a-1,b+1])
hold on;
area(linspace(a,b), hf1(linspace(a,b)))
hold off
```



built-in

```
ret1 = integral(hf1, a, b); %doc('integral')
disp(['built-in integral: ', num2str(ret1,14)])
```

```
built-in integral: 1.321971464861
```

naive Romberg integration

```
ret1 = my_naive_romberg_integral(hf1, a, b, 6);
disp(['naive Romberg integral: ', num2str(ret1,14)])
```

```
naive Romberg integral: 1.321971464861
```

Romberg Integration

```
ret1 = my_romberg_integral(hf1, a, b, 6);
disp(['Romberg integral: ', num2str(ret1,14)])
```

```
Romberg integral: 1.321971464861
```

function

```
function ret = my_naive_romberg_integral(hf1, a, b, m)
% naive Romberg Integration: solve linear equations directly
% reference: https://en.wikipedia.org/wiki/Romberg%27s_method
% hf1(function handle)
% a(float)
% b(float)
% m(int)
% ret(float)
R = zeros(1,m+1);
h = b-a;
R(1) = h/2*(hf1(a) + hf1(b));
for ind1 = 1:m
    tmp1 = hf1(a + h*(1:2:(2^ind1-1))/2^ind1);
    R(ind1+1) = R(ind1)/2 + h/2^ind1*sum(tmp1);
    %equivalent to my_trapezoidal_integral(hf1, a, b, 2^ind1+1)
end

matA = (h./(2.^(0:m)).').^(0:2:2*m);
tmp1 = matA\(R.');
ret = tmp1(1);
end

function R = my_romberg_integral(hf1, a, b, m)
% calculate integration using Romberg's method
% reference: https://en.wikipedia.org/wiki/Romberg%27s_method
% hf1(function handle)
% a(float)
% b(float)
% m(int)
% ret(float)
R = zeros(1,m+1);
h = b-a;
R(1) = h/2*(hf1(a) + hf1(b));
for ind1 = 1:m
    tmp1 = hf1(a + h*(1:2:(2^ind1-1))/2^ind1);
    R(ind1+1) = R(ind1)/2 + h/2^ind1*sum(tmp1);
    %equivalent to my_trapezoidal_integral(hf1, a, b, 2^ind1+1)
end
for ind1 = 1:m
    R = R(2:end) + (R(2:end) - R(1:end-1))/(4^ind1-1);
end
end
```