1. (a) Model can be rewritten as $(1-0.1B)^2 X_t = (1-1.2B) Z_t$

$\therefore X_t = (1-0.1B)^{-2}(1-1.2B)Z_t$

$= \left(1 + \frac{-2}{1!}(-0.1B) + \frac{(-2)(-3)}{2!}(-0.1B)^2 + \cdots\right)(1-1.2B)Z_t$

$= (1 + 0.2B + 0.03B^2 + \cdots)(1-1.2B)Z_t$

$= (1 - 1.2B + 0.2B - 0.24B^2 + 0.03B^2 + \cdots)Z_t$

$= (1 - B - 0.21B^2 + \cdots)Z_t$

$\therefore \psi_1 = 1 \quad \psi_2 = 0.21$

(b) Casual but not invertible

(c) $\gamma(0) = 0.2\gamma(1) - 0.01\gamma(2) + 1 - 1.2(0.2 - 1.2)$

$= 0.2\gamma(1) - 0.01\gamma(2) + 2.2$

$\gamma(1) = 0.2\gamma(0) - 0.01\gamma(1) - 1.2$

$\gamma(2) = 0.2\gamma(1) - 0.01\gamma(0)$

2. (a) $E(X_t) = \cos(\lambda t) EZ_1 + \sin(\lambda t) EZ_2 = 0$

$E(Y_t) = 2\cos(\lambda t) EZ_1 = 0$

(b) $\gamma_X(t,k) = \cos(\lambda t)\cos[\lambda(t+k)] + \sin(\lambda t)\sin[\lambda(t+k)]$

$= \frac{1}{2}\{\cos(2\lambda t + \lambda k) + \cos(\lambda k)\} - \frac{1}{2}\{\cos(2\lambda t + \lambda k) - \cos(\lambda k)\}$

$= \cos(\lambda k)$ which does not depend on t, So X_t is stationary.

(c) $\gamma_Y(t,0) = 4\cos^2(\lambda t)$

$\gamma_Y(t,k) = 4\cos(\lambda t)\cos[\lambda(t+k)]$

Y_t is not stationary since $\gamma_Y(t,k)$ depends on t.

d) $\eta(t,0) = Cov(X_t, Y_t) = 2\cos^2(\lambda t)$

$\eta(t,k) = Cov(X_t, Y_{t+k}) = 2\cos(\lambda t)\cos[\lambda(t+k)]$

3. (a) $\hat{Z}_{200}(1) = 0.5\times4 + 0.2\times5 + 0 + 0.2\times1 - 0.4\times0.5 = 3$

$\hat{Z}_{200}(2) = 0.5\times3 + 0.2\times4 + 0 + 0 - 0.4\times1 = 1.9$

$\hat{Z}_{200}(3) = 0.5\times1.9 + 0.2\times3 = 1.55$

$\hat{Z}_{200}(4) = 0.5\times1.55 + 0.2\times1.9 = 1.155$

$\hat{Z}_{200}(5) = 0.5\times1.155 + 0.2\times1.55 = 0.8875$

(b) $P_{201}^{200} = E(a_{201}^2 \mid Z_{200}, \cdots) = \left(\frac{4}{2\times1.96}\right)^2 = 1.0204^2 = 1.0412$

$P_{202}^{200} = E((Z_{202} - \hat{Z}_{202}^{200})^2 \mid Z_{200}, \cdots)$

$= E[(0.5Z_{201} + 0.2Z_{200} + a_{202} + 0.2a_{201} - 0.4a_{200} - 0.5\hat{Z}_{201}^{200} - 0.2Z_{200} + 0.4a_{200})^2 \mid Z_{200}, \cdots]$

$= E[(0.5a_{201} + a_{202} + 0.2a_{201})^2 \mid Z_{200}, \cdots]$

$= E[(0.7a_{201} + a_{202})^2 \mid Z_{200}, \cdots]$ $(*)$

Since $a_t \overset{iid}{\sim} N(0, \overset{1.0412}{\cancel{1.0204}})$

so $(*)$ can be rewritten as $= E[0.49a_{201}^2 + a_{202}^2] = 1.49\times1.0412 = 1.5514$

The 95% prediction interval of Z_{202} is $1.9 \pm 1.96\times\sqrt{1.5514}$

$P_{203}^{200} = E[(Z_{203} - \hat{Z}_{203}^{200})^2 \mid Z_{200}, \cdots]$

$= E[(0.5Z_{202} + 0.2Z_{201} + a_{203} + 0.2a_{202} - 0.4a_{201} - 0.5\hat{Z}_{202}^{200} - 0.2\hat{Z}_{201}^{200})^2 \mid Z_{200}, \cdots]$

$= E[(0.5\times0.7a_{201} + 0.5a_{202} + 0.2a_{202} + a_{203} + 0.2\times a_{201} - 0.4a_{201})^2 \mid Z_{200}, \cdots]$

$= E[(a_{203} + 0.7a_{202} + 0.15a_{201})^2 \mid Z_{200}, \cdots]$

$= E[a_{203}^2 + 0.49a_{202}^2 + 0.0225a_{201}^2 \mid Z_{200}, \cdots]$

$= 1.5125\times1.0412 = 1.5748$

The 95% Prediction interval of Z_{203} is $1.55 \pm 1.96\times\sqrt{1.5748}$

(c) $a_{201} = 18 - 0.5\times4 - 0.2\times5 - 0.2\times1 + 0.4\times0.5 = 15$

$\hat{Z}_{202} = 0.5\times18 + 0.2\times4 + 0.2\times15 - 0.4\times1 = 12.4$

$\hat{Z}_{203} = 0.5\times12.4 + 0.2\times18 - 0.4\times15 = 3.8$

$\hat{Z}_{204} = 0.5\times3.8 + 0.2\times12.4 = 4.38$

4 (a) $X_t^2 = \sigma_t^2 + X_t^2 - \sigma_t^2$

$= 0.1 + 0.2X_{t-1}^2 + 0.3X_{t-2}^2 + 0.35\sigma_{t-1}^2 + X_t^2 - \sigma_t^2$

$= 0.1 + 0.2X_{t-1}^2 + 0.3X_{t-2}^2 - 0.35(X_{t-1}^2 - \sigma_{t-1}^2) + 0.35X_{t-1}^2 + X_t^2 - \sigma_t^2$

$= 0.1 + 0.55X_{t-1}^2 + 0.3X_{t-2}^2 + V_t - 0.35V_{t-1}$

where $v_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\varepsilon_t^2 - 1)$ ARMA(2,1)

(b) $E(X_t^2) = 0.1 + 0.55E(X_{t-1}^2) + 0.3E(X_{t-2}^2)$

$\Rightarrow E(X_t^2) = \frac{0.1}{0.15} = 2/3$

$E(\sigma_t^2) = E(X_t^2) - E(V_t) = 2/3$

(c) $Cov(X_t^2, \sigma_{t-1}^2) = E(X_t^2\sigma_{t-1}^2) - E(X_t^2)E(\sigma_{t-1}^2)$ We only need to calculate $E(X_t^2\sigma_{t-1}^2)$

$E(X_t^2\sigma_{t-1}^2) = E(\varepsilon_t^2\sigma_t^2\sigma_{t-1}^2) = E(\sigma_t^2\sigma_{t-1}^2) = E[(0.1 + 0.2X_{t-1}^2 + 0.3X_{t-2}^2 + 0.35\sigma_{t-1}^2)\sigma_{t-1}^2]$

$= 0.1E(\sigma_{t-1}^2) + 0.2E(X_{t-1}^2\sigma_{t-1}^2) + 0.3E(X_{t-2}^2\sigma_{t-1}^2) + 0.35E(\sigma_{t-1}^4)$

$= 0.1E(\sigma_{t-1}^2) + 0.2E(\sigma_{t-1}^4) + 0.35E(\sigma_{t-1}^4) + 0.3E(X_{t-2}^2\sigma_{t-1}^2)$

$= 0.1E(\sigma_{t-1}^2) + 0.55E(\sigma_{t-1}^4) + 0.3E(X_{t-2}^2\sigma_{t-1}^2)$

We need to calculate $E(\sigma_{t-1}^4)$ and $E(X_{t-2}^2\sigma_{t-1}^2)$

From 4(a), we use Yule-Walker equation to deal with ARMA(2,1) model

① $E(X_t^2X_{t-1}^2) = 0.1E(X_{t-1}^2) + 0.55E(X_{t-1}^4) + 0.3E(X_{t-1}^2X_{t-2}^2) + 0.35E(V_{t-1}^2)$

② $E(X_t^4) = 0.1E(X_t^2) + 0.55E(X_t^2X_{t-1}^2) + 0.3E(X_t^2X_{t-2}^2) + E(V_t^2) - 0.35E(V_{t-1}^2)$

③ $E(X_t^2X_{t-2}^2) = 0.1E(X_{t-2}^2) + 0.55E(X_{t-1}^2X_{t-2}^2) + 0.3E(X_{t-2}^4)$

Since X_t^2 ~~and~~ is stationary, $E(X_t^4) = E(X_{t-1}^4) = E(X_{t-2}^4)$, $E(X_t^2X_{t-1}^2) = E(X_{t-1}^2X_{t-2}^2)$

For $E(V_t^4)$ we have $E(V_t^4) = E(\sigma_t^4(\varepsilon_t^2-1)^2) = E(\sigma_t^4)E(\varepsilon_t^4 - 2\varepsilon_t^2 + 1)$ ④

Since $\varepsilon_t \sim N(0,1)$ $\therefore E(\varepsilon_t^4) = 3$ $E(\varepsilon_t^2) = 1$

For $E(\sigma_t^4)$, we have $E(\sigma_t^4) = E[(0.1 + 0.2X_{t-1}^2 + 0.3X_{t-2}^2 + 0.35\sigma_{t-1}^2)^2]$

$= E[0.01 + 0.04X_{t-1}^2 + 0.04X_{t-1}^4 + 0.09X_{t-2}^4 + 0.21X_{t-2}^2\sigma_{t-1}^2 + 0.1225\sigma_{t-1}^4 + 0.06X_{t-2}^2 + 0.12X_{t-1}^2X_{t-2}^2$
$+ 0.07\sigma_{t-1}^2 + 0.14X_{t-1}^2\sigma_{t-1}^2]$

$= 0.01 + 0.1E(X_t^2) + 0.13E(X_t^4) + 0.21E(X_{t-2}^2X_{t-1}^2) + 0.1225E(\sigma_t^4) + 0.12E(X_{t-2}^2X_{t-1}^2) + 0.07E(\sigma_t^2)$
$+ 0.14E(\sigma_t^4)$ ⑤

To solve ① - ⑤, we can get $E(\sigma_t^4)$ and $E(X_{t-2}^2\sigma_{t-1}^2) = E(X_{t-2}^2X_{t-1}^2)$

(d) $\log\{f(X_1,\cdots,X_4|\mathcal{F}_0)\} = -\frac{4}{2}\log(2\pi) + \sum_{t=1}^{4} -\frac{1}{2}\log\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{\sigma_t^2}$

$\tilde{\sigma}_1 = 0.1$ $\tilde{\sigma}_2 = 0.1 + 0.2\times1.3^2 + 0.35\tilde{\sigma}_1^2 = 0.473$

$\tilde{\sigma}_3 = 0.1 + 0.2\times0.1^2 + 0.3\times1.3^2 + 0.35\times a\,\tilde{\sigma}_2^2 = 0.7746$

$\tilde{\sigma}_4 = 0.1 + 0.2\times2.2^2 + 0.3\times0.1^2 + 0.35\times\tilde{\sigma}_3^2 = 1.3421$

5. $W_t = \sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j} = \sum_{j=0}^{\infty} (-\theta)^{-j} [Z_{t-j} + \theta Z_{t-1-j}]$

$= \sum_{j=0}^{\infty} [(-\theta)^{-j} Z_{t-j} - (-\theta)^{-j+1} Z_{t-j-1}]$

$= Z_0 + \sum_{j=0}^{\infty} [-(-\theta)^{-j+1} Z_{t-j-1} + (-\theta)^{-j-1} Z_{t-j-1}]$

$= Z_0 + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}] Z_{t-j-1}$

since $Z_t \overset{iid}{\sim} WN(0,\sigma^2)$ $\therefore EW_t = EZ_0 + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}] E Z_{t-j-1} = 0$ 2 points

$Var(W_t) = Var(Z_0) + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}]^2 Var(Z_{t-j-1})$

σ_W^2

$= \sigma^2 + \sigma^2 (1-\theta^2)^2 \sum_{j=0}^{\infty} (-\theta)^{-2j-2}$

$= \sigma^2 + \sigma^2 (1-\theta^2)^2 \cdot \cancel{\frac{(-\theta)^{-2}(1-(-\theta)^{-2n})}{1-(-\theta)^{-2}}}$

$= \sigma^2 + \sigma^2 (1-\theta^2)^2 \frac{1}{\theta^2-1}$

$= \sigma^2 + \sigma^2(\theta^2-1) = \cancel{(\sigma^2+1)(\theta^2+1)}$ $\theta^2\sigma^2$ 2 points

For any integer k.

$Cov(W_t, W_{t+k}) = Cov[Z_t + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}] Z_{t-1-j}, Z_{t+k} + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}] Z_{t+k-j-1}]$

$= Cov(Z_t + \sum_{j=0}^{\infty} [(-\theta)^{-j-1} - (-\theta)^{-j+1}] Z_{t-j-1}, [(-\theta)^{-k} - (-\theta)^{-k+2}] Z_t + \sum_{j=k}^{\infty} \{(-\theta)^{-j-1} - (-\theta)^{-j+1}\} Z_{t+k-j-1})$

$= Cov(Z_t + \sum_{j=0}^{\infty} \{(-\theta)^{-j-1} - (-\theta)^{-j+1}\} Z_{t-j-1}, [(-\theta)^{-k} - (-\theta)^{-k+2}] Z_t + \sum_{j=0}^{\infty} [(-\theta)^{-j-k-1} - (-\theta)^{-j-k+1}] Z_{t-j-1})$

$= (-\theta)^{-k} - (-\theta)^{-k+2} + (-\theta)^{-k} (1-\theta^2)^2 \sum_{j=0}^{\infty} (-\theta)^{-2j-2}$

$= (-\theta)^{-k} - (-\theta)^{-k+2} + (-\theta)^{-k} (1-\theta^2)^2 \cdot \frac{1}{\theta^2-1}$

$= (-\theta)^{-k} - (-\theta)^{-k+2} + (-\theta)^{-k} (\theta^2-1)$

$= 0$

$\therefore W_t \sim WN(0, \sigma_W^2)$ 4 points