STAT 3007 Tutorial 2 Suggested solutions

Example 1. $\Pr(Y > y) = \Pr\left(\min\{X_1, \dots, X_n\} > y\right)$

$$= \Pr(X_1 > y, \dots, X_n > y)$$

$$= \Pr(X_1 > y) \cdots \Pr(X_n > y) = \left(1 - F_X(y)\right)^n$$

$$1 - F_X(y) = \Pr(X > y) = \int_y^\infty f_X(x)\,dx = \int_y^\infty 2e^{-2x}\,dx = e^{-2y}$$

$$F_Y(y) = 1 - e^{-2ny} \Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = 2n e^{-2ny} \qquad \square$$

Example 2. For $k = 2, 3, \dots, n+1$,

$$\Pr(N = k) = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-(k-2)}{n} \cdot \frac{k-1}{n}$$

$$\begin{array}{ccccc} 1 & 2 & \cdots & k-1 & k \\ x & x & \cdots & x & x \end{array}$$

$\underbrace{}_{\text{distinct}}$ $\uparrow$

Alternatively,

$$\Pr(N > k) = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-(k-1)}{n}$$

$$\begin{array}{ccccc} 1 & 2 & \cdots & k-1 & k \\ x & x & \cdots & x & x \end{array}$$

$\underbrace{}_{\text{distinct}}$ $\uparrow$

$$\Rightarrow \Pr(N = k) = \Pr(N > k-1) - \Pr(N > k)$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-(k-2)}{n} \cdot \left(1 - \frac{n-(k-1)}{n}\right)$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-(k-2)}{n} \cdot \frac{k-1}{n} \qquad \square$$

Example 3. For $k = 2, 3, \dots$

$$\Pr(N = k) = \Pr\left(X_2 > X_1, \dots, X_{k-1} > X_1, X_k \le X_1\right)$$

$$= \int_0^\infty \int_0^{X_1} \int_{X_1}^\infty \cdots \int_{X_1}^\infty f(x_2) \cdots f(x_{k-1}) f(x_k) f(x_1)\, dx_2 \cdots dx_{k-1}\, dx_k\, dx_1$$

How? $= \int_0^\infty (1 - F(x_1))^{k-2} F(x_1) f(x_1)\, dx_1$

$$= \int_0^\infty e^{-(k-2)x_1} (1 - e^{-x_1}) e^{-x_1}\, dx_1$$

$$= -\frac{1}{k-1} e^{-(k-1)x_1}\Big|_0^\infty - \frac{1}{k} e^{-kx_1}\Big|_0^\infty = \frac{1}{k-1} - \frac{1}{k}$$

Alternatively,

$$\Pr(N>k) = \Pr(X_2 > X_1, \ldots, X_k > X_1)$$

$$= \Pr(X_1 \text{ is the smallest among } X_1, X_2, \ldots, X_k)$$

$$\overset{\text{Why?}}{=} \frac{1}{k}$$

$$\Rightarrow \Pr(N=k) = \Pr(N>k-1) - \Pr(N>k) = \frac{1}{k-1} - \frac{1}{k} \qquad \square$$

Example 4. (1)

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix} \end{array}$$

(2) $\Pr(X_0 = 0, X_1 = 0, X_2 = 0) = \Pr(X_0 = 0)\Pr(X_1 = 0 \mid X_0 = 0)\Pr(X_2 = 0 \mid X_1 = 0, X_0 = 0)$

$$= (1-\alpha)^2$$

(3) $\Pr(X_2 = 0) = \Pr(X_0 = 0, X_1 = 0, X_2 = 0) + \Pr(X_0 = 0, X_1 = 1, X_2 = 0)$

$$= (1-\alpha)^2 + \Pr(X_0 = 0)\Pr(X_1 = 1 \mid X_0 = 0)\Pr(X_2 = 0 \mid X_1 = 1, X_0 = 0)$$

$$= (1-\alpha)^2 + \alpha^2 \qquad \square$$

Example 5. Analogous to Example 4.

Example 6. Let X_n be the ordered pair of the results of $(n-1)^{th}$ and n^{th} games for Michael.

(If the series has ended, interpret it as the results of the last two games.)

$$P = \begin{array}{c c} & \begin{array}{cccc} (W,W) & (W,L) & (L,W) & (L,L) \end{array} \\ \begin{array}{c} (W,W) \\ (W,L) \\ (L,W) \\ (L,L) \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1-\alpha \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \square$$