Solutions for Chapter 2 Exercises

1.

$$E(Z_t) = E(a_t a_{t-2}) = 0\,.$$

For $k \neq 0$,

$$\begin{aligned}
Cov(Z_t, Z_{t+k}) &= E(Z_t Z_{t+k}) - E(Z_t)E(Z_{t+k}) \\
&= E(a_t a_{t-2} a_{t+k} a_{a+k-2}) - 0 \\
&= 0 \\
Var(Z_t) &= E(Z_t^2) \\
&= E(a_t^2)E(a_{t-2}^2) \\
&= \sigma_a^4 < \infty\,.
\end{aligned}$$

Thus, the mean of $\{Z_t\}$ is constant, and the autocovariance function is independent of time t, so $\{Z_t\}$ is weakly stationary.

2.

$$\begin{aligned}
E(X_t) &= (-1)^t E(Z)\,, \\
\mathrm{Cov}(X_t, X_{t+k}) &= (-1)^{2t+k}\mathrm{Cov}(Z, Z) = (-1)^k \mathrm{Var(Z)}\,.
\end{aligned}$$

Note that the autocovariance function is independent of t. To make the mean constant we need $(-1)^t E(Z)$ for all t, which implies $E(Z) = -E(Z)$, i.e. $E(Z) = 0$. Therefore, we require $E(Z) = 0$ if $\{X_t\}$ is stationary.

3.

Yes, this type of i.i.d series $\{Z_t\}$ exists.

Consider $Z_t = t a_t$, where $a_t \overset{iid}{\sim} N(0,1)$. By straightforward algebra,

$$\begin{aligned}
E(Z_t) &= tE(a_t) = 0\,, \\
\mathrm{Var}(Z_t) &= t^2 \mathrm{Var}(a_t) = t^2\,, \\
\mathrm{Cov}(Z_t, Z_{t+k}) &= t(t+k)\mathrm{Cov}(a_t, a_{t+k}) = 0\,, \quad \text{for } k \neq 0, \\
\mathrm{Corr}(Z_t, Z_{t+k}) &= \frac{\mathrm{Cov}(Z_t, Z_{t+k})}{\sqrt{\mathrm{Var}(Z_t)\mathrm{Var}(Z_{t+k})}} \frac{t(t+k)\mathrm{Cov}(a_t, a_{t+k})}{\sqrt{t^2(t+k)^2}} = \begin{cases} 0\,, & \text{for } k \neq 0, \\ 1\,, & \text{for } k = 0, \end{cases}
\end{aligned}$$

Clearly, $E(Z_t)$ and $\text{Corr}(Z_t, Z_{t+k})$ are independent of time t. But $\{Z_t\}$ is not stationary since the variance $\text{Var}(Z_t) = t^2$ depends on time.

4.

(a) Note that $P(-Y_t = k) = 0.2, k = -2, -1, 0, 1, 2$ for all integers t. Thus Y_t and $-Y_t$ have the same distribution. Since $\{Y_t\}_{t=1,\dots}$ are independent, the series $\{(-1)^t Y_t\}_{t=1,\dots}$ are actually independent and identically distributed. Thus, $\{W_t\}$ is strictly stationary.

(b) Since $E(W_t^2) = E(Y_t^2) = 0.2((-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2) = 2 < \infty$, the second moment of W_t exists. Thus the strictly stationarity of $\{W_t\}$ implies its weakly stationarity.

5.

$$\text{Var}(X_t) = \text{Var}(\theta_0 a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \theta_3 a_{t-3} + \theta_4 a_{t-4}) = (\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2)\sigma_a^2.$$

$$\begin{aligned}
\text{Cov}(X_t, X_{t+1}) &= (\theta_0\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4)\sigma_a^2 \\
\text{Cov}(X_t, X_{t+2}) &= (\theta_0\theta_2 + \theta_1\theta_3 + \theta_2\theta_4)\sigma_a^2 \\
\text{Cov}(X_t, X_{t+3}) &= (\theta_0\theta_3 + \theta_1\theta_4)\sigma_a^2 \\
\text{Cov}(X_t, X_{t+4}) &= (\theta_0\theta_4)\sigma_a^2 .
\end{aligned}$$

Thus, the autocorrelation function is

$$\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{\theta_0\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4}{\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2} \\
\rho(2) &= \frac{\theta_0\theta_2 + \theta_1\theta_3 + \theta_2\theta_4}{\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2} \\
\rho(3) &= \frac{\theta_0\theta_3 + \theta_1\theta_4}{\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2} \\
\rho(4) &= \frac{\theta_0\theta_4}{\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2} \\
\rho(k) &= 0, \qquad |k| > 4
\end{aligned}$$

6.
(1)

$$\text{Var}(Z_t) = \text{Var}(a_t - 1.7a_{t-1} + 0.72a_{t-2}) = (1 + (1.7)^2 + (0.72)^2)\sigma_a^2 = 17.6$$

(2)

$$\begin{aligned}
\gamma(0) &= \text{Var}(Z_t) = 17.6 \\
\gamma(1) &= \text{Cov}(Z_t, Z_{t+1}) = -11.7 \\
\gamma(2) &= \text{Cov}(Z_t, Z_{t+2}) = 2.88 \\
\gamma(k) &= 0 \quad |k| > 2
\end{aligned}$$

(3)

$$\begin{aligned}
\text{Var}(\frac{1}{n}\sum_{t=1}^{15} Z_t) &= \frac{1}{n}\text{Var}(Z_t) + \frac{1}{n^2}2(n-1)r(1) + \frac{1}{n^2}2(n-2)r(2) \\
&= 0.05
\end{aligned}$$

where $n = 15$.

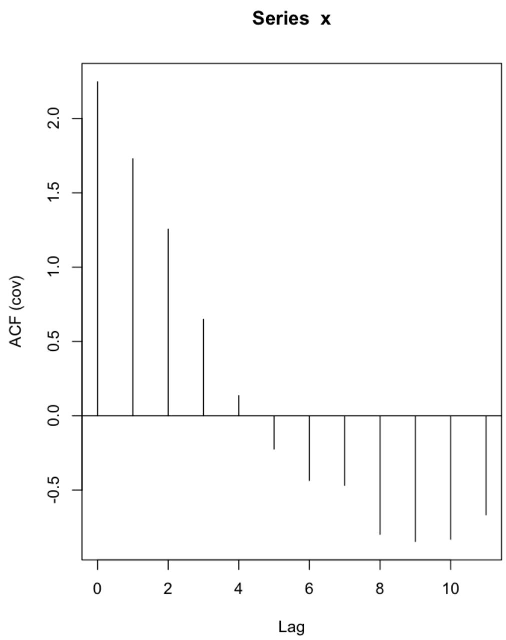
7.

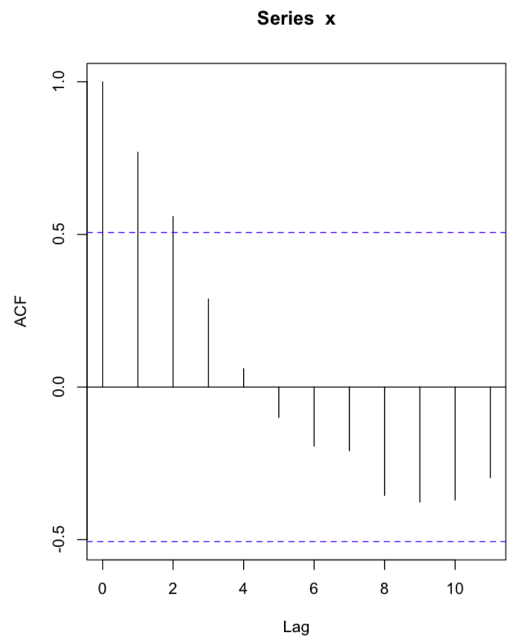
R code:

```
library(stats)
library(graphics)
x = c(-1.08,-0.33,0.18,0.42,0.18,-0.29,-0.03,-1.09,0.18,
-0.62,-2.18,-2.87,-3.61,-3.46,-3.92)
par(mfrow=c(1,2))
acf(x,type="covariance")
acf(x)
```

Remarks: The funciton acf in R is calculating the correlation in default. We can use **type = "covariance"** to change it to covariance.

Result of sample ACVF and ACF:





$\because \quad r(1) > \dfrac{2}{\sqrt{15}} \quad \text{and} \quad r(2) > \dfrac{2}{\sqrt{15}}$ (The blue dotted line)

$\therefore$ We conclude that this time series is not a white noise.