Summary of Chapter 3

1 Concepts

- **Probability mass function**: For a discrete random variable X with possible values $X_1, X_2, \cdots, X_n$, a probability mass function is a function such that

  (1) $P(X_i) \geq 0$

  (2) $\sum_{i=1}^{n} P(X_i) = 1$

  (3) $P(X_i) = P(X = X_i)$

- **Cumulative distribution function**: The cumulative distribution function of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{X_i \leq x} P(X_i)$$

  For a discrete random variable X, $F(x)$ satisfies the following properties.

  (1) $F(x) = P(X \leq x) = \sum_{X_i \leq x} P(X_i)$

  (2) $0 \leq F(x) \leq 1$

  (3) If $x \leq y$, then $F(x) \leq F(y)$

- **Summary measures**

$$\mu = E(X) = \sum_{i=1}^{N} X_i P(X_i)$$

$$\sigma^2 = Var(X) = \sum_{i=1}^{N} [X_i - E(X)]^2 P(X_i)$$

$$\sigma_{XY} = Cov(X,Y) = \sum_{i=1}^{N} [X_i - E(X)][(Y_i - E(Y)] P(X_i Y_i)$$

- **Properties**

$$E(aX + bY) = aE(X) + bE(Y)$$

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X,Y)$$

If X and Y are independent, then

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$$

- **Binomial Distribution**: A random experiment consists of n Bernoulli trials such that
  (1) The trial are independent
  (2) Each trial results in only two possible outcomes, labeled as "success" and "failure"
  (3) The probability of a success in each trial, denoted as p, remains constant

  Let X = number of successes in n Bernoulli trials, then X has a binomial distribution with probability mass function

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n$$

  Mean: $\mu = np$
  Variance: $\sigma^2 = np(1-p)$
  Standard Deviation: $\sigma = \sqrt{np(1-p)}$

  R functions:

$$P(X = x) = \text{dbinom}(x, n, p)$$
$$P(X \leq x) = \text{pbinom}(x, n, p)$$

- **Hypergeometric distribution**: A set of N objects contains A objects classified as successes, $N-A$ objects classified as failures. A sample of size n objects is selected randomly (without replacement) from the N objects.

  Let X = number of successes in the sample, then X has hypergeometric distribution with probability mass function

$$P(x) = \frac{\binom{A}{x}\binom{N-A}{n-x}}{\binom{N}{n}}$$

  Mean: $\mu = E(X) = \frac{nA}{N}$

  Variance: $\sigma^2 = \frac{nA(N-A)}{N^2} \cdot \frac{N-n}{N-1}$

  Standard Deviation: $\sigma = \sqrt{\frac{nA(N-A)}{N^2}} \cdot \sqrt{\frac{N-n}{N-1}}$

  R functions:

$$P(X = x) = \text{dhyper}(x, A, N-A, n)$$
$$P(X \leq x) = \text{phyper}(x, A, N-A, n)$$

- **Poisson distribution**: Given an interval of real numbers, assume events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

  (1) the probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval

  (2) the event in each subinterval is independent of other subintervals

  (3) the probability of more than one event in a subinterval is zero

  Let X = number of events in the interval, then X has Poisson distribution with probability mass function

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots$$

  where λ is the average number of events per unit.

  Mean: $\mu = E(X) = \lambda$
  Variance: $\sigma^2 = \lambda$
  Standard Deviation: $\sigma = \sqrt{\lambda}$

  R functions:

$$P(X = x) = \text{dpion}(x, \lambda)$$
$$P(X \leq x) = \text{ppion}(x, \lambda)$$

- **Geometric distribution**:

  Let X = number of Bernoulli trials until the first success, then X has geometric distribution with probability mass function

$$P(x) = (1-p)^{x-1}p, \quad x = 1, 2, \cdots$$

  where p $(0 < p < 1)$ is the probability of success.

  Mean: $\mu = E(X) = 1/p$
  Variance: $\sigma^2 = (1-p)/p^2$
  Standard Deviation: $\sigma = \sqrt{(1-p)/p^2}$

  R functions:

$$P(X = x) = \text{dgeom}(x^*, p)$$
$$P(X \leq x) = \text{pgeom}(x^*, p)$$

  where x^* = number of failure until the first success, and $x^* = x - 1$.

- **Negative binomial distribution**:

Let X = number of Bernoulli trials until r successes occur, then X has negative binomial distribution with probability mass function

$$P(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r, \quad x = r, r+1, r+2, \cdots$$

where p $(0 < p < 1)$ is the probability of success, r = 1, 2, 3, $\cdots$.

Mean: $\mu = E(X) = r/p$
Variance: $\sigma^2 = r(1-p)/p^2$
Standard Deviation: $\sigma = \sqrt{r(1-p)/p^2}$

R functions:

$$P(X = x) = \text{dnbinom}(x^*, r, p)$$
$$P(X \leq x) = \text{pnbinom}(x^*, r, p)$$

where x^* = number of failure until r successes, and $x^* = x - r$.

- **Discrete uniform distribution**: A random variable X has a discrete uniform distribution if each of the n values in its range, say, $X_1, X_2, \cdots, X_n$, has equal probability. The probability mass function is

$$P(X_i) = 1/n$$

Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, a+2, \cdots, b$, for $a \leq b$. Then,

Mean:

$$\mu = E(X) = \frac{b+a}{2}$$

Variance:

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Standard Deviation:

$$\sigma = \sqrt{\frac{(b-a+1)^2 - 1}{12}}$$

2 Examples

Example 1. A student is taking a multiple-choice exam in which each question has four choice. Assuming that she has no knowledge of the correct answers to any of the questions, she has decided on a strategy in which she will place four balls (marked A, B, C, and D) into a box. She randomly selects one ball for each question and replaces the ball in the box. The marking on the ball will determine her answer to the question. There are five multiple-choice questions on the exam. What is the probability that will she get

a. five questions correct?

b. at least four questions correct?

c. no questions correct?

d. no more than two questions correct?

Solution:

a.

$$P(X=5)=\binom{5}{5}\left(\frac{1}{4}\right)^5\left(1-\frac{1}{4}\right)^{5-5}=0.00098$$

R command: dbinom(5, 5, 1/4).

b.

$$P(X\geq 4)=P(X=4)+P(X=5)=0.01465+0.00098=0.01563$$

R command: dbinom(4, 5, 1/4) + dbinom(5, 5, 1/4).

c.

$$P(X=0)=\binom{5}{0}\left(\frac{1}{4}\right)^0\left(1-\frac{1}{4}\right)^{5-0}=0.23730$$

R command: dbinom(0, 5, 1/4)

d.

$$P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)=0.89648$$

R command: pbinom(2, 5, 1/4)

Example 2. In a shipment of 15 hard disks, 5 are defective. If 4 of the disks are inspected, what is the probability that

a. exactly 1 is defective?

b. at least 1 is defective?

c. no more than 2 are defective?

d. What is the average number of defective hard disks that you would expect to find in the sample of 4 hard disks?

Solution: Using hypergeometric distribution with $N = 15$, $A = 5$, and $n = 4$.

a. $P(X = 1) = \binom{5}{1}\binom{15-5}{4-1}/\binom{15}{4} = 0.43956.$

R command: dhyper(1,5,10,4))

b. $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{5}{0}\binom{15-5}{4-0}/\binom{15}{4} = 1 - 0.35185 = 0.84615.$

R command: 1 - dhyper(0,5,10,4))

c. $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.92308$

R command: phyper(2,5,10,4)

d. $E(X) = \mu = \frac{nA}{N} = \frac{4\times 5}{15} = 1.33333$

Example 3. The number of claims for missing baggage in a small city for a well-known airline follows the *Poisson distribution* and averages nine per day. What is the probability that, on a given day, there will be

a. fewer than three claims?

b. exactly three claims?

c. three or more claims?

d. more than three claims?

Solution:

a. $P(X < 3) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.00623.$

R command: ppois(2,9)

b. $P(X = 3) = \frac{9^3}{3!}e^{-9} = 0.01499.$

R command: dpois(3,9)

c. $P(X \geq 3) = 1 - P(X < 3) = 1 - 0.00623 = 0.99377.$

R command: 1 - ppois(2,9)

d. $P(X > 3) = P(X \geq 3) - P(X = 3) = 0.99377 - 0.01499 = 0.97878.$

Example 4. The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, what is the probability that exactly 125 wafers need to be analyzed before a large particle is detected?

Solution:

Let X denote the number of samples analyzed until a large particle is detected. X is a geometric random variable with $p = 0.01$.

The requested probability is

$$P(X = 125) = (0.99)^{124} 0.01 = 0.0029$$

R command: dgeom(124, 0.01)

Example 5. A Web site contains three identical computer servers. Only one is used to operate the site, and the other two are spares that can be activated in case the primary system fails. The probability of a failure in the primary computer (or any activated spare system) from a request for service is 0.0005. Assuming that each request represents an independent trial,

a. what is the mean number of requests until failure of all three servers?

b. What is the probability that all three servers fail within five requests?

Solution:

Let X denote the number of requests until all three servers fail, and let X_1, X_2, and X_3 denote the number of requests before a failure of the first, second, and third servers used, respectively. Then,

$$X = X_1 + X_2 + X_3$$

and X has a negative binomial distribution with $p = 0.0005$ and $r = 3$.

Thus,

a. The mean number of requests until failure of all three servers is

$$E(X) = 3/0.0005 = 6000 \text{ requests}$$

b. The probability is $P(X \leq 5)$ and

$$\begin{aligned}
P(X \leq 5) &= P(X=3) + P(X=4) + P(X=5) \\
&= 0.0005^3 + \binom{3}{2} 0.0005^3(0.9995) + \binom{4}{2} 0.0005^3(0.9995)^2 \\
&= 1.25 \times 10^{-10} + 3.75 \times 10^{-10} + 7.49 \times 10^{-10} \\
&= 1.249 \times 10^{-10}
\end{aligned}$$

R command:

dnbinom(0,3,0.0005) + dnbinom(1,3,0.0005) + dnbinom(2,3,0.0005)

= pnbinom(2,3,0.0005)

Example 6. Thickness measurements of a coating process are made to nearest hundredth of a millimeter. The thickness measurements are uniformly distributed with values 0.15, 0.16, 0.17, 0.18, and 0.19. Determine the mean and variance of the coating thickness for this process.

Solution:

$$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{[(b-a+1)^2 - 1]}{12}$$

Let $X = \frac{Y}{100}$, Y = 15, 16, 17, 18, 19, then $a = 15$ and $b = 19$, and

$$\mu = E(X) = \frac{1}{100}\left(\frac{15+19}{2}\right) = 0.17\text{mm}$$

$$\sigma^2 = \left(\frac{1}{100}\right)^2 \left[\frac{(19-15+1)^2 - 1}{12}\right] = 0.0002\text{mm}^2$$