Lecture notes 8
Selected Topics

1. Continuous Time Markov Chain

Definition:

Suppose we have a continuous-time stochastic process $\{X_t,\ t \geq 0\}$ taking on values in the set of nonnegative integers, we say that the process $\{X_t,\ t \geq 0\}$ is a continuous-time Markov chain if for all $s, t \geq 0$ and nonnegative integers $i, j, x(u),\ 0 \leq u < s$,

$$P\{X(t+s) = j \mid X(s) = i,\ X(u) = x(u),\ 0 \leq u < s\} = P\{X(t+s) = j \mid X(s) = i\}.$$

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future $X(t+s)$ given the present value $X(s)$ and the past $(X(u), 0 \leq u < s)$, depends only on the present and is independent of the past.
In addition, if

$$P\{X(t+s) = j \mid X(s) = i\}$$

is independent of s, then the process $\{X_t,\ t \geq 0\}$ is said to have stationary or homogeneous transition probability.

If we let T_i be the amount of time that the process stays in state i before making a transition into a different state, then

$$P\{T_i > t+s \mid T_i > s\} = P\{T_i > t\}.$$

Here, the random variable T_i is memoryless and is proved to be exponentially distributed.

In fact, this gives us another way to define a continuous-time Markov chain. Namely, it is a stochastic process having the properties that each time it enters state i

(a) the amount of time it spends in that state before making a transition into a different state is exponentially distributed with mean, say $1/v_i$,

(b) when the process leaves state i, it next enters state j with some probability, say P_{ij}. The P_{ij} must satisfy

$$P_{ii} = 0,\ \sum_j P_{ij} = 1, \quad for\ all\ i.$$

In other words, a continuous-time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed. In addition, the amount of time the process spends in state i, and the next state visited, must be independent random variables.

2. Birth and Death Processes

Consider a system whose state at any time is represented by the number of people in the system at that time. Suppose that whenever there are n people in the system, then
(i) new arrivals enter the system at an exponential rate λ_n, and
(ii) people leave the system at an exponential rate μ_n.

That is, whenever there are n persons in the system, then the time until the next arrival is exponentially distributed with mean $1/\lambda_n$ and is independent of the time until the next departure which is itself exponentially distributed with mean $1/\mu_n$.

Such a system is call a **birth and death process**. The parameters $\{\lambda_n,\ n = 0, \ldots, \infty\}$ and $\{\mu_n,\ n = 0, \ldots, \infty\}$ are called respectively the *arrival* (or birth) and *departure* (or death) rates.

Thus, a birth and death process is a continuous-time Markov chain with states $\{0, 1, \ldots\}$ for which transitions from state n may go only to either state $n-1$ or state $n+1$. The relation between the birth and death rates and the state transition rates and probabilities are

$$
\begin{aligned}
v_0 &= \lambda_0, \\
v_i &= \lambda_i + \mu_i,\ i > 0 \\
P_{01} &= 1, \\
P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i},\ i > 0 \\
P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i},\ i > 0
\end{aligned}
$$

The preceding follows, because if there are i in the system, then the next state will be $i+1$ if a birth occurs before a death, and the probability that an exponential random variable with rate λ_i will occur earlier than an (independent) exponential with rate μ_i is $\lambda_i/(\lambda_i + \mu_i)$. Moreover, the time until either a birth or a death occurs is exponentially distributed with rate $\lambda_i + \mu_i$ (and so, $v_i = \lambda_i + \mu_i$).

Example 1. (The Poisson Process) Consider a birth and death process for which

$$\mu_n = 0, \quad \text{for all} \quad n \geq 0,$$

$$\lambda_n = \lambda, \text{for all} \quad n \geq 0.$$

This is a process in which departures never occur, and the time between successive arrivals is exponential with mean $1/\lambda$. Hence, this is just the Poisson process.

A birth and death process for which $\mu_0 = 0$ for all n is called a pure birth process.

Example 2. (A birth process with Linear Birth rate) Consider a population whose members can give birth to new members but cannot die. If each member acts independently of the others and takes an exponentially distributed amount of time, with mean $1/\lambda$, to give birth, then if $X(t)$ is the population size at time t, then $\{X(t), t \geq 0\}$ is a pure birth process with $\lambda_n = n\lambda$, $n \geq 0$. This follows since if the population consists of n persons and each gives birth at an exponential rate λ, then the total rate at which births occur is $n\lambda$. This pure birth process is known as a Yule process after G. Yule, who used it in this mathematical theory of evolution.

3. Renewal Process

Definition:

Let $\{N(t), t > 0\}$ be a counting process and let X_n denote the time between the $(n-1)$th and nth event of this process, $n \geq 1$. If the sequence of nonnegative random variables $\{X_1, X_2, \ldots\}$ is independent and identically distributed, then the counting process $\{N(t), t > 0\}$ is said to be a **renewal process**.

When an event occurs, we say that a renewal has taken place.

For an example of a renewal process, suppose that we have an infinite supply of lightbulbs whose lifetime are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t), t > 0\}$ is a renewal process, when $N(t)$ is the number of lightbulbs that have failed by time t.

For a renewal process having interarrival times $X_1, X_2, X_3, \ldots$, let

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

In general, S_n denotes the time of the nth renewal.

Let F denote the interarrival distribution. To avoid trivialities, assume that $F(0) = P(X_n = 0) < 1$. Let $\mu = E(X_n),\ n \geq 1$ be the mean time between successive renewals.

Question: Can $N(t)$ be infinite for some finite value t?
Answer: No.

Reason: First note that

$$N(t) = \max\{n : S_n \leq t\} \tag{$**$}$$

and by the Strong Law of Large Number,

$$\frac{S_n}{n} \to \mu, \quad as\ n \to \infty.$$

Since $\mu > 0, S_n \to \infty$ as $n \to \infty$. But since $\mu > 0$ this means that S_n must be going to infinity as n goes to infinity. Thus, S_n can be less than or equal to t for at most a finite number of values of n, and hence by equation (**), $N(t)$ must be finite.

Distribution of $N(t)$

$$N(t) \geq n \quad if\ and\ only\ if\ S_n \leq t \tag{$***$}$$

By (***), we obtain

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \end{aligned}$$

Now since $X_i,\ i = 1, 2, 3, \ldots$ are independent and have a common distribution F, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as F_n, the n-fold convolution of F with itself. Therefore

$$P\{N(t) = n\} = F_n(t) - F_{n+1}(t).$$

4. Queues — A Single-Server Exponential Queuing System

Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ. That is, the times between successive arrivals are independent exponential random variables having mean $1/\lambda$. Each customer, upon arrival, goes directly into service if the server is free and, if not, the customer joins the queue.

When the server finishes serving a customer, the customer leaves the system, and the next customer in line, if there is any, enters service. The successive service times are assumed to be independent exponential random variables having mean $1/\mu$.

The above is called the $M/M/1$ queue. The two M's refer to the fact that both the interarrival and service distributions are exponential (and thus memoryless, or Markovian), and the 1 to the fact that there is a single server.

To analyze it, we shall begin by determining the limiting probabilities P_n, for $n = 0, 1, \ldots$

To do so, think along the following lines. Suppose that we have an infinite number of rooms numbered $0, 1, \ldots$, and suppose that we instruct an individual to enter room n whenever there are n customers in the system. That is, he would be in room 2 whenever there are two customers in the system; and if another were to arrive, then he would leave room 2 and enter room 3 Similarly, if a service would take place he would leave room 2 and enter room 1 (as there would now be only one customer in the system).

Now suppose that in the long-run our individual is seen to have entered room 1 at the rate of ten times an hour. Then at what rate must he have left room 1? Clearly, at this same rate of ten times an hour. For the total number of times that he enters room 1 must be equal to (or one greater than) the total number of times he leaves room 1. This sort of argument thus yields the general principle which will enable us to determine these probabilities. Namely, for each $n \geq 0$, *the rate at which the process enters state n equals the rate at which it leaves state n.*

Let us now determine these rates. Consider first state 0. When in state 0 the process can leave only by an arrival as clearly there cannot be a departure when the system is empty. Since the arrival rate is λ and the proportion of time that the process is in state 0 is P_0, it follows that the rate at which the process leaves state 0 is λP_0.

On the other hand, state 0 can only be reached from state 1 via a departure. That is, if there is a single customer in the system has exactly one customer is P_1, it follows that the rate at which the process enters state 0 is μP_1.

Hence, from our rate-equality principle we get our first equation,

$$\lambda P_0 = \mu P_1$$

Now consider state 1. The process can leave this state either by an arrival (which occurs at rate λ) or a departure (which occurs at rate μ). Hence, when in state 1, the process will leave this state at a rate of $\lambda + \mu$.

Since the proportion of time the process is in state 1 is P_1, the rate at which the process leaves state 1 is $(\lambda + \mu)P_1$. On the other hand, state 1 can be entered either from state 0 via an arrival or from state 2 via a departure. Hence, the rate at which the process enters state 1 is $\lambda P_0 + \mu P_2$. Because the reasoning for other states is similar, we obtain the following set of equations:

$$\begin{array}{cc} \textit{State} & \textit{Rate at which the process leaves} = \textit{rate at which it enters} \\ 0 & \lambda P_0 = \mu P_1 \\ n, n \geq 1 & (\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} \end{array} \qquad (*)$$

The set of Equation (*) which balances the rate at which the process enters each state with the rate at which it leaves that state is known as *balance equations.*

In order to solve Equation (*), we rewrite them to obtain

$$\begin{aligned} P_1 &= \frac{\lambda}{\mu} P_0, \\ P_{n+1} &= \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}), \ n \geq 1 \end{aligned}$$

Solving in terms of P_0 yields

$$\begin{aligned} P_0 &= P_0, \\ P_1 &= \frac{\lambda}{\mu} P_0, \\ P_2 &= \frac{\lambda}{\mu} P_1 + (P_1 - \frac{\lambda}{\mu} P_0) = \frac{\lambda}{\mu} P_1 = (\frac{\lambda}{\mu})^2 P_0, \\ P_3 &= \frac{\lambda}{\mu} P_2 + (P_2 - \frac{\lambda}{\mu} P_1) = \frac{\lambda}{\mu} P_2 = (\frac{\lambda}{\mu})^3 P_0, \\ P_4 &= \frac{\lambda}{\mu} P_3 + (P_3 - \frac{\lambda}{\mu} P_2) = \frac{\lambda}{\mu} P_3 = (\frac{\lambda}{\mu})^4 P_0, \\ P_{n+1} &= \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}) = \frac{\lambda}{\mu} P_n = (\frac{\lambda}{\mu})^{n+1} P_0. \end{aligned}$$

To determine P_0 we use the fact that the P_n must sum to 1, and thus

$$1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} (\frac{\lambda}{\mu})^n P_0 = \frac{P_0}{1 - \lambda/\mu}$$

or

$$P_0 = 1 - \frac{\lambda}{\mu},$$

$$P_n = (\frac{\lambda}{\mu})^n (1 - \frac{\lambda}{\mu}),\ n \geq 1. \qquad (**)$$

5. Brownian Motion

Let us start by considering the symmetric random walk which in each time unit is equally like to take a unit step either to the left or to the right. That is, it is a Markov chain with $P_{i,i+1} = 0.5 = P_{i,i-1},\ i = 0, \pm 1, \pm 2, \ldots$

Now, suppose that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. If we now go to the limit in the right manner, what we obtain is a Brownian Motion.

More precisely, suppose each $\triangle t$ time unit we take a step of size $\triangle x$ either to the left or the right with equal probabilities. If we let $X(t)$ denote the position at time t, then

$$X(t) = \triangle x (X_1 + \ldots + X_{[t/\triangle t]}) \qquad (*)$$

where

$$X_t = \begin{cases} +1, & if\ the\ ith\ step\ of\ length\ \triangle x\ is\ to\ the\ right \\ -1, & if\ the\ ith\ step\ of\ length\ \triangle x\ is\ to\ the\ left \end{cases}$$

and $[t/\triangle t]$ is the largest integer less than or equal to $[t/\triangle t]$, and where the X_i are assumed to be independent with

$$P(X_i = 1) = P(X_i = -1) = 0.5$$

As $E(X_i) = 0$, $Var(X_i) = E(X_i^2) = 1$, we have

$$E[X(t)] = 0,\ Var[X(t)] = (\triangle x)^2 (\frac{t}{\triangle t}).$$

If we let $\triangle x = \sigma\sqrt{\triangle t}$ for some positive constant σ, then when $\triangle t \to 0$, we have

$$E[X(t)] = 0,\ Var[X(t)] \to \sigma^2 t.$$

We now list some intuitive properties of this limiting process obtained by taking $\triangle x = \sigma\sqrt{\triangle t}$ and then letting $\triangle t \to 0$.

From (*) and the Central Limit Theorem, it seems reasonable that

(a) $X(t)$ is normal with mean 0 and variance $\sigma^2 t$.

(b) $\{X(t), t \geq 0\}$ has independent increments, in that for all $t_1 < t_2 < \ldots < t_n$,

$$X(t_n) - X(t_{n-1}) \ , \ X(t_{n-1}) - X(t_{n-2}) \ , \ \ldots \ , \ X(t_2) - X(t_1) \ , \ X(t_1)$$

are independent

(c) $\{X(t), t \geq 0\}$ has stationary increments, in that the distribution of $X(t+s) - X(t)$ does not depend on t.

Definition:
A stochastic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process if

(a) X(0) = 0;

(b) $\{X(t), t \geq 0\}$ has stationary and independent increments;

(c) For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

The Brownian motion process, sometimes called the Wiener process, is one of the most useful stochastic processes in applied probability theory. It originated in physics as a description of Brownian motion. This phenomenon, named after the English botanist Robert Brown who discovered it, is the motion exhibited by a small particle which is totally immersed in a liquid or gas. Since then, the process has been used beneficially in such areas as statistical testing of goodness of fit, analyzing the price levels on the stock market, and quantum machanics.

The first explanation of the phenomenon of Brownian motion was given by Einstein in 1905. He showed that Brownian motion could be explained by assuming that the immersed particle was continually being subjected to bombardment by the molecules of the surrounding medium. However, the preceding concise definition of this stochastic process underlying Brownian motion was given by Wiener in a series of papers originating in 1918.