STAT 3007 Introduction to Stochastic Processes
Tutorial 3 | Term 1, 2019−20

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1 Transition Probability Matrices

We focus on stationary discrete-time Markov chains with finite or countable state spaces.

- One-step transition probability: $p_{ij} := \Pr(X_{m+1} = j|X_m = i)$.

  n-step transition probability: $p_{ij}^{(n)} := \Pr(X_{n+m} = j|X_m = i)$.

- (One-step) transition probability matrix: $P = (p_{ij})$, which satisfies
  (1) all entries are nonnegative: $p_{ij} = \Pr(X_{n+1} = j|X_n = i) \geq 0$ for all i and j; and
  (2) each row sums to 1: $\sum_{j=0}^{\infty} p_{ij} = \sum_{j=0}^{\infty} \Pr(X_{n+1} = j|X_n = i) = 1$ for all i.

  n-step transition matrix: $P^{(n)} = (p_{ij}^{(n)})$.

- Chapman-Kolmogorov Equation:
  $p_{ij}^{(n)} = \sum_{k=0}^{\infty} p_{ik}p_{kj}^{(n-1)}$. In matrix form, $P^{(n)} = PP^{(n-1)} = \cdots = P^n$.
  $p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)}p_{kj}^{(n)}$. In matrix form, $P^{(m+n)} = P^{(m)}P^{(n)} = \cdots = P^mP^n = P^{m+n}$.

Example 1. Recall the signal (0 or 1) transmission problem. The transition probability matrix is given by P below. Suppose Yau sends $X_0 = 0$ with probability β and $X_0 = 1$ with probability $1 - \beta$. Find the probability that a signal 0 is sent and then correctly received at stage 2.

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix} \end{array}.$$

Example 2. (*Regular* transition probability matrix) For the transition probability matrix P of a Markov chain, show that if there exists a $k \in \mathbb{N}^+$ such that all entries of P^k are positive, then all entries of $P^{k+n}, n = 1, 2, ...,$ are positive.

Example 3. (Transitivity) We say state j is *accessible* from state i (denoted by $i \to j$) if $p_{ij}^{(n)} > 0$ for some integer $n \geq 0$. Show that if $i \to j$ and $j \to k$, then $i \to k$.

Example 4. (*Recurrence*) For states i and j of a Markov chain, suppose there exist $n, m \geq 1$ such that $p_{ij}^{(n)} > 0, p_{ji}^{(m)} > 0$. Show that if $\sum_{v=0}^{\infty} p_{ii}^{(v)}$ diverges, then $\sum_{v=0}^{\infty} p_{jj}^{(v)}$ also diverges.

2 First Step Analysis

Now we consider a Markov chain $\{X_n\}$ with finite state space $\{0, 1, 2, ..., N\}$.

- Absorbing states (labelled by $r, r+1, ..., N$): $p_{ii} = 1$ for $i = r, r+1, ..., N$.

- Transient states (labelled by $0, 1, 2, ..., r-1$): $p_{ij}^{(n)} \to 0$ as $n \to \infty$ for $0 \le i, j \le r-1$.
- Time of absorption: $T = \min\{n \ge 0 : X_n \ge r\}$.
- Absorption probabilities of an absorbing state $k \in \{r, r+1, \ldots, N\}$, starting at state i:
  $u_{ik} = \Pr(X_T = k | X_0 = i) = \sum_{j=0}^{N} \Pr(X_T = k | X_1 = j, X_0 = i) \Pr(X_1 = j | X_0 = i)$
  $\quad = p_{ik} + \sum_{j=r, j\neq k}^{N} 0 \cdot p_{ij} + \sum_{j=0}^{r-1} p_{ij} u_{jk} = p_{ik} + \sum_{j=0}^{r-1} p_{ij} u_{jk}$ for $i = 0, 1, \ldots, r-1$.
- Expected absorption times, starting at state i:
  $v_i = \mathrm{E}(T | X_0 = i) = \sum_{j=0}^{N} \mathrm{E}(T | X_1 = j, X_0 = i) \Pr(X_1 = j | X_0 = i)$
  $\quad = \sum_{j=0}^{r-1} (1 + v_j) p_{ij} + \sum_{j=r}^{N} 1 \cdot p_{ij} = 1 + \sum_{j=0}^{r-1} p_{ij} v_j$ for $i = 0, 1, \ldots, r-1$.
  Interpretation: initial step 1 + weighted average of additional steps.
- Expected total amount before absorption, starting at state i:
  $w_i = \mathrm{E}(\sum_{n=0}^{T-1} g(X_n) | X_0 = i) = g(i) + \sum_{j=0}^{r-1} p_{ij} w_j$ for $i = 0, 1, \ldots, r-1$.
- Matrix expressions: if P is (re)arranged as $P = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix}$, then

  $U_k = R_{[k]} + \boldsymbol{Q} U_k \Rightarrow U_k = (\boldsymbol{I} - \boldsymbol{Q})^{-1} R_{[k]}$, where $U_k = (u_{0k}, ..., u_{r-1,k})'$ and $R_{[k]}$ is the column of $\boldsymbol{R}$ corresponding to state k.

  $V = \mathbf{1}_r + \boldsymbol{Q} V \Rightarrow V = (\boldsymbol{I} - \boldsymbol{Q})^{-1} \mathbf{1}_r$, where $V = (v_0, ..., v_{r-1})'$ and $\mathbf{1}_r = (1, ..., 1)' \in \mathbb{R}^r$.

  $W = G + \boldsymbol{Q} W \Rightarrow W = (\boldsymbol{I} - \boldsymbol{Q})^{-1} G$, where $W = (w_0, ..., w_{r-1})'$ and $G = (g(0), ..., g(r-1))'$.

Example 5. A coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose that the first coin toss results in a head.
(1) Find the probability that the tosses end with two successive tails.
(2) Find the mean number of tosses (including the first toss) required.

Example 6. Yau and Ho are gambling. Suppose that in each gamble, win or lose are equally possible and there is a 20% chance to draw. Yau has m dollars and Ho has n dollars at the beginning. At the end of each gamble, the loser pays the winner 1 dollar (no payment if draw). They keep gambling until someone loses all. Find the probability that Yau loses all money to Ho.

Example 7. Consider a Markov Chain whose transition probability matrix is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

By interchanging the role of state 0 and state 2, the transition probability matrix becomes $P = \begin{pmatrix} Q & R \\ O & I \end{pmatrix}$.
(1) Identify the fundamental matrix Q.
(2) Find the inverse of the matrix $I - Q$.
(3) Find the probability of absorption into state 0 starting from state 1.
(4) Find the mean time spent in state 2 prior to absorption if the chain starts at state 1.

故不積跬步，無以至千里；不積小流，無以成江海。騏驥一躍，不能十步。
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