THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Introduction and Some Basics - Exercises Solutions

1. (Chevalier de Méré's Problem) The probability of having at least one six in 4 throws of die is $1-(5/6)^4 = 0.518$ where as the probability of having at least one double-six in 24 throws of 2 dice is $1-(35/36)^2 4 = 0.491$. The Chevalier thought these probabilities should be equal and lost a fair amount of money as a result.

2. (Exercises 1.2.1, 1.2.2 and 1.2.5 in Pinsky and Karlin) Recall the addition law: if X and Y are disjoint events, then
$$P(X \cup Y) = P(X) + P(Y)$$
Let $X = A \cap B$ and $Y = A \cap B^c$. The X and Y are disjoint - something can't be in B and B^c. Moreover, $X \cup Y = A$. Hence the addition law applies and we have
$$P(A) = P(A \cap B) + P(A \cap B^c).$$
Use this formula to calculate
$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= 2P(A \cap B) + P(A \cap B^c) + P(B \cap A^c) \\ &\quad - P(A \cap B) \\ &= P(A \cap B) + P(A \cap B^c) + P(B \cap A^c) \\ &= P(A) + P(A^c \cap B) \\ &= P(A \cup B). \end{aligned}$$
as required. Now let $X = B \cup C$. Then, using what we have shown we have
$$\begin{aligned} P(A \cup X) &= P(A) + P(X) - P(A \cap X) \\ &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C)) \end{aligned}$$
which is close to what we want. Concentrating on the last term:
$$\begin{aligned} P(A \cap (B \cup C)) &= P((A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C)) \\ &= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A \cap B \cap C) \end{aligned}$$
because they are disjoint and the addition law applies. Now take $P(A \cap B \cap C^c)$:
$$P(A \cap B \cap C^c) = P(A \cap B) - P(A \cap B \cap C)$$
by the addition law again. Hence
$$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

and we have the desired equality.

3. (Problem 1.2.4 In Pinsky and Karlin)

   (a) Draw a probability tree and you will notice a pattern: $P(N = 2) = 1/2$, $P(N = 3) = 1/4$, $P(N = 4) = 1/8$, etc. suggesting the p.m.f. is $P(N = n) = 1/2^{n-1}$ for $n = 2, 3, \ldots$. To prove this, consider the event $\{N = n\}$: it is equivalent to $\{HTH \cdots HTT\} \cup \{THT \cdots THH\}$, with n coin tosses. Hence

$$\begin{aligned} P(N = n) &= P(HTH \cdots HTT) + P(THT \cdots THH) \\ &= (1/2)^n + (1/2)^n = (1/2)^{n-1}. \end{aligned}$$

   (b) $P(N \text{ is even}) = \sum_{k=1}^{\infty} P(N = 2k) = 2/3$. $P(N \leq 6) = \sum_{k=2}^{6} (1/2)^{n-1} = 31/32$. $P(\{N \text{ is even}\} \cap \{N \leq 6\}) = P(N = 2, 4, 6) = 21/32$.

4. (Exercise 1.3.6 in Pinsky and Karlin)

   (a) Let X have this p.m.f. Then $\mathbb{E}[X] = \sum_{k=1}^{n} kp(k) = \frac{1}{n} \sum_{k=1}^{n} k$. Since $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)^*$, we find the mean is $\frac{n+1}{2}$. Note the variance is equal to $\mathbb{E}[X^2] - (\mathbb{E}[X])^2$. $\mathbb{E}[X^2] = \sum_{k=1}^{n} k^2 p(k) = \frac{1}{n} \sum_{k=1}^{n} k^2$. Since $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)^*$, we find $\mathbb{E}[X^2] = \frac{1}{6}(n+1)(2n+1)$ and the variance becomes $\frac{1}{12}(n^2 - 1)$. *To show these, consider

$$\sum_{k=1}^{n} [k^2 - (k-1)^2] = 2 \sum_{k=1}^{n} (k-1)$$

   and

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1.$$

   (b) Let $Z = X + Y$.

$$\begin{aligned} P(Z = m) &= P(X + Y = m) \\ &= \sum_{x=0}^{n} P(X + Y = m | X = x) P(X = x) \\ &= \sum_{x=0}^{n} P(Y = m - x | X = x) P(X = x) \end{aligned}$$

   Note that is $0 \leq m \leq n$ the sum becomes

$$\begin{aligned} P(Z = m) &= \sum_{x=0}^{m} P(Y = m - x | X = x) P(X = x) \\ &\quad \text{because } Y \text{ cannot take on negative values} \\ &= \sum_{x=0}^{m} P(Y = m - x) P(X = x) \\ &= \sum_{x=0}^{m} \frac{1}{n+1} \frac{1}{n+1} \\ &= \frac{m+1}{(n+1)^2} \end{aligned}$$

But if $n < m \leq 2n$ the sum becomes

$$\begin{aligned}
P(Z = m) &= \sum_{x=m-n}^{n} P(Y = m - x | X = x)P(X = x) \\
&\quad \text{because the smallest } Y \text{ can be is } m - n \\
&= \sum_{x=m-n}^{n} P(Y = m - x)P(X = x) \\
&= \frac{2n + 1 - m}{(n + 1)^2}
\end{aligned}$$

(c) Consider

$$\begin{aligned}
P(U \geq u) &= P(\min\{X, Y\} \geq u) \text{ for } u = 0, 1, \ldots, n \\
&= P(\{X \geq u\} \cap \{Y \geq u\}) \\
&= P(X \geq u)P(Y \geq u) \text{ by independence of } X, Y
\end{aligned}$$

Now

$$\begin{aligned}
P(X \geq u) &= P(X = u) + P(X = u + 1) + \cdots + P(X = n) \\
&= \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} \\
&= \frac{n - (u - 1)}{n + 1} = 1 - \frac{u}{n+1}
\end{aligned}$$

hence $P(U \geq u) = (1 - \frac{u}{n+1})^2$. Now for $u = 0, 1, \ldots, n$, $P(U = u) = P(U \geq u) - P(U \geq u + 1)$ (remembering $P(U \geq n + 1) = 0$). Therefore

$$\begin{aligned}
P(U = u) &= (1 - \frac{u}{n+1})^2 - (1 - \frac{u+1}{n+1})^2 \\
&= [1 - \frac{u}{n+1} - (1 - \frac{u+1}{n+1})][1 - \frac{u}{n+1} - (1 - \frac{u+1}{n+1})] \\
&= [\frac{u+1}{n+1} - \frac{u}{n+1}][2 - \frac{2u+1}{n+1}] \\
&= \frac{1}{(n+1)^2}[2(n + 1) - (2u + 1)] \\
&= \frac{1}{(n+1)^2}[2(n - u) + 1]
\end{aligned}$$

and we are done.

5. (Problem 1.3.11 in Pinsky and Karlin) First find the joint p.m.f. of U and W, $P(U = u, W = w)$. Consider $P(U = u, W = 0)$ first. If $W = 0$, $\max\{X, Y\} = \min\{X, Y\}$, that is $X = Y$. Hence $P(U = u, W = 0) = P(X = u, Y = u)$ and since X and Y are independent, this equals $(1 - \pi)^2\pi^{2u}$, $u \geq 0$.

Now consider $P(U = u, W = w > 0)$. Then we have two possibilities: either $X > Y$ or $Y > X$. Thus

$$\begin{aligned} P(U = u, W = w > 0) &= P(X = u, Y = u + w) + P(X = u + w, Y = u) \\ &= 2(1 - \pi)^2 \pi^{2u+w}. \end{aligned}$$

Sp, we have found the p.m.f. for the joint distribution of U and W. To show independence, we need to find the marginal distributions of U and W. Let's take U first.

$$\begin{aligned} P(U = u) &= \sum_{w=0}^{\infty} P(U = u, W = w) \\ &= (1 - \pi)^2 \pi^{2u} + 2(1 - \pi)^2 \pi^{2u} \sum_{w=1}^{\infty} \pi^w \\ &= (1 - \pi)^2 \pi^{2u} + 2(1 - \pi)^2 \pi^{2u} \times \frac{\pi}{1 - \pi} \\ &= (1 - \pi)^2 \pi^{2u} + 2(1 - \pi) \pi^{2u+1} \\ &= \pi^{2u}(1 - \pi^2). \end{aligned}$$

Now deal with W.

$$\begin{aligned} P(W = w) &= \sum_{u=0}^{\infty} P(U = u, W = w) \\ &= (1 - \pi)^2 \sum_{u=0}^{\infty} \pi^{2u} \text{ if } w = 0 \\ &\text{or } 2(1 - \pi)^2 \pi^w \sum_{u=0}^{\infty} \pi^{2u} \text{ if } w > 0 \\ &= \frac{(1 - \pi)^2}{1 - \pi^2} \text{ if } w = 0 \\ &\text{or } 2\frac{(1 - \pi)^2}{1 - \pi^2} \pi^w \text{ if } w > 0. \end{aligned}$$

Now let's check for independence.

$$\begin{aligned} P(U = u)P(W = w) &= \pi^{2u}(1 - \pi^2) \times \frac{(1 - \pi)^2}{1 - \pi^2} \text{ if } w = 0 \\ &\text{or } \pi^{2u}(1 - \pi^2) \times 2\frac{(1 - \pi)^2}{1 - \pi^2} \pi^w \text{ if } w > 0 \\ &= (1 - \pi^2)\pi^{2u} \text{ if } w = 0 \\ &\text{or } 2(1 - \pi^2)\pi^{2u+w} \text{ if } w > 0 \\ &= P(U = u, W = w) \end{aligned}$$

from before. Hence U and W are indeed independent.

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