Lecture notes 5
Markov Chain: Applications and Long run behavior of Markov Chain

Whatever happened in the past, be it glory or misery, be Markov!

5. **The Gamblers Ruin Problem**

Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gamblers fortune will reach N before reaching 0?

Solution: Let X_n be the players fortune at time n, then the process $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probabilities

$$P_{00} = P_{NN} = 1,\ P_{i,i+1} = p = 1 - P_{i,i-1},\quad i = 1, 2, 3, \ldots, N-1$$

This Markov chain has three classes, namely, $\{0\}$, $\{1, 2, \ldots, N-1\}$, $\{N\}$. The classes $\{0\}$ and $\{N\}$ are recurrent and the class $\{1, , 2, \ldots, N-1\}$ is transient. Since each transient state is visited only finitely often, it follows that, after some finite amount of time, the gambler will either attain his goal of N, or go broke.

Let $P_i,\ i = 0, 1, \ldots, N$ denote the probability that, starting with i, the gambler fortune will eventually reach N. By conditioning on the outcome of the initial play of the game we obtain

$$P_i = pP_{i+1} + qP_{i-1},\ i = 1, 2, \ldots, N-1$$

or equivalently (because $p + q = 1$)

$$pP_i + qP_i = pP_{i+1} + qP_{i-1},\ i = 1, 2, \ldots, N-1$$

or

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}),\ i = 1, 2, \ldots, N-1$$

Since $P_0 = 0$, we have

$$\begin{aligned}
P_2 - P_1 &= \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1 \\
P_3 - P_2 &= \frac{q}{p}(P_2 - P_1) = (\frac{q}{p})^2 P_1 \\
&\cdot \\
&\cdot \\
&\cdot \\
P_i - P_{i-1} &= \frac{q}{p}(P_{i-1} - P_{i-2}) = (\frac{q}{p})^{i-1} P_1
\end{aligned}$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = (\frac{q}{p})^{N-1}P_1$$

Adding the first $i-1$ of these equations, we have

$$P_i - P_1 = P_1[(\frac{q}{p}) + (\frac{q}{p})^2 + \ldots + (\frac{q}{p})^{i-1}]$$

or

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})}P_1, & if\ \frac{q}{p} \neq 1 \\ iP_1 & if\ \frac{q}{p} = 1. \end{cases}$$

Now using the fact that $P_N = 1$, we obtain

$$P_1 = \begin{cases} \frac{1-(\frac{q}{p})}{1-(\frac{q}{p})^N}, & if\ p \neq \frac{1}{2} \\ \frac{1}{N} & if\ p = \frac{1}{2}, \end{cases}$$

and hence

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & if\ p \neq \frac{1}{2} \\ \frac{i}{N} & if\ p = \frac{1}{2}. \end{cases}$$

Note that

$$\lim_{N\to\infty} P_i \to \begin{cases} 1-(\frac{q}{p})^i, & if\ p > \frac{1}{2} \\ 0, & if\ p \leq \frac{1}{2}. \end{cases}$$

Thus, if $p > \frac{1}{2}$, there is a positive probability that the gamblers fortune will increase indefinitely, while if $p \leq \frac{1}{2}$, the gambler will, with probability 1, go broke against an infinity rich adversary.

Example 1:

Suppose Max and Patty decide to flip pennies; the one coming closest to the wall wins. Patty, being the better player, has a probability 0.6 of winning on each flip.

(a) If Patty starts with five pennies and Max with ten, what is the probability that Patty will wipe Max out?

(b) What if Patty starts with 10 and Max with 20?

Solution:

(a) The desired probability is obtained by letting $i = 5$, $N = 15$, and $p = 0.6$. Hence, the desired probability is

$$\frac{1-(2/3)^5}{1-(2/3)^{15}} \approx 0.87.$$

(b) The desired probability is

$$\frac{1-(2/3)^{10}}{1-(2/3)^{30}} \approx 0.98.$$

6. The Long Run Behavior of Markov Chains

In the long run, we are all equal. —-with apology to John Maynard Keynes

6.1 Regular Markov chains.

Example 1. Let $\{X_n\}$ be a MC with two states 0 and 1, and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.33 & 0.67 \\ 0.75 & 0.25 \end{pmatrix}.$$

Then,

$$\mathbf{P}^2 = \begin{pmatrix} 0.611 & 0.389 \\ 0.438 & 0.562 \end{pmatrix}, \quad \mathbf{P}^5 = \begin{pmatrix} 0.524 & 0.476 \\ 0.536 & 0.464 \end{pmatrix},$$
$$\mathbf{P}^7 = \begin{pmatrix} 0.528 & 0.472 \\ 0.530 & 0.469 \end{pmatrix}, \quad .... \quad \mathbf{P}^{16} = \begin{pmatrix} 0.5294 & 0.4706 \\ 0.5294 & 0.4706 \end{pmatrix} \quad ....$$

Recall that $\mathbf{P}^{(n)} = \mathbf{P}^n$. We get the impression from this example that the n-step transition matrix converge, actually quite fast in this example, and that limits in the same columns are the same. The question is: is this a general phenomenon or only pertained to this MC? Note that all the above transition matrices, one-step or many step, have all positive entries.

Example 2 Let $\{X_n\}$ be a MC with two states 0 and 1, and the one-step transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In other words, from state 1, the MC sure goes to 0 the next step; and from state 0, it sure goes to 1 the next step. Then

$$\mathbf{P}^k = \mathbf{P} \quad \text{for all odd } k, \qquad \mathbf{P}^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for all even } k.$$

Clearly, $\mathbf{P}^n$ does not converge as $n \to \infty$.

Example 3. (Wealthiness does not last long.) According to an old Chinese saying: Fu Bu Guo San Dai (wealthiness does not last beyond third generation—a word-by-word translation). Indeed, suppose every family in a population assumes one of $N+1$ financial classes/states: State 0 is the poorest, 1 the second poorest, ..., N the richest. Let X_n the state of the n-th generation of a family. It is reasonable to assume a transition matrix with all $P_{ij} > 0$, $i, j = 0, 1, ..., N$. Such transition probability matrix and the MC are so called *regular*, to be defined below. It will be shown that, in the long run, the family, no matter it started rich or poor, will turn out ordinary after many generations. The same is true for, other than wealth, height or IQ or nearly all of the biological traits. The hundredth generation of Yao, Ming, the star basketball player, and that of Pan, Changjiang, a Chinese comedy star known for being short and funny, will have about same distribution of height. The former can very well be shorter than the latter, and, even if taller, it's largely by randomness.

Indeed, we shall show a large class of MCs, called regular MCs, the transition matrices do converge. Example 1 is one of those and Example 2 is not.

Definition (Regular Markov Chains)

(i). A transition probability matrix $\mathbf{P}$ is called a regular transition probability matrix, if there exists a $k > 0$ such that all entries of $\mathbf{P}^k$ are positive.

(ii). A MC is called regular if its transition probability matrix is regular.

The following is another example of three state regular transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 2/3 & 1/6 & 1/6 \end{pmatrix}$$

The following are three state non-regular transition matrices:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

DIY: verify that the two matrices are non-regular. Hint: for the former, $\mathbf{P}^k$ for all $k > 0$ can be figured out; for the latter, the third rows of $\mathbf{P}^k$ is always 0, 0, 1 for all $k > 0$.

Given a transition probability matrix, it is often not easy to identify whether it is regular. The following $sufficient$ condition offers some ease for.

Suppose (i), for any i and j in the state space, there exists a $k > 0$ (which depends on i and j) such that $P_{ij}^{(k)} > 0$; and (ii), there exists a state i such that $P_{ii} > 0$. Then the transition matrix is regular.

For example, consider

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.99 & 0 & 0 & 0 & 0.01 \end{pmatrix} \text{ is regular , but } \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is not.}$$

The former can be easily checked by the above sufficient condition, while the latter does not satisfy the condition since its diagonal elements are all 0.

*Remark 1.** In the definition of regular transition matrix, $\mathbf{P}^k$ has all positive elements for some $k > 0$. Then, for all $n > k$, $\mathbf{P}^n$ all have all positive elements (Please DIY). In other words, an equivalent definition is there exists a $k > 0$ such that, for all states i, j and all $n \geq k$, $P_{ij}^{(n)} > 0$. We also note that Part (i) of the above sufficient condition, in which k depends on i and j, is very different from the universal k in the definition of regular transition matrix. We shall later see that Part (i) of the above sufficient condition says all states are communicative and Part (ii) says one state is aperiodic. (Under communicativeness, all states are aperiodic.) In other words, communicativeness plus aperiodicity of all states ensures the regularity.

6.2. The limit of regular Markov chains.

The long run behavior of regular MCs is derived from the following theorem, which is the main result of this chapter.

Theorem 1. *Let* $\mathbf{P}$ *be a regular transition matrix with states* $\{0, 1, ..., N\}$. *Then, for all states* i *and* j, *as* $n \to \infty$,

$$P_{ij}^{(n)} \to \pi_j > 0 \quad \textit{and, as a result,} \quad P(X_n = j) \to \pi_j > 0, \tag{4.1}$$

where $(\pi_0, \pi_1, ..., \pi_N)$ *is the unique solution to*

$$\begin{cases} \pi_j = \sum_{j=0}^{n} \pi_k P_{kj}, \quad j = 0, 1, ..., N & (4.2) \\ \sum_{i=0}^{N} \pi_i = 1. & (4.3) \end{cases}$$

Equation (4.1) is also the definition of π_j. However its implication lies three folds: (a). the limit of $P_{ij}^{(n)}$ as $n \to \infty$ exists; (b). the limit is positive; and (c). the limit is irrelevant with the starting state i. Equations (4.2) can be conveniently written in matrix form

$$(\pi_0 \ \ \pi_1 \ldots \pi_N) = (\pi_0 \ \ \pi_1 \ldots \pi_N)\mathbf{P}. \tag{4.2$'$}$$

It appears to be odd that there are $N + 2$ equations in (4.2) and (4.3) to solve for $N + 1$ variables $\pi_0, ..., \pi_N$ (Please DIY).

A rigorous proof of Theorem 4.1 is technical, especially about the existence of the limit, and is beyond the pursuit of this course. Here we offer a brief explanation. First, should the limit π_j exist, they must satisfy the equation (4.2). This is because, for $j = 0, ..., N$,

$$\pi_j \equiv \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj} = \sum_{k=0}^{N} \lim_{n\to\infty} P_{ik}^{(n-1)} P_{kj} = \sum_{k=0}^{N} \pi_k P_{kj}.$$

The interpretation of limit theorem is, in my view, more important than the technical proofs. The interpretation of (4.1) has two faces that I call the *space average* and the *time average.* The space average interpretation is, at any fixed time long into the future, the MC has about probability π_j to be in state j. This gives the distribution of the MC X_n for a fixed large n in the state space. The time average interpretation is, over a long period of time, the relative frequency of the MC staying in state j is about π_j.

*Remark 2.** A little more elaboration on the time average. Suppose there is a payoff c_j whenever the MC visits state j once. Assume $X_0 = k$. The long run average of payoff is

$$A_n \equiv \sum_{i=1}^{n} c_{X_i}/n = \sum_{i=1}^{n} \sum_{j=0}^{N} c_j I_{\{X_i = j\}}/n.$$

Heuristically, for large n, A_n, which is random, would be close to its mean, which is

$$E(A_n) = \sum_{i=1}^{n}\sum_{j=0}^{N} c_j E(I_{\{X_i=j\}})/n = \sum_{i=1}^{n}\sum_{j=0}^{N} c_j P(X_i = j|X_0 = k)/n = \sum_{j=0}^{N} c_j [\sum_{i=1}^{n} P_{kj}^{(i)}/n] \approx \sum_{j=0}^{N} c_j \pi_j.$$

Now, consider the special case with $c_l = 1$ for a particular l and the rest $c_0, ...c_{l-1}, c_{l+1}..., c_N$ being all 0. Then A_n is the fraction of the MC visits of state l, which is indeed close to $\sum_{j=0}^{N} c_j \pi_j = \pi_l$.

Example 4. Ms. L spends her life time only in three places Home, Office and Gym. Assume every time she reaches H, O and G, she spends 12, 10 and 2 hours, respectively, and then transits to another location according to the following transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ \mathrm{H} \\ \mathrm{O} \\ \mathrm{G} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{H} & \mathrm{O} & \mathrm{G} \end{array} \\ \begin{pmatrix} 0 & 0.9 & .1 \\ .1 & 0 & .9 \\ 0.8 & 0.2 & 0 \end{pmatrix} \end{array}$$

It is easily verified that $\mathbf{P}^2$ has all entries positive and therefor $\mathbf{P}$ is regular, which is perhaps easier to see from the this figure.

Solving equation (4.2)-(4.3), we have

$$\pi_H = 0.309 \qquad \pi_O = 0.347 \qquad \pi_G = 0.343.$$

Let X_n be Ms L's whereabout after n transitions. $\{X_n : n \geq 0\}$ is a MC with state space $\{\mathrm{H}, \mathrm{O}, \mathrm{G}\}$ and transition matrix $\mathbf{P}$. Let n_H be the number of times she stays in state H in a long period with (a large) n transitions. Likewise define n_O and n_G. The long run fraction of her life time spent in office is

$$\begin{aligned} & \frac{n_O \times 10}{n_H \times 12 + n_O \times 10 + n_G \times 2} = \frac{10 n_O/n}{12 n_H/n + 10 n_O/n + 2 n_G/n} \\ \approx & \frac{10\pi_O}{12\pi_H + 10\pi_O + 2\pi_G} = 0.441. \end{aligned}$$

Translating into daily hours, it is $0.441 * 24 = 10.58$ hours in office per day. Likewise, the long run fractions of her life time spent in Home and Gym are, respectively, 0.472 and 0.087, which are 11.32 and 2.09 in daily hours.

6.3. More discussions on the limiting distribution of Markov chains.

Definition: State i is said to have a period d if $P_{ii}^n = 0$ whenever n is not divisible by d, and d is the largest integer with this property.

For instance, starting in state i, it may be possible for the process to enter state i only at times 2,4,6,8, $\ldots$, in which case state i has period 2.

Definition: A state with period 1 is said to be *aperiodic*.

Note that periodicity is a class property. That is, if state i has period d, and state i and j communicate, then state j also has period d.

Definition: If state i is recurrent, then it is said to be positive recurrent if, starting in i, the expected time until the process returns to state i is finite.

Note that positive recurrence is a class property. While there exist recurrent states that are not positive recurrent, it can be shown that in a finite-state Markov Chain, all recurrent states are positive recurrent.

Definition: Positive recurrent, aperiodic states are called *ergodic*.

With the above definitions, an alternative way to state the limiting distribution of Markov chain is as follows:

Theorem 1*: For any irreducible ergodic Markov chain,

$$\lim_{n\to\infty} P_{ij}^n,\ j \geq 0,$$

exists and is independent of i. Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^n,\ j \geq 0,$$

then π_i is the unique nonnegative solution to

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij},\ j \geq 0$$

and

$$\sum_{j=0}^{\infty} \pi_j = 1.$$

Remarks on the implications of the limiting probabilities:

(a) Given that $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exists and is independent of the initial state i, it is not difficult to see that the π's must satisfy the above equation. Let us derive an expression for $P(X_{n+1} = j)$ by conditioning on the state at time n. That is

$$\begin{aligned} P(X_{n+1} = j) &= \sum_{i=0}^{\infty} P(X_{n+1} = j \mid X_n = i) P(X_n = i) \\ &= \sum_{i=0}^{\infty} P_{ij} P(X_n = i). \end{aligned}$$

Letting $n \to \infty$ and assuming that we can bring the limit inside the summation, we have

$$\sum_{i=0}^{\infty} P_{ij}\pi_i = \pi_j.$$

(b) It can be shown that π_j, the limiting probability that the process will be in state j at time n, also equals the long-run proportion of time that the Markov chain is in state j. The long run proportions π_j, $j \geq 0$, are often called *stationary probabilities* as well. The reason being that if the initial state is chosen according to the probabilities π_j, $j \geq 0$, then the probability of being in state j at any time n is also equal to π_j. That is, if

$$P(X_0 = j) = \pi_j, \ j \geq 0,$$

then

$$P(X_n = j) = \pi_j, \quad for\ all\ n, j \geq 0.$$

(c) In the irreducible, positive recurrent, periodic case, we still have that the π_j, $j \geq 0$ are the unique non-negative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij},$$

and

$$\sum_{j=0}^{\infty} \pi_j = 1.$$

But now π_j must be interpreted as the long-run proportion of time that the Markov chain is in state j.

Example 4: Consider Example 1 in lecture notes 4, in which we assume that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. If we say that the state is 0 when it rains and 1 when it does not rain, then the limiting probabilities π_0 and π_1 are given by

$$\begin{aligned} \pi_0 &= \alpha\pi_0 + \beta\pi_1 \\ \pi_1 &= (1-\alpha)\pi_0 + (1-\beta)\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

which yields that

$$\pi_0 = \frac{\beta}{1+\beta-\alpha}, \quad \pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$$

For example if $\alpha = 0.7$ and $\beta = 0.4$, then the limiting probability of rain is $\pi_0 = 4/7 = 0.571$.

Example 5: Consider Example 3 in which the mood of an individual is considered as a three-state Markov chain having a transition probability matrix

$$P = \left\| \begin{array}{ccc} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{array} \right\|$$

In the long run, what proportion of time is the process in each of the three states?

Solution: The limiting probabilities π_i, i = 0,1,2, ... are obtained by solving the set of equations

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2 \\ \pi_1 &= 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2 \\ \pi_2 &= 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

Solving yields

$$\pi_0 = 21/62, \quad \pi_1 = 23/62, \quad \pi_2 = 18/62.$$

Example 6: (A Model of Class Mobility) A problem of interest to sociologists is to determine the proportion of society that has an upper- or lower-class occupation. One possible mathematical model would be to assume that transitions between social classes of the successive generations in a family can be regarded as transitions of a Markov chain. That is, we assume that the occupation of a child depends only on his or her parents occupation. Let us suppose that such a model is appropriate and that the transition probability matrix is given by

$$P = \begin{Vmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{Vmatrix}$$

That is, for instance, we suppose that the child of a middle-class worker will attain an upper-, middle, or lower-class occupation with respective probabilities 0.05, 0.70, 0.25.

The limiting probabilities π_i, thus satisfy

$$\begin{aligned} \pi_0 &= 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \\ \pi_1 &= 0.48\pi_0 + 0.70\pi_1 + 0.50\pi_2 \\ \pi_2 &= 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

Hence,

$$\pi_0 = 0.07, \quad \pi_1 = 0.62, \quad \pi_2 = 0.31$$

In other words, a society in which social mobility between classes can be described by a Markov chain with the above transition probability matrix has, in the long run, 7 percent of its people in upper-class jobs, 62 percent of its people in middle-class jobs, and 31 percent in lower-class jobs.