THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Markov Chains - Some Special Examples - Exercises Solutions

1. (Exercise 3.5.8 in Pinsky and Karlin) Since there is no limit on how many customers can arrive, the state space is $\{0, 1, 2, \dots\}$. Say we're in State 0. Then there are two possibilities: either no customer arrives (with probability $(1-\alpha)$) and we remain in State 0, or a single customer arrives (w.p. α) and we move to State 1. Now sa we're in State 1. There are 4 possibilities

   - the current customer is served, and a new customer arrives (w.p. $\beta\alpha$) and we stay in State 1;
   - the current customer is served, but no new customer arrives (w.p. $\beta(1-\alpha)$) and we move to State 0;
   - the current customer is not served, and a new customer arrives (w.p. $(1-\beta)\alpha$) and we move to State 2;
   - the current customer is not served, but no new customer arrives (w.p. $(1-\beta)(1-\alpha)$) and we stay in State 1.

   Combining the above gives us the transition probabilities p_{10}, p_{11}, p_{12}. Similar thinking will give us $p_{i,i-1}, p_{ii}, p_{i,i+1}$ for $i = 1, 2, \dots$. Hence the transition probability matrix is

$$\mathbb{P} = \begin{pmatrix} (1-\alpha) & \alpha & 0 & 0 & \cdots \\ \beta(1-\alpha) & (1-\beta-\alpha+2\alpha\beta) & \alpha(1-\beta) & 0 & \cdots \\ 0 & \beta(1-\alpha) & (1-\beta-\alpha+2\alpha\beta) & \alpha(1-\beta) & 0 \\ 0 & 0 & \beta(1-\alpha) & (1-\beta-\alpha+2\alpha\beta) & \alpha(1-\beta) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

2. (Exercise 3.5.9 in Pinsky and Karlin) Since there is no limit to how much the contestant could win, the state space is $\{0, 1, 2, \dots\}$. Say we're in State 0. Then we are out of the game: that is, we will remain there forever, thus $p_{00} = 1$. Now say we're in State 1. We play the game. W.p. q we win the game and move to State 2. W.p. p we lose the game and move to State 0. The same thinking applies for other states $i = 1, 2, \dots$: $p_{i0} = p$, $p_{i,i+1} = q$. Thus the transition probability matrix is

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ p & 0 & q & 0 & \cdots \\ p & 0 & 0 & q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

THE END