3-2 Probability Distributions and Probability Mass Functions

Definition

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, a probability mass function is a function such that

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$ (3-1)

X	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

3-3 Cumulative Distribution Functions

Definition

The cumulative distribution function of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

For a discrete random variable X, $F(x)$ satisfies the following properties.

(1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

(2) $0 \leq F(x) \leq 1$

(3) If $x \leq y$, then $F(x) \leq F(y)$ (3-2)

e.g. For a binomial random variable X

$F(2) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= f(0) + f(1) + f(2)$

3-4 Mean and Variance of a Discrete Random Variable

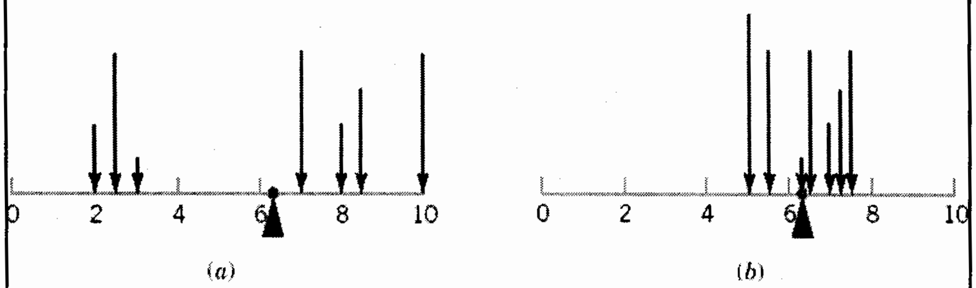


Figure 3-5 A probability distribution can be viewed as a loading with the mean equal to the balance point. Parts (a) and (b) illustrate equal means, but Part (a) illustrates a larger variance.

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

$$\sigma^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = EX^2 - (EX)^2$$

3-4 Mean and Variance of a Discrete Random Variable

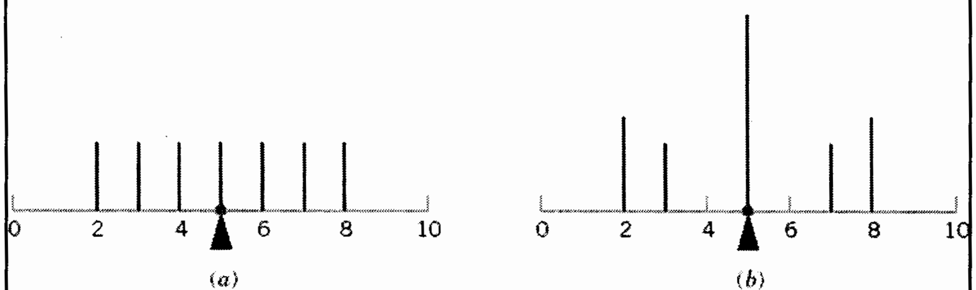


Figure 3-6 The probability distribution illustrated in Parts (a) and (b) differ even though they have equal means and equal variances.

3-5 Discrete Uniform Distribution

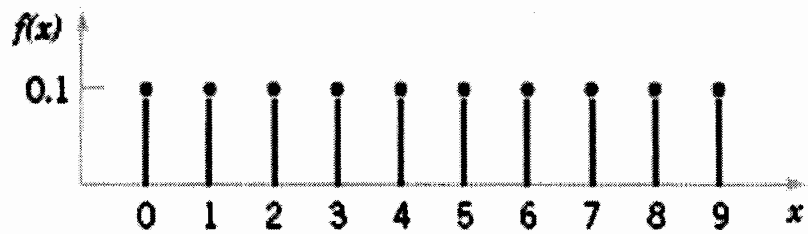


Figure 3-7 Probability mass function for a discrete uniform random variable.

$f(X_i) = \frac{1}{n}, \quad i = 1, 2, \cdots n$

3-5 Discrete Uniform Distribution

Mean and Variance

Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, a+2, \ldots, b$, for $a \le b$. The mean of X is

$$\mu = E(X) = \frac{b+a}{2}$$

The variance of X is

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12} \quad (3\text{-}6)$$

3-6 Binomial Distribution

Definition

A random experiment consists of n Bernoulli trials such that

(1) The trials are independent

(2) Each trial results in only two possible outcomes, labeled as "success" and "failure"

(3) The probability of a success in each trial, denoted as p, remains constant

The random variable X that equals the number of trials that result in a success has a binomial random variable with parameters $0 < p < 1$ and $n = 1, 2, \ldots$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \tag{3-7}$$

R functions: $dbinom(x, n, p) = P(X=x)$

$pbinom(x, n, p) = P(X \leq x)$

3-6 Binomial Distribution

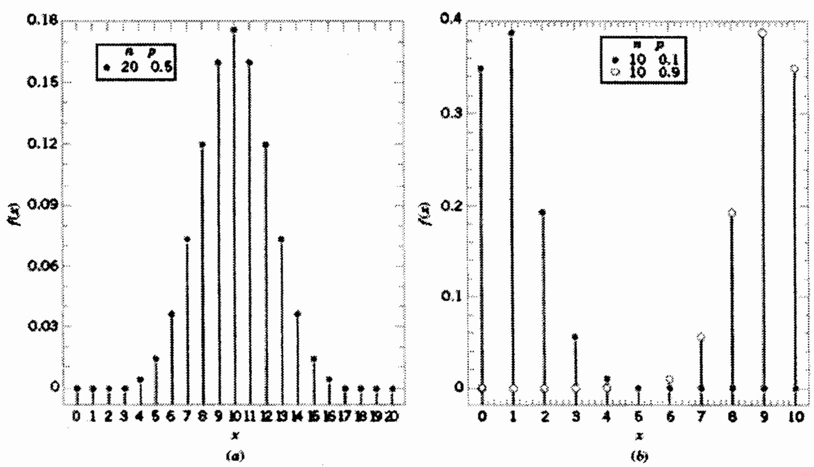
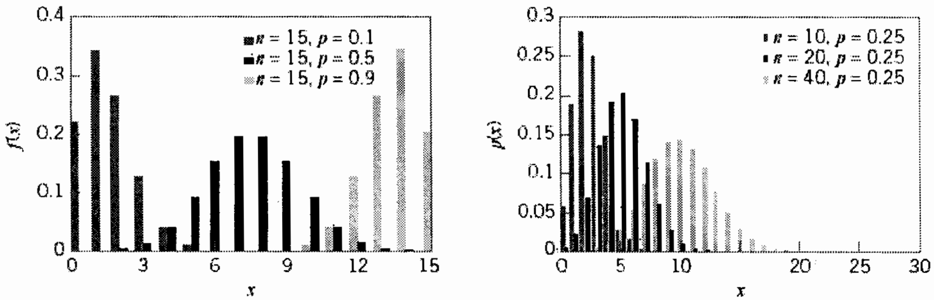


Figure 3-8 Binomial distributions for selected values of n and p.



(*a*) Binomial distributions for different values of p with $n = 15$.

(*b*) Binomial distributions for different values of n with $p = 0.25$.

Figure 2-14 Binomial distributions for selected values of n and p.

3-7 Geometric Distributions

Definition

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until the first success. Then X is a geometric random variable with parameter $0 < p < 1$ and

$$f(x) = (1-p)^{x-1}p \qquad x = 1, 2, \ldots \tag{3-9}$$

R functions: dgeom(x^*, p)

pgeom(x^*, p)

x^* — # of failure until the first success.

$x^* = x - 1$

3-7 Geometric Distributions

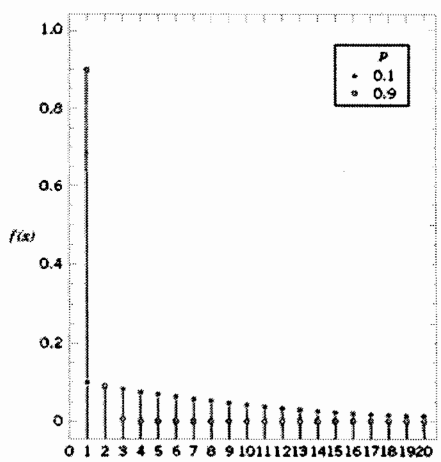


Figure 3-9. Geometric distributions for selected values of the parameter *p*.

$P(X=1 | P=0.9)$

$= \text{dgeom}(0, 0.9) = 0.9$

$P(X=1 | P=0.1)$

$= \text{dgeom}(0, 0.1) = 0.1$

3-7 Geometric Distributions

Definition

If X is a geometric random variable with parameter p,

$$\mu = E(X) = 1/p \quad \text{and} \quad \sigma^2 = V(X) = (1-p)/p^2 \tag{3-10}$$

3-7 Negative Binomial Distributions

3-7.2 Negative Binomial Distribution

A generalization of a geometric distribution in which the random variable is the number of Bernoulli trials required to obtain r successes results in the **negative binomial distribution**.

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until r successes occur. Then X is a **negative binomial random variable** with parameters $0 < p < 1$ and $r = 1, 2, 3, \ldots$, and

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r \qquad x = r, r+1, r+2, \ldots \qquad (3\text{-}11)$$

R functions: dnbinom (x^*, r, p)

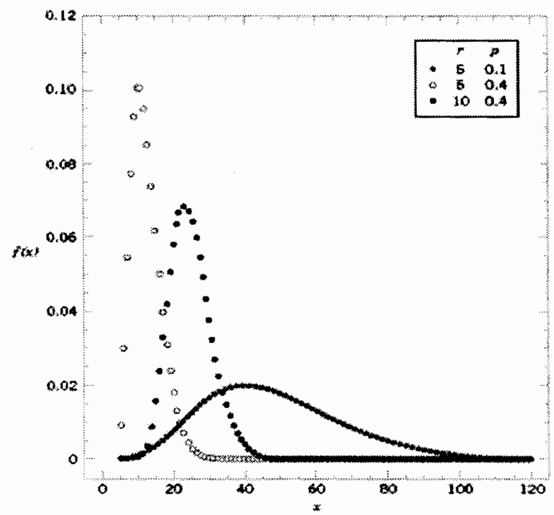
pnbinom (x^*, r, p)

x^* — # of failure until r successes

$x^* = x - r$

3-7 Negative Binomial Distributions

Figure 3-10. Negative binomial distributions for selected values of the parameters r and p.



dnbinom (0, 5, 0.4)

= 0.01024

dnbinom (1, 5, 0.4)

= 0.03072

⋮

dnbinom (5, 5, 0.4)

= 0.1003

3-7 Negative Binomial Distributions

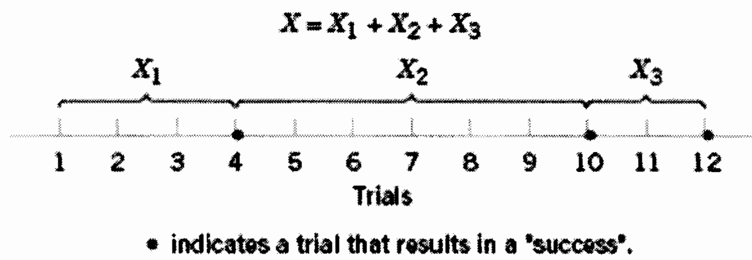


Figure 3-11. Negative binomial random variable represented as a sum of geometric random variables.

3-7 Negative Binomial Distributions

3-7.2 Negative Binomial Distribution

If X is a negative binomial random variable with parameters p and r,

$$\mu = E(X) = r/p \quad \text{and} \quad \sigma^2 = V(X) = r(1-p)/p^2 \qquad (3\text{-}12)$$

3-8 Hypergeometric Distribution

Definition

A set of N objects contains

K objects classified as successes

$N - K$ objects classified as failures

A sample of size n objects is selected randomly (without replacement) from the N objects, where $K \leq N$ and $n \leq N$.

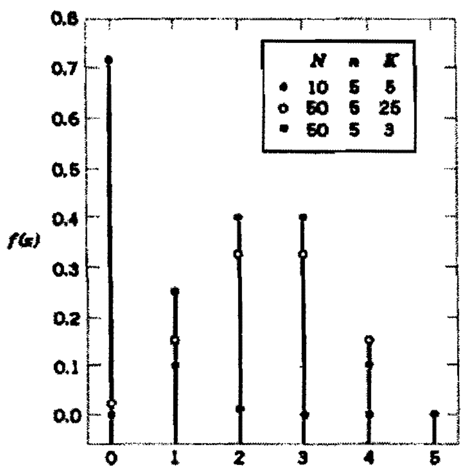
Let the random variable X denote the number of successes in the sample. Then X is a hypergeometric random variable and

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad x = \max\{0, n + K - N\} \text{ to } \min\{K, n\} \qquad (3\text{-}13)$$

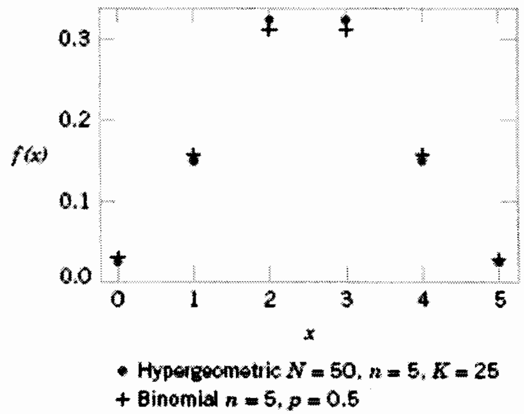
dhyper(x, K, N−K, n)

3-8 Hypergeometric Distribution

Figure 3-12. Hypergeometric distributions for selected values of parameters N, K, and n.



3-8 Hypergeometric Distribution



, $p = \frac{K}{N} = 0.5$, $\frac{n}{N} = 0.1$

	0	1	2	3	4	5
Hypergeometric probability	0.025	0.149	0.326	0.326	0.149	0.025
Binomial probability	0.031	0.156	0.321	0.312	0.156	0.031

Figure 3-13. Comparison of hypergeometric and binomial distributions.

3-9 Poisson Distribution

Definition

Given an interval of real numbers, assume events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

(1) the probability of more than one event in a subinterval is zero,

(2) the probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and

(3) the event in each subinterval is independent of other subintervals, the random experiment is called a Poisson process.

The random variable X that equals the number of events in the interval is a Poisson random variable with parameter $0 < \lambda$, and the probability mass function of X is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots \qquad \text{(3-16)}$$

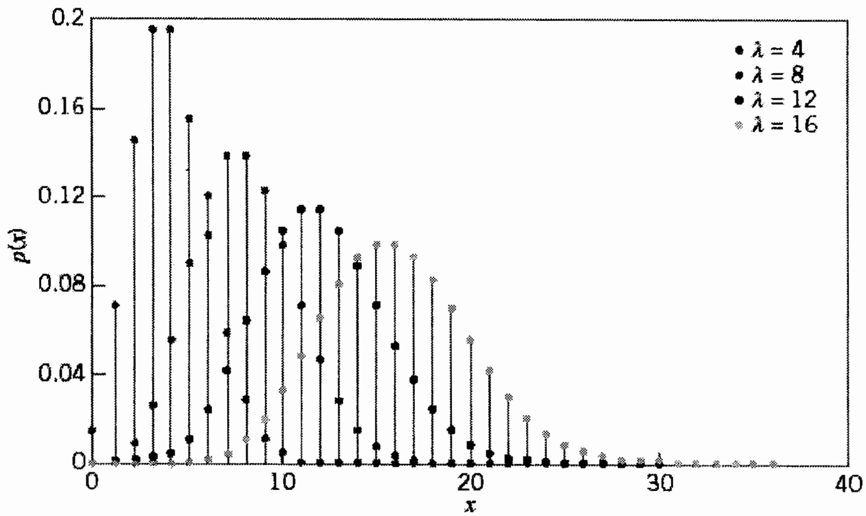


Figure 2-15 Poisson probability distributions for selected values of λ.