STAT 6106

Fall2018 Coding Mid-Term

1 Q1

1.1 Method 1

$$\int g(x)dx = \int \frac{g(x)}{f(x)} \cdot f(x)dx = E_x(\frac{g(x)}{f(x)}) \simeq \frac{1}{n}\sum_{i=1}^{m} \frac{g(x_i)}{f(x_i)}$$

where $f(x)$ is the density function of x, and $\{x_i, \cdots, x_n\}$ are its samples.

Because $x > 0$, thus, let x obey a log-normal distribution.

1.2 Method 2

Noitce that the pdf of log-normal is $f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\log(x)-\mu)}{2\sigma^2}}$, $x \in (0, \infty)$, which coincide with most of parts in the integral, so we can rewrite the integral as

$$\int_0^\infty \frac{|\cos(x)|\sqrt{\pi}}{x\sqrt{\pi}} e^{-\frac{(\log(x)-3)}{1}} dx = \int_0^\infty |\cos(x)|\sqrt{\pi} f(x)dx$$

where $\mu = 3, \sigma^2 = 0.5$ is the parameter of log-normal. Then the definition of the expectation shows us that integral is the value of $E(|\cos(x)|\sqrt{\pi})$ ($X \sim$ log-normal$(3, 0.5)$), which is the expectation of a bounded function.

2 Q2

2.1 Method 1

For target function

$$L(a, \mu, \theta, \sigma^2, \tau^2) = \prod_{i=1}^{n} f(x_i) \times p(a)p(\mu)p(\theta)p(\sigma^2)p(\tau^2)$$

Prior distributions for parameters are: $p(a) \propto 1; p(\mu) \propto 1; p(\theta) \propto 1; p(\sigma^2) \propto \frac{1}{\sigma^2}; p(\tau^2) \propto \frac{1}{\tau^2}$

We adopt Gibbs sampling + MH algorithm to sample these parameters:

During the i-th step

Step 1: Sample a.try $\sim$ U(0,1) and accept a.try as a_{i+1} with probability $\min(\frac{L(a.try,\mu_i,\theta_i,\sigma_i^2,\tau_i^2)}{L(a_i,\mu_i,\theta_i,\sigma_i^2,\tau_i^2)}, 1)$

Step 2: Sample μ.try $\sim$ TruncNorm$(\mu_i, \sigma_i^2, -\infty, 6)$ and accept μ.try as μ_{i+1} with probability $\min(\frac{L(a_{i+1},\mu.try,\theta_i,\sigma_i^2,\tau_i^2)}{L(a_{i+1},\mu_i,\theta_i,\sigma_i^2,\tau_i^2)}, 1)$

Step 3: Sample θ.try $\sim$ Norm(θ_i, τ_i^2) and accept θ.try as θ_{i+1} with probability $\min(\frac{L(a_{i+1},\mu_{i+1},\theta.tey,\sigma_i^2,\tau_i^2)}{L(a_{i+1},\mu_{i+1},\theta_i,\sigma_i^2,\tau_i^2)}, 1)$

Step 4: Sample σ^2.try $\sim$ log-Norm(σ_i^2, s) and accept σ^2.try as σ_{i+1}^2 with probability $\min(\frac{L(a_{i+1},\mu_{i+1},\theta_{i+1},\sigma^2.try,\tau_i^2)}{L(a_{i+1},\mu_{i+1},\theta_{i+1},\sigma_i^2,\tau_i^2)} \cdot \frac{dlnorm(\sigma_i^2,\sigma^2.try,s)}{dlnorm(\sigma^2.try,\sigma_i^2,s)}, 1)$

Step 5: Sample τ^2.try $\sim$ log-Norm(τ_i^2, s) and accept τ^2.try as τ_{i+1}^2 with probability $\min(\frac{L(a_{i+1},\mu_{i+1},\theta_{i+1},\sigma_{i+1}^2,\tau^2.try)}{L(a_{i+1},\mu_{i+1},\theta_{i+1},\sigma_{i+1}^2,\tau_i^2)} \cdot \frac{dlnorm(\tau_i^2,\tau^2.try,s)}{dlnorm(\tau^2.try,\tau_i^2,s)}, 1)$

2.2 Method 2

Introduce z_i for x_i to indicate which distribution it should obey.

$z_i = 1: \quad x_i \sim N(\mu, \sigma^2)$

$z_i = 0: \quad x_i \sim N(\theta, \tau^2)$

and $P(z_i = 1) = a$

thus: $f(x_i, z_i | a, \mu, \theta, \sigma^2, \tau^2) = \{a \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x_i-\mu)^2}{2\sigma^2}}\}^{z_i} \{(1-a) \frac{1}{\sqrt{2\pi\tau^2}} e^{\frac{(x_i-\theta)^2}{2\tau^2}}\}^{1-z_i}$

Prior distributions for parameters are:

$$p(a) \propto 1, \quad \mu \sim N(0, r^2), \quad \theta \sim N(0, r^2)$$

$$\sigma^2 \sim \text{Inverse-Gamma}(g, h), \quad \tau^2 \sim \text{Inverse-Gamma}(s, t)$$

The joint posterior distribution is:

$$\begin{aligned}
f(Z, a, \mu, \theta, \sigma^2, \tau^2 | X) &\propto \prod_{i=1}^{n} f(x_i, z_i | a, \mu, \theta, \sigma^2, \tau^2) \times p(a) p(\mu) p(\theta) p(\sigma^2) p(\tau^2) \\
&\propto \prod_{i=1}^{n} \{a \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x_i-\mu)^2}{2\sigma^2}}\}^{z_i} \{(1-a) \frac{1}{\sqrt{2\pi\tau^2}} e^{\frac{(x_i-\theta)^2}{2\tau^2}}\}^{1-z_i} \times e^{-\frac{\mu^2}{2r^2}} e^{-\frac{\theta^2}{2r^2}} \times (\sigma^2)^{-(g+1)} e^{-\frac{h}{\sigma^2}} (\tau^2)^{-(s+1)} e^{-\frac{t}{\tau^2}} \\
&\propto a^{\sum z_i} (1-a)^{\sum (1-z_i)} (\sigma^2)^{-(\frac{\sum z_i}{2} + g + 1)} (\tau^2)^{-(\frac{\sum (1-z_i)}{2} + s + 1)} \\
&\times e^{-\frac{1}{2\sigma^2} \sum z_i (x_i - \mu)^2 - \frac{h}{\sigma^2} - \frac{\mu^2}{2r^2}} e^{-\frac{1}{2\tau^2} \sum (1-z_i)(x_i - \theta)^2 - \frac{t}{\tau^2} - \frac{\theta^2}{2r^2}}
\end{aligned}$$

We can get the condistional posterior distributions:

$$
\begin{aligned}
a|\mu,\theta,\sigma^2,\tau^2,Z,X &\sim \text{Beta}(\sum z_i+1,\sum(1-z_i)+1)\\
\mu|a,\theta,\sigma^2,\tau^2,Z,X &\sim N(\frac{r^2\sum z_ix_i}{r^2\sum z_i+\sigma^2},\frac{r^2\sigma^2}{r^2\sum z_i+\sigma^2})\\
\theta|a,\mu,\sigma^2,\tau^2,Z,X &\sim N(\frac{r^2\sum(1-z_i)x_i}{r^2\sum(1-z_i)+\tau^2},\frac{r^2\sigma^2}{r^2\sum(1-z_i)+\tau^2})\\
\sigma^2|a,\mu,\theta,\tau^2,Z,X &\sim \text{Inverse-Gamma}(\frac{\sum z_i}{2}+g,\frac{\sum z_i(x_i-\mu)^2}{2}+h)\\
\tau^2|a,\mu,\theta,\sigma^2,Z,X &\sim \text{Inverse-Gamma}(\frac{\sum(1-z_i)}{2}+s,\frac{\sum(1-z_i)(x_i-\theta)^2}{2}+t)\\
z_i|a,\mu,\theta,\sigma^2,\tau^2,x_i &\sim \text{Bernoulli}(\frac{a(\sigma^2)^{-(\frac{3}{2}+g)}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}}{a(\sigma^2)^{-(\frac{3}{2}+g)}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}+(1-a)(\tau^2)^{-(\frac{3}{2}+s)}e^{-\frac{(x_i-\theta)^2}{2\tau^2}}})
\end{aligned}
$$

During the i-th step:

Step 1: Sample a_{i+1} given $\mu_i,\theta_i,\sigma_i^2,\tau_i^2,Z_i$

Step 2: Sample μ_{i+1} given $a_{i+1},\theta_i,\sigma_i^2,\tau_i^2,Z_i$

Step 3: Sample θ_{i+1} given $a_{i+1},\mu_{i+1},\sigma_i^2,\tau_i^2,Z_i$

Step 4: Sample σ_{i+1}^2 given $a_{i+1},\mu_{i+1},\theta_{i+1},\tau_i^2,Z_i$

Step 5: Sample τ_{i+1}^2 given $a_{i+1},\mu_{i+1},\theta_{i+1},\sigma_{i+1}^2,Z_i$

Step 6: If $\mu_{i+1}\geq 6$, then switch θ_{i+1} and μ_{i+1}, swith σ_{i+1}^2 amd τ_{i+1}^2, and $a_{i+1}=1-a_{i+1}$

If $\mu_{i+1}<6$, nothing changes

Step 7: Sample each z_j given $a_{i+1},\mu_{i+1},\theta_{i+1},\sigma_{i+1}^2,\tau_{i+1}^2$

In the code provided, $r^2=5,g=h=s=t=1$.