THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Markov Chains - First Step Analysis Exercises Solutions

1. (Slide 7 of the "Markov Chains - First Step Analysis" notes)

$$\begin{aligned}\mathbb{P}(T > k|X_0 = 1) &= \mathbb{P}(X_1 \neq 0, 2, X_2 \neq 0, 2, \ldots, X_k \neq 0, 2|X_0 = 1)\\ &= \mathbb{P}(X_1 = 1, X_2 = 1, \ldots, X_k = 1|X_0 = 1)\\ &= p_{11} \times p_{11} \times \cdots \times p_{11} (k \text{ times})\\ &= \beta^k\end{aligned}$$

2. (Exercise 3.4.2 in Pinsky and Karlin) Recall $u_{10} := P(X_T = 0|X_0 = 1)$ and $v_1 := \mathbb{E}[T|X_0 = 1]$, where T is absorption time. Using standard first step analysis, we arrive at these equations

$$\begin{aligned}u_{10} &= 1 \times 0.1 + u_{10} \times 0.6 + 0 \times 0.3\\ v_1 &= 1 \times 0.1 + (1 + v_1) \times 0.6 + 1 \times 0.1\end{aligned}$$

and these have solutions

(a) $u_{10} = 1/4$.

(b) $v_1 = 5/2$.

3. (Exercise 3.4.4 in Pinsky and Karlin) We follow the hint. We want to find the mean number of tosses, so let's make this the mean time to absorption. Let the states be $\{0, 1, 2\}$, so X_n is the running total of successive heads. Let State 2 be the absorption state, so that once we have 2 successive heads, we stop playing the game and stay in State 2 forever. Say we're in State 0. We move to State 1 w.p. 0.5 (we toss a Head) and stay in State 0 w.p. 0.5 (we toss a Tail). Say were in State 1. We move to State 2 w.p. 0.5 (we toss a Head) and move to State 0 w.p. 0.5 (we toss a Tail). The transition probability matrix is therefore

$$\begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}$$

and first step analysis yields these equations for $v_i = \mathbb{E}[T|X_0 = i]$, $i = 0, 1$

$$\begin{aligned}v_0 &= (1 + v_0) \times 0.5 + (1 + v_1) \times 0.5\\ v_1 &= (1 + v_0) \times 0.5 + 1 \times 0.5\end{aligned}$$

hence $v_0 = 6$.

4. (Exercise 3.4.7 in Pinsky and Karlin) We directly apply the formulae from Slide 25 from First Step Analysis notes, since asking for the mean time spent in State 1 until absorption is exactly $g(X_n) = \mathbf{1}_{\{X_n=1\}}$. We have

these equations for $w_{i1} = \mathbb{E}[\sum_{n=0}^{T-1} \mathbf{1}_{\{X_n=1\}}|X_0 = i]$

$$\begin{aligned} w_{11} &= 1 + 0.2w_{11} + 0.5w_{21} \\ w_{21} &= 0.2w_{11} + 0.6w_{21} \end{aligned}$$

and for $w_{i2} = \mathbb{E}[\sum_{n=0}^{T-1} \mathbf{1}_{\{X_n=2\}}|X_0 = i]$

$$\begin{aligned} w_{12} &= 0.2w_{12} + 0.5w_{22} \\ w_{22} &= 1 + 0.2w_{12} + 0.6w_{22} \end{aligned}$$

and for $v_i = \mathbb{E}[T|X_0 = i]$

$$\begin{aligned} v_1 &= 1 + 0.2v_1 + 0.5v_2 \\ v_2 &= 1 + 0.2v_1 + 0.6v_2 \end{aligned}$$

The solutions to these equations include are $w_{11} = \frac{20}{11}$, $w_{12} = \frac{25}{11}$, $v_1 = \frac{45}{11}$ thus the sum is verified.

5. (Problem 3.4.12 in Pinsky and Karlin) Let $T = \min\{n \geq 0; X_n = 2\}$. Then we want to find $\mathbb{P}(X_{T-1} = 1|X_0 = 0)$. More generally, let $z_i = \mathbb{P}(X_{T-1} = 1|X_0 = i)$ for $i = 1, 2$. Then first step analysis yields a system of equations

$$\begin{aligned} z_0 &= 0.3z_0 + 0.2z_1 \\ z_1 &= 0.5z_0 + 0.1z_1 + 0.4 \end{aligned}$$

from which we find $z_0 = \frac{8}{53}$.

THE END