Calculations for the Gambler's Ruin Problem

We consider a one-dimensional random walk with states $\{0, 1, 2, \ldots, N\}$, with states 0 and N as absorbing states. The transition probability matrix is

$$
\begin{array}{c|c}
\text{States} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N-1 & N \end{array} \\
\hline
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} &
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
q & 0 & p & \cdots & 0 & 0 \\
0 & q & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & 0 \\
0 & 0 & 0 & \cdots & 0 & p \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\end{array}
$$

We solve two problems, the first is about the probability we are ruined. We assume $p \neq q$.

Define $u_i = P\{X_n \text{ reaches } 0 \text{ before } N \mid X_0 = i\}$

Clearly $u_0 = 1$ and $u_N = 0$

First step analysis tells us $u_k = p u_{k+1} + q u_{k-1} \cdots$ ①

We will solve equation ① with conditions $u_0 = 1$ and $u_N = 0$.

Define $d_k = u_k - u_{k-1}$ ("d" for "difference").

Then equation ① can be re-written as

$$p d_{k+1} - q d_k = 0 \;\ldots\; ②$$

Equation ② tells us that

$$d_2 = \left(\frac{q}{p}\right) d_1, \quad d_3 = \left(\frac{q}{p}\right) d_2 = \left(\frac{q}{p}\right)^2 d_1, \quad \ldots, \quad d_N = \left(\frac{q}{p}\right)^N d_1$$

$\therefore$ So, in general, we have $d_k = \left(\frac{q}{p}\right)^{k-1} d_1 \ldots$ (3)

We need to relate u_k with the d's:

Notice

$$\begin{array}{rcl} u_k - u_{k-1} & & d_k \\ + \, u_{k-1} - u_{k-2} & & + \, d_{k-1} \\ + \quad \vdots \qquad \vdots & = & + \quad \vdots \\ + \, u_1 - u_0 & & + \, d_1 \\ \parallel & & \parallel \\ u_k - u_0 & & \sum_{j=1}^{k} d_j \\ \parallel & & \\ u_k - 1 & & \end{array}$$

(since $u_0 = 1$)

Hence
$$u_k = 1 + \sum_{j=1}^{k} d_j$$
$$= 1 + d_1 \sum_{j=1}^{k} \left(\frac{q}{p}\right)^{j-1} \quad \text{by equation (3)}$$

We are nearly finished, but we need to get rid of d_1.

Remember that $u_N = 0$, so

$$0 = u_N = 1 + d_1 \sum_{j=1}^{N} \left(\frac{q}{p}\right)^{k-1}$$

$$\Rightarrow \quad d_1 = - \frac{1}{\sum_{j=1}^{N} \left(\frac{q}{p}\right)^{k-1}}$$

We can now express u_k in terms of p, q, N only:

$$u_k = 1 - \frac{\sum_{j=1}^{k} \left(\frac{q}{p}\right)^{j-1}}{\sum_{j=1}^{N} \left(\frac{q}{p}\right)^{j-1}}$$, which simplifies to

$$u_k = 1 - \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad ■.$$

The second problem is about the expected time the walk lasts.

Define $v_i = \mathbb{E}[T \mid X_0 = i]$, where T is the absorbing time. Clearly $v_0 = 0$, and $v_N = 0$.

First step analysis tells us $v_k = 1 + p v_{k+1} + q v_{k-1} \quad \cdots (4)$,
which is very similar to equation (1).

Define $d_k = v_k - v_{k-1}$ ("d" for "difference") and equation (4) can be re-written as

$$p d_{k+1} - q d_k = -1 \quad \ldots (5)$$

which is very similar to equation (2).

The solution to equation (5) (<u>the derivation of which is non-examinable</u>) is

$$d_k = \left(\frac{q}{p}\right)^{k-1} C + \frac{1}{q-p} \quad \cdots (6)$$, C is a constant

which is similar to equation (3), but with an additional constant.

We need to relate the v_k with the d's.

Notice

$$\begin{array}{rcl} v_k - v_{k-1} & & d_k \\ + \; v_{k-1} - v_{k-2} & & + \; d_{k-1} \\ + \quad \vdots \qquad \vdots & = & + \; \vdots \\ + \; v_1 - v_0 & & + \; d_1 \\ \parallel & & \\ v_k - v_0 & & \\ \parallel & & \\ \text{(since } v_0 = 0\text{)} \quad v_k & = & \displaystyle\sum_{j=1}^{k} d_j \end{array}$$

Hence $$v_k = \sum_{j=1}^{k} \left(C \left(\frac{q}{p}\right)^{j-1} + \frac{1}{q-p} \right)$$

$$= C \frac{\left(1 - \left(\frac{q}{p}\right)^{k}\right)}{\left(1 - \left(\frac{q}{p}\right)\right)} + \frac{k}{q-p}$$

We are nearly finished, but we need to get rid of C

Remember that $v_N = 0$, so

$$0 = v_N = C \frac{\left(1 - \left(\frac{q}{p}\right)^{N}\right)}{\left(1 - \frac{q}{p}\right)} + \frac{N}{q-p}$$

$$\Rightarrow C = \frac{-N}{q-p} \times \frac{\left(1 - \frac{q}{p}\right)}{\left(1 - \left(\frac{q}{p}\right)^{N}\right)}$$

We can now express v_k in terms of p, q and N only:

$$v_k = \frac{k}{q-p} - \frac{N}{q-p} \frac{\left(1 - \frac{q}{p}\right)}{\left(1 - \left(\frac{q}{p}\right)^{N}\right)} \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^{k}\right)}{\left(1 - \frac{q}{p}\right)}$$

and if we define $\theta = \frac{q}{p}$ this simplifies to

$$v_k = \frac{1}{p(1-\theta)} \left[N . \frac{(1-\theta^k)}{(1-\theta^N)} - k \right].$$