THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Conditional Probabilities and Expectations - Exercises

1. (Problem 2.1.2 in Pinsky and Karlin) A card is picked at random from N cards labeled $1, 2, \ldots, N$ and the number that appears is X. A second card is picked at random from cards numbered $1, 2, \ldots, X$ and its number is Y. Determine the conditional distribution of X given $Y = y$, for $y = 1, 2, \ldots$.

2. (Problem 2.1.9 in Pinsky and Karlin) Let N have a Poisson distribution with parameter $\lambda = 1$. Conditioned on $N = n$, let X have a uniform distribution over the integers $0, 1, \ldots, n+1$. What is the marginal distribution for X?

3. (Problem 2.1.7 in Pinsky and Karlin) The probability that an airplane accident that is due to structural failure is correctly diagnosed is 0.85, and the probability that an airplane accident that is not due to structural failure is incorrectly diagnosed as being due to structural failure is 0.35. If 30% of all airplane accidents are due to structural failure, then find the probability that an airplane accident is due to structural failure given that it has been diagnosed as due to structural failure.

4. (Exercise 2.1.6 in Pinsky and Karlin) Suppose U and V are independent and follow the geometric distribution

   $$p(k) = \rho(1-\rho)^k, \text{ for } k = 0, 1, \ldots$$

   Define the random variable $Z = U + V$.

   (a) Determine the joint probability mass function $p_{U,Z}(u, z) = Pr\{U = u, Z = z\}$.

   (b) Determine the conditional p.m.f. for U given that $Z = n$.

5. (Problem 2.4.2 in Pinsky and Karlin) Let N have a Poisson distribution with parameter $\lambda > 0$. Suppose that, conditioned on $N = n$, the random variable X is binomially distributed with parameters $N = n$ and p. Set $Y = N - X$. Show that X and Y have Poisson distributions with respective parameters λp and $\lambda(1-p)$ and that X and Y are independent. *This shows that conditionally dependent random variables may become independent through randomization.*

6. (Problem 2.4.3 in Pinsky and Karlin) Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, following an exponential density with parameter θ.

   (a) What is the marginal distribution of X?

   (b) Determine the conditional density for λ given $X = k$.

7. (Problem 2.4.7 in Pinsky and Karlin) Suppose that X and Y are independent random variables, each having the same exponential distribution with parameter α. What is the conditional probability density function for X, given that $Z = X + Y = z$?

8. (Exercise 2.1.1 in Pinsky and Karlin) I roll a six-sided die and observe the number N on the uppermost face. I then toss a fair coin N times and observe X, the total number of heads to appear. What is the probability that $N = 3$ and $X = 2$? What is the probability that $X = 5$? What is $\mathbb{E}[X]$, the expected number of heads to appear?

9. (Problem 2.1.4 in Pinsky and Karlin) Suppose that X has a binomial distribution with parameters $p = 0.5$ and N, where N is also random and follows a binomial distribution with parameters $q = 0.25$; and $M = 20$. What is the mean of X?

10. (Problem 2.2.2 in Pinsky and Karlin) Consider a pair of dice that are unbalanced by the addition of weights in the following manner: Die 1 has a small piece of lead placed near the four side, causing the appearance of the outcome 3 more often than usual, while Die 2 is weighted near the three side, causing the outcome 4 to appear more often than usual. We assign the probabilities to Die 1: $p(1) = p(2) = p(5) = p(6) = 0.166667$ and $p(3) = 0.186666$ and $p(4) = 0.146666$. And we assign the following probabilities to Die 2: $p(1) = p(2) = p(5) = p(6) = 0.166667$ and $p(3) = 0.146666$ and $p(4) = 0.186666$. Determine the win probability if the game of craps is played with these loaded dice.

11. Given probability Q, random variable X has a binomial distribution with parameters n and Q. Q itself is uniformly distributed over $[0, 1]$. What is the distribution of X? What is $\mathbb{E}[X]$? *Hint:* $\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$.

12. (Exercise 2.1.5 in Pinsky and Karlin) Let X be a Poisson random variable with parameter λ. Find the conditional mean of X given that X is odd.

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