Lecture notes 6
Markov Chain: Applications of Markov Chain

7. Branching Processes

Branching process, as a typical discrete time Markov chain, is a very useful tool in epidemiologic and social studies, particularly in modeling disease spread or population growth.

Example 7.1. (The Confucius descendants)

Hypothetically, every male in the Kong family, since Confucius, produces his sons, independent of everything else, according to the same distribution

Number of sons:	0	1	2	3	4	5	6 or more
Probability :	0.17	0.50	0.25	0.05	0.02	.01	0

(The mean number of sons of any Kong male is about 1.2.) Let X_n be the number of the n-th generation of descendants of Confucius. (According to the old Chinese tradition, only males carrying the last name are counted as family descendants.) Then $\{X_n : n \geq 1\}$ is a so-called branching process. Here $X_0 = 1$ meaning that the 0-th generation is Confucius himself, alone.

Example 7.2 (The family tree of Confucius) (please refer to the family tree of Confucius on the slide)

The diagram shows the first few generations of the family tree of Confucius. Note that $\xi_i^{(n)}$ denotes the number of sons fathered by the i-th member of the n-generation of Kong Zi(Confucius), and X_n is the number of (male) descendants of Kong Zi at the n-th generation.

Allegedly, the 80-th generation of Kong Zi is around 1,300,000. Suppose the process is indeed a branching process.

Then, a reasonable estimate of μ, the mean number of sons of any male descendent of Kong Zi:

$$\hat{\mu}^{80} = 1,300,000,$$

Then, (based on some formula we shall learn later), $\hat{\mu} = 1.1924$, meaning that the average number of sons produced by any Kong father is about 1.2.

Taking one step further, if the process is indeed a branching process, the data is trustworthy and the male Kongs can be regarded as typical of Chinese males, the one might claim, on average, each Chinese male produces about 1.2 sons. In reality, the process may not be or even approximately a branching process—$\xi_i^{(n)}$ are not be iid, not over good/bad historical periods, and at least not over periods with/without birth control policies.

Example 7.3 (The 2003 Hong Kong SARS outbreak in Hong Kong)

(please refer to the graph on the slide)

Certainly and fortunately, the entire process of SARS epidemic in HK, beginning in mid-February and ending in May of 2003, does NOT follow a branching process, as defined above. The intervention/prevention measures took place that forever changed the course of the process. An accurate modeling of such a process requires more sophisticated probability/statistical tools than the basic branching process. On the other hand, the process may be broken into several phases, such as

phase 1: initial spread with no control measures;

phases 2: limited control measures enforced;

phase 3: public control measures enforced.

Each phase may be approximately follow a branching process. *Phase 3 with public control measures that made sure the extinction of the disease in Hong Kong.*

Consider a population consisting of individuals able to produce offspring of the same kind.

Suppose that each individual will, by the end of its lifetime, have produced j new offspring with probability $P_j,\ j \geq 0$, independently of the number produced by any other individual. Suppose that $P_j < 1$ for all $j \geq 0$.

Let

(a) X_0 be the number of individuals initially present and called the size of the zeroth generation.

(b) X_1 be the number of all offspring of the zeroth generation constitute the first generation.

(c) In general, let X_n denote the size of the nth generation.

It follows that $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a Markov chain having as its state space of the set of nonnegative integers.

Note that

(a) 0 is a recurrent state, since clearly $P_{00} = 1$.

(b) If $P_0 > 0$, all other states are transient. This follows since $P_{i0} = P_0^i$, which implies that starting with i individuals there is a positive probability of at least P_0^i that no later generation will ever consist of i individuals.

(c) Since any finite set of transient state $\{1, 2, \ldots, n\}$ will be visited only finitely often, this leads to the important conclusion that, if $P_0 > 0$, then the population will either die out or its size will converge to infinity.

Let

(a) $\mu = \sum_{j=0}^{\infty} jP_j$ denote the mean number of offspring of a single individual.

(b) $\sigma^2 = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$ be the variance of the number of offspring produced by a single individual.

Suppose $X_0 = 1$, we first want to calculate $E(X_n)$ and $Var(X_n)$.

Note that we may write

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

where Z_i represents the number of offspring of the ith individual of the $(n-1)$th generation. By conditioning on X_{n-1}, we obtain

$$\begin{aligned} E(X_n) &= E[E(X_n|X_{n-1})] \\ &= E[E(\sum_{i=1}^{X_n-1} Z_i \mid X_{n-1})] \\ &= E(\mu X_{n-1}) \\ &= \mu E(X_{n-1}) \end{aligned}$$

where we have used the fact that $E(Z_i) = \mu$. Since $E(X_0) = 1$, we have

$$\begin{aligned} E(X_1) &= \mu \\ E(X_2) &= \mu E(X_1) = \mu^2 \\ &\cdot \\ &\cdot \\ &\cdot \\ E(X_n) &= \mu E(X_{n-1}) = \mu^n. \end{aligned}$$

Similarly, $Var(X_n)$ may be obtained by using the conditional variance formula

$$Var(X_n) = E[Var(X_n|X_{n-1})] + Var[E(X_n|X_{n-1})].$$

Now, given X_{n-1}, X_n is just the sum of X_{n-1} independent random variables each having the distribution $\{P_j,\ j \geq 0\}$. Hence,

$$Var(X_n|X_{n-1}) = X_{n-1}\sigma^2.$$

Thus the conditional variance formula yields

$$\begin{aligned} Var(X_n) &= E(X_{n-1}\sigma^2) + Var(X_{n-1}\mu) \\ &= \sigma^2\mu^{n-1} + \mu^2 Var(X_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&= \sigma^2\mu^{n-1} + \mu^2(\sigma^2\mu^{n-2} + \mu^2 Var(X_{n-2})) \\
&= \sigma^2(\mu^{n-1} + \mu^n) + \mu^4 Var(X_{n-2}) \\
&= \sigma^2(\mu^{n-1} + \mu^n) + \mu^4\{\sigma^2\mu^{n-3} + \mu^2 Var(X_{n-3})\} \\
&= \sigma^2(\mu^{n-1} + \mu^n + \mu^{n+1}) + \mu^6 Var(X_{n-3}) = \ldots
\end{aligned}
$$

By mathematical induction and using the fact that $Var(X_0) = 0$, we can show that

$$
\begin{aligned}
Var(X_n) &= \sigma^2(\mu^{n-1} + \mu^n + \mu^{n+1} + \ldots \mu^{2n-1}) + \mu^{2n} Var(X_0) \\
&= \sigma^2(\mu^{n-1} + \mu^n + \mu^{n+1} + \ldots \mu^{2n-1}).
\end{aligned}
$$

Therefore,

$$
Var(X_n) = \begin{cases} \sigma^2\mu^{n-1}(\frac{\mu^n - 1}{\mu - 1}), & if\ \mu \neq 1 \\ n\sigma^2, & if\ \mu = 1. \end{cases}
$$

Let π_0 denote the probability that the population will eventually die out (under the assumption that $X_0 = 1$). That is,

$$
\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1).
$$

To determine the value of π_0, we first claim that $\pi_0 = 1$ if $\mu < 1$.

Note that

$$
\begin{aligned}
\mu^n &= E(X_n) \\
&= \sum_{j=1}^{\infty} jP(X_n = j) \\
&\geq \sum_{j=1}^{\infty} 1P(X_n = j) \\
&\geq P(X_n \geq 1).
\end{aligned}
$$

Since μ^n converges to 0 when $\mu < 1$, it follows that $P(X_n \geq 1)$ converges to 0. Hence $\pi_0 = 1$.

In fact, it can be shown that $\pi_0 = 1$ even when $\mu = 1$. When $\mu > 1$, it turns out that $\pi_0 < 1$, and an equation determining π_0 may be derived by conditioning on the number of offspring of the initial individual, as follows:

$$
\begin{aligned}
\pi_0 &= P(\text{population die out}) \\
&= \sum_{j=0}^{\infty} P(\text{population die out} \mid X_1 = j)P_j.
\end{aligned}
$$

Now, given $X_1 = j$, the population will eventually die out if and only if each of the j families started by the members of the first generation eventually die out. Since each family is assumed to act independently, and since the probability that any particular family dies out is just π_0, this yields that

$$P(\text{population die out} \mid X_1 = j) = \pi_0^j$$

and thus π_0 satisfies

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j \tag{$*$}$$

When $\mu > 1$, it can be shown that π_0 is the smallest positive number satisfy (*).

Example 7.4.
If $P_0 = 1/2,\ P_1 = 1/4,\ P_2 = 1/4$, then determine π_0.

Solution: Since $\mu = 3/4 \leq 1$, it follows that $\pi_0 = 1$.

Example 7.5.
If $P_0 = 1/4,\ P_1 = 1/4,\ P_2 = 1/2$, then determine π_0.

Solution: π_0 satisfies

$$\pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2$$

or

$$2\pi_0^2 - 3\pi_0 + 1 = 0$$

The smallest positive solution of this quadratic equation is $\pi_0 = 1/2$.

Example 7.6.
In Examples 2 and 3, what is the probability that the population will die out if it initially consists of n individuals?

Solution: Since the population will die out if and only if the families of each of the members of the initial generation die out, the desired probability is π_0^n. For Example 1, this yields $\pi_0^n = 1$, and for Example 2, $\pi_0^n = (\frac{1}{2})^n$.