STAT 3007 Introduction to Stochastic Processes
Tutorial 2 | Term 1, 2019−20

ZHAN Zebang

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1 Expectations and Tail Probabilities: A Second Review

- $\mathrm{E}(X) = \sum_i x_i \Pr(X = x_i)$ (discrete X); $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx$ (continuous X).
- Nonnegative integer-valued X: $\mathrm{E}(X) = \sum_{k=1}^{\infty} \Pr(X \geq k)$.
- Nonnegative continuous X: $\mathrm{E}(X) = \int_0^{\infty} \Pr(X \geq x) dx$.
- It is sometimes easier to find the tail probabilities than to find the pmf/pdf directly.

Example 1. (Distribution of the minimum) Let $X_1, X_2, \ldots, X_n$ be independent Exponential(2) random variables. Find the probability density function of $Y = \min\{X_1, X_2, \ldots, X_n\}$.
(If $X \sim$ Exponential(λ), then its pdf is given by $f_X(x) = \lambda e^{-\lambda x}, x > 0$.)

Example 2. To find a lucky student among n students in an event, Yau conducts lot drawing from a box with n balls numbered $1, 2, \ldots, n$. Yau draws a ball, records the number, returns it, and continues. The lucky student is the first one who has been drawn <u>for a second time</u>. Let N be the number of drawings to find a lucky student. Find the probability distribution of N.

Example 3. Let $X_1, X_2, \ldots$ be a series of independent Exponential(1) random variables. Define $N = \min\{k \geq 2 : X_k \leq X_1\}$. Find the probability mass function of N.

2 Markov Chains

- Stochastic process: a family of random variables X_t, where t runs over an index set T.
- Discrete time: T=$\{0, 1, 2, \ldots\}$ / Continuous time: T=$[0, \infty)$ (to be covered in the future).
- State space: the range of possible values for the random variables X_t.
  We focus our discussions on **finite or countable** state spaces.
  Without loss of generality, we usually label a state space by $\{0, 1, 2, ..., N\}$ or $\{0, 1, 2, ...\}$.
- Discrete-time Markov chain: a stochastic process with a finite or countable state space and index set $T = \{0, 1, 2, \ldots\}$, and satisfies the **Markov property**
  $\Pr(X_{n+1} = j | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, \underline{X_n = i}) = \Pr(X_{n+1} = j | X_n = i)$
  for all time points n and all states $i_0, \ldots, i_{n-1}, i, j$.

  A more general form of the Markov property (they are equivalent):
  $\Pr(X_{t_{n+1}} = j | X_{t_0} \in B_0, \ldots, X_{t_{n-1}} \in B_{n-1}, \underline{X_{t_n} = i}) = \Pr(X_{t_{n+1}} = j | X_{t_n} = i)$
  for all time points $t_0 < \ldots < t_n < t_{n+1}$, all states i, j and collections of states $B_0, \ldots, B_{n-1}$.
  (The "present" state should be known while all the "past" states can be arbitrary.)

- One-step transition probability: $p_{ij}^{n,n+1} := \Pr(X_{n+1} = j | X_n = i)$.

- Stationary (*temporally homogeneous*) Markov chain: $p_{ij}^{n,n+1} = p_{ij}$ is independent of n. We focus our discussions on stationary Markov chains.

- n-step transition probability: $p_{ij}^{(n)} := \Pr(X_{n+m} = j | X_m = i)$.

- (One-step) transition probability matrix: $P = (p_{ij})$, which satisfies
  (1) all entries are nonnegative: $p_{ij} = \Pr(X_{n+1} = j | X_n = i) \geq 0$ for all i and j; and
  (2) each row sums to 1: $\sum_{j=0}^{\infty} p_{ij} = \sum_{j=0}^{\infty} \Pr(X_{n+1} = j | X_n = i) = 1$ for all i.

  n-step transition matrix: $P^{(n)} = (p_{ij}^{(n)})$.

Example 4. Yau tries to contact aliens and sends a binary message (0 or 1) through a signal channel consisting of several stages. The channel is poorly made, so the transmission through each stage may become wrong with a probability $\alpha \in (0, 1)$. Suppose that Yau sends $X_0 = 0$. Let X_n be the signal received at the n-th stage. Assume that $\{X_n\}$ is a Markov chain.
(1) Find the transition probability matrix P.
(2) Find the probability that no error occurs up to stage 2.
(3) Find the probability that a correct signal is received at stage 2.

Example 5. As a millionaire, Yau owns a production line where a sequence of items are produced, with each item being graded as either good or defective. Suppose that a good item is followed by another good one with probability $\alpha \in (0, 1)$, and a defective item is followed by another defective one with probability $\beta \in (0, 1)$.
(1) Model the setting by a Markov chain $\{X_n\}$ and find its transition probability matrix P.
(2) Suppose that the first item is good. Find the probability that the first defective item to appeared is the third item.

The next example illustrates that it is sometimes insufficient to describe a Markov chain by one-dimensional states.

Example 6. Michael plays badminton with Wilson. A series of games are played. Suppose that the outcomes of successive games are independent, and each game is won by Michael with probability $\alpha \in (0, 1)$ and won by Wilson with probability $1 - \alpha$. One wins the series if he wins two consecutive games. They stop when one wins.
Model the setting by a Markov chain $\{X_n\}$ and find its transition probability matrix P.

3 Chapman-Kolmogorov Equation

By the Law of Total Probability, we can prove the Chapman-Kolmogorov Equation.

- $p_{ij}^{(n)} = \sum_{k=0}^{\infty} p_{ik} p_{kj}^{(n-1)}$. In matrix form, $P^{(n)} = PP^{(n-1)}$.

- $P^{(n)} = PP^{(n-1)} = P^2 P^{(n-2)} = \cdots = P^n$.

- A more general form of the Chapman-Kolmogorov Equation: $p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} p_{kj}^{(n)}$.