Solution of Midterm

1.a) $\sum_{r=-2}^{2} a_r = 0.1+0.2+0.4+0.2+0.1 = 1$

$\sum_{r=-2}^{2} r a_r = -2\times 0.1 + (-1)\times 0.2 + 0\times 0.4 + 1\times 0.2 + 2\times 0.1 = 0$

$\therefore$ The filter pass through a linear trend without distortion.

$\sum_{r=-2}^{2} r^2 a_r = 4\times 0.1 + 1\times 0.2 + 0\times 0.4 + 1\times 0.2 + 4\times 0.1 = 1.2 \neq 0$

$\therefore$ The filter doesn't pass through a linear trend without distortion.

(b) $T_3 = a_{-2}X_1 + a_{-1}X_2 + a_0X_3 + a_1X_4 + a_2X_5 = 0.1\times 3.8 + 0.2\times 3.1 + 0.4\times 2.1 + 0.2\times 1 + 0.1\times 0$
$= 2.04$

$T_5 = 0.1\times 2.1 + 0.2\times 1 + 0.4\times 0 + 0.2\times 0.9 + 0.1\times 1.9 = 0.78$

T_9 does not exist.

(c) The given filter cannot be used for estimating the trend because the length of the filter should be 3 but not 5.

The filter we proposed are $(1/3, 1/3, 1/3)$

$\hat{T}_2 = 3$ $\hat{T}_3 = 2.07$ $\hat{T}_4 = 1.03$ $\hat{T}_5 = 0.63$ $\hat{T}_6 = 0.93$ $\hat{T}_7 = 2$ $\hat{T}_8 = 3.03$

$D_2 = X_2 - \hat{T}_2 = 3.1 - 3 = 0.1$

$D_3 = X_3 - \hat{T}_3 = 2.1 - 2.07 = 0.03$

$D_4 = X_4 - \hat{T}_4 = 1 - 1.03 = -0.03$

$D_5 = X_5 - \hat{T}_5 = 0 - 0.63 = -0.63$

$D_6 = X_6 - \hat{T}_6 = 0.9 - 0.93 = -0.03$

$D_7 = X_7 - \hat{T}_7 = 1.9 - 2 = -0.1$

$D_8 = X_8 - \hat{T}_8 = 3.2 - 3.03 = 0.17$

$\bar{D} = \frac{1}{7}\sum_{i=2}^{8} D_i = -0.07$

$S_1 = [(D_4 - \bar{D}) + (D_7 - \bar{D})]/2 = 0.005$

2. a) $Var(X_t) = Var(2 + a_t + 0.3a_{t-1}) = Var(a_t) + 0.09\,Var(a_{t-1}) = 2 + 0.18 = 2.18$

since a_ts are independent

b) $Cov(X_t, X_{t+k}) = Cov(2 + a_t + 0.3a_{t-1},\ 2 + a_{t+k} + 0.3a_{t+k-1})$

$= Cov(a_t, a_{t+k}) + 0.3\,Cov(a_t, a_{t+k-1}) + 0.3\,Cov(a_{t-1}, a_{t+k})$
$+ 0.09\,Cov(a_{t-1}, a_{t+k-1})$

$= \begin{cases} 0.6, & |k| = 1 \\ 0, & \text{else} \end{cases}$

(c) $Var(\bar{X}) = Var(\sum_{t=1}^{10} X_t/10) = \frac{1}{100}\left[\sum_{t=1}^{10} Var(X_t) + \sum_{\substack{i\neq j \\ 1\le i,j\le 10}} Cov(X_i, X_j)\right]$

$= \frac{1}{100}\left[10\times 2.18 + 2\times(10-1)\times 0.6\right]$

$= 0.326$

3. (a) According to Yule-Walker Equation.

Multiply both side by Y_t, Y_{t-1} and Y_{t-k} $(k\geq 2)$ and take expectation on both side. we have the following equation:

$$\begin{cases} \gamma(0) = 0.5\gamma(1) + 1 + 0.36 + 0.3 \\ \gamma(1) = 0.5\gamma(0) + 0.6 \\ \gamma(k) = 0.5\gamma(k-1) \end{cases} \Rightarrow \begin{cases} \gamma(0) = 2.6133 \\ \gamma(1) = 1.9067 \\ \gamma(k) = 0.5\gamma(k-1) \end{cases}$$

(b) $\hat{\gamma}(0) = \frac{1}{9}\sum_{t=1}^{9}(X_t-\bar{X})(X_t-\bar{x}) = \frac{1}{9}\times 78.09 = 8.6767$

$\hat{\gamma}(1) = \frac{1}{9}\sum_{t=1}^{8}(X_t-\bar{x})(X_{t+1}-\bar{x}) = \frac{1}{9}\times 58.26 = 6.4733$

$\hat{\gamma}(2) = \frac{1}{9}\sum_{t=1}^{7}(X_t-\bar{X})(X_{t+2}-\bar{X}) = \frac{1}{9}\times 22.62 = 2.5133$

$\hat{\gamma}(3) = \frac{1}{9}\sum_{t=1}^{6}(X_t-\bar{X})(X_{t+3}-\bar{X}) = \frac{1}{9}\times(-7.03) = -0.7811$

$\hat{\gamma}(4) = \frac{1}{9}\sum_{t=1}^{5}(X_t-\bar{x})(X_{t+4}-\bar{X}) = \frac{1}{9}\times(-27.68) = -3.0756$

$\hat{\rho}(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = \frac{6.4733}{8.6767} = 0.7461$

$\hat{\rho}(2) = \frac{\hat{\gamma}(2)}{\hat{\gamma}(0)} = \frac{2.5133}{8.6767} = 0.2897$

$\hat{\rho}(3) = \frac{\hat{\gamma}(3)}{\hat{\gamma}(0)} = \frac{-0.7811}{8.6767} = -0.0900$

$\hat{\rho}(4) = \frac{\hat{\gamma}(4)}{\hat{\gamma}(0)} = \frac{-3.0756}{8.6767} = -0.3545$

We can trust up to $h = 9/3 = 3$, otherwise r_h is not accurate.

Draw horizontal lines at levels $\pm\frac{2}{\sqrt{9}}$

$|\hat{\rho}(1)|, |\hat{\rho}(2)| > \frac{2}{\sqrt{9}}$

So lag 1 and lag 2 are reliable

(c) $(1-0.5B)Y_t = (1+0.6B)a_t \Rightarrow Y_t = (1-0.5B)^{-1}(1+0.6B)a_t$

$\Rightarrow Y_t = (1+0.5B+0.25B^2+0.5^3B^3+\cdots)(1+0.6B)a_t$

$= (1+1.1B+0.55B^2+0.275B^3+\cdots)a_t$

$\therefore \psi_0 = 1, \quad \psi_1 = 1.1, \quad \psi_2 = 0.55, \quad \psi_3 = 0.275$

4. a) $(1-1.4B+0.4B^2)Z_t = (1-2B^2)a_t$

$\Rightarrow (1-0.4B)(1-B)Z_t = (1-\sqrt{2}B)(1+\sqrt{2}B)a_t$ ARIMA (1, 1, 2)

$\therefore$ The model is casual but invertible

b) $(1+0.5B-0.14B^2)Z_t = (1-0.49B^2)a_t$

$\Rightarrow (1-0.2B)(1+0.7B)Z_t = (1+0.7B)(1-0.7B)a_t$

$\Rightarrow (1-0.2B)Z_t = (1-0.7B)a_t$ ARIMA (1, 0, 1)

The model is casual and invertible

c) $(1-1.6B+0.6B^2)Z_t = (1-3B+3B^2-B^3)a_t$

$\Rightarrow (1-0.6B)(1-B)Z_t = (1-B)^3 a_t$

$\Rightarrow (1-0.6B)Z_t = (1-B)^2 a_t$ ARIMA (1, 0, 2)

The model is casual but not invertible

d) $(1+0.8B-0.2B^2)Z_t = (2-3.2B+1.28B^2)a_t$

$\Rightarrow (1+B)(1-0.2B)Z_t = 2(1-0.8B)^2 a_t$ ARIMA (2, 0, 2)

The model is not casual but invertible.

e) $(1-6B+0.59B^2)Z_t = (1-B-0.25B^2)a_t$

$\Rightarrow (1-5.9B)(1-0.1B)Z_t = (1-\frac{1+\sqrt{2}}{2}B)(1-\frac{1-\sqrt{2}}{2}B)a_t$

ARIMA (2, 0, 2)

The model is neither casual nor invertible

5. $E(Y_t) = E[Z_t \mid Y_t = Z_t] + E[1 + 0.8Y_{t-1} + Z_t \mid Y_t = 1 + 0.8Y_{t-1} + Z_t]$

$= 0.3E(Z_t) + 0.7E(1 + 0.8Y_{t-1} + Z_t)$

$= 0.7 + 0.56E(Y_{t-1})$

$\Rightarrow \quad E(Y_t) = \frac{35}{22} \approx 1.5909$

$Cov(Y_t, Y_{t+1}) = E[Y_t Y_{t+1}] - EY_t EY_{t+1}$

$= E[Y_t(0.3Z_{t+1} + 0.7(1 + 0.8Y_t + Z_{t+1}))] - (EY_t)^2$

$= E[0.3Y_t Z_{t+1} + 0.7Y_t + \overset{0.56}{\cancel{0.7}} Y_t^2 + 0.7Y_t Z_{t+1}] - (EY_t)^2 \quad \cdots\cdots (1)$

$E(Y_t Z_{t+1}) = E[Z_{t+1}(0.3Z_t + 0.7(1 + 0.8Y_{t-1} + Z_t))]$

$= 0.3E(Z_t Z_{t+1}) + 0.7E(Z_{t+1}) + 0.56E(Z_{t+1}Y_{t-1}) + 0.7E(Z_t Z_{t+1})$

$= 0.7 \times 0.8^n E(Z_{t+1} Y_{t-n}) \longrightarrow 0$

$\therefore$ (1) becomes $Cov(Y_t, Y_{t+1}) = 0.7E(Y_t) + 0.56E(Y_t^2) - (EY_t)^2 \quad \cdots\cdots (2)$

$E(Y_t^2) = E[0.3Z_t^2 + 0.7(1 + 0.8Y_{t-1} + Z_t)^2]$

$= 0.3 + 0.7E[1 + 1.6Y_{t-1} + 0.64Y_{t-1}^2 + Z_t^2 + 2Z_t + 1.6Z_tY_{t-1}]$

$= 1.7 + 1.12E(Y_{t-1}) + 0.448E(Y_{t-1}^2)$

$\Rightarrow \quad E(Y_t^2) = \frac{1.7 + 1.12 \times 1.5909}{1 - 0.448} = 6.3076$

So (2) equals to $0.7 \times 1.5909 + 0.56 \times 6.3076 - 1.5909^2 = 2.1149$