Solutions for Chapter 4 Exercises

1) The the ACVF decays to zero rapidly, so the series is stationary and should not have "integrated" part. Since the sample estimator of autocorrelation function alternates between negative and positive signs, an AR(1) model with negative coefficient may be tentatively specified.

2) For an AR(1) model,

$$X_t = \phi X_{t-1} + a_t \,,$$

the lag-1 autocorrelation function is $\rho_1 = \frac{\gamma(1)}{\gamma(0)} = \phi$. So testing whether $\rho_1 = 0.9$ is equivalent to testing whether $\phi = 0.9$. Note that we have $r_1 = \frac{\sum_{t=1}^{n-1}(X_t-\bar{X})(X_{t+1}-\bar{X})}{\sum_{t=1}^{n}(X_t-\bar{X})^2} = 0.5$. Since $\frac{\sum_{t=1}^{n}(X_t-\bar{X})^2}{\sum_{t=1}^{n-1}(X_t-\bar{X})^2} \to 1$ as $n \to \infty$, r_1 is very close to the least squares estimate

$$\hat{\phi} = \frac{\sum_{t=1}^{n-1}(X_t - \bar{X})(X_{t+1} - \bar{X})}{\sum_{t=1}^{n-1}(X_t - \bar{X})^2} \,.$$

In other words, we have $\hat{\phi} \approx 0.5$.

Using the asymptotic distribution of LSE,

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{\mathcal{L}} N\left(0, \frac{\sigma_a^2}{\gamma(0)}\right)$$

and $\gamma(0) = \frac{\sigma_a^2}{1-\phi^2}$ for AR(1) model, we have the test statistic

$$S = \sqrt{\frac{n}{1-\phi^2}}(\hat{\phi} - \phi) \sim N(0,1) \tag{1}$$

Under $H_o : \phi = 0.9$, the test statistic

$$|S| = \left|\sqrt{\frac{200}{1-0.9^2}}(0.5 - 0.9)\right| = 12.98 > 1.96 \,.$$

Therefore, we reject the null hypothesis.

3) From the idea of method of moments, we can construct several estimation equation from the model.
First, we take expectation to the two-sides of the model.
Second, we take covariance with Z_t on both sides.
Third, we take covariance with Z_{t-1} on both sides.
Fourth, we take covariance with Z_{t-2} on both sides.
The above four operations yield the following four equations:

(1) $\mu = \theta_0 + \phi_1\mu + \phi_2\mu$
(2) $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_a^2$
(3) $\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1)$
(4) $\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0)$

where $\gamma(k)$ is the lag k autocovariance function. To use the Method of Moment, we replace $\gamma(0), \gamma(1), \gamma(2), \mu$ by $\hat{\gamma}(0), \hat{\gamma}(1), \hat{\gamma}(2), \hat{\mu}$ in (1)-(4) to solve for $\theta_0, \phi_1, \phi_2, \sigma_a^2$. Since

$$
\begin{aligned}
&\hat{\mu} = \overline{Z} = 2 \\
&\hat{\gamma}(0) = S^2 = 5 \\
&\hat{\gamma}(1) = r_1\hat{\gamma}(0) = 3.5 \\
&\hat{\gamma}(2) = r_2\hat{\gamma}(0) = 2.25\,,
\end{aligned}
$$

we have the estimating equations

$$
\begin{aligned}
&2 = \theta_0 + 2\phi_1 + 2\phi_2 \\
&5 = 3.5\phi_1 + 2.25\phi_2 + \sigma_a^2 \\
&3.5 = 5\phi_1 + 3.5\phi_2 \\
&2.25 = 3.5\phi_1 + 5\phi_2\,.
\end{aligned}
$$

Solving the above equations, we have

$$
\begin{aligned}
&\hat{\phi}_1 = 0.7549 \text{ and } \hat{\phi}_2 = -0.0784 \\
&\hat{\sigma}_a^2 = 2.534 \text{ and } \hat{\theta}_0 = 0.6471\,.
\end{aligned}
$$

4) From the model, we get $(1-\phi B)Z_t = (1-\theta B)a_t$, where B is the back-shift operator. Multiplying $(1-\phi B)^{-1} = \sum_{k=0}^{\infty}\phi^k B^k$ on both sides yields

$$Z_t = [1 + (\phi-\theta)B + \phi(\phi-\theta)B^2 + \phi^2(\phi-\theta)B^3 + \cdots]a_t\,.$$

From this we obtain the autocovariance function as

$\gamma(0) = \frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_a^2$
$\gamma(1) = \frac{(\theta-\phi)(\phi\theta-1)}{1-\phi^2}\sigma_a^2$
$\gamma(2) = \frac{\phi(\theta-\phi)(\phi\theta-1)}{1-\phi^2}\sigma_a^2$.

Thu, the autocorrelation function is given by

$$\begin{aligned} \rho(1) &= \frac{\gamma(1)}{\gamma(0)} = \frac{(\theta-\phi)(\phi\theta-1)}{1-2\phi\theta+\theta^2}, \\ \rho(2) &= \frac{\gamma(2)}{\gamma(0)} = \frac{\phi(\theta-\phi)(\phi\theta-1)}{1-2\phi\theta+\theta^2}. \end{aligned}$$

Using the method of moment, we match $(\rho(1), \rho(2))$ to $(r(1), r(2))$ to solve for ϕ and θ. First, note that $\frac{\rho(1)}{\rho(2)} = 1/\phi$. As $\frac{r(1)}{r(2)} = -2$ we get $\hat{\phi} = -0.5$. Then, solving

$$r(1) = \frac{(\theta-\hat{\phi})(\hat{\phi}\theta-1)}{1-2\hat{\phi}\theta+\theta^2}$$

for θ, we get $\hat{\theta} = -1.3$ or -0.77. Since the series is stationary and invertible, the roots of characteristic function should lie outside the unit circle. So, we take $\hat{\theta} = -0.77$.

5) First, the autocovariance function of $\gamma(k)$ can be readily found as

$$\begin{aligned} \gamma(0) &= \alpha^2 + \beta^2 \\ \gamma(1) &= \alpha\beta. \end{aligned}$$

Multiply 2 or -2 to the second equation and add to the first equation, the method of moment estimator can be obtained by solving

$$\begin{aligned} (\hat{\alpha}+\hat{\beta})^2 &= \hat{\gamma}(0) + 2\hat{\gamma}(1) = \frac{1}{n}\sum_{t=1}^{n}(Y_t - \overline{Y})^2 + \frac{2}{n}\sum_{t=1}^{n-1}(Y_t-\overline{Y})(Y_{t+1}-\overline{Y}) =: S_1 \\ (\hat{\alpha}-\hat{\beta})^2 &= \hat{\gamma}(0) - 2\hat{\gamma}(1) = \frac{1}{n}\sum_{t=1}^{n}(Y_t - \overline{Y})^2 - \frac{2}{n}\sum_{t=1}^{n-1}(Y_t-\overline{Y})(Y_{t+1}-\overline{Y}) =: S_2 \end{aligned}$$

Note that there is no solution if $S_2 < 0$. For $S_2 \geq 0$, we have the following four pairs of possible solutions:

(1) $\hat{\alpha} = \frac{1}{2}\left(\sqrt{S_1} + \sqrt{S_2}\right)$, $\hat{\beta} = \frac{1}{2}\left(\sqrt{S_1} - \sqrt{S_2}\right)$
(2) $\hat{\alpha} = \frac{1}{2}\left(\sqrt{S_1} - \sqrt{S_2}\right), \hat{\beta} = \frac{1}{2}\left(\sqrt{S_1} + \sqrt{S_2}\right)$
(3) $\hat{\alpha} = -\frac{1}{2}\left(\sqrt{S_1} - \sqrt{S_2}\right), \hat{\beta} = -\frac{1}{2}\left(\sqrt{S_1} + \sqrt{S_2}\right)$
(4) $\hat{\alpha} = -\frac{1}{2}\left(\sqrt{S_1} + \sqrt{S_2}\right), \hat{\beta} = -\frac{1}{2}\left(\sqrt{S_1} - \sqrt{S_2}\right)$.

6) From the computer outputs, we can see immediately that $\hat{\theta}_0 = 5.6421$ and $\hat{\phi} = 0.6250$. Next, the residual standard error is defined as

$$\sqrt{\hat{\sigma}_a^2} = 2.862\,.$$

So $\hat{\sigma}_a^2 = 2.862^2 = 8.19$. Finally, from the definition of the AR(1) model, we can see that

$$\mu = \theta_0 + \phi\mu\,.$$

Thus, we may define the estimator $\hat{\mu} = \frac{\hat{\theta}_0}{1-\hat{\phi}} = 15.0456$.

7) (a) Condition on $a_0 = 0$, for a given θ we estimate the white noise a_t by

$\hat{a}_1 = Z_1,$
$\hat{a}_2 = Z_2 + \theta\hat{a}_1 = Z_2 + \theta Z_1,$
$\hat{a}_3 = Z_3 + \theta\hat{a}_2 = Z_3 + \theta Z_2 + \theta^2 Z_1,$
$\hat{a}_4 = Z_4 + \theta\hat{a}_3 = Z_4 + \theta Z_3 + \theta^2 Z_2 + \theta^3 Z_1.$

The conditional least squares criterion is

$$\begin{aligned}\sum_{t=1}^{4} a_t^2 &= Z_1^2 + (Z_2 + \theta Z_1)^2 + (Z_3 + \theta Z_2 + \theta^2 Z_1)^2 \\ &\quad +(Z_4 + \theta Z_3 + \theta^2 Z_2 + \theta^3 Z_1)^2 \\ &= 2^2 + (1 + 2\theta)^2 \\ &= 4\theta^2 + 4\theta + 5\,.\end{aligned}$$

The conditional least squares estimator of θ is computed by minimizing the $\sum_{t=1}^{4} a_t^2$. By differentiating $\sum_{t=1}^{4} a_t^2$, $\hat{\theta}$ is obtained by solving

$8\theta + 4 = 0$. Thus we have $\hat{\theta} = -\frac{1}{2}$.

(b) With the estimate $\hat{\theta} = -\frac{1}{2}$, the white noise a_ts can be estimated by

$$\begin{aligned}
\hat{a}_1 &= Z_1 = 0 \\
\hat{a}_2 &= Z_2 + \hat{\theta} Z_1 = 0 \\
\hat{a}_3 &= Z_3 + \hat{\theta} Z_2 + \hat{\theta}^2 Z_1 = 2 \\
\hat{a}_4 &= Z_4 + \hat{\theta} Z_3 + \hat{\theta}^2 Z_2 + \hat{\theta}^3 Z_1 = 0
\end{aligned}$$

The conditional least squares estimate for the variance σ_a^2 can be obtained by

$$\hat{\sigma_a}^2 = \frac{1}{4} \sum_{t=1}^{4} \hat{a}_t^2 = 1\,.$$

8) Condition on $Z_0 = 0$ and $Y_0 = 0$, we have

$$\begin{aligned}
Z_1 &= Y_1 \\
Z_2 &= Y_2 - \phi Y_1 - \theta Z_1 = Y_2 - (\phi + \theta) Y_1 \\
Z_3 &= Y_3 - \phi Y_2 - \theta Z_2 = Y_3 - (\phi + \theta) Y_2 + \theta(\phi + \theta) Y_1\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{t=1}^{3} Z_t^2 &= (1 + (\phi+\theta)^2 + \theta^2(\phi+\theta)^2) Y_1^2 + (1 + (\phi+\theta)^2) Y_2^2 + Y_3^2 \\
&\quad -2((\phi+\theta) + \theta(\phi+\theta)^2) Y_1 Y_2 + 2\theta(\phi+\theta) Y_1 Y_3 - 2(\phi+\theta) Y_2 Y_3\,.
\end{aligned}$$

9) a) First note that

$$\sum_{t=2}^{n} Y_t^2 = \sum_{t=1}^{n-1} Y_t^2 + Y_n^2 - Y_1^2 = 416.96\,.$$

Thus, the Least Squares estimate of parameters ϕ and σ^2 are

$$\begin{aligned}
\hat{\phi} &= \frac{\sum_{t=2}^{n} Y_{t-1}Y_t}{\sum_{t=1}^{n-1} Y_t^2} = 0.794 \\
\hat{\sigma}^2 &= \frac{\sum_{t=2}^{n}(Y_t - \hat{\phi}Y_{t-1})^2}{n} \\
&= \frac{1}{199}\sum_{t=2}^{200} Y_t^2 - \frac{2}{199}\hat{\phi}\sum_{t=2}^{200} Y_tY_{t-1} + \frac{1}{199}\hat{\phi}^2\sum_{t=1}^{199} Y_t^2 \\
&= 0.786\,.
\end{aligned}$$

b) Since $\sqrt{n}(\hat{\phi}-\phi) \sim N(0, \sigma^2/\gamma(0))$ and $\gamma(0) = 1/(1-\phi^2)$, the confidence interval for ϕ is $\left(\hat{\phi} - 1.96\sqrt{(1-\hat{\phi}^2)/n}, \hat{\phi} + 1.96\sqrt{(1-\hat{\phi}^2)/n}\right) = (0.7097, 0.8783)$.

10) Multiplying suitable X_{t-k} on both sides and take expectation, we have the Yule-Walker equations

$$\begin{aligned}
\gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) \\
\gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0)
\end{aligned}$$

Let $\boldsymbol{\phi} = (\phi_1, \phi_2)'$, the Yule-Walker estimate is

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\
&= \begin{pmatrix} 0.317 \\ 0.238 \end{pmatrix}.
\end{aligned}$$

11) Note that the ACVF function is $\gamma(0) = \sigma^2(1+\theta^2)$, $\gamma(1) = \sigma^2\theta$ and $\gamma(k) = 0$ for $k \geq 2$. Thus the likelihood function is

$$l(\theta, \sigma^2) = f(y_1, y_2, y_3) = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}Y'\Sigma^{-1}Y},$$

where $Y = (y_1, y_2, y_3)'$ and

$$\Sigma = \sigma^2 \begin{pmatrix} 1+\theta^2 & \theta & 0 \\ \theta & 1+\theta^2 & \theta \\ 0 & \theta & 1+\theta^2 \end{pmatrix}.$$

12) Lag-2 PACF is the β_2 that achieve the minimum of

$$E(Y_t - \beta_1 Y_{t-1} - \beta_2 Y_{t-2})^2 .$$

Differentiating the above expectation with respect to β_1 and β_2, the best (β_1, β_2) is the (ϕ_{21}, ϕ_{22}) that satisfies

$$\begin{aligned}
\gamma(1) - \phi_{21}\gamma(0) - \phi_{22}\gamma(1) &= 0 , \\
\gamma(2) - \phi_{21}\gamma(1) - \phi_{22}\gamma(0) &= 0 ,
\end{aligned}$$

or

$$\begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} \begin{pmatrix} \phi_{21} \\ \phi_{21} \end{pmatrix} = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} .$$

Solving the above equation, the lag-2 PACF ϕ_{22} is

$$\phi_{22} = \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)} . \tag{2}$$

For an MA(2) model with $\sigma^2 = 1$, the autocovariance functions is

$$\begin{aligned}
\gamma(0) &= 1 + \theta_1^2 + \theta_2^2, \\
\gamma(1) &= \theta_1(1 + \theta_2), \\
\gamma(2) &= \theta_2, \\
\gamma(k) &= 0 \quad \text{for} \quad |k| \geq 3.
\end{aligned}$$

Substitute the autocovariance functions into (2) yields

$$\phi_{22} = \frac{\theta_2(1 + \theta_1^2 + \theta_2^2) - \theta_1^2(1 + \theta_2)^2}{(1 + \theta_1^2 + \theta_2^2)^2 - \theta_1^2(1 + \theta_2)^2} .$$

13) The ACF plot has a cutoff at lag one, which suggests an MA(1) model. Also, the PACF is exponentially decaying, which further supports the MA(1) model. Nevertheless, the ACF and PACF at lag-12 are marginally significant, suggesting that there could be a seasonal effect. Therefore, we may study models such as

- MA(1): $Y_t = (1 - \theta B)Z_t$,
- SARIMA(1,0,0)×(0,0,1): $(1 - \Phi_s B^{12})Y_t = (1 - \theta B)Z_t$,

- SARIMA(0,0,1)×(0,0,1): $Y_t = (1 - \Theta_s B^{12})(1 - \theta B)Z_t$,

where $Z_t \sim WN(0, \sigma^2)$. Model selection criteria such as AIC/BIC can be employed to select the final model.

14) By the AICC formula

$$-2 \log L_x + \frac{2n(p+q+1)}{n-p-q-2},$$

the AICC value for the four models are computed respectively as

a) AICC = 1276
b) AICC = 1284
c) AICC = 1298
d) AICC = 1266

So we choose model ARMA(1,1) which has the smallest value of AICC.