1. Functions

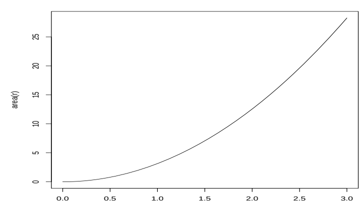
- The relation between two variables

$$y = f(x)$$

- x is called the independent variable and y is called the dependent variable
- The area of a circle and its radius

$$A = \pi r^2$$

Note that radius is easier to measure than the area of the circle



- If the radius is 1, then the area of the circle will be

$$A(1) = \pi(1)^2 = \pi$$

- How about r=1.1,

$$A(1.1) = \pi(1.1)^2 = 1.21\pi$$

- Therefore, the area will increase

$$A(1.1) - A(1) = 0.21\pi$$

- How much more area if the radius increase 0.01

$$A(1.01) - A(1) = 0.0201\pi$$

- How about increase 0.001

Increment (Δr)	$A(1+\Delta r) - A(1)$	$\frac{A(1+\Delta r)-A(1)}{\Delta r}$
0.1	0.21π	2.1π
0.01	0.0201π	2.01π
0.001	0.002001π	2.001π
0.0001	0.00020001π	2.0001π
0.00001	0.0000200001π	2.00001π
0.000001	0.000002000001π	2.000001π

- From the table, we can see the difference will get smaller and smaller but the ratio is more or less constant

2. Definition of Limit

- Definition: We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a, but $x \neq a$ equals L", or, symbolically,

$$f(x) \to L \quad \text{as} \quad x \to a$$

- Example: Guess the value of

$$\lim_{x \to 1} \frac{x-1}{x^2-1}.$$

0.9	0.99	0.999	1.001	1.01	1.1
0.5263	0.5025	0.5003	0.4998	0.4997	0.4762

- One sided limit

Definition: We write

$$\lim_{x \to a^-} f(x) = L$$

and say "the left-hand limit of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a from the left] is equal to L"

Similarly we can define the right-hand limit of $f(x)$ as x approaches a is equal to L and we write

$$\lim_{x \to a^+} f(x) = L$$

The following is true

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

- Limit Laws

  Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

  exist. Then

$$\begin{aligned}
\lim_{x\to a}[f(x)+g(x)] &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
\lim_{x\to a}[f(x)-g(x)] &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\
\lim_{x\to a}[cf(x)] &= c\lim_{x\to a} f(x) \\
\lim_{x\to a}[f(x)g(x)] &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0 \\
\lim_{x\to a}[f(x)]^n &= [\lim_{x\to a} f(x)]^n \text{where } n \text{ is a positive integer}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to a} c &= c \\
\lim_{x\to a} x &= a \\
\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x\to a} f(x)}
\end{aligned}$$

  Direct Substitution Property: If f is a polynomial or a rational function and a is in the domain of f. Then

$$\lim_{x\to a} f(x) = f(a)$$

- Examples:

(a)

$$\lim_{x \to 5}(2x^2 - 3x + 4)$$

(b)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{-3x}$$

(c)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

(d)

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

- Some Theorems
  - $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$
    Example:
    $$\lim_{x\to 0} |x| = 0$$
  - If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approach a, then
    $$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$
  - Squeeze (Sandwiches) Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and
    $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
    then
    $$\lim_{x\to a} g(x) = L$$
    Example: Show that
    $$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

– If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Example: Evaluate

$$\lim_{x \to 1} \exp\left(\frac{1 - \sqrt{x}}{1 - x}\right)$$

- Exercise:

(a)

$$\lim_{t\to -2}(t+1)^9(t^2-1)$$

(b)

$$\lim_{x\to 2}\frac{2x^2+1}{x^2+x-4}$$

(c)

$$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$$

(d)

$$\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3}$$

(e)

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h}$$

(f)

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$$

(g)

$$\lim_{x\to -4}\frac{\frac{1}{4}-\frac{1}{x}}{4+x}$$

(h)

$$\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]$$

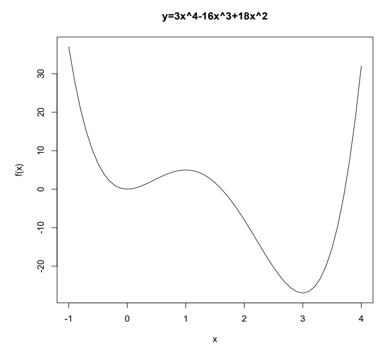
(i)

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

3. Definition of the Derivative

- Let $f(x)$ be a function defined in a neighborhood (a,b) of a point x_0
  Suppose the limit
  $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$
  exists.
  Then we say that f is differentiable at x_0 and its derivative at x_0 is equal to the limit in (1).
- Denote by
  $$\frac{df(x_0)}{dx}, \qquad f'(x_0)$$
- $f(x)$ has a derivative $f'(x)$ at x_0
- Other notations
  $$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$
- if the function $f(x)$ be a function defined on an interval I and if $f(x)$ is differentiable at each point of I, then f is differentiable on I

- Remarks:
  - The derivative of the function f at x_0 is the slope of the tangent line of the function f at x_0
  - It can be thought as the instantaneous rate of change of the function f at x_0
  - If $f'(x_0) > 0$, the function is increasing at x_0
  - If $f'(x_0) < 0$, the function is decreasing at x_0
  - If $f'(x_0) = 0$, the function is constant at x_0



- Example 1:
  $f(x) = x^2$ is differentiable at each point x_0

- Example 2: Find the derivative of $f(x) = \sqrt{x}, x > 0$.

- Example 3: For any values of n, x^n is differentiable and

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

Binomial Theorem: For a positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Remarks:
(a)

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

(b)

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Generalized binomial theorem: For any number r

$$\begin{aligned} (x+y)^r &= \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \\ &= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots . \end{aligned}$$

- Example 4. Derivative of $f(x) = \log x$ is $\frac{1}{x}$

$$\begin{aligned}
\frac{d \log x}{dx} &= \lim_{h \to 0} \frac{\log(x+h) - \log x}{h} \\
&= \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{x\left(\frac{h}{x}\right)} \\
&= \frac{1}{x} \lim_{h \to 0} \log\left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \\
&= \frac{1}{x} \log e \\
&= \frac{1}{x}
\end{aligned}$$

The definition of e

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

For all functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has slope exactly 1

4. Operations

- Multiply by a constant c

$$\frac{d}{dx}[cf(x)] = cf'(x) \tag{3}$$

- Addition and Subtraction

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \tag{4}$$

- Product Rule

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x) \tag{5}$$

Examples:

(a) $\frac{d}{dx}[x(1-x)]$

(b) $\frac{d}{dx}(x \log x - x)$

- Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \tag{6}$$

Example:

$$\frac{d}{dx}\frac{x}{1-x}$$

- Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \tag{7}$$

Examples:

(a) $\frac{d}{dx}(1+x^2)^3$

(b) $\frac{d}{dx}(x^4+3x^3+5)^{-1/3}$

(c) $\frac{d}{dx}\frac{x^2}{4x^4+2x^3}$

(d) $\frac{d}{dx}[(x^3+2x+3)(3x^4+2x^3)\sqrt{2x^5-3x^2}]$

5. Inverse Functions

- $f(x)$ a strictly monotonic function in an open interval I
- Its derivative $f'(x)$ exists
- Let $x_0 \in I$ and $f'(x_0) \neq 0$
- g is the inverse function of f if $y_0 = f(x_0), \quad y_0 + k = f(x_0 + h)$, then $g(y_0) = x_0, \quad g(y_0 + k) = x_0 + h$.
- Theorem: Let $f(x)$ be a strictly monotone, differentiable function in an open interval I. Then, for each point x_0 where $f'(x_0) \neq 0$, the inverse function $g(y)$ is differentiable at $y_0 = f(x_0)$, and

$$g'(y_0) = \frac{1}{f'(x_0)} \tag{8}$$

- Example 1:

$$\frac{d}{dx} e^x = e^x$$

- Example 2:

$$\frac{d}{dx} a^x = a^x \log a, a > 0.$$

6. Formula

$f(x)$	$f'(x)$
constant	0
x	1
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$

7. Implicit Differentiation

- So far we find dy/dx where $y = f(x)$
- Sometimes, some functions are defined implicitly by a relation between x and y.
- Example: $x^2 + y^2 = 25$ or $x^3 + y^3 = 6xy$
- How can we find dy/dx
- we differentiate the relation with respect to x

$$
\begin{aligned}
&\frac{d}{dx}[x^2 + y^2 = 25] \\
&\quad \Rightarrow \quad \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}(25) \\
&\quad \Rightarrow \quad 2x + \frac{d}{dy}y^2\frac{dy}{dx} = 0 \\
&\quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \\
&\quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}
\end{aligned}
$$

Exercise: Find the derivatives of the following functions:

(a) $\frac{1}{\sqrt{x^2+x+1}}$

(b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1+x^2}}$

(d) x^x

(e) $e^{\sqrt{1+x^2}}$

(f) $\log(x + \sqrt{x^2+1})$

(g) $\exp\left(\frac{x^2+1}{x^2-1}\right)$

(h) $\log(\log x)$

8. Higher Derivatives

- Given a function $f(x)$ in some interval I, we denote its derivative by $f'(x)$ or df/dx.
- The derivative of $f'(x)$ is called the second derivative of $f(x)$ and denoted by

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2}$$

- The derivative of $f''(x)$ is called the third derivative of $f(x)$ and denoted by

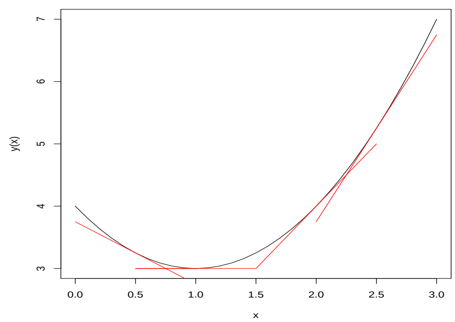
$$f'''(x) \quad \text{or} \quad \frac{d^3 f(x)}{dx^3}$$

- If we differentiate f n times, we get the nth derivative of $f(x)$.
- It is denoted by

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n f(x)}{dx^n}$$

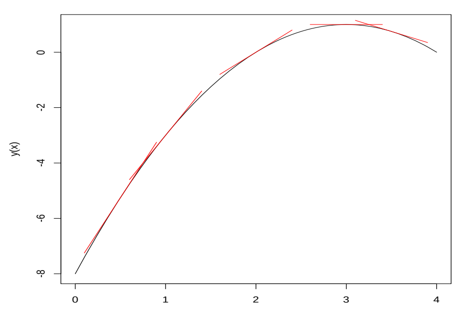
- What does f'' say about f

  Since $f'' = (f')'$, then f'' positive means f' increasing



  we say the function is concave upward at this interval

  f'' negative means f' decreasing



  we say the function is concave downward at this interval

  Example: $f(x) = e^{-x}$

9. Applications

- Minimum and Maximum Values

  Optimization problem

  Defn: A function f has an absolute maximum (minimum) at c if $f(c) \geq (\leq) f(x)$ for all x in D, where D is the domain of f. The maximum and minimum values of f are called the extreme values of f.

  Defn: A function f has a local maximum (minimum) at c if $f(c) \geq (\leq) f(x)$ for all x in some open interval containing c.

  Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem: If f has a local minimum or maximum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Defn: A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local maximum or minimum at c, then c is a critical number.

Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

(a) Find the values of f at the critical numbers of f in $[a, b]$.

(b) Find the values of f at the end points of the intervals

(c) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of the values is the absolute minimum value.

The First Derivative Test: Suppose that c is a critical number of a continuous function f:

(a) If f' changes from positive to negative at c, then f has a local maximum

(b) If f' changes from negative to positive, then f has a local minimum at c

(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

Example:

$$f(x) = x^4 + 4x^3$$

The Second Derivative Test: Suppose f" is continuous near c

(a) $f'(c) = 0$ and $f"(c) > 0$, then f has a local minimum at c

(b) $f'(c) = 0$ and $f"(c) < 0$, then f has a local maximum at c

This result can be further extended.

Theorem: Assume $f'(x), \ldots, f^{(n+1)}(x)$ exist in the open interval I and that $f^{(n+2)}(x)$ continuous. Let c be a point of I for which

$$f'(c) = f"(c) = \cdots = f^{(n)}(c) = 0$$

and $f^{(n+1)}(c) \neq 0$. Then

(a) If n is even, then c is not an extreme point

(b) If n is odd, then c is an extreme point: a local maximum if if $f^{(n+1)}(c) < 0$ and local maximum if $f^{(n+1)}(c) > 0$

- L'Hospital Rule
  - If $f(a) = g(a) = 0$ or if $f(a) = g(a) = \infty$, and if further,

$$A = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

    exits, then also

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

    exits and equals A.
  - A may be a real number of or $\pm\infty$.
  - It is called Indeterminate form of type $\frac{0}{0}$
  - Examples:
    (a) $\lim_{x \to 0} \frac{log(1+x)}{x}$
    (b) $\lim_{x \to \infty} \frac{x^n}{e^x}$

- How about $\lim fg$
- It is called Indeterminate form of type $0 \cdot \infty$
- if $\lim f = 0$ and $\lim g = \infty$ then

$$\lim fg = \lim \frac{f}{\frac{1}{g}}$$

- Example: Find the limit of

$$\frac{\frac{4}{(1+x)^5}}{x^3 e^{-x}}$$

as $x \to \infty$

- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$

- Example:

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

- Indeterminate Powers

  Several indeterminate forms arises from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

  (a) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0

  (b) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0

  (c) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

- Write the function as

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

  and determine the indeterminate form $g(x)\ln f(x)$

- Example: Find $\lim_{x\to 0^+} x^x$.

- Taylor's Theorem
  - Let $f(x)$ be a function defined on a closed bounded interval $a \leq x \leq b$. Assume that its derivatives $f'(x), f''(x), \ldots, f^{(n+1)}(x)$ exist for all $a \leq x \leq b$. Then, for any x, $a < x \leq b$, there is a point $\eta, a < \eta \leq x$, such that

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &\quad + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1} \end{aligned} \tag{9}$$

  - Examples:
    For any positive x,

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + R_{n+1}$$

where $|R_{n+1}| < e^x \frac{x^{n+1}}{(n+1)!}$.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Note that

$f'(x) = \frac{1}{1+x}$	$f'(0) = 1$
$f''(x) = -\frac{1}{(1+x)^2}$	$f''(0) = -1$
$f'''(x) = \frac{2}{(1+x)^3}$	$f'(0) = 2$
$f^{(4)}(x) = -\frac{2\times 3}{(1+x)^4}$	$f'(0) = -2 \times 3$
$\vdots$	$\vdots$
$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$	$f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Remarks:

  - For any function f continuous on an interval I and its derivative $f'(x)$ exists on I, then the function can be approximated by

$$f(x) = a + bx$$

or in general

$$f(x) = a + \sum_{i=1}^{n} b_i x^i$$

where $x \in I$.

10. Newton's Method

- One of the most powerful and well known numerical methods for solving a root-finding problem $f(x) = 0$.
- Suppose f is twice continuously differentiable on the interval $[a, b]$. Let $p \in [a, b]$ is the root of the function f, i.e., $f(p) = 0$ and let $\bar{x} \in [a, b]$ be an approximation to p such that $f(\bar{x}) \neq 0$ and $|\bar{x} - p|$ is small. Consider the first-degree Taylor polynomial for $f(x)$, expanded about $\bar{x}$,

$$f(x) = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + \frac{(x - \bar{x})^2}{2} f"(\eta(x)),$$

where $\eta(x)$ lies between x and $\bar{x}$. Since $f(p) = 0$, with $x = p$, we have

$$0 = f(\bar{x}) + (p - \bar{x})f'(\bar{x}) + \frac{(p - \bar{x})^2}{2} f"(\eta(p)).$$

Assuming $(p - \bar{x})^2$ is negligible and that

$$0 \simeq f(\bar{x}) + (p - \bar{x})f'(\bar{x}).$$

Therefore,

$$p \simeq \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}$$

which should be a better approximation to p than is $\bar{x}$.

- Newton' Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

- Example: Suppose we want to x such that

$$f(x) = e^{x^2} - 2 = 0$$

Note that $f'(x) = 2xe^{x^2}$. Use $p_0 = 1$ then

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{0.7182818}{5.436564} = 0.8678794$$

Keep going

$$\begin{aligned} p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} = 0.8342924 \\ p_3 &= p_2 - \frac{f(p_2)}{f'(p_2)} = 0.8325589 \\ p_4 &= p_3 - \frac{f(p_3)}{f'(p_3)} = 0.8325546 \end{aligned}$$

Usually we will stop the iteration procedure if two consecutive guesses are close enough.

11. Antiderivative

- A function F is called an antiderivative (primitive) of f on an interval I if

$$F'(x) = f(x)$$

  for all x in I.

- Theorem: If F is an aniderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

  where C is an arbitrary constant.

- Examples:
  (a) $f(x) = 1/x$

  (b) $f(x) = x^n, n \neq 1$

  (c) Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

  An equation that involves the derivatives of a function is called a differential equation

- Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec^2(x)$	$\tan x$
$x^n, n \neq 1$	$\frac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\log x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

12. Definite Integral

- Definition:

  If f is continuous function defined for $a \leq x \leq b$, we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, \ldots, x_n(=b)$ be the endpoints of these intervals and we choose sample points $x_1^*, x_2^*, \ldots, x_n^*$ in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If $f \geq 0$ in $[a, b]$, then $\int_a^b f(x)dx$ is the area bounded between x-axis, $f(x)$ from $x = a$ to $x = b$.

We need the following formula for evaluating the definite interval

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

Proved by Mathematical Induction.

Example: Evaluate

$$\int_0^3 (x^3 - 6x)dx$$

Divide the interval $[0, 3]$ into n subintervals and the width of the interval is

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

and using the right end-points as x_i^*, then

$$\begin{aligned}
\int_0^3 (x^3 - 6x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\} \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 = -6.75
\end{aligned}$$

13. Properties of The Integral

- If f is integrable over $[a, b]$, then it is also integrable over any sub-interval $[\alpha, \beta]$.
- If f is integrable over $[a, b]$ and over $[b, c]$, then it is also integrable over $[a, c]$ and

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

- Let f and g are integrable over $[a, b]$ and so are $f + g$ and $f - g$ and

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

- If f is integrable over $[a, b]$, then for any constant λ, λf is also integrable over $[a, b]$
- Let f is integrable over $[a, b]$ and

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

-

$$M = \sup_{a \le x \le b} f(x) \quad \text{and} \quad m = \inf_{a \le x \le b} f(x)$$

Then

$$\int_a^b f(x)dx = \mu(b - a)$$

for some $m \le \mu \le M$

The difference between minimum and infimum:

Minimum of a set of number is the element in the set smaller than or equal to any members in the set

Infimum of a set is the greatest value which smaller than or equal to any members in the set

Examples: $A = \{1, 2, 3, 4, 5\}$

$$\min A = 1 = \inf A$$

$A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots, \}$, then there is no minimum in this set but the infimum is 0

14. The Fundamental Theorem of Calculus

- Fix a point c in $[a, b]$, and consider the integral

$$g(x) = \int_c^x f(t)dt$$

for each x in $[a, b]$. This is a function of x and it is called the indefinite integral of f.

Theorem: An indefinite integral of an integrable function f is a continuous function.

- Fundamental Theorem of Calculus Part I

Theorem: At each point $x_0 \in (a, b)$, where $f(x)$ is continuous , the derivative $g'(x_0)$ of the indefinite integral $g(x)$ exists, and it is equal to $f(x_0)$.

Thus,

$$\frac{d}{dx}\int_c^x f(t)dt = f(x) \tag{10}$$

at each point x where f is continuous. Examples:

(a) Let $F(x)$ be given as

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

Then what is $F'(x)$

(b) Let $F(x)$ be given as

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt$$

Then what is $F'(x)$

(c) Find the derivative of the function $\int_a^x \frac{t^2}{1+t^2} dt$ for all real x.

- Fundamental Theorem of Calculus Part II

  Theorem Let $f(x)$ be a continuous function in $[a, b]$, and let $F(x)$ be an antiderivative (a primitive) of f. Then

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a) \tag{11}$$

- A powerful method for computing integrals. Instead of trying to compute them from the definition of the integral, all one has to do is to find a primitive function.
- Examples

  (a) $\int_a^b x^n dx$ with n positive
  The function $\frac{x^{n+1}}{n+1}$ is a primitive of x^n for n positive. Hence,

$$\int_a^b x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

  (b) $\int \frac{dx}{x}$

  (c) $\int \frac{dx}{x^n}$

  (d) $\int \log x dx$

15. Indefinite Integral

    Because the relation between antiderivaties and integrals, the notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called indefinite integral

$$\int f(x)dx = F(x) \quad \text{menas} \quad F'(x) = f(x)$$

    Difference between the definite integral and indefinite integral:

    Definite integral

$$\int_a^b f'(x)dx = f(b) - f(a)$$

    is a number and indefinite integral

$$\int f'(x)dx = f(x)$$

    Also note that if F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is a constant.

16. Methods of integration

(a) Change of variables formula

Theorem: Let $f(x)$ be a continuous function in a closed interval $[a, b]$. Let $\phi(t)$ be a continuous function defined on a closed interval $[c, d]$ and assume that $\phi'(t)$ exists and is continuous on $[c, d]$. Finally, assume that the values of $\phi(t)$ lie in the interval $[a, b]$, and that $\phi(c) = a, \phi(d] = b.$, Then

$$\int_c^d f(\phi(t))\phi'(t)dt = \int_a^b f(x)dx \tag{12}$$

The change of variables formula (12) can be wriiten in an abbreviated form

$$\int_c^d f(x)\frac{dx}{dt}dt$$

Examples

i. $\int_e^{e^2} \frac{logt}{t}dt$

ii. $\int_0^1 t\sqrt{1-t^2}dt$

Integrals of symmetric functions

Suppose f is continuous on $[-a, a]$.

i. If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

ii. If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$

This is very useful in probability, e.g., mean of a normal random variable

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

(b) Integration by parts

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) \tag{13}$$

Examples:

i. $\int xe^{-x}dx$

ii. $\int_0^1 \frac{y}{e^{2y}}dx$

(c) Partial Fraction

We integrate rational function (ratio of two polynomials) by expressing them as sums of simpler fractions, called partial fraction.

Example: Find $\int \frac{5x-4}{2x^2+x-1}dx$

- If the degree in the numerator have been same as that of denominator, or higher, performancing a long division
  $\frac{2x^3-11x^2-2x+2}{2x^2+x-1}$

- if the denominator has more than two linear factors, we need to include a term corresponding to each factor
  $\frac{x+6}{x(x-3)(4x+5)}$

- If a linear factor is repeated, we need to include extra terms in the partial fraction expansion
  $\frac{x}{(x+2)^2(x-1)}$

(d) Exercises:

- $\int_1^2 x\sqrt{x-1}dx$
- $\int re^{r/2}dr$
- $\int \frac{e^x+1}{e^x+x}dx$
- $\int x^3\sqrt{1+x^2}dx$
- $\int \frac{dx}{x^2+2x+5}$
- $\int \frac{x+2}{x^2+4x-5}dx$
- $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}}$
- $\int x^2(1+2x^3)^5dx$

17. Differential Equations

- Most important application of calculus
- An equation containing an unknown function and some of its derivative
- Example: Let P be the population and the rate of growth of the population is proportional to the population size

$$\frac{dP}{dt} = kP$$

- Assume $P > 0$, then if $k < 0$, then P is decreasing and $k > 0$, P is increasing

$$\begin{aligned} & \frac{dP}{dt} = kP \\ \Rightarrow\ & \frac{dP}{P} = kdt \\ \Rightarrow\ & \int \frac{dP}{P} = \int kdt \\ \Rightarrow\ & \log P = kt + C \\ \Rightarrow\ & P = e^{kt+C} = Ae^{kt} \end{aligned}$$

  where A is a constant.
- Sometimes, we want to find the particular solution satisfies

$$P(t_0) = P_0$$

  We call this the initial value problem.
- For the above problem

$$P(t) = P_0 e^{kt}, t > 0$$

18. Function of several variables

- Partial Derivative

  Let $f(x,y)$ be a function of two variables defined on an open set G

  Let (a,b) be a point in G

  If

  $$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} \quad \text{exists}$$

  then we say that $f(x,y)$ has a partial derivative with respect to x at the point (a,b)

  Denote it by

  $$f_x(a,b), \quad \frac{\partial f}{\partial x}(a,b), \quad \left(\frac{\partial f}{\partial x}\right)_{(a,b)}$$

  Similarly, if the limit

  $$\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k} \quad \text{exists}$$

  then we say that $f(x,y)$ has a partial derivative with respect to y at the point (a,b)

  Denote it by

  $$f_y(a,b), \quad \frac{\partial f}{\partial y}(a,b), \quad \left(\frac{\partial f}{\partial y}\right)_{(a,b)}$$

  Example:

  $$f(x,y) = 2x^3 + y^2 + 4x\sqrt{y}$$

If f_x has a partial derivative with respect to x, then we denote it by

$$f_{xx}, \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2}$$

If f_x has a partial derivative with respect to y, then we denote it by

$$f_{xy}, \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x}$$

If f_y has a partial derivative with respect to y, then we denote it by

$$f_{yy}, \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2}$$

If f_y has a partial derivative with respect to x, then we denote it by

$$f_{yx}, \quad \text{or} \quad \frac{\partial^2 f}{\partial x \partial y}$$

Theorem: Let f defines on an open set G of $\mathbf{R}^2$. If f_x, f_y and f_{xy} exist and continuous in G, then f_{yx} also exists and $f_{yx} = f_{xy}$.

- Derivatives of functions of several variables

$$\frac{\partial^n f}{\partial x^i \partial y^j \partial z^{n-i-j}} \text{ for some } i, j, \text{ where } 0 \leq i + j \leq n.$$

- Examples: Compute the first and second partial derivatives of the following functions
  $e^{y\sqrt{1+x^2}}$

  $\sqrt{x^2+y^2+z^2}$

  $x^2\log(y^2+z^2)$

- Taylor Theorem

  Let $f(x, y)$ be a function having continuous partial derivatives of all order $\leq n+1$ in an open set G. Let the line segment connecting two points (a, b) and $(a+h, b+k)$ be contained in G. Then

$$f(a+h, b+k) = f(a, b) + \sum_{m=1}^{n} \frac{1}{m!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^m f(x, y) \Bigg|_{(a,b)} + R_{n+1}$$

  where

$$R_{n+1} = \frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(x, y) \Bigg|_{(a+\theta h, b+\theta k)}, 0 < \theta < 1$$

  In particular, when $n = 2$,

$$f(a+h, b+k) = f(a, b) + h\frac{\partial f}{\partial x}(a, b) + k\frac{\partial f}{\partial y}(a, b) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(a, b) + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2}(a, b) + hk\frac{\partial^2 f}{\partial x \partial y}(a, b) + R_3$$

  Now write $x = a + h, y = b + k$, then

$$f(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{f_{xx}(a, b)}{2}(x-a)^2 + \frac{f_{yy}(a, b)}{2}(y-b)^2 + f_{xy}(a, b)(x-a)(y$$

- Computation of Extremum

  Theorem: Let $f(x_1, x_2, \ldots, x_n)$ have first derivatives in a domain D in $\mathbf{R}^n$. If $f(x_1, x_2, \ldots, x_n)$ has a relative extremum at a point α of D, then

$$\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} = 0, i = 1, 2, \ldots, n.$$

  Theorem: Let $f(x, y)$ have continuous first two partial derivatives in a domain D of $\mathbf{R}^2$ and let $f(a, b)$ be a critical point of f in D. Set

$$A = f_{xx}(a, b), \quad B = f_{xy}(a, b), \quad C = f_{yy}(a, b).$$

  Then:

  (a) If $B^2 - AC < 0$ and $A > 0$, then f has relative minimum at (a, b)
  (b) If $B^2 - AC < 0$ and $A < 0$, then f has relative maximum at (a, b)
  (c) If $B^2 - AC > 0$, then f has neither relative maximum or relative minimum at (a, b)
  (d) If $B^2 - AC = 0$, then no conclusion can be drawn and higher derivative is needed to analyze the nature of the critical point.

  Find the critical points of the following functions and test the nature of the critical points:

  Example: $x^2 - 8x^2y + 3y^2$

  Exercise: $(ax^2 + by^2)e^{-x^2-y^2}$

- Lagrange Multiplier (Constrainted Optimazation)

  To find the maximum or minimum values of $f(x, y)$ subject to constraint $g(x, y) = k$ (assuming that these extreme values exist):

  (a) Consider the function

  $$h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - k)$$

  (b) Find the values of x, y, λ such that

  $$\frac{\partial h}{\partial x} = 0; \frac{\partial h}{\partial y} = 0; \frac{\partial h}{\partial \lambda} = 0$$

  (c) Evaluate f at all points (x, y) that result from last step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

  Example: Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
  Now consider the function

  $$h(x, y, \lambda) = x^2 + 2y^2 - \lambda(x^2 + y^2 - 1)$$

  and

  $$\begin{aligned} \frac{\partial h}{\partial x} &= 2x - 2\lambda x = 0 && (14) \\ \frac{\partial h}{\partial y} &= 4y - 2\lambda y = 0 && (15) \\ \frac{\partial h}{\partial \lambda} &= x^2 + y^2 = 1 && (16) \end{aligned}$$

  From (14), we have $x = 0$ or $\lambda = 1$. If $x = 0$, then $y = \pm 1$, If $\lambda = 1$, then $y = 0$ from (15), so $x = \pm 1$.
  So f has possible extreme values at points $(0, 1), (0, -1), (1, 0), (-1, 0)$ and

  $$f(0, 1) = 2, f(0, -1) = 2, f(1, 0) = 1, f(-1, 0) = 1$$

  Therefore, the maximum values of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum values is $f(\pm 1, 0) = 1$.

This result can be extended to n variables situation.

To find the maximum or minimum values of $f(x_1, x_2, \ldots, x_n)$ subject to constraint $g(x_1, x_2, \ldots, x_n) = k$ (assuming that these extreme values exist):

(a) Consider the function

$$h(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) - \lambda(g(x_1, x_2, \ldots, x_n) - k)$$

(b) Find the values of $x_1, x_2, \ldots, x_n, \lambda$ such that

$$\frac{\partial h}{\partial x_i} = 0, i = 1, 2, \ldots, n; \frac{\partial h}{\partial \lambda} = 0$$

(c) Evaluate f at all points $(x_1, x_2, \ldots, x_n)$ that result from last step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

19. Double integral

- Definition: The double integral of f over the rectangle R is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

  if the limit exists.

- Properties of Double integral

  (a)

$$\int\int_R [f(x,y)+g(x,y)]dA = \int\int_R f(x,y)dA + \int\int_R g(x,y)dA$$

  (b)

$$\int\int_R cf(x,y)dA = c\int\int_R f(x,y)dA$$

  where c is a constant

  (c) If $f(x,y) \geq g(x,y)$ in R, then

$$\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA$$

20. Iterated Integral

- Theorem: Let $f(x,y)$ be integrable in a rectangle R, and suppose that for each fixed x in $[a,b]$ the function $f(x,y)$ is integrable in y on the interval $[c,d]$. Then the function
$$g(x) = \int_c^d f(x,y)dy$$
is integrable in the interval $a \leq x \leq b$ and
$$\int \int_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$
This theorem tells us that we can first integrate with respect to y (holding x fixed) and then integrate the resulting function with respect to x
- $\int_1^2 \int_2^3 x^2 + y^3 + 3xydxdy$

- Fubini's Theorem: If f is continuous on the rectangle $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

  More generally, this is true if f is bounded on R, f is discontinuous only a finite number of smooth curves, and the iterated integrals exist.

  This theorem tell us we can change the order of integration

- $\int_1^2 \int_0^\pi y \sin(xy) dy dx$

- Suppose $f(x, y)$ can be factorized into product of a function of x only and a function of y only, i.e.,

$$f(x, y) = g(x)h(y)$$

then

$$\int\int_R f(x, y)dxdx = \int_a^b g(x)dx \int_c^d h(y)dy \text{ where } R = [a, b] \times [c, d]$$

- very important in the independence of two random variables

Example: Find

$$\int_1^3 \int_2^3 xydydx$$

- Theorem: Let D be the domain consisting of all the points (x, y) with

$$\alpha < x < \beta, \quad \gamma_1(x) < y < \gamma_2(x)$$

and if f is continuous in D, then

$$\int\int_D f(x,y)dA = \int_\alpha^\beta \int_{\gamma_1(x)}^{\gamma_2(x)} f(x,y)dydx$$

- $\int_0^1 \int_x^1 3xy + y^2 dydx$

- $\int_0^1 \int_y^1 e^{-x} dxdy$

- Theorem: Let D be the domain consisting of all the points (x, y) with

$$\alpha < x < \beta, \quad \gamma_1(x) < y < \gamma_2(x)$$

or

$$a < y < b, \delta_1(y) < x < \delta_2(y)$$

and if f is continuous in D, then

$$\int_\alpha^\beta \int_{\gamma_1(x)}^{\gamma_2(x)} f(x, y)dydx = \int_a^b \int_{\delta_1(y)}^{\delta_2(y)} f(x, y)dxdy$$

Example: Find

$$\int\int_D (x^2 + y^2)dxdy$$

where

$$D = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

$$\begin{aligned}
\int\int_D (x^2 + y^2)dydx &= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)dydx \\
&= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3}\right] dx \\
&= \int_0^2 \left[-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right] dx \\
&= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big]_0^2 \\
&= \frac{216}{35}
\end{aligned}$$

Alternatively the region D can be expressed as follows.

$$D = \{(x, y) : 0 \leq y \leq 4, \frac{y}{2} \leq y \leq \sqrt{y}\}$$

Therefore,

$$\begin{aligned}
\int\int_D (x^2 + y^2)dydx &= \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2)dxdy \\
&= \int_0^4 \left[\frac{x^3}{3} + xy^2\right]_{\frac{y}{2}}^{\sqrt{y}} dy \\
&= \int_0^4 \left[\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right] dy \\
&= \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4\Big]_0^4 \\
&= \frac{216}{35}
\end{aligned}$$

- Change of variables in Double Integrals

  Let

$$\left.\begin{array}{l} u = g_1(x,y) \\ v = g_2(x,y) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = f_1(u,v) \\ y = g_2(u,v) \end{array}\right.$$

  and

$$J = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \neq 0$$

  over the domain of integration. Then

$$\int\int_D h(x,y)dxdy = \int\int_{D^*} h(f_(u,v), f_2(u,v))Jdudv$$

  Example: Evaluate

$$\int\int_D xydxdy$$

  where D is bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $x^2 + y^2 = 9$, $x^2 + y^2 = 16$ in the first quadrant $x \geq 0, y \geq 0$.

  Now let

$$\left.\begin{array}{l} u = x^2 - y^2 \\ v = x^2 + y^2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \sqrt{\frac{u+v}{2}} \\ y = \sqrt{\frac{v-u}{2}} \end{array}\right.$$

  Then

$$\begin{array}{ll} \dfrac{\partial x}{\partial u} = \dfrac{1}{4\sqrt{\frac{u+v}{2}}} & \dfrac{\partial x}{\partial v} = \dfrac{1}{4\sqrt{\frac{u+v}{2}}} \\ \dfrac{\partial y}{\partial u} = -\dfrac{1}{4\sqrt{\frac{v-u}{2}}} & \dfrac{\partial y}{\partial v} = \dfrac{1}{4\sqrt{\frac{v-u}{2}}} \end{array}$$

  and

$$J = \frac{1}{4\sqrt{v^2 - u^2}}$$

  Therefore,

$$\begin{aligned} \int\int_D xydxdy &= \int_9^{16}\int_1^4 \sqrt{\frac{u+v}{2}}\sqrt{\frac{v-u}{2}}\frac{1}{4\sqrt{v^2-u^2}}dudv \\ &= \int_9^{16}\int_1^4 \frac{1}{8}dudv \\ &= \frac{1}{8}(4-1)(16-9) = \frac{21}{8} \end{aligned}$$

Change to Polar Coordinates:
Let

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}\right.$$

and

$$\begin{array}{ll} \dfrac{\partial x}{\partial r} = \cos\theta & \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial y}{\partial r} = \sin\theta & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{array}$$

and

$$J = r\cos^2\theta + r\sin^2\theta = r$$

Then we have the following theorem:
If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\int\int_R f(x,y)dxdy = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta) r d\theta dr$$

Example: Evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$

Note that the polar rectangle is $\{(r,\theta) : 0 \le \theta \le 2\pi, 0 \le r \le \infty\}$. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy &= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r d\theta dr \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{r^2}{2}} r dr \\ &= 2\pi \int_0^{\infty} e^{-u} du \\ &= 2\pi \end{aligned}$$

That is why

$$\int_{\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$