**Solutions for Chapter 5 Exercises**

1. a) First, we compute $a_1, a_2, a_3, a_4$ given $a_0 = 0$ using $Z_1, Z_2, Z_3, Z_4$.

$$\begin{aligned} a_1 &= Z_1 = 0.5 \\ a_2 &= Z_2 - 0.4a_1 = 1.9 \\ a_3 &= Z_3 - 0.4a_2 = 0.44 \\ a_4 &= Z_4 - 0.4a_3 = -0.976\,. \end{aligned}$$

The one-step ahead forecast is

$$\begin{aligned} Z_5^4 &= E(Z_5|Z_1, Z_2, Z_3, Z_4) \\ &= E(a_5 + 0.4a_4|Z_1, Z_2, Z_3, Z_4) \\ &= 0.4E(a_4|Z_1, Z_2, Z_3, Z_4) \\ &= 0.4a_4 = -0.3904\,. \end{aligned}$$

The one-step ahead forecast error is

$$e_4(1) = Z_5 - Z_5^4 = a_5\,.$$

The forecast error variance is given by

$$P_5^4 = \mathrm{Var}(e_4(1)|Z_1, Z_2, Z_3, Z_4) = E(e_4(1)^2|Z_1, Z_2, Z_3, Z_4) = E(a_5^2) = 4\,.$$

Since $a_t$ is assumed to be normally distributed, we have $e_4(1) \sim N(0, 4)$. Thus, the prediction interval for the value $Z_5$ is constructed as

$$(Z_5^4 - 1.96 \times \sqrt{P_5^4}, Z_5^4 + 1.96 \times \sqrt{P_5^4}) = (-4.31, 3.53)\,.$$

b) First, we have the two-step ahead forecast

$$Z_6^4 = E(a_6 + 0.4a_5|Z_1, Z_2, Z_3, Z_4) = 0\,,$$

and forecast error

$$e_4(2) = Z_6 - Z_6^4 = a_6 + 0.4a_5\,.$$

Therefore, the forecast error variance is

$$P_6^4 = \mathrm{Var}(e_4(2)|Z_1, Z_2, Z_3, Z_4) = E(e_4^2(2)|Z_1, Z_2, Z_3, Z_4) = 1.16{\times}4 = 4.64\,,$$

so the prediction interval is constructed as

$$(Z_6^4 - 1.96 \times \sqrt{P_6^4}, Z_6^4 + 1.96 \times \sqrt{P_6^4}) = (-4.22, 4.22)\,.$$

c) The forecast is $Z_{100}^{4} = 0$ and the forecast error is $e_4(96) = Z_{100} = a_{100} + 0.4a_{99}$. The variance of the forecast error is

$$P_{100}^{4} = E(e_4(96)^2 | Z_1, Z_2, Z_3, Z_4) = 1.16 \times 4\,.$$

So the prediction interval for $Z_{100}$ is the same as that of $Z_6$, i.e. $(-4.22, 4.22)$.

2. a) Using the Box and Jenkins forecasting approach, we have

$$\begin{aligned}
Z_{201}^{200} &= E(Z_{201}|Z_{200}) = 0.1Z_{200} + 0.2Z_{199} = 34 \\
Z_{202}^{200} &= E(Z_{202}|Z_{200}) = 0.1Z_{201}^{200} + 0.2Z_{200} = 23.4 \\
Z_{203}^{200} &= E(Z_{203}|Z_{200}) = 0.1Z_{202}^{200} + 0.2Z_{201}^{200} = 9.14 \\
Z_{204}^{200} &= E(Z_{204}|Z_{200}) = 0.1Z_{203}^{200} + 0.2Z_{202}^{200} = 5.6 \\
Z_{205}^{200} &= E(Z_{205}|Z_{200}) = 0.1Z_{204}^{200} + 0.2Z_{203}^{200} = 2.39\,.
\end{aligned}$$

b) The variance of forecast error is

$$\begin{aligned}
P_{201}^{200} &= \mathrm{Var}(e_{200}(1)|Z_{200}, \ldots, Z_1) \\
&= E(e_{200}(1)^2|Z_{200}, \ldots, Z_1) \\
&= E[(Z_{201} - \hat{Z}_{200}(1))^2|Z_{200}, \ldots, Z_1] \\
&= E(a_{201}^2|Z_{200}, \ldots, Z_1) \\
&= E(a_{201}^2) \\
&= 25\,.
\end{aligned}$$

Thus, the prediction interval of $Z_{201}$ can be constructed as

$$(Z_{201}^{200} - 1.96 \times \sqrt{P_{201}^{200}}, Z_{201}^{200} + 1.96 \times \sqrt{P_{201}^{200}} = (24.2, 43.8)\,.$$

For the 95% prediction interval for $Z_{203}$, we need to first find the forecast error of $Z_{202}^{200}$, then find the forecast error of $Z_{203}^{200}$. First, as $e_{200}(1) = Z_{201} - Z_{201}^{200} = a_{201}$, the forecast error of $Z_{202}^{200}$ is

$$\begin{aligned}
e_{200}(2) &= Z_{202} - Z_{202}^{200} \\
&= a_{202} + 0.1(Z_{201} - Z_{201}^{200}) \\
&= a_{202} + 0.1a_{201}\,.
\end{aligned}$$

Next, forecast error of $Z_{203}^{200}$ is

$$\begin{aligned}
e_{200}(3) &= Z_{203} - Z_{203}^{200} \\
&= a_{203} + 0.1(Z_{202} - Z_{202}^{200}) + 0.2(Z_{201} - Z_{201}^{200}) \\
&= a_{203} + 0.1e_{200}(2) + 0.2e_{200}(1) \\
&= a_{203} + 0.1a_{202} + 0.21a_{201}\,.
\end{aligned}$$

Hence, the variance of forecast is

$$\begin{aligned}
P_{203}^{200} = \mathrm{Var}(e_{200}(3)|Z_{200},\ldots,Z_1) &= E[(a_{203} + 0.1a_{202} + 0.21a_{201})^2|Z_{200},\ldots,Z_1] \\
&= E(a_{203}^2) + 0.1^2E(a_{202}^2) + 0.21^2E(a_{201}^2) \\
&= (1 + 0.01 + 0.0441) \times 25 \\
&= 26.35
\end{aligned}$$

Finally, the 95% prediction interval for $Z_{203}$ is constructed as

$$(Z_{203}^{200} - 1.96\sqrt{P_{203}^{200}}, Z_{203}^{200} + 1.96\sqrt{P_{203}^{200}}) = (-0.92, 19.2)\,.$$

c) Following the steps in part (a), we have

$$\begin{aligned}
Z_{202}^{201} &= 0.1Z_{201} + 0.2Z_{200} = 21.1 \\
Z_{203}^{201} &= 0.1Z_{202}^{201} + 0.2Z_{201} = 4.31 \\
Z_{204}^{201} &= 0.1Z_{203}^{201} + 0.2Z_{202}^{201} = 4.65 \\
Z_{205}^{201} &= 0.1Z_{204}^{201} + 0.2Z_{203}^{201} = 1.327\,.
\end{aligned}$$

d) Rewrite the AR model as

$$a_t = (1 + 0.4B)(1 - 0.5B)Z_t\,.$$

The causal (MA) representation is given by

$$\begin{aligned}
Z_t &= (1-0.4B+0.4^2B^2...)(1+0.5B+0.5^2B^2+...)a_t \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-0.4)^j 0.5^k B^{j+k}a_t \\
&= \sum_{i=0}^{\infty}\left(\sum_{j\geq 0, k\geq 0, j+k=i}(-0.4)^j 0.5^k\right)B^i a_t \\
&= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}(-0.4)^j 0.5^{i-j}\right)B^i a_t \\
&= \sum_{i=0}^{\infty}\psi_i a_{t-i}\,,
\end{aligned}$$

where

$$\psi_i = \sum_{j=0}^{i}(-0.4)^j(0.5)^{i-j} = (0.5)^i\frac{1-(-0.8)^{i+1}}{1.8}\,.$$

Therefore, the forecast error is

$$e_t(k) = \sum_{i=0}^{k-1}a_{t-i+k}\psi_i\,.$$

Without loss of generality, suppose $k \geq m$, we have

$$\begin{aligned}
&\mathrm{Cov}(e_t(k), e_t(m)) \\
= &\sum_{i=0}^{m-1}\psi_i\psi_{i+k-m}\sigma_a^2 \\
= &25\sum_{i=0}^{m-1}(0.5)^i\frac{1-(-0.8)^{i+1}}{1.8}(0.5)^{i+k-m}\frac{1-(-0.8)^{i+k-m+1}}{1.8} \\
= &25\frac{0.5^{k-m}}{1.8^2}\sum_{i=0}^{m-1}(0.5)^{2i}(1-(-0.8)^{i+1})(1-(-0.8)^{i+k-m+1}) \\
= &25\frac{0.5^{k-m}}{1.8^2}\sum_{i=0}^{m-1}\left(0.25^i+0.8(-0.2)^i+0.8(-0.8)^{k-m}(-0.2)^i+(-0.8)^{k-m+2}(0.16)^i\right) \\
= &25\frac{0.5^{k-m}}{1.8^2}\left(\frac{1-0.25^m}{0.75}+0.8((-0.8)^{k-m}+1)\frac{1-(-0.2)^m}{1.2}+(-0.8)^{k-m+2}\frac{1-0.16^m}{0.84}\right)\,.
\end{aligned}$$