THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Conditional Probabilities and Expectations - Exercises Solutions

1. (Problem 2.1.2 in Pinsky and Karlin) From the information given in the question, we know

$$\begin{aligned} P(X=x) &= \frac{1}{N} \text{ for } x=1,2,\ldots,N = \frac{1}{N}\mathbf{1}_{\{\leq x \leq N\}} \\ P(Y=y|X=x) &= \frac{1}{x} \text{ for } y=1,2,\ldots,x = \frac{1}{x}\mathbf{1}_{\{1\leq y\leq x\}} \end{aligned}$$

then the joint distribution of X and Y is given by

$$\begin{aligned} P(X=x,Y=y) &= P(Y=y|X=x)P(X=x) \\ &= \frac{1}{xN}\mathbf{1}_{\{1\leq y\leq x\leq N\}} \end{aligned}$$

We also find the marginal distribution of Y from

$$P(Y=y) = \sum_x P(X=x,Y=y) = \sum_x \frac{1}{xN}\mathbf{1}_{\{1\leq y\leq x\leq N\}} = \frac{1}{N}\sum_{x=y}^{N}\frac{1}{x}$$

hence we can find

$$\begin{aligned} P(X=x|Y=y) &= P(X=x,Y=y)/P(Y=y) \\ &= \frac{\frac{1}{xN}\mathbf{1}_{\{1\leq y\leq x\leq N\}}}{P(Y=y)} \\ &= \frac{\frac{1}{x}\mathbf{1}_{\{1\leq y\leq x\leq N\}}}{\frac{1}{y}+\frac{1}{y+1}+\cdots+\frac{1}{N}}. \end{aligned}$$

2. (Problem 2.1.9 in Pinsky and Karlin) From the information given in the question, we know

$$\begin{aligned} P(N=n) &= \frac{e^{-1}}{n!} \text{ for } n=0,1,\cdots = \frac{e^{-1}}{n!}\mathbf{1}_{\{n\geq 0\}} \\ P(X=x|N=n) &= \frac{1}{n+2} \text{ for } x=0,1,\ldots,n+1 = \frac{1}{n+2}\mathbf{1}_{\{0\leq x\leq n+1\}} \end{aligned}$$

then the law of total probability tells us

$$\begin{aligned} P(X=x) &= \sum_n P(X=x|N=n)P(N=n) \\ &= \sum_n \frac{e^{-1}}{n!}\mathbf{1}_{\{n\geq 0\}}\frac{1}{n+2}\mathbf{1}_{\{0\leq x\leq n+1\}} \\ &= \sum_n \frac{e^{-1}}{n!}\frac{1}{n+2}\mathbf{1}_{\{\max\{0,x-1\}\leq n\}} \end{aligned}$$

so for the case $x = 0$, $P(X = 0) = e^{-1} \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}$. Now notice $\frac{n+1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$ hence

$$P(X = 0) = e^{-1} \left[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right] = e^{-1} \frac{1}{1!} = e^{-1}$$

and for all other values of $x = 1, 2, \ldots$

$$\begin{aligned} P(X = x) &= e^{-1} \left[\sum_{n=x-1}^{\infty} \frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right] \\ &= e^{-1} \frac{1}{x!}. \end{aligned}$$

Thus $P(X = x) = e^{-1} \frac{1^x}{x!}$ for $x = 0, 1, 2, \ldots$. That is, X has the Poisson distribution with parameter 1.

3. (Problem 2.1.7 in Pinsky and Karlin) Let TSF = True Structural Failure, NSF = Not a Structural Failure, DSF = Diagnosed as Structural Failure. From the information given in the question, we know

$$P(DSF|TSF) = 0.85, P(DSF|NSF) = 0.35, P(TSF) = 0.3$$

and we want to find $P(TSF|DSF)$. This situation lends itself perfectly to Bayes's formula:

$$\begin{aligned} P(TSF|DSF) &= \frac{P(DSF|TSF)P(TSF)}{P(DSF|TSF)P(TSF) + P(DSF|NSF)(1 - P(TSF))} \\ &= \frac{0.85 \times 0.3}{0.85 \times 0.3 + 0.35 \times (1 - 0.3)} = 0.51. \end{aligned}$$

4. (Exercise 2.1.6 in Pinsky and Karlin)

   (a) We are asked to find $P(U = u, Z = z)$. We know this equals $P(Z = z|U = u)P(U = u)$ so we find $P(Z = z|U = u)$ first.

$$\begin{aligned} P(Z = z|U = u) &= P(U + V = z|U = u) \\ &= P(V = z - u|U = u) \\ &= P(V = z - u) \text{ from independence of } U, V \end{aligned}$$

   Thus $p(Z = z|U = u) = \rho(1 - \rho)^{z-u}$ for $z - u \geq 0$. Hence

$$\begin{aligned} P(U = u, Z = z) &= \rho(1 - \rho)^{z-u} \mathbf{1}_{\{0 \leq u \leq z\}} \rho(1 - \rho)^u \mathbf{1}_{\{0 \leq u\}} \\ &= \rho^2 (1 - \rho)^z \mathbf{1}_{\{0 \leq u \leq z\}}. \end{aligned}$$

   (b) We are asked to find $P(U = u|Z = n)$. We know this equals $P(U = u, Z = n)/P(Z = n)$, so we find

$P(Z = n)$, the marginal distribution of Z first.

$$\begin{aligned} P(Z = n) &= \sum_u P(U = u, Z = n) \\ &= \sum_{u=0}^{n} \rho^2 (1-\rho)^n \\ &= (n+1)\rho^2(1-\rho)^n \mathbf{1}_{\{n\geq 0\}} \end{aligned}$$

hence

$$\begin{aligned} P(U = u|Z = n) &= \frac{\rho^2(1-\rho)^z \mathbf{1}_{\{0\leq u\leq z\}}}{(n+1)\rho^2(1-\rho)^n \mathbf{1}_{\{n\geq 0\}}} \\ &= \frac{1}{n+1}\mathbf{1}_{\{0\leq u\leq z\}} \end{aligned}$$

thus conditioned on the value of Z, U has a uniform distribution.

5. (Problem 2.4.2 in Pinsky and Karlin) From the information given in the question, we know

$$\begin{aligned} P(N = n) &= e^{-\lambda}\frac{\lambda^n}{n!}\mathbf{1}_{\{n\geq 0\}} \\ P(X = x|N = n) &= \binom{n}{x} p^x (1-p)^{n-x} \mathbf{1}_{\{0\leq x\leq n\}} \end{aligned}$$

since $P(X = x) = P(X = x|N = n)P(N = n)$

$$\begin{aligned} P(X = x) &= \sum_n \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)} e^{-\lambda}\frac{\lambda^n}{n!}\mathbf{1}_{\{0\leq x\leq n\}} \\ &= \frac{1}{x!}(\lambda p)^x e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{m!}((1-p)\lambda)^m \\ &= \frac{1}{x!}(\lambda p)^x e^{-\lambda} e^{(1-p)\lambda} \end{aligned}$$

that is X has the Poisson distribution with parameter λp.

For Y, $P(Y = y) = \sum_{n=0}^{\infty} P(Y = y|N = n)P(N = n)$, so $P(Y = y) = \sum_{n=0}^{\infty} P(X = n-y|N = n)P(N = n)$ and the rest of the calculation is similar to before to show Y has the Poisson distribution with parameter $\lambda(1-p)$.

To show independence, note that $P(X = x)P(Y = y) = \lambda^{x+y} e^{-\lambda} \frac{p^x(1-p)^y}{x!y!}$. Now

$$\begin{aligned} P(X = x, Y = y) &= P(X = x, N = x+y) \\ &= P(X = x|N = x+y)P(N = x+y) \\ &= \frac{(x+y)!}{x!y!} p^x q^{x+y-x} e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!} \end{aligned}$$

which simplifies to $\lambda^{x+y} e^{-\lambda} \frac{p^x q^y}{x!y!}$.

6. (Problem 2.4.3 in Pinsky and Karlin) We are given $X|\lambda \sim Po(\lambda)$ and $\lambda \sim Exp(\theta)$.

(a) Consider $P(X = x)$.

$$\begin{aligned}
P(X = x) &= \mathbb{E}[\mathbf{1}_{\{X=x\}}] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{X=x\}}|\lambda]] \\
&= \mathbb{E}[P(X = x|\lambda)] = \mathbb{E}[e^{-\lambda}\frac{\lambda^x}{x!}] \\
&= \int_0^\infty e^{-\lambda}\frac{\lambda^x}{x!}\theta e^{-\theta\lambda}d\lambda \\
&= \frac{\theta}{x!}\sum_0^\infty \lambda^x e^{-(1+\theta)}d\lambda.
\end{aligned}$$

Substitute $u = (1+\theta)\lambda$, hence ...$du = (1+\theta)\lambda$ and

$$P(X = x) = \frac{\theta}{x!}\frac{1}{(1+\theta)^{x+1}}\int_0^\infty e^{-u}u^x du = \frac{\theta}{x!}\frac{1}{(1+\theta)^{x+1}}\Gamma(x+1)$$

where $\Gamma(n) = \int_0^\infty e^{-u}u^{n-1}du$ is the Gamma function. For a positive integer n, $\Gamma(n) = (n-1)!$, hence

$$P(X = x) = \frac{\theta}{x!}\frac{1}{(1+\theta)^{x+1}}x! = \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^x$$

as required. Note this is the geometric distribution.

(b) Note the conditional density we want is given by

$$\frac{d}{d\lambda_1}\left[\frac{P(X = k \cap \lambda \geq \lambda_1)}{P(X = k)}\right].$$

We have

$$P(X = k \cap \lambda \leq \lambda_1) = \int_0^{\lambda_1} P(X = k|\lambda)\theta e^{-\theta\lambda}d\lambda = \int_0^{\lambda_1} e^{-\lambda}\frac{\lambda^k}{k!}\theta e^{-\theta\lambda}d\lambda.$$

Hence our conditional density is given by

$$\frac{d}{d\lambda_1}\left[\frac{\int_0^{\lambda_1} e^{-\lambda}\frac{\lambda^k}{k!}\theta e^{-\theta\lambda}d\lambda}{\frac{\theta}{1+\theta}(\frac{1}{1+\theta})^k}\right] = \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!}\theta e^{-\theta\lambda_1}}{\frac{\theta}{1+\theta}(\frac{1}{1+\theta})^k} = \frac{(1+\theta)^{k+1}\lambda_1^k e^{-(1+\theta)\lambda_1}}{k!}$$

as required. Note this is the gamma distribution.

7. (Problem 2.4.7 in Pinsky and Karlin) Find $f_{XZ}(x,z)$ first. Note $f_{XZ}(x,z) = f_{XZ}(z|x)f_X(x)$ and

$$\begin{aligned}
f_{XZ}(z|x) &= \frac{d}{dz}P(Z \leq z|X = x) = \frac{d}{dz}P(X + Y \leq z|X = x) = \frac{d}{dz}P(Y \leq z - x) \\
&= \frac{d}{dz}(1 - e^{-\alpha(z-x)}) = \alpha e^{-\alpha(z-x)} \text{ for } z \geq x \\
&\text{and } \frac{d}{dz}(0) = 0 \text{ for } z < x \\
\Rightarrow f_{XZ}(z|x) &= \alpha e^{-\alpha(z-x)}\mathbf{1}_{\{z\geq x\}}.
\end{aligned}$$

Then

$$f_{XZ} = \alpha e^{-\alpha(z-x)}\mathbf{1}_{\{z\geq x\}} \times \alpha e^{-\alpha x}\mathbf{1}_{\{x\geq 0\}} = \alpha^2 e^{-\alpha z}\mathbf{1}_{\{0\leq x\leq z\}}.$$

We find $f_Z(z)$:

$$f_Z(z) = \int f_{XZ}(x,z)dx = \int_0^z \alpha^2 e^{-\alpha z} dx = \alpha^2 z e^{-\alpha z} \mathbf{1}_{\{z \geq 0\}}$$

and since $f_{XZ}(x|z) = f_{XZ}(x,z)/f_Z(z)$, we have

$$f_{XZ}(x|z) = \frac{\alpha^2 e^{-\alpha z} \mathbf{1}_{\{0 \leq x \leq z\}}}{\alpha^2 z e^{-\alpha z} \mathbf{1}_{\{z \geq 0\}}} = \frac{1}{z} \mathbf{1}_{\{0 \leq x \leq z\}}$$

and we are done. Note this is the uniform distribution over $[0, z]$.

8. (Exercise 2.1.1 in Pinsky and Karlin) From the information given in the question, we know

$$P(N = n) = \frac{1}{6} \text{ for } n = 1, 2, \ldots, 6$$
$$P(X = x | N = n) = \binom{n}{x} (\frac{1}{2})^x (\frac{1}{2})^{n-x} \text{ for } 0 \leq x \leq n$$

therefore

$$P(N = 3 \cap X = 2) = P(X = 2 | N = 3) P(N = 3) = \frac{3}{8} \cdot \frac{1}{6} = \frac{1}{16}.$$

To find $P(X = 5)$ use the law of total probability

$$\begin{aligned} P(X = 5) &= \sum_{n=1}^{6} P(X = 5 | N = n) P(N = n) \\ &= 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + (\frac{1}{2})^5 \cdot \frac{1}{6} + 6 \cdot (\frac{1}{2})^6 \cdot \frac{1}{6} \\ &= \frac{1}{48}. \end{aligned}$$

To find $\mathbb{E}[X]$, use the law of total expectation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|N]] = \mathbb{E}[X \cdot \frac{1}{2}] = \frac{7}{4}.$$

9. (Problem 2.1.4 in Pinsky and Karlin) Again use the property of conditional expectation: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|N]]$, so $\mathbb{E}[X] = \mathbb{E}[Np] = Mpq = \frac{5}{2}$. Note this is precisely the example given on Slide 12 of the "Conditional Probabilities and Expectations" notes.

10. (Problem 2.2.2 in Pinsky and Karlin) The weighting of the dice changes the distribution of the total from them both: The probabilities have changed, but the game hasn't. The formula from the notes still applies

Table 1: Distribution of total from the weighted dice

Total	2	3	4	5	6	7
Probability	0.027778	0.055556	0.083334	0.111112	0.138489	0.167467
Total	8	9	10	11	12	
Probability	0.138489	0.111112	0.083334	0.055556	0.027778	

$$P(\text{Win}) = P(7) + P(11) + \sum_{k=4,5,6,8,9,10} \frac{P(k)^2}{P(k)+P(7)}.$$

After plugging in the numbers we find the probability of winning is around 49.24%.

11. Use the law of total probability:

$$P(X=k) = \int P(X=k|Q=q) f_Q(q) dq$$

where $f_Q(q)$ is the probability density function of the continuous random variable, Q. In this case, because Q is uniformly distributed on $[0,1]$, the $f_Q(q)$ is very simple - it equals one on $[0,1]$ and zero elsewhere. Hence

$$P(X=k) = \int_0^1 \frac{n!}{k!(n-k)!} q^k (1-q)^{(n-k)} \cdot 1 dq$$

which, using the hint, gives

$$P(X=k) = \frac{n!}{k!(n-k)!} \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}$$

so X is in fact uniformly distributed on the values $0, 1, \ldots, n$. You could use this marginal distribution to find $\mathbb{E}[X]$, but perhaps this way is quicker: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|P]] = \mathbb{E}[nP] = n \cdot \frac{1}{2}$.

12. (Exercise 2.1.5 in Pinsky and Karlin) We'll find the conditional expectation by finding the conditional p.m.f. and then calculating the expectation. By definition of conditional p.m.f.s

$$P(X=x|X \text{ is odd}) = \frac{P(X=x \text{ and is odd})}{P(X \text{ is odd})}$$

Now, the probability X is odd is simply the sum of the probabilities X takes the values 1, 3, 5 etc.

$$P(X \text{ is odd}) = e^{-\lambda}(\lambda + \lambda^3/3! + \lambda^5/5! + \cdots)$$

Recall that

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$

and so we can see that

$$P(X \text{ is odd}) = e^{-\lambda}(1/2)(e^{\lambda} - e^{-\lambda})$$

and using the fact that $P(X \text{ is odd}) + P(X \text{ is even}) = 1$ we see that

$$P(X \text{ is even}) = e^{-\lambda}(1/2)(e^{\lambda} + e^{-\lambda}).$$

Now, $P(X = x \text{ and is odd})$ is zero if x is even and is $e^{-\lambda}\lambda^x/x!$ if x is odd. Putting all together, we have that the conditional p.m.f. of X given X is odd is zero if $x = 2k$ for $k = 0, 1, 2, \ldots$ and for $x = 2k+1$, $k = 0, 1, 2, \ldots$ it is

$$\frac{e^{-\lambda}\lambda^x/x!}{e^{-\lambda}(1/2)(e^{\lambda} - e^{-\lambda})}$$

hence the expected value of X given X is odd is

$$\frac{\sum_{k=0}^{\infty}(2k+1)e^{-\lambda}\lambda^{2k+1}/(2k+1)!}{e^{-\lambda}(1/2)(e^{\lambda}-e^{-\lambda})}$$

which is

$$\frac{\lambda\sum_{k=0}^{\infty}e^{-\lambda}\lambda^{2k}/(2k)!}{e^{-\lambda}(1/2)(e^{\lambda}-e^{-\lambda})}$$

or

$$\frac{\lambda P(X \text{ is even})}{e^{-\lambda}(1/2)(e^{\lambda}-e^{-\lambda})}$$

which, finally, is equal to

$$\frac{\lambda(e^{\lambda}+e^{-\lambda})}{(e^{\lambda}-e^{-\lambda})}.$$

THE END