4-2 Probability Distributions and Probability Density Functions

Definition

For a continuous random variable X, a probability density function is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx =$ area under $f(x)$ from a to b

for any a and b **(4-1)**

4-2 Probability Distributions and Probability Density Functions

If X is a continuous random variable, for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

$$P(X = x) = 0$$

4-3 Cumulative Distribution Functions

Definition

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

$0 \leq F(x) \leq 1$

If $x \leq y$, then $F(x) \leq F(y)$

4-4 Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The mean or expected value of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad (4\text{-}4)$$

The variance of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The standard deviation of X is $\sigma = \sqrt{\sigma^2}$.

4-4 Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx \qquad (4\text{-}5)$$

1.4.3 The gamma distribution

$$f(x) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}, \qquad x \geq 0$$

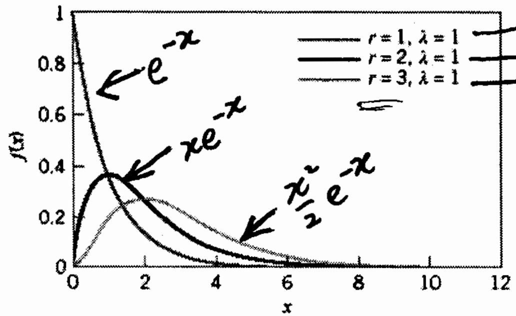
with shape parameter $r > 0$ and scale parameter $\lambda > 0$. The mean and variance of the gamma distribution are

$$\mu = \frac{r}{\lambda}, \qquad \sigma^2 = \frac{r}{\lambda^2}.$$

The gamma distribution can assume many different shapes, depending on the values chosen for r and λ. This makes it useful as a model for a wide variety of continuous random variables.

1. If $r = 1$, the gamma distribution reduces to the exponential distribution with parameter λ.
2. if r is an integer, $x_1, x_2, \cdots, x_r$ are exponential with parameter λ and independent, then $y = x_1 + x_2 + \cdots + x_r$ is distributed as gamma with parameter r and λ.

- When r is an integer, the gamma distribution is the result of summing r independently and identically exponential random variables each with parameter λ



$f(x) = e^{-x}$

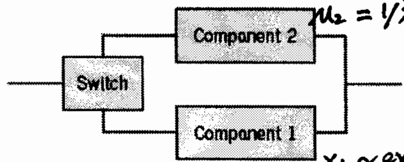
$f(x) = xe^{-x}$

$f(x) = \frac{x^2}{2}e^{-x}$

Figure 2-23 Gamma distributions for selected values or r and $\lambda = 1$.

$X_2 \sim exp(\lambda)$

$\mu_2 = 1/\lambda$



$X_1 + X_2 \sim Gamma(2, \lambda)$

$\mu = \frac{r}{\lambda} = \frac{2}{\lambda}$

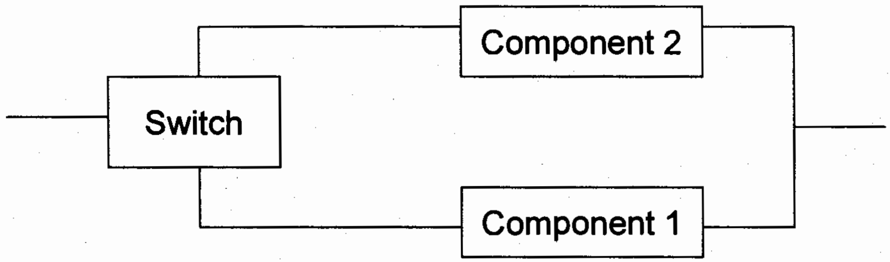
$X_1 \sim exp(\lambda)$

Figure 2-24 The standby redundant system

$\mu_1 = 1/\lambda$

- The gamma distribution has many applications in reliability engineering; see Example 2-121, text page 71

Example 1.7: Consider the system shown in the following figure:



This is called a *standby redundant system*, because while component 1 is on, component 2 is off, and when component 1 fails, the switch automatically turns component 2 on.

If each component has a life described by an exponential distribution with $\lambda = 10^{-4}$/h, then the system life is gamma distribution with parameters $r = 2$ and $\lambda = 10^{-4}$. Thus, the mean time to failure is $\mu = r / \lambda = 2/10^{-4} = 2 \times 10^{4}$h.

1.4.4 The Weibull distribution

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\theta}\right)^{\beta}\right], \qquad x \geq 0$$

where $\theta > 0$ is the scale parameter, and $\beta > 0$ is the shape parameter.

The mean and variance of the Weibull distribution are

$$\mu = \theta\Gamma\left(1+\frac{1}{\beta}\right), \quad \sigma^2 = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right].$$

The Weibull distribution is very flexible, and by appropriate selection of the parameters θ and β, the distribution can assume a wide variety of shapes.

- When $\beta = 1$, the Weibull distribution reduces to the exponential distribution

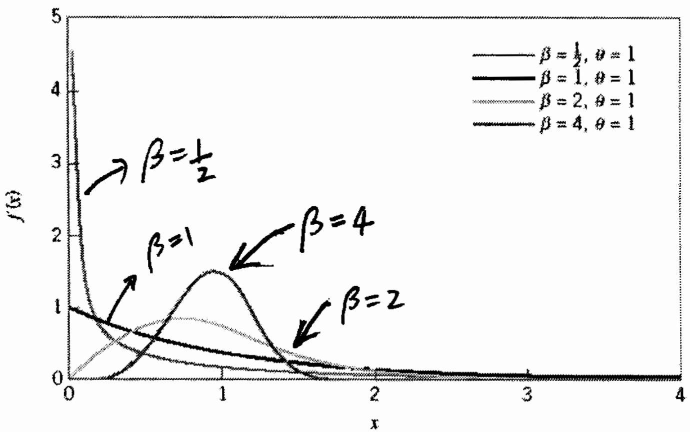


Figure 2-25 Weibull distributions for selected values of the shape parameter β and scale parameter $\theta = 1$.

If $\beta = 1$, the Weibull distribution reduces to the exponential distribution with mean $1/\theta$.

$$f(x) = \frac{1}{\theta} \exp\left[-\frac{1}{\theta} x\right]$$

The Weibull distribution has been used extensively in reliability engineering as a model of time to failure for electrical and mechanical components and systems.

The cumulative Weibull distribution is

$$F(a) = 1 - \exp\left[-\left(\frac{a}{\theta}\right)^{\beta}\right].$$

<u>Example 1.8</u>: The time to failure for an electronic subassembly used in RISC workstation is satisfactorily modeled by a Weibull distribution with $\beta = ½$ and $\theta = 1000$.

The mean time to failure is

$$\mu = \theta\Gamma\left(1+\frac{1}{\beta}\right) = 1000\Gamma\left(1+\frac{1}{1/2}\right)$$

$$= 1000\Gamma(3) = 2000h.$$

The fraction of subassemblies expected to survive a = 4000 h is

$$P\{x > a\} = 1 - F(a) = \exp\left[-\left(\frac{a}{\theta}\right)^{\beta}\right]$$

$$= \exp\left[-\left(\frac{4000}{1000}\right)^{1/2}\right] = e^{-2} = 0.1353.$$

That is, all but about 13.53% of the subassemblies will fail by 4000h.