# Solutions for Chapter 3 Exercises

1.i) The process is stationary if $|\phi_1| \neq 1$. The process is causal if $|\phi_1| < 1$.

ii) The stationary conditions are $|r_1| \neq 1$ and $|r_2| \neq 1$. The causal conditions are $|r_1| < 1$ and $|r_2| < 1$.

iii) Note that the assumed causality implies that the characteristic polynomial $\phi(x) = 1 - \phi_1 x - \phi_2 x^2$ has no root inside the unit circle. Since $\phi(0) = 1 - 0 - 0 = 1 > 0$, we must have $\phi(x) > 0$ for $|x| \leq 1$. In particular, we have $\phi(1) = 1 - \phi_1 - \phi_2 > 0$ and $\phi(-1) = 1 + \phi_1 - \phi_2 > 0$, giving the first two inequalities. Finally, as $|r_1| < 1$ and $|r_2| < 1$, we have $|\phi_2| = |r_1 r_2| < 1$.

2. First, by definition of white noise, $\text{Cov}(Z_k, a_t) = 0$ for $k < t$. Therefore,

$$\text{Cov}(Z_t, a_t) = \text{Cov}(\alpha Z_{t-1} + \beta Z_{t-2} + a_t, a_t) = \text{Cov}(a_t, a_t) = \sigma_a^2 .$$

Multiplying $Z_t$ on both sides and take expectation, and using $\text{Cov}(Z_t, a_t) = \sigma_a^2$, we have

$$\gamma(0) = \alpha\gamma(1) + \beta\gamma(2) + \sigma_a^2 .$$

Dividing by $\text{Var}(Z_t) = \gamma(0)$ on both sides gives

$$1 = \alpha\rho_1 + \beta\rho_2 + \frac{1}{2} ,$$

i.e., $\alpha\rho_1 + \beta\rho_2 = 0.5$.

3.(a) ARMA(1,2) model (given that the characteristic polynomials have no common roots.

(b) (i) The process $\{Z_t\}$ is stationary if $|\theta| \neq 1$. However, if $|\theta| = 1$, $Z_t$ cannot be represented by $a_t$s with decaying coefficients. Thus $Z_t$ is not stationary.

(ii) The process $\{Z_t\}$ is causal if $\phi(x) \neq 0$ for $|x| \leq 1$, i.e., $|\theta| < 1$.

(iii) The process $\{Z_t\}$ is invertible if $\theta(x) \neq 0$ for $|x| \leq 1$.

(c) (i) For $|\theta| < 1$, the stationary solution is given by

$$\begin{aligned}
Z_t &= \theta Z_{t-1} + a_t + \alpha a_{t-1} + \beta a_{t-2} \\
&= (1-\theta B)^{-1}(1+\alpha B + \beta B^2)a_t \\
&= \left(\sum_{j=0}^{\infty} \theta^j B^j\right)(1+\alpha B + \beta B^2)a_t \\
&= \left(\sum_{j=0}^{\infty} \theta^j B^j + \sum_{j=0}^{\infty} \alpha\theta^j B^{j+1} + \sum_{j=0}^{\infty} \beta\theta^j B^{j+2}\right) a_t \\
&= \left(1 + (\theta+\alpha)B + \sum_{j=2}^{\infty}(\theta^2+\alpha\theta+\beta)\theta^{j-2}B^j\right) a_t \\
&= a_t + (\theta+\alpha)a_{t-1} + \sum_{j=2}^{\infty}(\theta^2+\alpha\theta+\beta)\theta^{j-2}a_{t-j}\,.
\end{aligned}$$

(ii) Using the first principle, we have

$$\begin{aligned}
Z_t &= \theta Z_{t-1} + a_t + \alpha a_{t-1} + \beta a_{t-2} \\
&= a_t + (\alpha+\theta)a_{t-1} + (\beta+\alpha\theta)a_{t-2} + \beta\theta a_{t-3} + \theta^2 Z_{t-2} \\
&= a_t + (\alpha+\theta)a_{t-1} + (\beta+\alpha\theta+\theta^2)a_{t-2} + (\alpha\theta^2+\beta\theta)a_{t-3} + \beta\theta^2 a_{t-4} + \theta^3 Z_{t-3} \\
&= a_t + (\alpha+\theta)a_{t-1} + (\beta+\alpha\theta+\theta^2)a_{t-2} + (\beta+\alpha\theta+\theta^2)\theta a_{t-3} + (\beta\theta^2+\alpha\theta^3)a_{t-4} \\
&\quad +\beta\theta^3 a_{t-5} + \theta^4 Z_{t-4} \\
&= \cdots \\
&= a_t + (\theta+\alpha)a_{t-1} + \sum_{i=2}^{\infty}(\theta^2+\alpha\theta+\beta)\theta^{i-2}\beta a_{t-i}\,.
\end{aligned}$$

(d) For $|\theta| > 1$, the stationary solution is given by

$$
\begin{aligned}
Z_t &= \theta Z_{t-1} + a_t + \alpha a_{t-1} + \beta a_{t-2} \\
\Rightarrow -\left(1 - \frac{1}{\theta}B^{-1}\right) Z_{t-1} &= \frac{1}{\theta}(1 + \alpha B + \beta B^2) a_t \\
\Rightarrow Z_{t-1} &= -\frac{1}{\theta}\left(\sum_{j=0}^{\infty} \theta^{-j} B^{-j}\right)(1 + \alpha B + \beta B^2) a_t \\
&= -\frac{1}{\theta}\left(\sum_{j=0}^{\infty} \theta^{-j} B^{-j} + \sum_{j=0}^{\infty} \alpha \theta^{-j} B^{1-j} + \sum_{j=0}^{\infty} \beta \theta^{-j} B^{2-j}\right) a_t \\
&= -\left(\frac{\beta B^2}{\theta} + \frac{\beta + \theta\alpha}{\theta^2} B + \sum_{j=0}^{\infty} \frac{\beta + \alpha\theta + \theta^2}{\theta^3} \theta^{-j} B^{-j}\right) a_t \\
\Rightarrow Z_t &= -\left(\frac{\beta B^2}{\theta} + \frac{\beta + \theta\alpha}{\theta^2} B + \sum_{j=0}^{\infty} \frac{\beta + \alpha\theta + \theta^2}{\theta^3} \theta^{-j} B^{-j}\right) a_{t+1} \\
&= -\frac{\beta}{\theta} a_{t-1} - \frac{\beta + \theta\alpha}{\theta^2} a_t - \sum_{j=0}^{\infty} \frac{\beta + \alpha\theta + \theta^2}{\theta^3} \theta^{-j} a_{t+1+j} \,.
\end{aligned}
$$

(e) From the result $Z_t = a_t + (\theta + \alpha) a_{t-1} + \sum_{j=2}^{\infty} (\theta^2 + \alpha\theta + \beta)\theta^{j-2} a_{t-j}$ in (a), the zero autocovariance of white noise implies that

$$
\begin{aligned}
\mathrm{Var}(Z_t) &= \left(1 + (\theta + \alpha)^2 + (\theta^2 + \alpha\theta + \beta)^2 + \theta^2(\theta^2 + \alpha\theta + \beta)^2 + \cdots\right) \sigma_a^2 \\
&= \sigma_a^2 + (\theta + \alpha)^2 \sigma_a^2 + \frac{(\theta^2 + \alpha\theta + \beta)^2}{1 - \theta^2} \sigma_a^2 \,.
\end{aligned}
$$

The autocovariance function is given by

$$
\gamma(k) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|k|}
$$

where

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \theta + \alpha \\
\psi_j &= \theta^{j-2}(\theta^2 + \alpha\theta + \beta) \,, j \geq 2
\end{aligned}
$$

In particular,

$$\begin{aligned}\gamma(1) &= \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+1} = \sigma_a^2 \left( \psi_0 \psi_1 + \psi_1 \psi_2 + \sum_{j=2}^{\infty} \psi_j \psi_{j+1} \right) \\ &= \sigma_a^2 \left( (\theta + \alpha)(\theta^2 + \alpha\theta + \beta) + \frac{\theta(\theta^2 + \alpha\theta + \beta)^2}{1 - \theta^2} \right),\end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned}\gamma(k) &= \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k} = \sigma_a^2 \left( \psi_0 \psi_k + \psi_1 \psi_{k+1} + \sum_{j=2}^{\infty} \psi_j \psi_{j+k} \right) \\ &= \sigma_a^2 \left( (\theta^2 + \alpha\theta + 1)\theta^{k-2}(\theta^2 + \alpha\theta + \beta) + \frac{\theta^k(\theta^2 + \alpha\theta + \beta)^2}{1 - \theta^2} \right).\end{aligned}$$

The autocorrelation function can be obtained by $\rho_k = \gamma(k)/\gamma(0)$. Alternatively, Yule-Walker equations can be used to obtain the same answer.

4. From the assumed stationarity, we have $E(Z_t) = E(Z_{t-1})$, thus $E(Z_t) = \mu + \phi E(Z_{t-1}) = \mu + \phi E(Z_t)$, giving

$$E(Z_t) = \frac{\mu}{1 - \phi}.$$

Taking variance on both sides of $Z_t = \mu + \phi Z_{t-1} + a_t$, and using the stationarity assumption $\mathrm{Var}(Z_t) = \mathrm{Var}(Z_{t-1})$, we have

$$\begin{aligned}\mathrm{Var}(Z_t) &= \mathrm{Var}(\mu + \phi Z_{t-1} + a_t) = \phi^2 \mathrm{Var}(Z_{t-1}) + \sigma_a^2 \\ \Rightarrow \qquad \mathrm{Var}(Z_t) &= \gamma(0) = \frac{\sigma_a^2}{1 - \phi^2}.\end{aligned}$$

Considering the covariance of $Z_{t-k}$ $(k = 1, 2, \ldots)$ and each of the both sides of $Z_t = \mu + \phi Z_{t-1} + a_t$, we have the autocovariance function $\gamma(k)$:

- $\gamma(1) = \mathrm{Cov}(Z_t, Z_{t-1}) = \mathrm{Cov}(\mu + \phi Z_{t-1} + a_t, Z_{t-1}) = \phi\gamma(0) = \frac{\phi\sigma_a^2}{1-\phi^2}$.
- For $k \geq 2$, $\gamma(k) = \mathrm{Cov}(Z_t, Z_{t-k}) = \mathrm{Cov}(\mu + \phi Z_{t-1} + a_t, Z_{t-k})$ $= \phi\gamma(k-1) = \phi^2\gamma(k-2) = \cdots = \phi^k\gamma(0) = \frac{\phi^k\sigma_a^2}{1-\phi^2}$.

5. Let $B$ be the back-shift operator satisfying $BZ_t = Z_{t-1}$, the process can be written as

$$\begin{aligned}
(1-0.5B+0.06B^2)Z_t &= a_t \\
\Rightarrow (1-0.2B)(1-0.3B)Z_t &= a_t \\
\Rightarrow \qquad\qquad\qquad Z_t &= (1-0.2B)^{-1}(1-0.3B)^{-1}a_t \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 0.2^j 0.3^k B^{j+k} a_t \\
&= \sum_{m=0}^{\infty}\sum_{j,k\geq 0, j+k=m} \left(0.2^j 0.3^k\right) B^m a_t \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{m} \left(0.2^j 0.3^{m-j}\right) B^m a_t \\
&= \sum_{m=0}^{\infty} 0.3^m \frac{1-\left(\frac{0.2}{0.3}\right)^{m+1}}{1-\frac{0.2}{0.3}} a_{t-m}\,.
\end{aligned}$$

That is, $\psi_j = 0.3^j \frac{1-\left(\frac{0.2}{0.3}\right)^{j+1}}{1-\frac{0.2}{0.3}}$.

6. Using $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{j=0}^{\infty}(-x)^j$, we have the AR representation

$$\begin{aligned}
(1+0.6B)Z_t &= (1+0.5B)a_t \\
\Rightarrow \qquad a_t &= (1+0.5B)^{-1}(1+0.6B)Z_t \\
&= \sum_{j=0}^{\infty}(-0.5)^j B^j (1+0.6B)Z_t \\
&= \left(\sum_{j=0}^{\infty}(-0.5)^j B^j + 0.6\sum_{j=0}^{\infty}(-0.5)^j B^{j+1})\right) Z_t \\
&= \left(\sum_{j=0}^{\infty}(-0.5)^j B^j + 0.6\sum_{j=1}^{\infty}(-0.5)^{j-1} B^{j})\right) Z_t \\
&= \left(1 + 0.1\sum_{j=1}^{\infty}(-0.5)^{j-1} B^j\right) Z_t \\
&= Z_t + 0.1\sum_{j=1}^{\infty}(-0.5)^{j-1} Z_{t-j}\,.
\end{aligned}$$

Similarly, we have the MA representation

$$\begin{aligned}
(1+0.6B)Z_t &= (1+0.5B)a_t \\
\Rightarrow \qquad Z_t &= (1+0.6B)^{-1}(1+0.5B)a_t \\
&= \sum_{j=0}^{\infty}(-0.6)^j B^j (1+0.5B)a_t \\
&= \left(\sum_{j=0}^{\infty}(-0.6)^j B^j + 0.5\sum_{j=0}^{\infty}(-0.6)^j B^{j+1})\right) a_t \\
&= \left(\sum_{j=0}^{\infty}(-0.6)^j B^j + 0.5\sum_{j=1}^{\infty}(-0.6)^{j-1} B^{j})\right) a_t \\
&= \left(1 - 0.1\sum_{j=1}^{\infty}(-0.6)^{j-1} B^j\right) a_t \\
&= a_t - 0.1\sum_{j=1}^{\infty}(-0.6)^{j-1} a_{t-j}\,.
\end{aligned}$$

7.(a) First, note that $\mathrm{Cov}(Z_k, a_t) = 0$ for $k < t$. Thus

$$
\begin{aligned}
\mathrm{Cov}(Z_t, a_t) &= \mathrm{Cov}(0.6Z_{t-1} + a_t - 0.2a_{t-1}, a_t) \\
&= 0.6\mathrm{Cov}(Z_{t-1}, a_t) + \mathrm{Cov}(a_t, a_t) - 0.2\mathrm{Cov}(a_{t-1}, a_t) \\
&= \mathrm{Cov}(a_t, a_t) = 4\,, \qquad (1) \\
\mathrm{Cov}(Z_t, a_{t-1}) &= \mathrm{Cov}(0.6Z_{t-1} + a_t - 0.2a_{t-1}, a_{t-1}) \\
&= 0.6\mathrm{Cov}(Z_{t-1}, a_{t-1}) + \mathrm{Cov}(a_t, a_{t-1}) - 0.2\mathrm{Cov}(a_{t-1}, a_{t-1}) \\
&= 0.6 \times 4 + 0 - 0.2 \times 4 = 1.6\,. \qquad (2)
\end{aligned}
$$

Multiply both sides by $Z_t$, $Z_{t-1}$ and take expectation respectively, we have from (1) and (2) that

$$
\begin{aligned}
\gamma(0) &= 0.6\gamma(1) + \mathrm{Cov}(Z_t, a_t) - 0.2\mathrm{Cov}(Z_t, a_{t-1}) = 0.6\gamma(1) + 3.68 \\
\gamma(1) &= 0.6\gamma(0) + \mathrm{Cov}(Z_{t-1}, a_t) - 0.2\mathrm{Cov}(Z_{t-1}, a_{t-1}) = 0.6\gamma(0) - 0.8\,.
\end{aligned}
$$

Substituting the second equation to the first one we get $\gamma(0) = 0.6^2\gamma(0) - 0.48 + 3.68$, giving $\gamma(0) = 3.2/0.64 = 5$ and thus $\gamma(1) = 2.2$.

For $k \geq 2$, multiply both sides by $Z_{t-k}$ and take expectation gives

$$
\begin{aligned}
\gamma(k) &= 0.6\gamma(k-1) + \mathrm{Cov}(Z_{t-k}, a_t) - 0.2\mathrm{Cov}(Z_{t-k}, a_{t-1}) \\
&= 0.6\gamma(k-1) = \cdots = 0.6^{k-1}\gamma(1) = 2.2(0.6)^{k-1}\,.
\end{aligned}
$$

In summary, the ACVF is

$$
\gamma(k) = \begin{cases} 5 & k = 0\,, \\ 2.2 & k = \pm 1\,, \\ 2.2(0.6)^{|k|-1} & |k| \geq 2\,. \end{cases}
$$

The ACF is $\rho(k) = \gamma(k)/\gamma(0)$, which is

$$
\rho(k) = \begin{cases} 1 & k = 0\,, \\ 0.44 & k = \pm 1\,, \\ 0.44(0.6)^{|k|-1} & |k| \geq 2\,. \end{cases}
$$

(b) By directly counting the number terms of autocovaraince with different lags, we have

$$
\begin{aligned}
\mathrm{Var}(\sum_{t=1}^{4} Z_t) &= 4\gamma(0) + 6\gamma(1) + 4\gamma(2) + 2\gamma(3) \qquad \text{(16 terms in total)} \\
&= 4(5) + 6(2.2) + 4(2.2)0.6 + 2(2.2)0.6^2 = 40.064\,.
\end{aligned}
$$

8.

i) ARIMA(0,1,1), no stationary solution, not causal, not invertible.

ii) $(1 - 1.7B + 0.72B^2) = (1 - 0.9B)(1 - 0.8B)$;
ARIMA(1,0,2) or ARMA(1,2), stationary solution exists, not causal but invertible.

iii) $(1 - 1.2B + 0.2B^2) = (1 - B)(1 - 0.2B)$;
ARIMA(1,0,2) or ARMA(1,2), stationary solution exists, causal, not invertible.

iv) $(1 - 0.5B - 0.5B^2) = (1 - B)(1 + 0.5B)$;
$(1 - 1.2B + 0.2B^2) = (1 - B)(1 - 0.2B)$. Note that the $(1 - B)$ on both sides can be canceled, resulting in the process

$$(1 + 0.5B)Z_t = (1 - 0.2B)a_t\,.$$

ARIMA(1,0,1) or ARMA(1,1), stationary solution exists, causal and invertible.

v) $(1 - 0.4B - 0.45B^2) = (1 - 0.9B)(1 + 0.5B)$;
$(1 + B + 0.25B^2) = (1 + 0.5B)(1 + 0.5B)$; Note that the $(1 + 0.5B)$ on both sides can be canceled, resulting in the process

$$(1 - 0.9B)Z_t = (1 + 0.5B)a_t\,.$$

ARIMA(1,0,1) or ARMA(1,1), stationary solution exists, causal and invertible.

vi) $(1 - 1.25B + 0.25B^2) = (1 - B)(1 - 0.25B)$;
ARIMA(1,1,0), no stationary solution, not causal but invertible.

9. First note that $\mathrm{Cov}(Z_k, a_t) = 0$ for $k < t$. Now, multiply both sides by $Z_{t-k}$, $k = 1, 2, 3$, then take expectation give

$$\begin{aligned}
\gamma(1) &= 0.5\gamma(0) - 0.06\gamma(1) \\
\gamma(2) &= 0.5\gamma(1) - 0.06\gamma(0) \\
\cdots &= \cdots \\
\gamma(k) &= 0.5\gamma(k-1) - 0.06\gamma(k-2)\,.
\end{aligned}$$

Diving by $\gamma(0)$ on both sides gives

$$\begin{aligned}
\rho(1) &= 0.5 - 0.06\rho(1) \\
\rho(2) &= 0.5\rho(1) - 0.06 \\
\cdots &= \cdots \\
\rho(k) &= 0.5\rho(k-1) - 0.06\rho(k-2)\,.
\end{aligned}$$

Solving the first equation gives $\rho(1) = 0.5/1.06 = 0.472$. Generally, $\rho(k)$, with $k \geq 2$, can be computed recursively using $\rho(0) = 1, \rho(1) = 0.472$ and

$$\rho(k) = 0.5\rho(k-1) - 0.06\rho(k-2)\,.$$

10.(a) From the definition of $Z_t$ we have

$$\mathrm{Var}(Z_t) = [1 + C^2(1 + 1 + 1 + \cdots)]\sigma_a^2 = \infty\,,$$

so $Z_t$ is not weakly stationary.
(b) Since $W_t = Z_t - Z_{t-1} = a_t + (C-1)a_{t-1}$, we have $E(W_t) = 0$ and autocovariance function $\gamma(k) = Cov(W_t, W_{t-k})$ satisfying

$$\begin{aligned}
\gamma(0) &= (1 + (C-1)^2)\sigma_a^2 \\
\gamma(1) &= (C-1)\sigma_a^2 \\
\gamma(k) &= 0 \quad \text{for} \quad |k| > 1\,.
\end{aligned}$$

Thus, $\{W_t\}$ is a stationary MA(1) process.
(c) From the above autocovariance function, we get the autocorrelation function

$$\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{C-1}{1 + (C-1)^2} \\
\rho(k) &= 0 \quad \text{for} \quad |k| > 1\,.
\end{aligned}$$