THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Markov Chains - Introduction - Exercises Solutions

1. (Problem 3.2.3 in Pinsky and Karlin) The first item appears at time zero. The fourth item appears at time 3. So we want to find

$$P(X_3 = \text{defecive}|X_0 = \text{defective}) := P(X_3 = 1|X_0 = 1) = p_{11}^{(3)}.$$

This by Chapman-Kolmogorov we need to find the $(1,1)$ entry in $\mathbb{P}^3$:

$$\mathbb{P}^3 = \begin{pmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{pmatrix}$$

so $P(X_3 = 1|X_0 = 1) = P_{11}^3 = 0.6848$.

2. (Exercise 3.3.1 in Pinsky and Karlin) This is the inventory model from Slides 13 to 20 of the Markov Chains - Introduction notes. Recall the (s, S) policy meant the chain $\{X_n\}$ represented the quantity of hand at the end of period n just prior to re-stocking and

$$\begin{aligned} X_{n+1} &= X_n - \xi_{n+1} \text{ if } s < X_n \leq S \\ &= S - \xi_{n+1} \text{ if } X_n \leq s \end{aligned}$$

where ξ_{n+1} is the demand on day $n+1$. The state space is $\{-1, 0, 1, 2, 3\}$, as we could start the day with 1 units and have 2 units demanded (meaning State -1 is possible). We work out the transition probabilities one-by-one. Say we are in State -1 now. Then tomorrow morning $S = 3$ units will arrive and by the end of the day we will have 3 left if $\xi_1 = 0$ (w.p 0.4); 2 left if $\xi_1 = 1$ (w.p. 0.3); 1 left if $\xi_1 = 2$ (w.p. 0.3). Say we are in State 0 now. Then tomorrow morning $S = 3$ units will arrive and by the end of the day we will have 3 left if $\xi_1 = 0$ (w.p 0.4); 2 left if $\xi_1 = 1$ (w.p. 0.3); 1 left if $\xi_1 = 2$ (w.p. 0.3). Say we are in State 1 now... etc. etc. The transition probability matrix is therefore

$$\begin{pmatrix} 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \end{pmatrix}$$

3. (Exercise 3.3.2 in Pinsky and Karlin) Let the chain model how many balls are in Urn A. Thus the state space will be $\{0, 1, \ldots, N\}$. Say at time n, Urn A has i balls in it. On the next draw, there are four possible outcomes:

- Choose a ball from Urn A, ball is placed in Urn A, w.p. $(i/N) * p$ and $X_{n+1} = i$;

- Choose a ball from Urn A, ball is placed in Urn B, w.p. $(i/N) * q$ and $X_{n+1} = i - 1$;
- Choose a ball from Urn B, ball is placed in Urn A, w.p. $(1 - i/N) * p$ and $X_{n+1} = i + 1$;
- Choose a ball from Urn B, ball is placed in Urn B, w.p. $(1 - i/N) * q$ and $X_{n+1} = i$.

Thus the transition probability matrix will have entries $p_{ii} = (i/N)p + (1 - i/N)q$; $p_{i,i+1} = (1 - i/N)p$; $p_{i,i-1} = (i/N)q$ for $i = 0, 1, \ldots, N$ and zero everywhere else.

4. (Exercise 3.1.4 in Pinsky and Karlin) Use the definition of conditional probability and the probability of a path :

$$\begin{aligned} P(X_1 = 1, X_2 = 1 | X_0 = 0) &= \frac{P(X_0 = 0, X_1 = 1, X_2 = 1)}{P(X_0 = 0)} \\ &= \frac{q_0 p_{01} p_{11}}{q_0} = 0.1 \times 0.2 = 0.02 \end{aligned}$$

The argument for the second term is extremely similar. Use the law of total probability

$$P(X_1 = 0, X_2 = 1, X_3 = 1) = \sum_{i=0}^{2} P(X_1 = 0, X_2 = 1, X_3 = 1 | X_0 = i) P(X_0 = i)$$

hence $Pr(X_2 = 1, X_3 = 1 | X_1 = 0) = 0.02$.

5. (Exercise 3.1.5 in Pinsky and Karlin) This involves the direct use of the probability of a path, as given on Slide 8 of the Markov Chains - Introduction notes. For example, $P(X_0 = 1, X_1 = 1, X_2 = 0) = q_0 p_{01} p_{10}$. Hence $P(X_0 = 1, X_1 = 1, X_2 = 0) = 0.025$ and $P(X_1 = 1, X_2 = 1, X_3 = 0) = 0.0075$.

6. (Problem 3.2.2 in Pinsky and Karlin) We are given

$$\mathbb{P} = \begin{pmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{pmatrix}$$

We want to find $\mathbb{P}_{00}^{(5)} = \mathbb{P}_{00}^5$. We follow the hint and prove by induction that

$$\mathbb{P}^n = \frac{1}{2} \begin{pmatrix} 1 + (1 - 2\alpha)^n & 1 - (1 - 2\alpha)^n \\ 1 - (1 - 2\alpha)^n & 1 + (1 - 2\alpha)^n \end{pmatrix}.$$

Clearly the statement holds for $n = 1$. Assume it is true for $n = k$. We need to prove it for $n = k + 1$. We show it holds for the $(1, 1)$th entry of $\mathbb{P}^{k+1}$ and leave the (very similar) calculations to the reader. Now $\mathbb{P}^{k+1} = \mathbb{P} \cdot \mathbb{P}^k$ and the $(1, 1)$th entry of $\mathbb{P}^{k+1}$ is given by

$$\frac{1}{2}[(1 - \alpha)(1 + (1 - 2\alpha)^k) + \alpha - \alpha(1 - 2\alpha)^k]$$

which equals $\frac{1}{2}[(1 - 2\alpha)^k(1 - \alpha - \alpha) + 1 - \alpha + \alpha]$ which simplifies to $\frac{1}{2}(1 - 2\alpha)^{k+1}$. The other entries hold similarly and the statement holds for $n = k + 1$ and thus true for all $n = 1, 2, \ldots,$.

Consequently, we find $\mathbb{P}_{00}^5 = 1/2(1 + (1 - 2\alpha)^5)$.

7. (Exercise 3.2.2 in Pinsky and Karlin) We could calculate $\mathbb{P}^k$ for $k = 1, 2, 3, 4$ and then just read offthe $(0, 0)$ entry. Doing so yields: $Pr(X_0 = 0|X_0 = 0) = 1$, $Pr(X_1 = 0|X_0 = 0) = 0$, $Pr(X_2 = 0|X_0 = 0) = 1/2$, $Pr(X_3 = 0|X_0 = 0) = 1/4$, $Pr(X_4 = 0|X_0 = 0) = 3/8$.

8. (From Slide 8 of the "Markov Chains - Introduction" notes) We begin with

$$Pr(X_{n+1} = j_1, \ldots, X_{n+m} = j_m|X_0 = i_0, \ldots, X_n = i_n)$$

$$\begin{aligned}
&= \frac{Pr(X_{n+1} = j_1, \ldots, X_{n+m} = j_m)}{Pr(X_0 = i_0, \ldots, X_n = i_n)} \text{ [Defn. of cond. prob.]} \\
&= \frac{q_{i_0} p_{i_0,i_1} \cdots p_{i_{n-1},i_n} p_{i_n,j_1} \cdots p_{j_{m-1},j_m}}{q_{i_0} p_{i_0,i_1} \cdots p_{i_{n-1},i_n}} \text{ [Slide 9]} \\
&= q_{i_0} p_{i_0,i_1} \cdots p_{i_{n-1},i_n} p_{i_n,j_1} \\
&= Pr(X_{n+1} = j_1|X_n = i_n) Pr(X_{n+2} = j_2|X_{n+1} = j_1) \cdots \\
&\cdots Pr(X_{n+m} = j_m|X_{n+m-1} = j_{m-1}) \text{ [Defn. of trans. prob.]} \\
&= Pr(X_{n+1} = j_1|X_n = i_n) Pr(X_{n+2} = j_2|X_{n+1} = j_1, X_n = i_n) \cdots \\
&\cdots Pr(X_{n+m} = j_m|X_{n+m-1} = j_{m-1}, X_{n+m-2} = j_{m-2}, \ldots, X_n = i_n) \\
&\times \frac{Pr(X_n = i_n)}{Pr(X_n = i_n)} \text{ [Markov property].}
\end{aligned}$$

Let's use some notation to clarify what this last line means. Let the event A_k be the event $\{X_{n+k} = j_k\}$ for $k = 1, \ldots, m$ and A_0 be the event $\{X_n = i_n\}$. Then last line maybe written as

$$\begin{aligned}
&= Pr(A_0) Pr(A_1|A_0) Pr(A_2|A_0 \cap A_1) Pr(A_3|A_0 \cap A_1 \cap A_2) \cdots \\
&\cdots Pr(A_m|A_{m-1} \cap A_{m-2} \cap \cdots \cap A_0) \frac{1}{Pr(A_0)} \\
&= \frac{Pr(A_0 \cap A_1 \cdots \cap A_m)}{Pr(A_0)} \\
&= Pr(A_1 \cap A_2 \cap \cdots \cap A_m|A_0)
\end{aligned}$$

Since the RHS of the last line is

$$Pr(X_{n+1} = j_1, \ldots, X_{n+m} = j_m|X_n = i_n)$$

we are done.

THE END