STAT 6104 Exercise 7 solution

1) a) Taking expectation on both sides of the ARCH(1) process, we have

$$\mathrm{E}(\sigma_t^2) = \alpha_0 + \alpha_1 \mathrm{E}(\epsilon_{t-1}^2)\mathrm{E}(\sigma_{t-1}^2) = \alpha_0 + \alpha_1 \mathrm{E}(\sigma_t^2)\,,$$

since $\mathrm{E}(\epsilon_{t-1}^2) = 1$ and $\mathrm{E}(\sigma_{t-1}^2) = \mathrm{E}(\sigma_t^2)$. Thus $\mathrm{E}(\sigma_t^2) = \alpha_0/(1-\alpha_1)$. Now

$$\begin{aligned}
\mathrm{E}(\sigma_t^4) &= \mathrm{E}(\alpha_0^2 + 2\alpha_0\alpha_1\sigma_{t-1}^2\epsilon_{t-1}^2 + \alpha_1^2\sigma_{t-1}^4\epsilon_{t-1}^4) \\
&= \alpha_0^2 + 2\alpha_0\alpha_1\mathrm{E}(\sigma_{t-1}^2)\mathrm{E}(\epsilon_{t-1}^2) + \alpha_1^2\mathrm{E}(\sigma_{t-1}^4)\mathrm{E}(\epsilon_{t-1}^4)) \\
&= \alpha_0^2 + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1} + 3\alpha_1^2\mathrm{E}(\sigma_t^4))\,,
\end{aligned}$$

since $\mathrm{E}(\epsilon_{t-1}^4) = 3$ and $\mathrm{E}(\sigma_{t-1}^4) = \mathrm{E}(\sigma_t^4)$. Solving gives

$$\mathrm{E}(\sigma_t^4) = \frac{\alpha_0^2}{1-\alpha_1}\frac{1+\alpha_1}{1-3\alpha_1^2}$$

b)

$$\begin{aligned}
\mathrm{E}(X_t^4) &= \mathrm{E}(\sigma_t^4)\mathrm{E}(\epsilon_t^4) = 3\mathrm{E}(\sigma_t^4) \\
&= 3\frac{\alpha_0^2}{1-\alpha_1}\frac{1+\alpha_1}{1-3\alpha_1^2}
\end{aligned}$$

2)

$$\begin{aligned}
\mathrm{E}(\sigma_{t+j}^2|\mathcal{F}_{t-1}) &= \mathrm{E}[\mathrm{E}(\sigma_{t+j}^2|\mathcal{F}_{t+j-2})|\mathcal{F}_{t-1}] = \alpha_0 + \mathrm{E}(\sigma_{t+j-1}^2|\mathcal{F}_{t-1}) \\
&= 2\alpha_0 + \mathrm{E}(\sigma_{t+j-2}^2|\mathcal{F}_{t-1}) = \dots \\
&= j\alpha_0 + \mathrm{E}(\sigma_t^2|\mathcal{F}_{t-1}) \\
&= j\alpha_0 + \sigma_t^2\,,
\end{aligned}$$

since σ_t^2 is $\mathcal{F}_{t-1}$ measurable.

3) Let X_t be a GARCH(2,3) model satisfying

$$\sigma_t^2 = \alpha_0 + \textstyle\sum_{i=1}^2 \beta_i\sigma_{t-i}^2 + \sum_{j=1}^3 \alpha_j X_{t-j}^2.$$

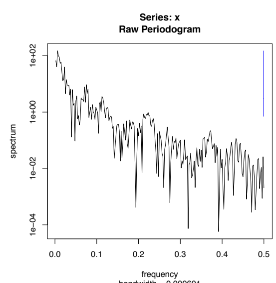
Then,

$$
\begin{aligned}
X_t^2 &= \sigma_t^2 + (X_t^2 - \sigma_t^2) \\
&= \alpha_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \alpha_3 X_{t-3}^2 + \\
&\quad X_t^2 - \sigma_t^2 \\
&= \alpha_0 + (\alpha_1 + \beta_1) X_{t-1}^2 + (\alpha_2 + \beta_2) X_{t-2}^2 + \alpha_3 X_{t-3}^2 - \\
&\quad \beta_1 (X_{t-1}^2 - \sigma_{t-1}^2) - \beta_2 (X_{t-2}^2 - \sigma_{t-2}^2) + \\
&\quad X_t^2 - \sigma_t^2 \\
&= \alpha_0 + (\alpha_1 + \beta_1) X_{t-1}^2 + (\alpha_2 + \beta_2) X_{t-2}^2 + \alpha_3 X_{t-3}^2 - \beta_1 \nu_{t-1} \\
&\quad -\beta_2 \nu_{t-2} + \nu_t,
\end{aligned}
$$

where $\nu_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\epsilon_t^2 - 1)$.
Now, we can identify the equation of an ARMA(3,2) model with noise ν_t.

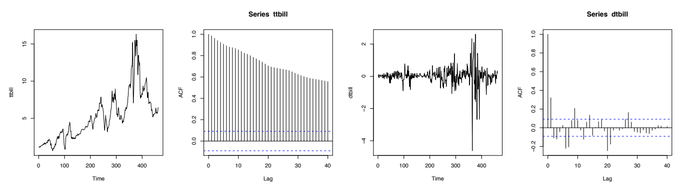
4) a) If we look at the periodogram of the 3-month Treasury Bill data we notice that the peak is for frequency $\omega = 0$. This suggests that there is probably no periodical effect in the data.

```
 ttbill=read.table(‘‘D://ustbill.dat")
ttbill=ttbill[,-1]
ttbill=as.vector(t(ttbill))
spectrum(ttbill)
```



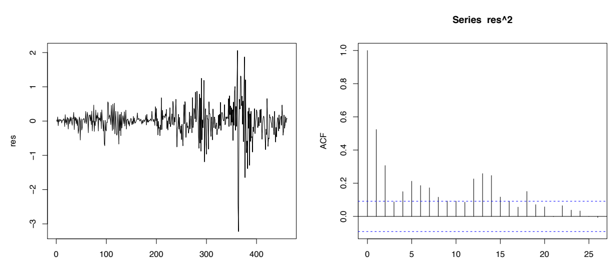
b) From the time series plot and the ACF plot it is clear that the series is not stationary. A differencing might be appropriate in this case. The differenced series looks stationary but the ACF shows several significant lag correlations (6,7,9,12,14,20,21,27).

```
 par(mfrow=c(1,4))
ts.plot(ttbill)
acf(ttbill,40)
dtbill=diff(ttbill)
ts.plot(dtbill)
acf(dtbill,40)
```



For simplicity we consider an AR(27) model. The residual plot of the AR(27) model shows some clusters of large residuals and clusters of small residuals. Moreover, the acf plot of the squared residuals indicates that serial correlation exists.

```
 fit=arima(dtbill,order=c(27,0,0))
res=fit$residuals
par(mfrow=c(1,2))
ts.plot(res^2);acf(res^2)
```



These can be interpreted as nonconstant variance and would thus fit into the assumptions of GARCH model.

c) We fit a GARCH(p,q) model to the residual series for $p, q \leq 5$ and compare the AIC:

```
 ans=matrix(0,6,6)
for (i in 0:5){
for (j in 0:5){
if ((i==0)&(j==0)){ ans[i+1,j+1]=0}
else{ ans[i+1,j+1]=AIC(garch(res,order=c(i,j)))}
}
}
```

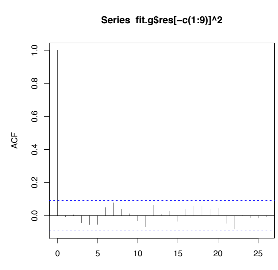
It is found that GARCH(1,1) gives the smallest AIC of 153.32.

d) Looking closely into the GARCH(1,1) model. The fitted model is

$$e_t = \eta_t \sigma_t^2; \quad \sigma_t^2 = 0.0020247 + 0.2580783\sigma_t^2 + 0.7592271 e_t^2 \,.$$

where e_t is the residual of the AR(27) model and $\eta_t \sim N(0,1)$. The Box-Ljung test (from `summary(fit.g)`) does not reject the null hypothesis, suggesting that the fitting is appropriate. Moreover, acf plot of the squared residuals of the GARCH fitting shows no serial correlations, indicating that the non-constant variance effect has been removed.

```
 fit.g=garch(res,order=c(1,1))
summary(fit.g)
acf(fit.g$res[-c(1:9)]∧2)   # The first 9 NA entries should
be removed.
```



e) To conclude, an AR(27) model with GARCH(1,1) effect for the residuals adequately explains the Treasury Bill data.