A Calculation for Slide 28 of Week2.ppt

We showed in the lecture that if we let $M_n = \mathbb{E}[X_n]$ and $\mu = \mathbb{E}[\xi]$, then $M(n+1) = \mu M(n)$ ...... ①

We also said that if $V(n) = \mathrm{Var}[X_n]$, then

$$V(n+1) = \sigma^2 M(n) + \mu^2 V(n).$$

Let's prove that statement.

$$\begin{aligned}
\mathrm{Var}[X_{n+1}] &= \mathbb{E}[X_{n+1}^2] - (\mathbb{E}[X_{n+1}])^2 && \text{(Definition of variance)} \\
&= \mathbb{E}[X_{n+1}^2] - \mu^2 M^2(n) && \text{(using Equation ①)} \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\Big(\sum_{i=1}^{N} \xi_i\Big)^2 \,\Big|\, X_n = N\Big]\Big] - \mu^2 M^2(n) && \text{(using total expectation and defn. of } X_{n+1}) \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N} \xi_i^2 + 2\sum_{i<j}^{N} \xi_i \xi_j \,\Big|\, X_n = N\Big]\Big] - \mu^2 M^2(n) && \text{(expanding the squared sum)} \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N} \xi_i^2 \,\Big|\, X_n = N\Big]\Big] + \mathbb{E}\Big[\mathbb{E}\Big[2\sum_{i<j}^{N} \xi_i \xi_j \,\Big|\, X_n = N\Big]\Big] - \mu^2 M^2(n) \\
&= \mathbb{E}[N(\mu^2 + \sigma^2)] + \mathbb{E}\Big[\tfrac{1}{2}N(N-1) \cdot 2\mu\mu\Big] - \mu^2 M^2(n)
\end{aligned}$$

(since $\mathbb{E}(Y^2) = \mathrm{Var}(Y) + (\mathbb{E}(Y))^2$; there are $\frac{1}{2}n(n-1)$ terms in $\sum_{i<j}^{n} y_i y_j$; the ξ_i, ξ_j are independent for $i \neq j$)

$$\begin{aligned}
&= (\mu^2 + \sigma^2) M(n) + \underbrace{\mu^2 \mathbb{E}[N^2] - \mu^2 M(n) - \mu^2 M^2(n)}_{=\ \mu^2 \mathrm{Var}(X_n)} \\
&= \sigma^2 M(n) + \mu^2 \mathrm{Var}(X_n)
\end{aligned}$$

$$\Rightarrow \quad V(n+1) = \mu^2 V(n) + \sigma^2 M(n)$$