## (1) Linear Algebra: a brief overview (I)

Basic notation

1. Size, row, column, entry

   - Let number of rows $= m$
   - Let number of columns $= n$
   - Let the $(i, j)$-entry be $a_{i,j}$

2. For two matrices, $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} = \mathbf{B}$ if and only if they have the same size and $[a_{ij}] = [b_{ij}]$.

3. Vector: when $m$ or $n$ equal to 1.

4. transpose

5. Square matrix: $m = n$

   - main diagonal
   - symmetric matrix
   - diagonal matrix, identity matrix, upper triangular matrix and lower triangular matrix

6. Addition, Subtraction and Multiplication

7. Properties

   Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be matrices such that $\mathbf{AB}$, $\mathbf{AC}$ and $\mathbf{BC}$ are defined. Assume that $\mathbf{B}$ and $\mathbf{C}$ have the same dimension. Also, let $r$ be a scalar.

   - $\mathbf{IA} = \mathbf{A} = \mathbf{AI}$
   - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
   - $\mathbf{A}(\mathbf{B} \pm \mathbf{C}) = \mathbf{A}(\mathbf{B} \pm \mathbf{C})$
   - $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B}$
   - $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

8. Linear Equations in matrix notation

   Example:

$$4x_1 - 2x_2 + x_3 = 11$$

$$2x_1 + 3x_2 + 2x_3 = 6$$

   Express the above in the form of

$$\mathbf{AX} = \mathbf{b}$$

   What are $\mathbf{A}, \mathbf{X}$ and $\mathbf{b}$? Is there any solution?

9. Inverse (Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices with the same size)

- If $\mathbf{B}$ is an inverse of $\mathbf{A}$, then $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$.
- If $\mathbf{A}$ has an inverse ($\mathbf{A}^{-1}$), then $\mathbf{A}$ is an invertible matrix.
- If $\mathbf{A}$ is invertible, the system of linear equations $\mathbf{AX} = \mathbf{b}$ has the unique solution $\mathbf{X} = \mathbf{A^{-1}b}$
- If $\mathbf{A}$ is invertible, $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- If $\mathbf{A}$ and $\mathbf{B}$ are invertible, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- The above can be generalized to product of more than two matrices.
- Let $k$ be an integer $\geq 1$. Then if $\mathbf{A}$ is invertible, $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$
- Let $a \neq 0$ be a scalar and if $\mathbf{A}$ is invertible, $(a\mathbf{A})^{-1} = \frac{1}{a}\mathbf{A}^{-1}$
- If $\mathbf{A}$ is invertible, $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- Rank and inverse

10. LU factorization

- For an $m \times n$ matrix $\mathbf{A}$, the entries $a_{11}, a_{22}, ...$ are called the main diagonal of $\mathbf{A}$.
- Let $\mathbf{A}$ be $m \times n$. LU factorization of $\mathbf{A}$: $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is a lower triangular $m \times m$ matrix and $\mathbf{U}$ is an upper $m \times n$ triangular matrix.
- Consider the system of linear equations $\mathbf{AX} = \mathbf{b}$. If $\mathbf{A}$ can be factored by LU factorization, the system can be solved by the following two steps.
  (a) Solve $\mathbf{LY} = \mathbf{b}$ for $\mathbf{Y}$ by forward substitution.
  (b) Solve $\mathbf{UX} = \mathbf{Y}$ for $\mathbf{X}$ by back substitution.
- Example:

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -4 & 2 \\ 1 & -2 & 1 & 3 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

11. Determinant ($|\mathbf{A}|$)

    - Square matrix
    - If $\mathbf{A}$ has a row or column of zeros, $|\mathbf{A}| = 0$
    - The determinant of the resulting matrix is $-|\mathbf{A}|$ if two rows (or columns) are interchanged
    - If a row (or a column) of $\mathbf{A}$ is multiplied by $c$ (a scalar), the resulting determinant is $c|\mathbf{A}|$
    - If two rows (or columns) of $\mathbf{A}$ are identical, $|\mathbf{A}| = 0$
    - Determinant of identity matrix, diagonal matrix and triangular matrix.
    - $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
    - $|\mathbf{A}^{'}| = |\mathbf{A}|$
    - If $\mathbf{A}$ is not invertible (singular), $|\mathbf{A}| = 0$.

12. Trace

    - Square matrix
    - The trace is the sum of the main diagonal elements
    - $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$

13. Eigenvalues and eigenvectors

- Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\mathbf{x} \neq \mathbf{0}$ be an $n \times 1$ vector such that

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

Then, $\lambda$ is the eigenvalue (characteristic root) and $\mathbf{x}$ is the corresponding eigenvector (characteristic vector).

- For the above equation $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$, we have

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}.$$

However, $\mathbf{x} \neq 0$, implying that $(\lambda \mathbf{I} - \mathbf{A})$ is not invertible. Therefore

$$|(\lambda \mathbf{I} - \mathbf{A})| = \mathbf{0}$$

which is called the characteristic equation.

- $|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$
- $\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$
- Examples:

(a)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ -1 & 3 & -2 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

14. Diagonalization of a square matrix

- A matrix is diagonalizable if and only if every eigenvalue of multiplicity $m$ yields $m$ basic eigenvectors.
- Let $\mathbf{A}$ be diagonalizable with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and corresponding eigenvectors $\mathbf{x_1}, ..., \mathbf{x_n}$. Let $\mathbf{P} = [\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}]$. Then

$$\mathbf{P^{-1}AP} = \mathbf{D}$$

  where $\mathbf{D}$ is a diagonal matrix with the eigenvalues on the diagonal.
- If $\mathbf{A}$ is symmetric
  - $\mathbf{P}$ always exists and it is orthogonal.
  - (spectral decomposition)

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{e_i} \mathbf{e_i}'$$

    where $\mathbf{e_i}$ is the normalized eigenvector.
- Example:

$$\mathbf{A} = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$$