Summary of Chapter 7

1 Concepts

- **Hypothesis Test for μ:**

1. σ is known (Z test): Test statistic $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

Two-Tail Test: H_0: $\mu = \mu_0$, H_1: $\mu \neq \mu_0$

1) Use critical value: If $Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$, reject H_0.

2) Use p-value: If p-value = $2[1 - \Phi(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}})] < \alpha$, reject H_0.

3) Use C.I.: If $\mu_0 \notin (\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$, reject H_0.

Upper-Tail Test: H_0: $\mu \leq \mu_0$, H_1: $\mu > \mu_0$

1) Use critical value: If $Z > Z_\alpha$, reject H_0.

2) Use p-value: If p-value = $1 - \Phi(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}) < \alpha$, reject H_0.

Lower-Tail Test: H_0: $\mu \geq \mu_0$, H_1: $\mu < \mu_0$

1) Use critical value: If $Z < -Z_\alpha$, reject H_0.

2) Use p-value: If p-value = $\Phi(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}) < \alpha$, reject H_0.

2. σ **is unknown (**t **test):** Test statistic $t = \frac{\bar{X}-\mu}{S/\sqrt{n}}$

Two-Tail Test: H_0: $\mu = \mu_0$, H_1: $\mu \neq \mu_0$

1) Use critical value: If $t > t_{\alpha/2,n-1}$ or $t < -t_{\alpha/2,n-1}$, reject H_0.

2) Use p-value: If p-value = $2[1 - T_{n-1}(\frac{\bar{X}-\mu_0}{S/\sqrt{n}})] < \alpha$, reject H_0.

T_{n-1} is the cumulative t-distribution with degree of freedom $n-1$.

R function for calculating p-value: $2[1 - pt(\frac{\bar{X}-\mu_0}{S/\sqrt{n}}, n-1, \alpha/2)]$

3) Use C.I.: If $\mu_0 \notin (\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}})$, reject H_0.

Upper-Tail Test: H_0: $\mu \leq \mu_0$, H_1: $\mu > \mu_0$

1) Use critical value: If $t > t_{\alpha,n-1}$, reject H_0.

2) Use p-value: If p-value = $1 - T_{n-1}(\frac{\bar{X}-\mu_0}{S/\sqrt{n}}) < \alpha$, reject H_0.

Lower-Tail Test: H_0: $\mu \geq \mu_0$, H_1: $\mu < \mu_0$

1) Use critical value: If $t < -t_{\alpha,n-1}$, reject H_0.

2) Use p-value: If p-value = $T_{n-1}(\frac{\bar{X}-\mu_0}{S/\sqrt{n}}) < \alpha$, reject H_0.

- **Hypothesis Test for** p: $(np \geq 5$ and $n(1-p) \geq 5)$

Z test: Test statistic $Z = \frac{p_s - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{X - np}{\sqrt{np(1-p)}}$

Two-Tail Test: H_0: $p = p_0$, H_1: $p \neq p_0$

1) Use Critical value: If $Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$, reject H_0.

2) Use p-value: If p-value = $2[1 - \Phi(\frac{p_s - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}})] < \alpha$, reject H_0.

Upper-Tail Test: H_0: $p \leq p_0$, H_1: $p > p_0$

1) Use critical value: If $Z > Z_\alpha$, reject H_0.

2) Use p-value: If p-value = $1 - \Phi(\frac{p_s - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}) < \alpha$, reject H_0.

Lower-Tail Test: H_0: $p \geq p_0$, H_1: $p < p_0$

1) Use critical value: If $Z < -Z_\alpha$, reject H_0.

2) Use p-value: If p-value = $\Phi(\frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}) < \alpha$, reject H_0.

- **Type I and Type II Errors**

$$\alpha = P\{\text{type I error}\} = P\{\text{reject } H_0 | H_0 \text{ is true}\}$$

— probability of *incorrectly* rejecting a true H_0

$$\beta = P\{\text{type II error}\} = P\{\text{fail to reject } H_0 | H_0 \text{ is false}\}$$

— probability of *incorrectly* fail to reject a false H_0

$$\text{Power} = 1 - \beta = P\{\text{reject } H_0 | H_0 \text{ is false}\}$$

— probability of *correctly* rejecting a false H_0

Usually, α is decided previously by researchers, and β can be calculated when δ is known:

$$\beta = \Phi(Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}) - \Phi(-Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}),$$

where δ is the difference between the true value and hypothesized value of the parameter.

2 Examples

Example 1. The quality-control manager at a light bulb factory needs to determine whether the mean life of a large shipment of light bulbs is equal to the specified value of 375 hours. The process standard deviation is known to be 100 hours. A random sample of 64 light bulbs indicates a sample mean life of 350 hours.

a. State the null and alternative hypotheses.

b. At the 0.05 level of significance is there evidence that the mean life is different from 375 hours?

c. Compute the p-value and interpret its meaning.

d. Set up a 95% confidence interval estimate of the population mean life of the light bulbs.

e. Compare the results of (b) and (d). What conclusions do you reach?

a. $H_0: \ \mu = 375$ hours vs $H_1: \ \mu \neq 375$ hours.

b. $\alpha = 0.05, \ n = 64, \ \bar{X} = 350$, and $\sigma = 100$. Since $Z_{\alpha/2} = Z_{0.025} = 1.96$, we reject H_0 when $Z < -1.96$ or $Z > 1.96$. And the Test statistic is:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{350 - 375}{100/\sqrt{64}} = -2.00 < -1.96.$$

Then H_0 is rejected. There is enough evidence to conclude that the average life of the manufacturer's light bulbs differs from 375 hours.

c. p-value = $P(Z < -2.00 \text{ or } Z > 2.00) = 2[1-\Phi(2)] = 2\Phi(-2) = 0.04550 < 0.05$ (R function: p-value = 2[1-pnorm(2)]=2pnorm(-2)). It means that under null hypothesis, the probability of obtaining a sample whose mean is further away from the hypothesized value of 375 is less than 0.05.

d.

$$\bar{X} \pm Z \cdot \frac{\sigma}{\sqrt{n}} = 350 \pm (1.96) \cdot \frac{100}{\sqrt{64}} = 350 \pm 24.50$$

$$325.50 \leq \mu \leq 374.50$$

e. The results are the same. The confidence interval formed does not include the hypothesized value of 375 hours.

Example 2. The director of manufacturing at a clothing factory needs to determine whether a new machine is producing a particular type of cloth according to the manufacturer's specifications, which indicate that the cloth should have a mean breaking strength of at least 70 pounds and a standard deviation of 3.5 pounds. The director is concerned that if the mean breaking strength is actually *less* than 70 pounds, the company will face too many lawsuits. A sample of 49 pieces of cloth reveals a sample mean of 69.1 pounds.

a. State the null and alternative hypotheses.

b. At the 0.05 level of significance, using the critical value approach to hypothesis testing, is there evidence that the mean breaking strength is less than 70 pounds?

c. At the 0.05 level of significance, using the p-value approach to hypothesis testing, is there evidence that the mean breaking is less than 70 pounds?

d. Interpret the meaning of the p-value in this problem.

e. Compare your conclusions in (b) and (c).

a. $H_0: \ \mu \geq 70$ pounds vs $H_1: \mu < 70$ pounds.

b. $\alpha = 0.05, \ n = 49, \ \bar{X} = 69.1$, and $\sigma = 3.5$. Since $Z_\alpha = Z_{0.05} = 1.645$, we reject H_0 when $Z < -1.645$. And the Test statistic is:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{69.1 - 70}{3.5/\sqrt{49}} = -1.80 < -1.645.$$

Then H_0 is rejected. There is enough evidence to conclude that the new machine does not produce cloth that meets the manufacturer's specifications.

c. p-value = $P(Z < -1.80) = \Phi(-1.8) = 0.0359 < 0.05$. According to the p-value, we reject H_0.

d. Under the null hypothesis, the probability of obtaining a sample whose mean is 69.1 pounds is 0.0359.

e. **b** and **c** give same conclusion of rejecting H_0.

Example 3. In an article (Nanci Hellmich, "'Supermarket Guru' Has a simple Mantre", *USA Today*, June 19, 2002, 7D) it was claimed that the average supermarket trip takes 22 minutes. Suppose that, in an effort to test this claim, a sample of 50 shoppers at a local supermarket were studied. The mean shopping time for the sample of 50 shoppers was 25.36 minutes with a sample standard deviation of 7.24 minutes. Using 0.10 level of significance, is there evidence that the mean shopping time at the local supermarket is different from the claimed value of 22 minutes.

Solution: $H_0: \ \mu = 22$ vs $H_1: \ \mu \neq 22$.

$\alpha = 0.10, \ n = 50$, df = n - 1 = 49, $\bar{X} = 25.36$, and $S = 25.36$.

Since $t_{\alpha/2,49} = 1.68$, we reject H_0 with $t < -1.68$ or $t > 1.68$.

The Test statistic is:

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{25.36 - 22}{7.24/\sqrt{50}} = 3.2816 > 1.68.$$

So H_0 is rejected. There is enough evidence to conclude that the mean shopping time at the local supermarket is different from the claimed value of 22 minutes.

Example 4. In a national poll of 811 personal computer owners, Peter D. Hart Research Associates found that 44% (357) of the PC owners ranked sharing their credit card information as the number one concern in online shopping. The survey also indicated that when given an actual dollar amount the added security would cost, 57% (462) indicated that they would pay an extra $75 for a new PC delivering more secure online experience.

a. Test the null hypothesis that 50% of all PC owners in the US rank sharing their credit card information as the number one concern in online shopping versus the alternative that the percentage is not equal to 50% (use $\alpha = 0.05$).

b. Compute the p-value in (a) and interpret its meaning.

c. At the 0.05 level of significance, is there evidence that more than half of all PC owners in the US would pay an extra $75 for a new PC delivering a more secure online experience?

d. Compute the p-value in (c) and interpret its meaning.

e. At the 0.05 level of significance, is there evidence that more than 55% of all PC owners in the US would pay an extria $75 for a new PC delivering a more secure online experience?

f. Compute the p-value in (e) and interpret its meaning.

a. $H_0: \ p = 0.50$ vs $H_1: \ p \neq 0.50$.

$\alpha = 0.05, p_s = 0.44$, $n = 811$, and

$$\sigma_{p_s} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.50(1-0.50)}{811}} = 0.01756.$$

Since $Z_{\alpha/2} = Z_{0.025} = 1.96$, we reject H_0 when $Z < -1.96$ or $Z > 1.96$.

The test statistic is:

$$Z = \frac{p_s - p}{\sigma_{p_s}} = \frac{0.44 - 0.50}{0.01756} = -3.4169 < -1.96.$$

So H_0 is rejected. There is enough evidence to show that the percentage of all PC owners in the US who rank sharing their credit card information as the number on concern in on-line shopping is not 50%.

b. p-value= $2P(Z < -3.4169) = 2\Phi(-3.4169) = 0.00063$.

Under null hypothesis, the probability of obtaining a sample proportion further away from the hypothesized value of 0.50 is 0.00066.

c. $H_0: \ p \leq 0.50$ vs $H_1: \ p > 0.50.$

$\alpha = 0.05, \ n = 811, \ p_s = 0.57, \ \sigma_{p_s} = 0.01756.$

Since $Z_\alpha = Z_{0.05} = 1.65$, we reject H_0 when $Z > 1.65$.

The test statistic is:

$$Z = \frac{p_s - p}{\sigma_{p_s}} = \frac{0.57 - 0.50}{0.01756} = 3.9863 > 1.65.$$

So H_0 is rejected. There is enough evidence to show that the percentage of all PC owners in the US who would pay an extra \$75 for a new PC delivering a more secure on-line expericnce is more than 50%.

d. p-value = $P(Z > 3.9863) = 1 - \Phi(3.9863) = 3.356 \times 10^{-5}$.

Under null hypothesis, the probability of obtaining a sample which will yield higher than 57% of all PC owners in the US who would pay an extra \$75 for a new PC delivering a more secure on-line expericnce is near zero.

e. $H_0: \ p \leq 0.55$ vs $H_1: \ p > 0.55.$

$\alpha = 0.05, \ n = 811, \ p_s = 0.57,$

$$\sigma_{p_s} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.55(1-0.55)}{811}} = 0.01747.$$

Since $Z_\alpha = Z_{0.05} = 1.65$, so we reject H_0 when $Z > 1.65$.

The test statistic is:

$$Z = \frac{p_s - p}{\sigma_{p_s}} = \frac{0.57 - 0.55}{0.01747} = 1.145 < 1.65.$$

Then we cannot reject H_0. There is not enough evidence to show that the percentage of all PC owners in the US who would pay an extra \$75 for a new PC delivering a more secure on-line expericnce is more than 55%.

f. p-value = $P(Z > 1.145) = 1 - \Phi(1.145) = 0.1261.$

Under null hypothesis, the probability of obtaining a sample which will yield higher than 57% of all PC owners in the US who would pay an extra \$75 for a new PC delivering a more secure on-line expericnce is 0.1261.

Example 5. The mean contents of coffee cans filled on a particular production line are being studied. Standards specify that the mean contents must be 16.0 oz, and from past experience it is known that the standard deviation of the can contents is 0.1 oz. The hypothesis are

$$H_0: \quad \mu = 16.0 \quad \text{vs} \quad H_1: \quad \mu \neq 16.0$$

A random sample of nine cans is to be used, and the type I error probability is specified as $\alpha = 0.05$. What is the test statistic? If the true mean contents are $\mu_1 = 16.1$ oz, What is the probability of type II error?

Solution: The test statistic is $Z = \frac{\bar{X}-16.0}{0.1/\sqrt{9}}$. H_0 is rejected if $|Z| > Z_{\alpha/2} = Z_{0.025} = 1.96$. Since $\delta = \mu_1 - \mu_0 = 16.1 - 16.0 = 0.1$, we have

$$\begin{aligned}\beta &= \Phi(Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}) - \Phi(-Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}) \\ &= \Phi(1.96 - \frac{(0.1)(3)}{0.1}) - \Phi(-1.96 - \frac{(0.1)(3)}{0.1}) \\ &= \Phi(-1.04) - \Phi(-4.96) = 0.1496.\end{aligned}$$

That is, the probability that we will incorrectly fail to reject H_0 if the true mean contents are 16.1 oz is 0.1492. Equivalently, we can say that the power of the test is $1 - \beta = 1 - 0.1492 = 0.8508$.