Solutions for Chapter 1 Exercises

1) Suppose that $X_t = 10 + a_t + 0.3a_{t-1}$, $a_t \sim N(0,1)$.

a)

$$Var(X_t) = Var(a_t) + (0.3)^2 Var(a_{t-1}) = 1.09\,.$$
$$Cov(X_t, X_{t+1}) = Cov(a_t + 0.3a_{t-1}, a_{t+1} + 0.3a_t) = Cov(a_t, 0.3a_t) = 0.3\,.$$

$$\begin{aligned} Corr(X_t, X_{t-1}) &= Cov(X_t, X_{t-1})/(\sqrt{Var(X_t)}\sqrt{Var(X_{t-1})}) \\ &= Cov(a_t + 0.3a_{t-1}, a_{t-1} + 0.3a_{t-2})/Var(X_t) \\ &= Cov(0.3a_{t-1}, a_{t-1})/Var(X_t) \\ &= 0.275\,. \end{aligned}$$

$$\begin{aligned} Cov(X_t, X_{t+k}) &= Cov(a_t + 0.3a_{t-1}, a_{t+k} + 0.3a_{t+k-1}) \\ &= 0\,. \end{aligned}$$

b) Since $\{X_t\}$ is stationary, we have $Cov(X_t, X_{t-l}) = \gamma(l)$, where $\gamma(l)$ is independent of time t. So

$$\begin{aligned} Var\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right) &= \frac{1}{n^2}\left(\sum_{t=1}^{n} Var(X_t) + \sum_{i\neq j} Cov(X_i, X_j)\right) \\ &= \frac{1}{n^2}\left(\sum_{t=1}^{n} Var(X_t) + 2\sum_{k=1}^{n-1}(n-k)\gamma(k)\right) \\ &= \frac{1}{n^2}\left(n \times 1.09 + 2(n-1) \times 0.3\right) \\ &= 0.01684\,, \end{aligned}$$

where $n = 100$.

c) Without assuming dependence, we would have $X_t = 10 + a_t$ and thus $Var(\overline{X}) = \frac{1}{n^2}\sum_{t=1}^{n} Var(a_t) = \frac{1}{n} = 0.01$. For example, if $\overline{X} = 9.78$, without accounting for the dependence, the CI is $\overline{X} \pm 2\sqrt{Var(\overline{X})} = 9.78 \pm 2 \times \sqrt{0.01} = (9.58, 9.98)$, which does not cover 10, thus H_0 is rejected. However, when the dependence is accounted for, the CI is $\overline{X} \pm 2\sqrt{Var(\overline{X})} = 9.78 \pm 2 \times \sqrt{0.01684} = (9.52, 10.04)$, which covers 10, thus H_0 is not rejected.

2) We design a filter which has order $s = 2$ does not distort a quadratic trend as follows,

$$\begin{aligned} a_r &= a_{-r} \\ a_r &= (a_0 = 1/4, a_1 = 1/2, a_2 = -1/8) \end{aligned}$$

satisfy conditions

$$\sum_r a_r = 1, \quad \sum_r r a_r = 0, \quad \sum_r r^2 a_r = 0.$$

any other filter with different weights $\{a_r\}$ are acceptable

3) a) The length of sequence is 6.

b)

$$\hat{x}_6 = a_{-2}x_4 + a_{-1}x_5 + a_0 x_6 + a_1 x_7 + a_2 x_8.$$

c) The filter can pass any linear trend since

$$\begin{aligned} \sum_{r=-2}^{2} a_r &= 1, \\ \sum_{r=-2}^{2} r a_r &= 0. \end{aligned}$$

But since

$$\sum_{r=-2}^{2} r^2 a_r \neq 0,$$

the filter does not pass through quadratic trend.

d) The filter is not useful because $\sum_{r=-2}^{2} a_r = 0.9 \neq 1$.

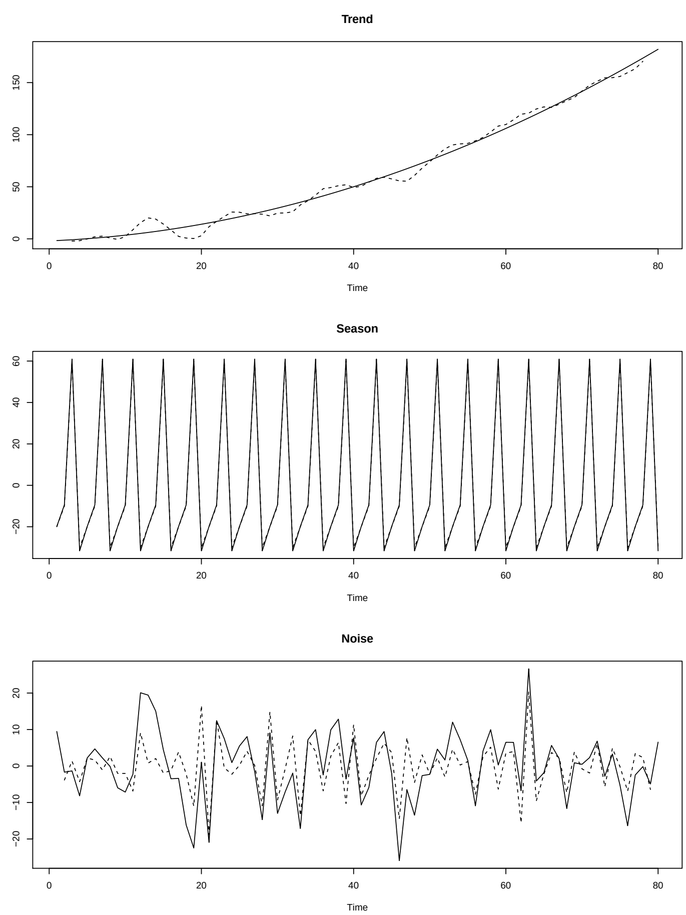
4) a) R code of filtering approach:

```
library(stats)
library(graphics)

# this step is used to generate data
set.seed(6104)
season.effect<-c(-20,-10,60,-30)
sea.com<-rep(season.effect,20)
t<-1:80
trend<--2+0.3*t+0.025*t^2
noise<-rnorm(80,0,10)
data<-trend+sea.com+noise

# this step give the estimate of trend, season and noise terms
data.ma<-filter(data,c(1/3,1/3,1/3))
data.sea<-rep(0,4)
for(i in 1:2){
for(j in 1:19){
data.sea[i]<-data.sea[i]+(data[i+4*j][[1]]-data.ma[i+4*j][[1]])
}
print(i)
}
for(i in 3:4){
for(j in 1:19){
data.sea[i]<-data.sea[i]+(data[i+4*(j-1)][[1]]-data.ma[i+4*(j-1)][[1]])
}
print(i)
}
data.sea<-(data.sea-mean(data.sea))/19
data.sea1<-rep(data.sea,20)
data.nosea<-data-data.sea
data.ma2<-filter(data.nosea,c(1/3,1/3,1/3))
data.res<-data-data.ma2-data.sea
write(data.sea1,file='out.dat')
data.seatime<-ts(scan('out.dat'))
leg.names<-c('data','estimated data')
```

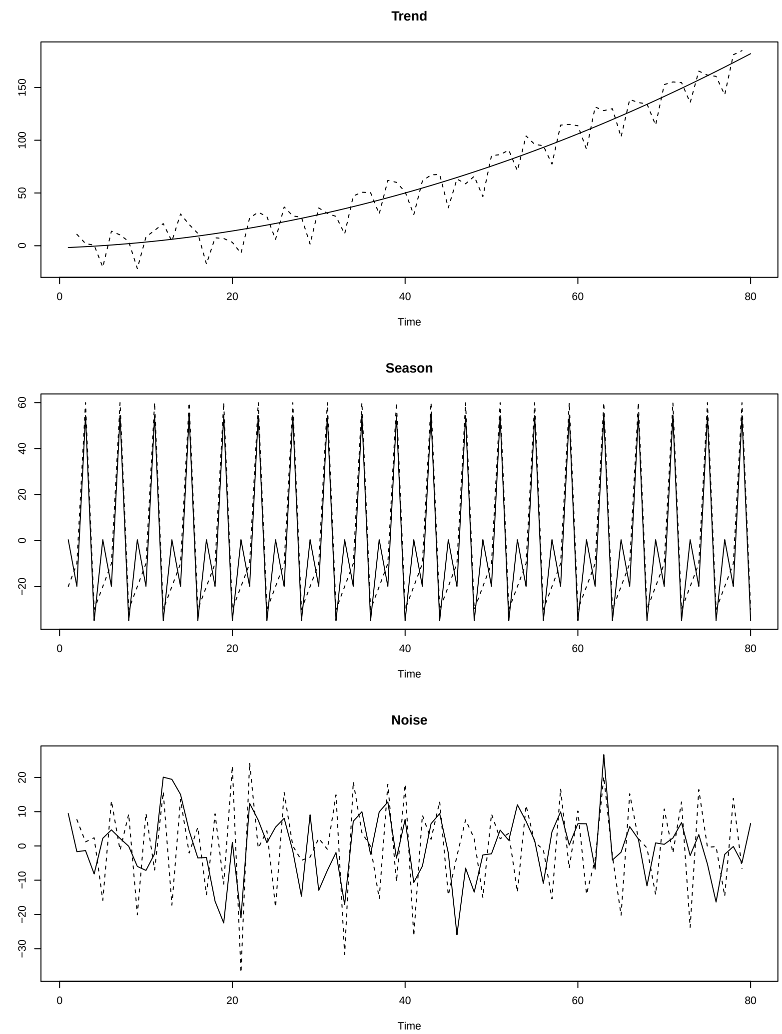
```
# we plot the two series
par(mfrow=c(3,1))
ts.plot(trend,data.ma,main='Trend',lty=c(1,2))
legend(locator(1),leg.names,lty=c(1,2))
ts.plot(data.seatime,sea.com,main='Season',lty=c(1,2))
ts.plot(noise,data.res,main='Noise',lty=c(1,2))
```

Remarks: If we use $(1/3, 1/3, 1/3)$ to filter the data, the results are inferior compared with using $(1/8, 1/4, 1/4, 1/4, 1/8)$. The corresponding pictures are given as follows:

Result for the filter $(1/3, 1/3, 1/3)$



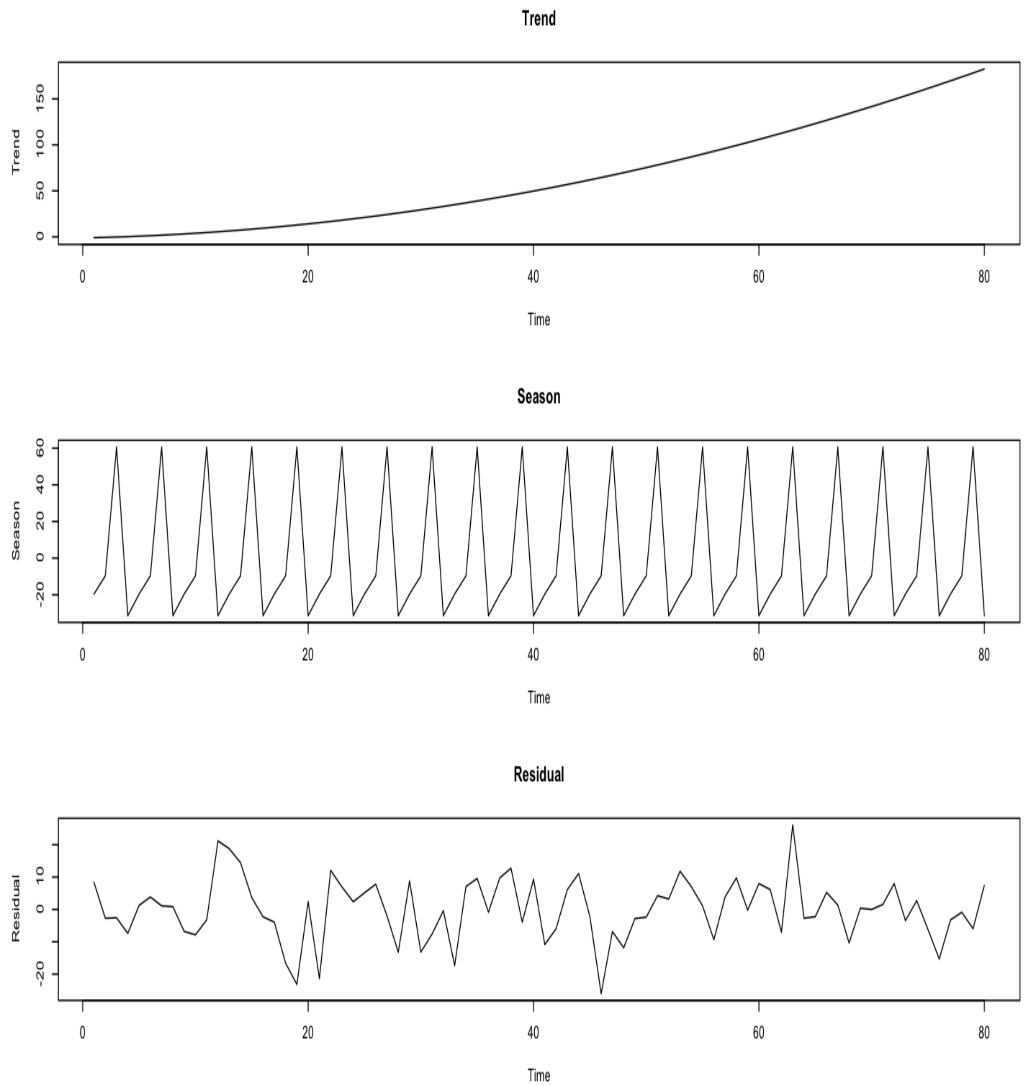
Result for the filter $(1/8, 1/4, 1/4, 1/4, 1/8)$



R code of least square approach: (Set $d = 4$ and $k = 2$)

```
library(stats)
library(graphics)

# this step is used to generate data
set.seed(6104)
season.effect<-c(-20,-10,60,-30)
sea.com<-rep(season.effect,20)
t<-1:80
trend<--2+0.3*t+0.025*t^2
noise<-rnorm(80,0,10)
data<-trend+sea.com+noise

# this step give the estimate of trend, season and noise terms
s1=rep(c(1,0,0,0),20)
s2=rep(c(0,1,0,0),20)
s3=rep(c(0,0,1,0),20)
s4=rep(c(0,0,0,1),20)
power=cbind(1:80,(1:80)^2)
X=cbind(s1,s2,s3,s4,power)
Reg.coef=solve(t(X)%*%X,t(X)%*%data)
s=rep(0,4)
for (i in 1:4){s[i]=Reg.coef[i]-mean(Reg.coef[1:4])}
Season=rep(s,20)
tre.coef=c(mean(Reg.coef[1:4]),Reg.coef[5],Reg.coef[6])
Power=cbind(rep(1,80),power)
Trend=Power%*%tre.coef
Residual=data-Trend-Season


# we plot the two series
par(mfrow=c(3,1))
ts.plot(Trend,main='Trend')
ts.plot(Season,main='Season')
ts.plot(Residual,main='Residual')
```

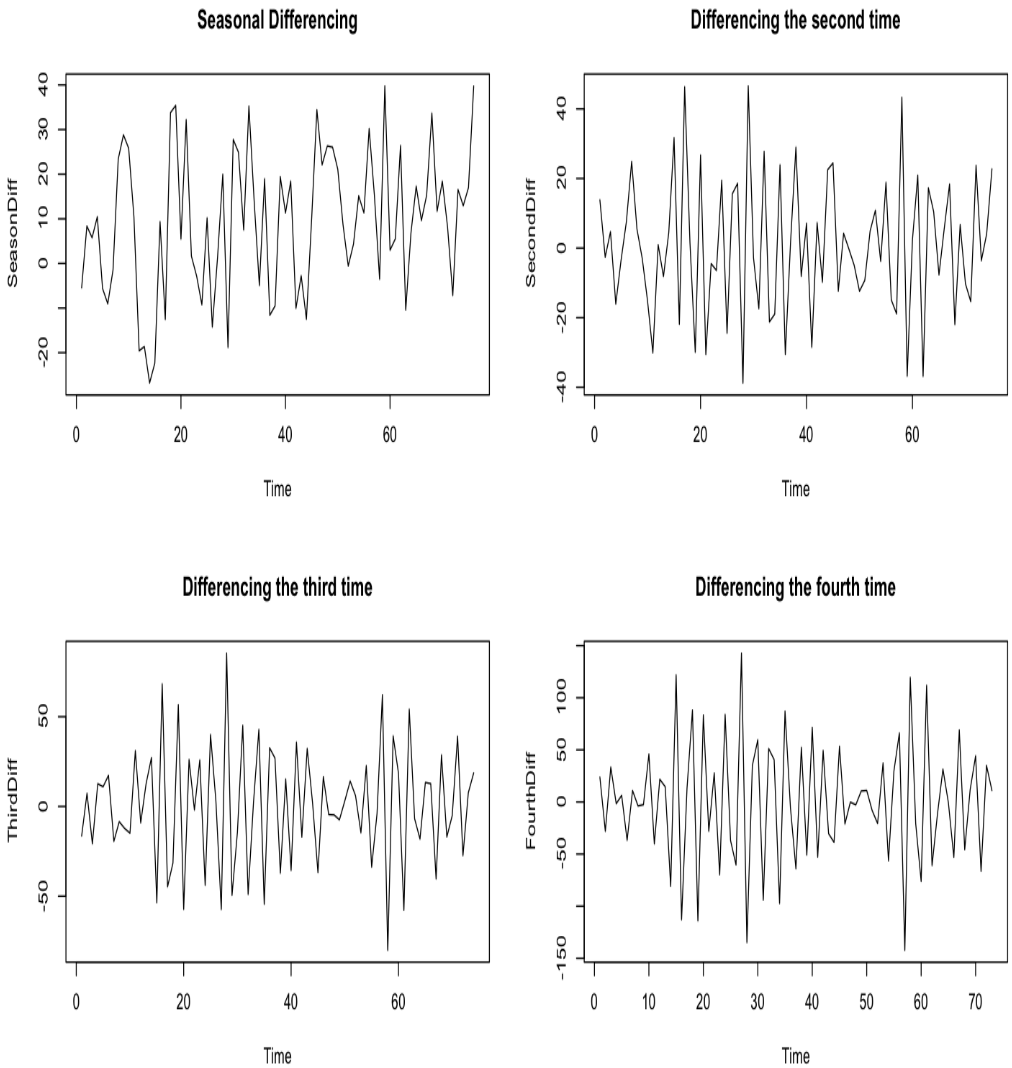
Result for the least square approach



b) R code for differencing approach:

```
library(stats)
library(graphics)

# this step is used to generate data
set.seed(6104)
season.effect<-c(-20,-10,60,-30)
sea.com<-rep(season.effect,20)
t<-1:80
trend<--2+0.3*t+0.025*t∧2
noise<-rnorm(80,0,10)
data<-trend+sea.com+noise

# this step give the method of differencing
SeasonDiff=rep(0,76)
for (i in 1:76){SeasonDiff[i]=data[i+4]-data[i]}
SecondDiff=diff(SeasonDiff)
ThirdDiff=diff(SecondDiff)
FourthDiff=diff(ThirdDiff)

# we plot the two series
par(mfrow=c(2,2))
ts.plot(SeasonDiff,main='Seasonal Differencing')
ts.plot(SecondDiff,main='Differencing the second time')
ts.plot(ThirdDiff,main='Differencing the third time')
ts.plot(FourthDiff,main='Differencing the fourth time')
```

Result for the differencing approach



From the graph, we conclude that two times of differencing is enough to obtain a stationary sequance since the second graph is already random enough and the mean is close to 0.