THE CHINESE UNIVERSITY OF HONG KONG
Department of Statistics

STAT3007: Introduction to Stochastic Processes
Markov Chains - Introduction - Exercises

1. (Problem 3.2.3 in Pinsky and Karlin) Let X_n denote the quality of the nth item produced by a production system with $X_n = 0$ meaning 'good' and $X_n = 1$ meaning 'defective'. Suppose that $\{X_n\}$ evolves as a Markov chain whose transition probability matrix is

$$\mathbb{P} = \begin{pmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{pmatrix}$$

   What is the probability that the fourth item is defective given that the first item is defective?

2. (Exercise 3.3.1 in Pinsky and Karlin) Consider a spare parts inventory model in which either $0, 1,$ of 2 repair parts are demanded in any period, with $Pr(\xi_n = 0) = 0.4$, $Pr(\xi_n = 1) = 0.3$, $Pr(\xi_n = 2) = 0.3$ and suppose $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end-of-period n.

3. (Exercise 3.3.2 in Pinsky and Karlin) Consider two urns A and B containing a total of N balls. An experiment is performed in which a ball is selected at random (all selections equally likely) at time t ($t = 1, 2, \ldots$) from among the totality of N balls. Then, an urn is selected at random (A is chosen with probability p and B is chosen with probability q) and the ball previously drawn is placed in this urn. The state of the system at each trial is represented by the number of balls in A. Determine the transition matrix for this Markov chain.

4. (Exercise 3.1.4 in Pinsky and Karlin) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbb{P} = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

   Determine the conditional probabilities $Pr(X_1 = 1, X_2 = 1 | X_0 = 0)$ and $Pr(X_2 = 1, X_3 = 1 | X_1 = 0)$.

5. (Exercise 3.1.5 in Pinsky and Karlin) A Markov chain $X_0, X_1, X_2, \ldots$ has

the transition probability matrix

$$\mathbb{P} = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $Pr(X_0 = 1, X_1 = 1, X_2 = 0)$ and $Pr(X_1 = 1, X_2 = 1, X_3 = 0)$.

6. (Problem 3.2.2 in Pinsky and Karlin) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Let X_0 be the signal that is sent, and let X_n be the signal that is received at the nth stage. Suppose X_n is a Markov chain with transition probabilities $\mathbb{P}_{00} = \mathbb{P}_{11} = 1 - \alpha$ and $\mathbb{P}_{01} = \mathbb{P}_{10} = \alpha$, $(0 < \alpha < 1)$. Determine $Pr(X_5 = 0|X_0 = 0)$, the probability of correct transmission through five stages. *Hint: prove by induction that*

$$\mathbb{P}^n = \frac{1}{2} \begin{pmatrix} 1 + (1-2\alpha)^n & 1 - (1-2\alpha)^n \\ 1 - (1-2\alpha)^n & 1 + (1-2\alpha)^n \end{pmatrix}.$$

7. (Exercise 3.2.2 in Pinsky and Karlin) A particle moves among the states 0, 1, 2 according to a Markov process whose transition probability matrix is

$$\mathbb{P} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$$

Let X_n denote the position of the particle at the nth move. Calculate $Pr(X_n = 0|X_0 = 0)$ for $n = 0, 1, 2, 3, 4$.

8. (From Slide 8 of the "Markov Chain - Introduction" notes) Show that if X is a stationary Markov chain, then $Pr(X_{n+1} = j_1, \dots, X_{n+m} = j_m | X_0 = i_0, \dots, X_n = i_n) = Pr(X_{n+1} = j_1, \dots, X_{n+m} = j_m | X_n = i_n)$.

THE END