STAT 3007 Tutorial 3 Suggested solutions

Example 1. $P^2 = \begin{pmatrix} (1-\alpha)^2+\alpha^2 & 2\alpha(1-\alpha) \\ 2\alpha(1-\alpha) & (1-\alpha)^2+\alpha^2 \end{pmatrix}$

$$\Pr(X_0=0, X_2=0) = \Pr(X_0=0)\Pr(X_2=0 \mid X_0=0) = \beta\left((1-\alpha)^2+\alpha^2\right) \quad \square$$

Compare: $\Pr(X_2=0 \mid X_0=0) = \Pr(X_2=0 \mid X_1=0, X_0=0)\Pr(X_1=0 \mid X_0=0)$

$\qquad + \Pr(X_2=0 \mid X_1=1, X_0=0)\Pr(X_1=1 \mid X_0=0)$

$\qquad = (1-\alpha)^2+\alpha^2$

Example 2. $P_{ij}^{k+n} = \sum_{r=0}^{\infty} P_{ir}^{n} P_{rj}^{k}$

$\sum_{r=0}^{\infty} P_{ir}^{n} = 1$, $P_{ir}^{n} \geq 0 \Rightarrow$ there exists an s such that $P_{is}^{n} > 0$

all entries of P^k are positive $\Rightarrow P_{sj}^{k} > 0$

Therefore, $P_{ij}^{k+n} = \sum_{r=0}^{\infty} P_{ir}^{n} P_{rj}^{k} \geq P_{is}^{n} P_{sj}^{k} > 0$ for all i, j $\square$

Example 3. There exist $n \geq 0$, $m \geq 0$ such that $P_{ij}^{(n)} > 0$, $P_{jk}^{(m)} > 0$

Therefore, $P_{ik}^{(m+n)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rk}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)} > 0$ $\square$

Example 4. Analogous to Example 3.

Example 5. Let X_n be the ordered pair of the results of $(n-1)^{th}$ & n^{th} tosses

$$P = \begin{array}{c} \\ HH \\ HT \\ TH \\ TT \end{array} \begin{array}{c} \begin{array}{cccc} HH & HT & TH & TT \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$T = \min\{n \geq \underset{\sim}{2} : X_n = HH \text{ or } X_n = TT\}$$

(1) $\Pr(X_T = TT \mid \text{first toss} = H)$

$= \Pr(X_T = TT \mid X_2 = HH)\Pr(\text{second toss} = H \mid 1\text{st} = H)$

$+ \Pr(X_T = TT \mid X_2 = HT)\Pr(\text{second toss} = T \mid 1\text{st} = H)$

$= 0.5\Pr(X_T = TT \mid X_2 = HT)$

Now, $u_{HT} = \Pr(X_T = TT \mid X_2 = HT)$

$= \Pr(X_T = TT \mid X_3 = TT, X_2 = HT)\Pr(X_3 = TT \mid X_2 = HT)$

$+ \Pr(X_T = TT \mid X_3 = TH, X_2 = HT)\Pr(X_3 = TH \mid X_2 = HT)$

$= 0.5 + 0.5\,u_{TH}$

$u_{TH} = \Pr(X_T = TT \mid X_2 = TH)$

$= \Pr(X_T = TT \mid X_3 = HT, X_2 = TH)\Pr(X_3 = HT \mid X_2 = TH)$

$+ \Pr(X_T = TT \mid X_3 = HH, X_2 = TH)\Pr(X_3 = HH \mid X_2 = TH)$

$= 0.5\,u_{HT}$

$\Rightarrow u_{HT} = \frac{2}{3}$, then $\Pr(X_T = TT \mid \text{first toss} = H) = \frac{1}{3}$

(2) $E(T \mid \text{first toss} = H)$

$= E(T \mid X_2 = HH)\Pr(\text{second toss} = H \mid 1\text{st} = H)$

$+ E(T \mid X_2 = HT)\Pr(\text{second toss} = T \mid 1\text{st} = H)$

$= 2 \times 0.5 + 0.5\,V_{HT}$

Now, $V_{HT} = E(T \mid X_2 = HT) = 3 + 0.5(V_{TH} - 2)$

$V_{TH} = E(T \mid X_2 = TH) = 3 + 0.5(V_{HT} - 2)$

$\Rightarrow V_{HT} = 4$, then $E(T \mid \text{first toss} = H) = 3$ $\square$

Example 6. Let X_k be Yau's money after k^{th} gamble.

The state space of $\{X_k\}$ is $\{0, 1, 2, \ldots, m+n\}$

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & m+n-1 & m+n \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ m+n-1 \\ m+n \end{array} & \begin{pmatrix} 1 & & & & & \\ 0.4 & 0.2 & 0.4 & & & \\ & 0.4 & 0.2 & 0.4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0.4 & 0.2 & 0.4 \\ & & & & & 1 \end{pmatrix} \end{array}$$

$T = \min\{k \geq 0 : X_k = 0\}$, $u_i = \Pr(X_T = 0 \mid X_0 = i)$

$$u_i = \Pr(X_T = 0 \mid X_0 = i) = \sum_{j=i-1}^{i+1} \Pr(X_T = 0 \mid X_1 = j, X_0 = i) \Pr(X_1 = j \mid X_0 = i)$$

$$= 0.4\, u_{i-1} + 0.2\, u_i + 0.4\, u_{i+1}$$

$$\Rightarrow u_i - u_{i-1} = u_{i+1} - u_i =: A \quad \text{for } i = 1, 2, \ldots, m+n-1$$

Note that $u_0 = \Pr(X_T = 0 \mid X_0 = 0) = 1$

$$u_{m+n} = \Pr(X_T = 0 \mid X_0 = m+n) = 0$$

then $u_{m+n} = (u_{m+n} - u_{m+n-1}) + (u_{m+n-1} - u_{m+n-2}) + \cdots + (u_1 - u_0) + u_0$

$$0 = (m+n)A + 1 \quad \Rightarrow \quad A = -\frac{1}{m+n}$$

$$\Rightarrow u_m = (u_m - u_{m-1}) + (u_{m-1} - u_{m-2}) + \cdots + (u_1 - u_0) + u_0$$

$$= mA + 1 = \frac{n}{m+n}$$

□

Example 7. Direct application of matrix expressions