Exercises for Chapter 3

1. i) For an AR(1) process with characteristic polynomial $\phi(x) = 1 - \phi_1 x$, what is the stationary condition for ϕ_1? What is the causal condition for ϕ_1?

   ii) For an AR(2) process with characteristic polynomial $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = (1 - r_1 x)(1 - r_2 x)$, what is the stationary conditions for r_1 and r_2? What is the causal conditions for r_1 and r_2?

   iii) Show that if the AR(2) process in ii) is causal, then it is necessarily that $\phi_1 + \phi_2 < 1$, $\phi_1 - \phi_2 > -1$ and $|\phi_2| < 1$.

2. Consider a causal AR(2) process $Z_t = \alpha Z_{t-1} + \beta Z_{t-2} + a_t$, where $\mathrm{Var}(Z_t) = 4$ and $\sigma_a^2 = 2$. Find $\alpha\rho_1 + \beta\rho_2$ where ρ_1 and ρ_2 are lag 1 and 2 autocorrelations, respectively.

3. Consider the model $Z_t = \theta Z_{t-1} + a_t + \alpha a_{t-1} + \beta a_{t-2}$ where α, β, θ are constants and $a_t \sim WN(0, \sigma_a^2)$.

   (a) What is the name of this model?

   (b) Give the conditions for $\{Z_t\}$ to be *i*)stationary, *ii*) causal *iii*) invertible.

   (c) Using *i*) $(1-x)^{-1} = \sum_{j=0}^{\infty} x^j$ and *ii*) first principle, find the stationary solution of $\{Z_t\}$ for the case $|\theta| < 1$.

   (d) Find the stationary solution of $\{Z_t\}$ for the case $|\theta| > 1$.

   (e) Let $|\theta| < 1$. Find the variance and autocorrelation function of $\{Z_t\}$.

4. Consider the stationary process $Z_t = \mu + \phi Z_{t-1} + a_t$, where $a_t \sim WN(0, \sigma_a^2)$, and $|\phi| < 1$. Find $E(Z_t)$ and $\mathrm{Cov}(Z_t, Z_{t-k})$ for k =0,1,2,.....

5. Consider the process $Z_t = 0.5Z_{t-1} - 0.06Z_{t-2} + a_t$. Find the values of ψ_j, $j = 0, 1, 2, 3...$ if the process is written in the form of general linear model

$$Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \,.$$

6. Find the AR and MA representations of the following process:

$$Z_t + 0.6Z_{t-2} = a_t + 0.5a_{t-1}\,, \quad a_t \sim WN(0, \sigma^2)\,.$$

7. Consider the process

$$Z_t = 0.6Z_{t-1} + a_t - 0.2a_{t-1}\,, \quad a_t \sim WN(0, 4)\,.$$

   (a) Using Yule-Walker equations, find the ACVF and ACF of $\{Z_t\}$.

   (b) Find $\mathrm{Var}(\sum_{t=1}^{4} Z_t)$.

8. Identifying each of the following models as a specific ARIMA model, determine whether it is stationary, causal, or invertible.

i) $(1-B)Z_t = (1-1.5B)a_t$

ii) $(1-1.1B)Z_t = (1-1.7B+0.72B^2)a_t$

iii) $(1-0.6B)Z_t = (1-1.2B+0.2B^2)a_t$

iv) $(1-0.5B-0.5B^2)Z_t = (1-1.2B+0.2B^2)a_t$

v) $Z_t = 0.4Z_{t-1} + 0.45Z_{t-2} + a_t + a_{t-1} + 0.25a_{t-2}$

vi) $Z_t = 1.25Z_{t-1} - 0.25Z_{t-2} + a_t$

9. Consider the process $Z_t = 0.5Z_{t-1} - 0.06Z_{t-2} + a_t$, $a_t \sim WN(0,1)$. Find the ACF $\rho(k)$ for k=0,1,2,3,4....

10. Consider an MA(∞) process $Z_t = a_t + C(a_{t-1} + a_{t-2} + \cdots)$, where C is a fixed constant, $E(a_t) = 0$, and $\mathrm{Var}(a_t) = \sigma_a^2$.

    (a) Is Z_t weakly stationary?

    (b) Let $W_t = Z_t - Z_{t-1}$. Show that $\{W_t\}$ is a stationary MA(1) model.

    (c) Find the autocorrelation function of $\{W_t\}$.