STAT3007: Introduction to Stochastic Processes

Markov Chains - Some Special Examples

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Markov Chains with I.I.D. R.V.s

- Let X denote a discrete valued random variable (r.v.) whose values are non-negative integers and where $P(X = i) = a_i$ for $i = 0,1, \ldots$ and the sum of all a_i is 1.
- Let $X_1, X_2, \ldots, X_n, \ldots$ represent independent observations of X
- Several interesting Markov Chains can be created using this sequence.

Successive Maxima

- Given a process $\{X_n\}$, we consider the realized maximum

$$Y_n = \max\{X_1, X_2, \dots, X_n\}.$$

- We may write

$$Y_{n+1} = \max\{Y_n, X_{n+1}\}$$

- What is the transition probability matrix for the process $\{Y_n\}$?

Successive Maxima

- $\boldsymbol{P} = \begin{pmatrix} A_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & A_1 & a_2 & a_3 & \cdots \\ 0 & 0 & A_2 & a_3 & \cdots \\ 0 & 0 & 0 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$
- Where $A_k = a_0 + a_1 + \cdots + a_k$

Bids and Sales

- Suppose $X_1, X_2, \ldots$represent successive bids on a certain asset that is offered for sale.
- Then, $Y_n = \max\{X_1, \ldots, X_n\}$ is the maximum that is the bid up to stage n.
- Suppose that the bid that is accepted is the first bid that equals or exceeds a prescribed level M.
- The time of sale is $T = \min\{n \geq 1; Y_n \geq M\}$

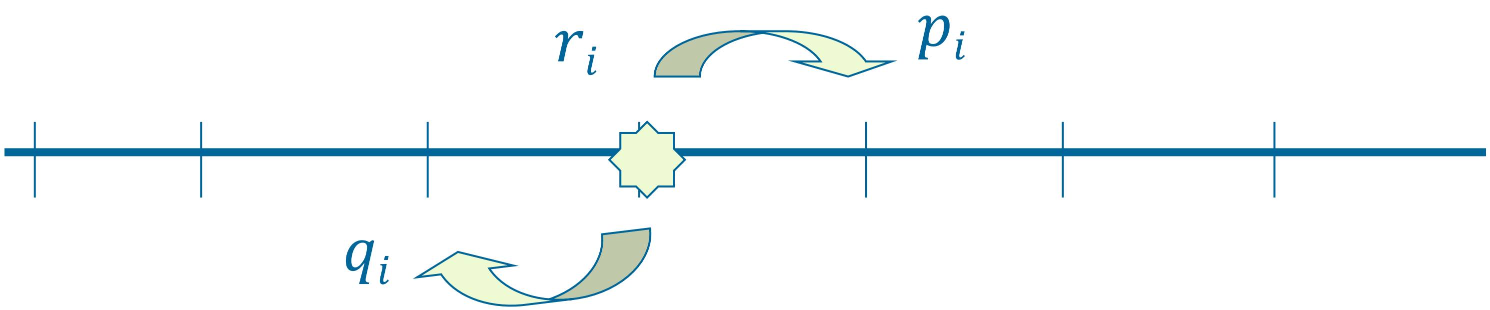
Bids and Sales

- What is the expected time of sale?
- Denote $v = E[T]$.
- Then, by first step analysis (conditional on $X_1 = k$), we have
  - $v = 1 + vP(X_1 < M)$
  - $\Rightarrow v = \frac{1}{P(X_1 \geq M)} = \frac{1}{a_M + a_{M+1} + \cdots}$

Partial Sums

- Given the same process of i.i.d. r.v.s $\{X_n\}$, define $Z_n = X_1 + \cdots + X_n$
- What is the transition probability matrix for the process $\{Z_n\}$?

One-Dimensional Random Walk



$$\boldsymbol{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \end{array} \begin{pmatrix} r_0 & p_0 & 0 & \cdots & 0 & \cdots & \cdots & \\ q_1 & r_1 & p_1 & \cdots & 0 & \cdots & \cdots & \\ 0 & q_2 & r_2 & \cdots & 0 & \cdots & \cdots & \\ & \ddots & & & & & & \\ & & 0 & \cdots & q_i & r_i & p_i & \cdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Gambler's Ruin

- Consider two players in successive games.
- Player A has an initial fortune k and player B has a limited fortune h. Let $N = k + h$.
- Denote X_n be player A's fortune after the nth game. The event of reaching state $k = 0$ is known as the "gambler's ruin".
- Suppose A has probability p_k of winning one unit, r_k getting even and $q_k = 1 - p_k - r_k$ of losing one unit when the game is in state k. Then the transition probability matrix is similar to the previous matrix (with 0 and N as absorbing states).

Gambler's Ruin

- Take $r_k = 0$, p_k and r_k independent of k

- $$\boldsymbol{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Gambler's Ruin

- $u_i = P(X_n$ reaches 0 before $N | X_0 = i)$
- Obviously, we have $u_0 = 1$ and $u_N = 0$.
- By first step analysis, we have
  - $u_k = pu_{k+1} + qu_{k-1}$
- If we consider $d_k = u_k - u_{k-1}$, then we have (for $k = 1, \dots, N$)
  - $0 = pu_{k+1} + qu_{k-1} - u_k$
  - $= p(u_{k+1} - u_k) - q(u_k - u_{k-1})$
  - $\Rightarrow 0 = pd_{k+1} - qd_k$

Gambler's Ruin

- Hence
  - $d_2 = \left(\frac{q}{p}\right) d_1, d_3 = \left(\frac{q}{p}\right) d_2 = \left(\frac{q}{p}\right)^2 d_1$, etc. until $d_N = \left(\frac{q}{p}\right)^{N-1} d_1$
- Moreover, we have
  - $u_k - u_0 = \sum_{j=1}^{k} (u_j - u_{j-1}) = \sum_{j=1}^{k} d_j = \sum_{j=0}^{k-1} \left(\frac{q}{p}\right)^j d_1 \Rightarrow u_k = 1 + d_1 \sum_{j=0}^{k-1} \left(\frac{q}{p}\right)^j$

Gambler's Ruin

- Since $u_N = 0$
  - $0 = u_N = 1 + d_1 \sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j \Rightarrow d_1 = \frac{-1}{\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j}$
- Hence
  - $u_k = 1 - \frac{\sum_{j=0}^{k-1} \left(\frac{q}{p}\right)^j}{\sum_{j=0}^{N-1} \left(\frac{q}{p}\right)^j} = \begin{matrix} 1 - \frac{k}{N} \text{ if } p = q = \frac{1}{2} \\ 1 - \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N} \text{ if } p \neq q \end{matrix}$

Gambler's Ruin

- Now we find the expected duration of the game. Let $v_i = E[T|X_0 = i]$
- Using similar calculations as for u_i, we can find
  - $v_i = \frac{1}{p(1-\theta)}\left[N\left(\frac{1-\theta^i}{1-\theta^N}\right) - i\right]$, where $\theta = \frac{q}{p} \neq 1$
  - When $p = q = \frac{1}{2}$, $v_k = k(N-k)$

Gambler's Rule

- If we allow the transition probabilities to depend on the state of the game, the transition matrix becomes
- $$\boldsymbol{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & r_1 & p_1 & \cdots & 0 & 0 \\ 0 & q_2 & r_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & \cdots & r_{N-1} & p_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Gambler's Ruin

- And the same arguments can be repeated to find the probabilities of ruin
  - $u_0 = 1, u_N = 0$
  - $u_k = \frac{\rho_k + \cdots + \rho_{N-1}}{1 + \rho_1 + \rho_2 + \cdots + p_{N-1}}$, for $k = 1, \ldots N-1$, where
    $\rho_k = \frac{q_1 q_2 \cdots q_k}{p_1 p_2 \cdots p_k}$

Gambler's Ruin

- And we also find the expected time to ruin
  - $v_k = \left(\frac{\Phi_1+\cdots+\Phi_{N-1}}{1+\rho_1+\cdots+\rho_{N-1}}\right)(1+\rho_1+\cdots+\rho_{k-1}) - (\Phi_1+\cdots+\Phi_{k-1})$, for $k = 1, \ldots, N-1$
  - Where $\Phi_k = \left(\frac{1}{q_1}+\frac{1}{q_2\rho_1}+\cdots+\frac{1}{q_k\rho_{k-1}}\right)\rho_k$, $k = 1, \ldots, N-1$