STAT 6104 Time Series
2017 First Semester Final Exam

1) (45 marks) Given the observations $\{X_1, X_2, X_3, X_4\} = (-1.2, 2, -1.6, 0.8)$. Suppose that an MA(2) model

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}\,, \quad Z_t \sim N(0, \sigma^2)\,,$$

is proposed to model the data.

a) (6 marks) Find the sample autocovariance C_k and sample autocorrelation r_k for $k = 0, 1, 2$.

b) (6 marks) Write down the log-likelihood function for the data set (in terms of a covariance matrix).

c) (6 marks) Give a system of equations such that the solution of the system gives the method of moment estimators for θ_1, θ_2 and σ^2.

d) (8 marks) Conditioned on $Z_t = X_t = 0$ for $t \le 0$, write each of Z_1 to Z_4 in terms of X_t and θ_1, θ_2. Discuss how to obtain the conditional least square estimators for θ_1, θ_2 and σ^2.

e) (7 marks) Given $(\hat\theta_1, \hat\theta_2, \hat\sigma^2) = (-1.57, 0.93, 0.4)$, find the residuals of the fitted model. Is the fitted model causal?

f) (6 marks) Conduct a Portmanteau test using $Q(3)$. (Hints: the 95 percentiles of χ^2_k, $k = 1, \ldots, 5$ are $3.84, 5.99, 7.81, 9.49, 11.1$, respectively.)

g) (6 marks) Find the 95% prediction intervals for the forecast of X_5 and X_6.

2) (10 marks) Given $a_t \overset{i.i.d.}{\sim} N(0, 1)$. Identify the following SARIMA models (state the order of the model):

a) $Z_t = -0.2Z_{t-1} + Z_{t-6} + 0.2Z_{t-7} + a_t + 1.2a_{t-6}$.

b) $Z_t = -4Z_{t-2} + a_t + 6a_{t-1} + 12a_{t-2} + 8a_{t-3}$.

State whether the models have stationary solutions. State whether the models are causal and invertible.

3) (15 marks) Consider the ARMA(2,1) model $X_t = 0.2X_{t-1} - 0.01X_{t-2} + Z_t + 0.3Z_{t-1}$, $Z_t \overset{iid}{\sim} N(0, 1)$. Find the ACVF $\gamma(k)$ for all integers k using Yule-Walker equations.

4) (9 marks) Suppose that for a time series of length 100, the AIC of the models AR(1), AR(2), MA(1), MA(2), ARMA(1,1), ARMA(1,2), ARMA(2,1) and ARMA(2,2) are 287, 286, 317, 303, 285, 287, 287, 286, respectively. Select the best model using AICC.

5) (6 marks) Consider the ARIMA(1,1,0) model $(1 - \phi B)(1 - B)X_t = Z_t$, where $Z_t \sim WN(0, \sigma^2)$. Give the expressions for the h-step ahead forecast X^n_{n+h} and the forecast variance P^n_{n+h} for $h = 1, 2, 3$.

6) (9 marks) Consider the GARCH model $(a + b < 1)$

$$\begin{aligned} X_t &= \sigma_t \epsilon_t\,, \quad \epsilon_t \overset{iid}{\sim} N(0, 1)\,, \\ \sigma_t^2 &= \omega + a\sigma_{t-1}^2 + bX_{t-1}^2\,. \end{aligned}$$

a) (2 marks) Express the GARCH model in terms of an ARCH model. What is the order of the ARCH model?

b) (4 marks) Prove or disproof: $\{X_t^2 - \sigma_t^2\}$ is a white noise sequence.

c) (3 marks) Given the estimates $(\hat\omega, \hat a, \hat b)$, discuss how to find the 95% prediction interval for X_{n+1} using the data $\{X_1, \ldots, X_n\}$.

7) (6 marks) Consider the model

$$Y_t = Y_{t-1}^{\alpha} e^{Z_t}\,, \quad Z_t \overset{iid}{\sim} N(0, \sigma^2)\,,$$

for the time series $\{Y_1, \ldots, Y_n\}$. Using the idea of least-squares estimation, derive an estimator for each of α and σ^2 (write down the expression of the estimator in terms of $\{Y_1, \ldots, Y_n\}$.

–End–