# Solution to assignment 1

**Problem 2.1, 20'**

Table 1: Joint distribution of occupational categories of fathers and sons

| | son's occupation ($Y_2$) | | | | |
|---|---|---|---|---|---|
| father's occupation($Y_1$) | farm | operatives | craftsmen | sales | professional |
| farm | 0.018 | 0.035 | 0.031 | 0.008 | 0.018 |
| operatives | 0.002 | 0.112 | 0.064 | 0.032 | 0.069 |
| craftsmen | 0.001 | 0.066 | 0.094 | 0.032 | 0.084 |
| sales | 0.001 | 0.018 | 0.019 | 0.010 | 0.051 |
| professional | 0.001 | 0.029 | 0.032 | 0.043 | 0.130 |

Notation: $A =$ farm, $B =$ operatives, $C =$ craftsmen, $D =$ sales, $E =$ professional

a) $P(Y_1 = \text{A}) = P(Y_1 = A, Y_2 = A) + P(Y_1 = A, Y_2 = B) + P(Y_1 = A, Y_2 = C) + P(Y_1 = A, Y_2 = D) + P(Y_1 = A, Y_2 = E)$, i.e, the marginal distribution of father to be a farm is the sum of the first row of the Table 1, so, $P(Y_1 = \text{A}) = 0.11$.

Similarly, we have $P(Y_1 = \text{B}) = 0.279$, $P(Y_1 = \text{C}) = 0.277$, $P(Y_1 = \text{D}) = 0.099$, $P(Y_1 = \text{E}) = 0.235$.

b) Similar to a), we have $P(Y_2 = \text{A}) = 0.023$, $P(Y_2 = \text{B}) = 0.26$, $P(Y_2 = \text{C}) = 0.24$, $P(Y_2 = \text{D}) = 0.125$, $P(Y_2 = \text{E}) = 0.352$.

c) $P(Y_2 = A|Y_1 = A) = \frac{P(Y_2=A,Y_1=A)}{P(Y_1=A)} = \frac{0.018}{0.11} = 0.164$.

$P(Y_2 = B|Y_1 = A) = \frac{P(Y_2=B,Y_1=A)}{P(Y_1=A)} = \frac{0.035}{0.11} = 0.318$.

$P(Y_2 = C|Y_1 = A) = \frac{P(Y_2=C,Y_1=A)}{P(Y_1=A)} = \frac{0.031}{0.11} = 0.282$.

$P(Y_2 = D|Y_1 = A) = \frac{P(Y_2=D,Y_1=A)}{P(Y_1=A)} = \frac{0.008}{0.11} = 0.073$.

$P(Y_2 = E|Y_1 = A) = \frac{P(Y_2=E,Y_1=A)}{P(Y_1=A)} = \frac{0.018}{0.11} = 0.164$.

d) $P(Y_1 = A|Y_2 = A) = \frac{P(Y_2=A,Y_1=A)}{P(Y_2=A)} = \frac{0.018}{0.023} = 0.783$.

$P(Y_1 = B|Y_2 = A) = \frac{P(Y_2=A,Y_1=B)}{P(Y_2=A)} = \frac{0.002}{0.023} = 0.087$.

$P(Y_1 = C|Y_2 = A) = \frac{P(Y_2=A,Y_1=C)}{P(Y_2=A)} = \frac{0.001}{0.023} = 0.043$.

$P(Y_1 = D|Y_2 = A) = \frac{P(Y_2=A,Y_1=D)}{P(Y_2=A)} = \frac{0.001}{0.023} = 0.043$.

$P(Y_1 = E|Y_2 = A) = \frac{P(Y_2=A,Y_1=E)}{P(Y_2=A)} = \frac{0.001}{0.023} = 0.043$.

**Problem 2.2, 20'**

Assume that $X$ and $Y$ are independent random variables with density function $f(x)$ and $f(y)$, respectively, $a$ and $b$ are arbitrary constants. So the joint density function of $(X, Y)$ is $f(x, y) = f(x)f(y)$, we have,

**Property of expectation**:

$E(aX + bY) = \int(aX + bY)f(x, y)dxdy = \int(aX + bY)f(x)f(y)dxdy = aE(X) + bE(Y)$.

**Property of variance:**

$Var(aX + bY) = E(aX + bY)^2 - (E(aX + bY))^2 = a^2Var(X) + b^2Var(Y)$, by using the property of expectation, and a simple arrangement.

a) Let $a = a_1, b = a_2, X = Y_1, Y = Y_2$, we have,

$E(a_1Y_1 + a_2Y_2) = E(aX + bY) = aE(X) + bE(Y) = a\mu_1 + b\mu_2 = a_1\mu_1 + a_2\mu_2$.

$Var(a_1Y_1 + a_2Y_2) = Var(aX + bY) = a^2Var(X) + b^2Var(Y) = a^2\sigma_1^2 + b^2\sigma_2^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

b) Similar to a) we have, $E(a_1Y_1 - a_2Y_2) = a_1\mu_1 - a_2\mu_2$, $Var(a_1Y_1 - a_2Y_2) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

**Problem 2.3, 20'**

According to the assumption, we have $f(x, y, z) = Cf(x, z)g(y, z)h(z)$, where $C$ is a constant.

a) $p(x|y, z) = \frac{f(x,y,z)}{f(y,z)} = \frac{f(x,y,z)}{\int f(x,y,z)dx} = \frac{f(x,z)g(y,z)h(z)}{\int f(x,z)g(y,z)h(z)dx} = \frac{f(x,z)}{\int f(x,z)dx} \propto f(x, z)$.

b) $p(y|x, z) = \frac{f(x,y,z)}{f(x,z)} = \frac{f(x,y,z)}{\int f(x,y,z)dy} = \frac{f(x,z)g(y,z)h(z)}{\int f(x,z)g(y,z)h(z)dy} = \frac{g(y,z)}{\int g(y,z)dy} \propto g(y, z)$.

c) $p(x, y|z) = \frac{f(x,y,z)}{f(z)} = \frac{f(x,y,z)}{\int f(x,y,z)dxdy} = \frac{f(x,z)g(y,z)h(z)}{\int f(x,z)g(y,z)h(z)dxdy} = \frac{f(x,z)g(y,z)}{\int f(x,z)dx \int g(y,z)dy} = f(x|z)g(y|z)$, so, $X$ and $Y$ are conditionally independent given $Z$.

**Problem 2.5, 20'**

a)

$P(X = 1, Y = 1) = P(Y = 1|X = 1)P(X = 1) = 0.5 \times 0.4 = 0.2$

$P(X = 1, Y = 0) = P(Y = 0|X = 1)P(X = 1) = 0.5 \times 0.6 = 0.3$

$P(X = 0, Y = 1) = P(Y = 1|X = 0)P(X = 0) = 0.5 \times 0.6 = 0.3$

$P(X = 0, Y = 0) = P(Y = 0|X = 0)P(X = 0) = 0.5 \times 0.4 = 0.2$

Table 2: Joint distribution of $X$ and $Y$

| Y \ X | 1 | 0 |
|---|---|---|
| 1 | 0.2 | 0.3 |
| 0 | 0.3 | 0.2 |

b) $P(Y = 1) = 0.2 + 0.3 = 0.5$, $P(Y = 0) = 0.3 + 0.2 = 0.5$, $E(Y) = 0.5 \times 1 + 0.5 \times 0 = 0.5$.

c) $E(Y^2) = 0.5 \times 1 + 0.5 \times 0 = 0.5$, $Var(Y) = E(Y^2) - (EY)^2$=0.5-0.25=0.25.

$E(Y^2|X = 1) = P(Y = 1|X = 1) \times 1 = 0.4$, $E(Y|X = 1) = P(Y = 1|X = 1) \times 1 = 0.4$, $Var(Y|X = 1) = E(Y^2|X = 1) - (E(Y|X = 1))^2$=0.4-0.16=0.24.

$E(Y^2|X = 0) = P(Y = 1|X = 0) \times 1 = 0.6$, $E(Y|X = 1) = P(Y = 1|X = 1) \times 1 = 0.6$,

$Var(Y|X=1) = E(Y^2|X=1) - (E(Y|X=1))^2$=0.6-0.36=0.24.

Since $X$ and $Y$ are dependent, if $X$ is given, we get more information about $Y$, then we have less uncertainty about $Y$.

d) $P(X=0|Y=1) = \frac{P(X=0,Y=1)}{P(Y=1)} = \frac{0.3}{0.5} = 0.6$.

**Problem 3.1, 20'**

a)

$$\begin{aligned} P(Y_1 = y_1, \cdots, Y_{100} = y_{100}|\theta) &= \prod_{i=1}^{100} P(Y_i = y_i|\theta) \\ &= \prod_{i=1}^{100} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{\sum_{i=1}^{100} y_i}(1-\theta)^{100-\sum_{i=1}^{100} y_i} \end{aligned}$$

$P(\sum_{i=1}^{100} y_i = y|\theta) = C_{100}^{y}\theta^{y}(1-\theta)^{100-y}$.

b) For $y = 57$, let $g(\theta) = P(\sum_{i=1}^{100} y_i = 57|\theta) = C_{100}^{57}\theta^{57}(1-\theta)^{43}$, we have,

$g(0) = 0, g(1) = 0$
$g(0.1) = 4.107157 \times 10^{-31}, g(0.2) = 3.738459 \times 10^{-16}, g(0.3) = 1.306895 \times 10^{-8}$
$g(0.4) = 2.285792 \times 10^{-4}, g(0.5) = 3.006864 \times 10^{-2}, g(0.6) = 6.672895 \times 10^{-2}$
$g(0.7) = 1.853172 \times 10^{-3}, g(0.8) = 1.003535 \times 10^{-7}, g(0.9) = 9.395858 \times 10^{-18}$

Plot of $g$ see Figure 1.

c) According to the assumption, for $\theta = (0, 0.1, \cdots, 1)$, $\theta_0 \in \theta$, we have

$$\begin{aligned} h(\theta_0) &= P(\theta_0|\sum_{i=1}^{100} y_i = 57) \\ &= \frac{P(\sum_{i=1}^{100} y_i = 57|\theta_0)P(\theta_0)}{P(\sum_{i=1}^{100} y_i = 57)} \\ &= \frac{P(\sum_{i=1}^{100} y_i = 57|\theta_0)P(\theta_0)}{\sum_{i=1}^{11} P(\sum_{i=1}^{100} y_i = 57|\theta_i)P(\theta_i)} \\ &= \frac{P(\sum_{i=1}^{100} y_i = 57|\theta_0)}{\sum_{i=1}^{11} P(\sum_{i=1}^{100} y_i = 57|\theta_i)} \\ &= \frac{g(\theta_0)}{\sum_{i=1}^{11} g(\theta_i)} \end{aligned}$$

$h(0) = 0, h(1) = 0$
$h(0.1) = 4.153701 \times 10^{-30}, h(0.2) = 3.780824 \times 10^{-15}, h(0.3) = 1.321705 \times 10^{-7}$
$h(0.4) = 2.311695 \times 10^{-3}, h(0.5) = 3.040939 \times 10^{-1}, h(0.6) = 6.748515 \times 10^{-1}$
$h(0.7) = 1.874172 \times 10^{-2}, h(0.8) = 1.014907 \times 10^{-6}, h(0.9) = 9.502335 \times 10^{-17}$

Plot of $h$ see Figure 1.

d) $k(\theta) = p(\theta) \times P(\sum_{i=1}^{100} y_i = 57|\theta) = P(\sum_{i=1}^{100} y_i = 57|\theta) = C_{100}^{57}\theta^{57}(1-\theta)^{43}$, Plot of $k$ see Figure 1.

e) $B(\theta) = \text{Beta}(58, 44)$, Plot of $B$ see Figure 1.

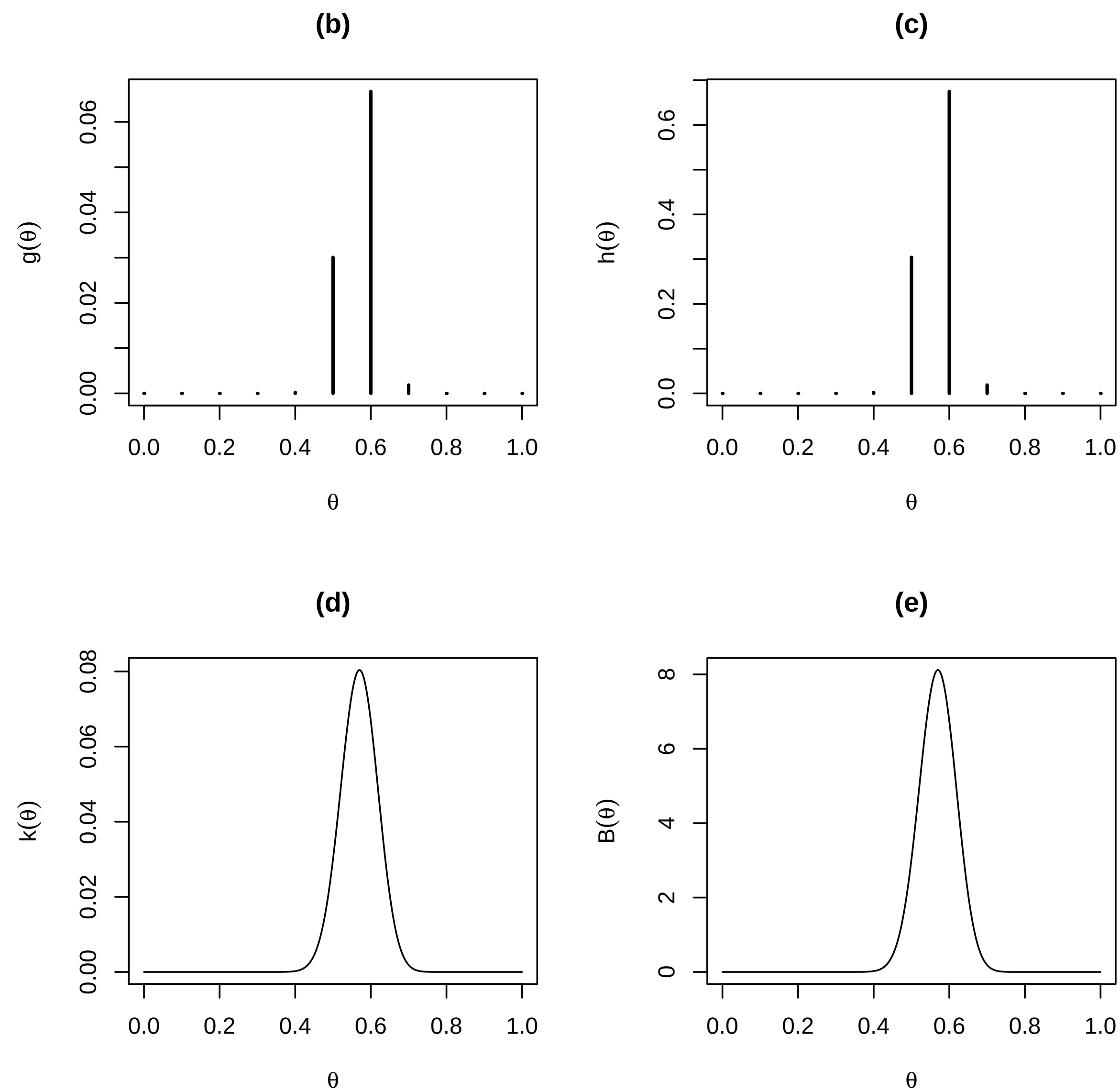


Figure 1: Plots of problem 3.1 (b-e). As $g(\theta) = C_1 h(\theta)$, (b) and (c) give the same estimation of $\theta$, denoted by $\hat{\theta}_1 = 0.6$. In fact, $k(\theta) = C_2 B(\theta)$, so, (d) and (e) give the same estimation of $\theta$, denoted by $\hat{\theta}_2 = 0.57$. Here $C_1$ and $C_2$ are constants.