STAT 6104 Final Exam Solution. 2015.

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（答題不得寫在紅綫外） 第 頁

1. A → 2
B → 5
C → 1
D → 4
F → 3

2. (a) $EW_t = EX_t + EY_t = 0 + E a_t + \theta E a_{t-2} = 0$

(b) $\gamma(k) = Cov(W_t, W_{t+k}) = Cov[X_t + Y_t, X_{t+k}, Y_{t+k})$
$= Cov(X_t, X_{t+k}) + Cov(Y_t, Y_{t+k})$ [X_t and Y_t are independent].
$= \gamma_1(k) + \gamma_2(k)$

$\begin{cases} \gamma_1(0) = \phi\gamma_1(1) + 1 \\ \gamma_1(1) = \phi\gamma_1(0) \end{cases} \Rightarrow \gamma_1(0) = \frac{1}{1-\phi^2} \quad \gamma_1(k) = \frac{\phi^{|k|}}{1-\phi^2}, \ |k| \geq 1$

$\gamma_2(0) = 1 + \theta^2$
$\gamma_2(1) = 0$
$\gamma_2(2) = \theta$
$\gamma_2(k) = 0, \ k \geq 3.$

$$\therefore \gamma(k) = \begin{cases} 1 + \theta^2 + \frac{1}{1-\phi^2}, & k = 0 \\ \theta + \frac{\phi^2}{1-\phi^2}, & k = \pm 2 \\ \frac{\phi^{|k|}}{1-\phi^2}, & \text{other} \end{cases}$$

3. (a) AR(1), casual, invertible.

(b) $EY_1 = \mu + 0.3EY_0 + 0 \quad \because EY_1 = EY_0 \quad \therefore EY_1 = \frac{\mu}{0.7}$

(c) $Z_3^2 + Z_2^2 = (Y_3 - \mu - 0.3Y_2)^2 + (Y_2 - \mu - 0.3Y_1)^2$
$= (6 - \mu - 0.3\times4)^2 + (4 - \mu - 0.3\times3)^2$
$= 2\mu^2 - 15.8\mu + 3.1^2 + 4.8^2$
$\therefore \hat{\mu} = \frac{-15.8}{-2\times2} = 3.95$

4. $\begin{cases} \hat{\gamma}(0) = \theta_1^2 + \theta_2^2 \\ \hat{\gamma}(2) = \theta_1\theta_2 \end{cases} \Rightarrow \begin{cases} \theta_1^2 + \theta_2^2 = 2.81 \\ \theta_1\theta_2 = 1.4 \end{cases}$

$\Rightarrow \begin{cases} \theta_1 = \pm 1.134 \\ \theta_2 = \pm 1.234 \end{cases}$ Or $\begin{cases} \theta_1 = \pm 1.234 \\ \theta_1 = \pm 1.134 \end{cases}$

5. (a) $a_{2014} = Y_{2014} - 0.4Y_{2013} + 0.04Y_{2012} - 0.3a_{2013}$
$= 12.41$
$a_{2015} = Y_{2015} - 0.4Y_{2014} + 0.04Y_{2013} - 0.3a_{2014}$
$= 1.677$
$\therefore Y_{2016}^{2015} = 0.4Y_{2015} - 0.04Y_{2014} + 0.3a_{2015}$
$= 4.5831$

$Y_{2017}^{2015} = 0.4Y_{2016}^{2015} - 0.04Y_{2015}$ ~~$+0.04Y_{2013}$~~
$= 1.3532$

(b) $e_{2015}^{(1)} = Y_{2016} - Y_{2016}^{2015} = a_{2016}$
$\therefore Var(e_{2015}^{(1)}) = Var(a_{2016}) = 9$
$\therefore$ 95% prediction interval of Y_{2016} is $(4.5831 - 1.96\times 3, 4.5831 + 1.96\times 3)$
s.t $(-1.2969, 10.4631)$

6. (a) $\gamma(0) = (1+\theta^2)\sigma^2$
$\gamma(1) = \theta\sigma^2$
$\gamma(2) = 0$
$\therefore g_Y(z) = \theta\sigma^2 z^{-1} + (1+\theta^2)\sigma^2 + \theta\sigma^2 z$

(b) $\begin{cases} \gamma(0) = \phi\gamma(1) + \sigma^2 \\ \gamma(1) = \phi\gamma(0) \end{cases}$

$\Rightarrow \gamma(0) = \phi^2\gamma(0) + \sigma^2 \Rightarrow \gamma(0) = \frac{\sigma^2}{1-\phi^2}$ $\gamma(k) = \frac{\phi^{|k|}\sigma^2}{1-\phi^2}$

$\therefore g_Y(z) = \sum_{j=-\infty}^{+\infty} \frac{\sigma^2\phi^{|j|}}{1-\phi^2} z^j$