6 marks

1a. $C_0 = 2.16$ $\qquad r_1 = 1$

$C_1 = -1.72$ $\qquad r_2 = -0.7963$

$C_2 = 0.88$ $\qquad r_3 = 0.4074$

1b 6 marks

$$X = \begin{pmatrix} -1.2 \\ 2 \\ -1.6 \\ 0.8 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}$$

likelihood function:

$$L(\theta, \sigma^2) = \frac{1}{(2\pi)^2 |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} X' \Sigma^{-1} X\right)$$

loglikelihood $\quad \ell(\theta, \sigma^2) = \ln(L(\theta, \sigma^2))$

$$= -2\ln(2\pi) - \frac{1}{2}\ln|\Sigma| - \frac{1}{2} X' \Sigma^{-1} X$$

6

1c. $\gamma_0 = (1+\theta_1^2+\theta_2^2)\sigma^2$

$\gamma_1 = (\theta_1+\theta_1\theta_2)\sigma^2$

$\gamma_2 = \theta_2\sigma^2$

$$\begin{cases} C_0 = (1+\theta_1^2+\theta_2^2)\sigma^2 \\ C_1 = (\theta_1+\theta_1\theta_2)\sigma^2 \\ C_2 = \theta_2\sigma^2 \end{cases}$$

8

1d. $Z_t = X_t = 0$, for $t \le 0$ $\qquad$ (0.5 each)

$Z_1 = -1.2$ $\qquad Z_1 = X_1$

$Z_2 = 1.2\theta_1 + 2$ $\qquad Z_2 = X_2 - \theta_1 Z_1 = X_2 - \theta_1 X_1$

$Z_3 = 1.2\theta_2 - 1.2\theta_1^2 - 2\theta_1 - 1.6$ $\qquad Z_3 = X_3 - \theta_1 Z_2 - \theta_2 Z_1 = X_3 - \theta_1(X_2 - \theta_1 X_1) - \theta_2 X_1$

$Z_4 = 1.2\theta_1^3 + 2\theta_1^2 - 2.4\theta_1\theta_2 + 1.6\theta_1 - 2\theta_2 + 0.8$

$= X_3 - \theta_1 X_2 + (\theta_1^2 - \theta_2) X_1$

Minimize $\sum_{i=1}^{4} Z_i^2$ to obtain $\hat\theta_1$, $\hat\theta_2$

$\hat\sigma^2 = \frac{1}{4}\sum_{i=1}^{4} Z_i^2$

$Z_4 = X_4 - \theta_1 Z_3 - \theta_2 Z_2$

$= X_4 - \theta_1(X_3 - \theta_1 X_2 + (\theta_1^2 - \theta_2)X_1) - \theta_2(X_2 - \theta_1 X_1)$

$= X_4 - \theta_1 X_3 + (\theta_1^2 - \theta_2) X_2 + (-\theta_1(\theta_1^2 - \theta_2) + \theta_1\theta_2) X_1$

7 marks

1e $X_t = Z_t - 1.57 Z_{t-1} + 0.93 Z_{t-2}$

Residual : $Z_t \sim N(0, 0.4)$

$Z_1 = -1.2$

$Z_2 = 0.116$ 0.5 each if wrong

$Z_3 = -0.30188$

$Z_4 = 0.2181684$

Causual.

6 marks

1f. $\bar{Z} = -0.2919279$

$C_0 = 0.3128243$ $C_1 = -0.09489105$

$C_2 = 0.05427993$ $C_3 = -0.1158011$

$r(0) = 1$ $r(1) = -0.3033365$

$r(2) = 0.1735175$ $r(3) = -0.3701792$

$Q(3) = 4 \times (4+2) \times \sum_{j=1}^{3} \frac{r_z^2(j)}{4-j} \rightarrow 2$

$= 4.386179 > 3.84 = \chi^2_{0.05}(4-3)$

$\therefore$ Not white noise

6 marks

1g. $\hat{X}_5 = Z_5 - 0.6232728$

$X_5^4 = -0.6232728$ $e_4(1) = Z_5$ $P_5^4 = Var(Z_5) = 0.4$

$\therefore$ 95% CI for X_5^4 $(-0.6232728 - 1.96\sqrt{0.4}, -0.6232728 + 1.96\sqrt{0.4})$

$= (-1.862886, 0.61634)$

$\hat{X}_6 = 0.2028966 + Z_6 - 1.57 Z_5$

$X_6^4 = 0.2028966$ $e_4(2) = Z_6 - 1.57 Z_5$ $P_6^4 = 0.4(1 + 1.57^2)$

$= 1.38596$

$\therefore$ 95% CI for X_6^4 is $(0.2028966 \pm 1.96\sqrt{1.38596})$

$= (-2.104549, 2.510342)$

5

$(1+0.2B)(1-B^6)$

2a. $Z_t(1+0.2B-B^6-0.2B^7) = a_t(1+1.2B^6)$

Model : SARIMA $(1,0,0) \times (0,1,1)_6$

not stationary , not causual , not invertible.

5

2b. $Z_t(1+4B^2) = a_t(1+2B)^3$

$\therefore$ SARIMA $(2,0,3) \times (0,0,0)_0$

stationary , not causual , not invertible.

15

3. $cov(X_t, Z_t) = 1$

$cov(X_t, Z_{t-1}) = 0.5$

$$\left.\begin{aligned}\gamma_0 &= 0.2\gamma_1 - 0.01\gamma_2 + 1 + 0.3\times 0.5\\ \gamma_1 &= 0.2\gamma_0 - 0.01\gamma_1 + 0.3\\ \gamma_2 &= 0.2\gamma_1 - 0.01\gamma_0\end{aligned}\right\} \quad 2 \text{ each}$$

$\gamma_k = 0.2\gamma_{k-1} - 0.01\gamma_{k-2}$ for $k \geq 3$

$$\Rightarrow \left.\begin{aligned}\gamma_0 &= 1.25827\\ \gamma_1 &= 0.546192\\ \gamma_k &= 0.2\gamma_{k-1} - 0.01\gamma_{k-2} \quad k\geq 2\end{aligned}\right\} - ① \text{ each}$$

9

4. $$\left.\begin{aligned}AIC &= -2\log L + 2(p+q+1)\\ AICc &= -2\log L + \frac{2(p+q+1)n}{n-p-q-2}\end{aligned}\right\} -5 \text{ if wrong}$$

AICc : AR(1) : 287.1237 AR(2) : 286.25

MA(1) : 317.1237 MA(2) : 303.25

-3 { ARMA(1,1) : 285.25 ARMA(1,2) : 287.4211

ARMA(2,1) : 287.4211 ARMA(2,2) : 286.6383

-1 $\therefore$ ARMA(1,1) should be chosen.

6 marks

5. $X_t = (\phi+1)X_{t-1} - \phi X_{t-2} + Z_t$

$h=1$

$X_{n+1} = (\phi+1)X_n - \phi X_{n-1} + Z_{n+1}$

$X^n_{n+1} = (\phi+1)X_n - \phi X_{n-1}$

$e_n(1) = Z_{n+1}$ $\quad P^n_{n+1} = \sigma^2$

$h=2$

$X_{n+2} = (\phi^2+\phi+1)X_n - \phi(\phi+1)X_{n-1} + (\phi+1)Z_{n+1} + Z_{n+2}$

$X^n_{n+2} = (\phi^2+\phi+1)X_n - \phi(\phi+1)X_{n-1}$

$e_n(2) = (\phi+1)Z_{n+1} + Z_{n+2}$

$P^n_{n+2} = ((\phi+1)^2+1)\sigma^2 = (\phi^2+2\phi+2)\sigma^2$

$h=3$

$X_{n+3} = (\phi+1)(\phi^2+1)X_n - \phi(\phi^2+\phi+1)X_{n-1} + (\phi^2+\phi+1)Z_{n+1}$
$+ (\phi+1)Z_{n+2} + Z_{n+3}$

$X^n_{n+3} = (\phi+1)(\phi^2+1)X_n - \phi(\phi^2+\phi+1)X_{n-1}$

$e_n(3) = (\phi^2+\phi+1)Z_{n+1} + (\phi+1)Z_{n+2} + Z_{n+3}$

$P^n_{n+3} = ((\phi^2+\phi+1)^2 + (\phi+1)^2 + 1)\sigma^2$

2 marks

6a. $\sigma_t^2 = w + a\sigma_{t-1}^2 + bX_{t-1}^2$

$$\sigma_t^2(1-aB) = w + bX_{t-1}^2$$

$$\sigma_t^2 = \frac{w + bX_{t-1}^2}{1-aB}$$

$$= \frac{w}{1-a} + \sum_{i=1}^{\infty} b a^{i-1} X_{t-i}^2$$

$\therefore$ ARCH(∞)

4 marks

6b. Proof:

$$E(X_t^2 - \sigma_t^2) = 0$$

$$Var(X_t^2 - \sigma_t^2) = E[(X_t^2 - \sigma_t^2)^2] - E[(X_t^2 - \sigma_t^2)]^2$$

$$= E(X_t^4 - 2X_t^2\sigma_t^2 + \sigma_t^4)$$

$$= 2E(\sigma_t^4)$$

$$E(\sigma_t^2) = w + aE(\sigma_t^2) + bE(\sigma_t^2)$$

$$E(\sigma_t^2) = \frac{w}{1-a-b}.$$

$$E(\sigma_t^4) = E(w^2 + a^2\sigma_{t-1}^4 + b^2X_{t-1}^4 + 2wa\sigma_{t-1}^2 + 2wbX_{t-1}^2 + 2ab\sigma_{t-1}^2X_{t-1}^2)$$

$$E(\sigma_t^4) = \frac{w^2 + \frac{2w^2(a+b)}{1-a-b}}{1-a^2-3b^2-2ab}.$$

3 marks

6c. Condition on σ_0^2 and $X_0^2 = 0$

Find $\sigma_1^2 = \hat{w} + \hat{a}\sigma_0^2 + \hat{b}X_0^2 = \hat{w}$

Find $\sigma_2^2 \to \sigma_3^2 \to \cdots \to \sigma_n^2$ recursively

Find $\sigma_{n+1}^2 = \hat{w} + \hat{a}\sigma_n^2 + bX_n^2$

$$E(X_{n+1}) = E(\sigma_{n+1}\varepsilon_{n+1}) = 0$$

$$Var(X_{n+1}) = E(\sigma_{n+1}^2\varepsilon_{n+1}^2) = E(\sigma_{n+1}^2) = \sigma_{n+1}^2$$

95% PI : $(-1.96\sqrt{\sigma_{n+1}^2},\ 1.96\sqrt{\sigma_{n+1}^2})$

6 marks

7. $Y_t = Y_{t-1}^{\alpha} e^{Z_t}$

$\log Y_t = \alpha \log Y_{t-1} + Z_t$

$Z_t = \log Y_t - \alpha \log Y_{t-1}$ $\qquad X_t = \log Y_t$

① Find $Z_2, Z_3, \dots Z_n$ in terms of α by $Y_1, \dots Y_n$

② Minimize $\sum_{i=2}^{n} Z_i^2$ to find $\hat{\alpha}$

③ Calculate $\hat{Z}_t = \log Y_t - \hat{\alpha} \log Y_{t-1}$ for $t = 2, 3, \dots, n$

④ $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=2}^{n} \hat{Z}_i^2$

$$\min_{\alpha} \sum_{t=2}^{n} (X_t - \alpha X_{t-1})^2 = \min_{\alpha} (X_2 - \alpha X_1)^2 + (X_3 - \alpha X_2)^2 + \dots + (X_n - \alpha X_{n-1})^2$$

$$2(X_2 - \alpha X_1)(-X_1) + 2(X_3 - \alpha X_2)(-X_2) + \dots + 2(X_n - \alpha X_{n-1})(-X_{n-1}) = 0$$

$$\hat{\alpha} = \frac{X_1 X_2 + \dots + X_{n-1} X_n}{X_1^2 + \dots + X_{n-1}^2} = \frac{\sum_{i=1}^{n-1} X_i X_{i+1}}{\sum_{i=1}^{n-1} X_i^2}$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=2}^{n} (X_i - \hat{\alpha} X_{i-1})^2$$

$$\therefore \quad \hat{\alpha} = \frac{\sum_{i=1}^{n-1} (\log Y_i)(\log Y_{i+1})}{\sum_{i=1}^{n-1} (\log Y_i)^2}$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=2}^{n} \left[(\log Y_i - \hat{\alpha} \log Y_{i-1})^2 \right]$$