STAT 3007 Tutorial 1 Suggested solutions

Example 1. View $1(A)$ as a discrete random variable

$1(A)$	1	0
$\Pr$	$\Pr(A)$	$1-\Pr(A)$

$$E\big(1(A)\big) = 1 \times \Pr(A) + 0 \times \big(1-\Pr(A)\big) = \Pr(A) \qquad \square$$

Example 2.
$$E(X) = E\Big(\sum_{k=1}^{X} 1\Big) = E\Big(\sum_{k=1}^{\infty} 1(k \le X)\Big)$$
$$\overset{(*)}{=} \sum_{k=1}^{\infty} E\big(1(X \ge k)\big) = \sum_{k=1}^{\infty} \Pr(X \ge k) \qquad \square$$

(*) A remark for mathematical rigour:

Fubini-Tonelli Theorem guarantees that we can exchange the order of integration / summation and expectation when the integrand / summand is a nonnegative measurable function in $\mathbb{R}$.

Example 3. Analogous to Example 2.

Example 4.
$$E(T) = \sum_t t \Pr(T=t)$$
$$= \sum_t t\big(\Pr(T=t, \xi \ge r) + \Pr(T=t, \xi < r)\big)$$
$$= \underbrace{\sum_t t \Pr(T=t \mid \xi \ge r)}\, \Pr(\xi \ge r) + \underbrace{\sum_t t \Pr(T=t \mid \xi < r)}\, \Pr(\xi < r)$$
$$= \underbrace{E(T \mid \xi \ge r)}\, \Pr(\xi \ge r) + \underbrace{E(T \mid \xi < r)}\, \Pr(\xi < r) \qquad \square$$

Example 5. Analogous to Example 4.

Example 6. $\Pr(X\in A) = \Pr(X\in A, Y\in A) + \Pr(X\in A, Y\notin A)$

$$\leq \Pr(Y\in A) + \Pr(X\neq Y)$$

By symmetry, $\Pr(Y\in A) \leq \Pr(X\in A) + \Pr(Y\neq X)$

$$\Rightarrow |\Pr(X\in A) - \Pr(Y\in A)| \leq \Pr(X\neq Y) \qquad \square$$

Example 7. $E(X) = E(E(X|p))$ $\qquad X \sim \text{Binomial}(8, p)$

$$= E(8p) = 8E(p) \qquad p \sim \text{Uniform}(0,1)$$

$$= 4$$

$$\mathrm{Var}(X) = E(\mathrm{Var}(X|p)) + \mathrm{Var}(E(X|p))$$

$$= E(8p(1-p)) + \mathrm{Var}(8p)$$

$$= 8E(p) - 8E(p^2) + 64\,\mathrm{Var}(p)$$

$$\mathrm{Var}(p) = E(p^2) - (E(p))^2$$

$$= 8E(p) - 8(E(p))^2 + 56\,\mathrm{Var}(p)$$

$$= 4 - 2 + \frac{14}{3} = \frac{20}{3} \qquad \square$$

Example 8. Analogous to Example 7.