STAT 3007 Introduction to Stochastic Processes
Tutorial 1 | Term 1, 2019−20

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18/19 September 2019

1 Expectations

The expectation of a random variable can be viewed as its "average performance".

- Discrete case: $\mathrm{E}(X) = \sum_i x_i \Pr(X = x_i)$, $\mathrm{E}[g(X)] = \sum_i g(x_i) \Pr(X = x_i)$.
- Continuous case with pdf $f(x)$: $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$, $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$.

We now consider a special type of functions/random variables: the indicator function.

The indicator function of an event A is defined as a boolean function, where $\mathbf{1}(A) = 1$ if A occurs and $\mathbf{1}(A) = 0$ otherwise. We can then easily obtain a useful result.

Example 1. (Expectation of an indicator) For an arbitrary event A and its indicator function $\mathbf{1}(A)$, show that $\mathrm{E}[\mathbf{1}(A)] = \Pr(A)$.

The use of indicator functions sometimes help simplify the problems. For example, the sum of a series of indicator functions can be used for counting according to some criteria.

Example 2. (Expectation in terms of tail probabilities, discrete case) Let X be a positive integer-valued random variable. Show that $\mathrm{E}(X) = \sum_{k=1}^{\infty} \Pr(X \geq k)$.
(The condition "positive" can be released to "nonnegative" by observing that $0 \times \Pr(X = 0) = 0$ makes no effect on the expectation.)

Example 3. (Expectation in terms of tail probabilities, continuous case) Let X be a nonnegative continuous random variable with pdf $f(x)$. Show that $\mathrm{E}(X) = \int_0^{\infty} \Pr(X \geq x) dx$.

2 Conditional Probabilities and Law of Total Probability

- Conditional probability: $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ if $\Pr(B) > 0$.
- The conditional probability is also additive for disjoint events: $\Pr(A|B) = \sum_{i=1}^{\infty} \Pr(A_i|B)$ if $\cup_{i=1}^{\infty} A_i = A$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

  CAUTION. $\Pr(A|B) \neq \sum_{i=1}^{\infty} \Pr(A|B_i)$ (refer to the Law of Total Probability).

- (Discrete case) Conditional pmf of X given $Y{=}y$: $p(x|y) = \Pr(X = x|Y = y)$ if $\Pr(Y{=}y) > 0$. Conditional expectation of X given $Y = y$: $\mathrm{E}(X|Y = y) = \sum_i x_i p(x_i|y)$.
- Similar terms can be defined for the continuous and mixed cases.
- Law of total probability: $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap B_i) = \sum_{i=1}^{\infty} \Pr(A|B_i)\Pr(B_i)$ if $\cup_{i=1}^{\infty} B_i = \Omega$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$.

Example 4. (Breaking down an expectation) Let T be a discrete random variable and ξ a continuous random variable (T and ξ are not necessarily independent). Show that for any $r \in \mathbb{R}$, $\mathrm{E}(T) = \mathrm{E}(T|\xi \geq r)\Pr(\xi \geq r) + \mathrm{E}(T|\xi < r)\Pr(\xi < r)$.

Example 5. (Breaking down a conditional expectation) Let T, M, K be three discrete random variables. Conditional on $M = m$, K can only take its value of either $m + 1$ or $m - 1$. Show that $\mathrm{E}(T|M = m) = \mathrm{E}(T|K = m + 1, M = m)\Pr(K = m + 1|M = m) + \mathrm{E}(T|K = m - 1, M = m)$ $\Pr(K = m - 1|M = m)$.

(These kinds of decompositions are very useful for the First Step Analysis in subsequent chapters.)

Example 6. (Comparing probabilities) Let X and Y be two random variables and A an arbitrary subset of $\mathbb{R}$. Show that $|\Pr(X \in A) - \Pr(Y \in A)| \leq \Pr(X \neq Y)$.

3 Hierarchical Structures and Tower Rules

- Hierarchical structures appear when the distribution of a random variable X depends on another random component Y. We usually consider the tower rules.
- Law of total expectation: $\mathrm{E}(X) = \mathrm{E}[\mathrm{E}(X|Y)]$.
- Law of total variance: $\mathrm{Var}(X) = \mathrm{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}[\mathrm{E}(X|Y)]$.
- "Consider X conditional on Y" $\approx$ "consider X by viewing Y as fixed".

Example 7. (Hierarchical structure) Suppose that the outcome of X is generated according to Binomial$(8, p)$, where p is randomly chosen from Uniform$(0, 1)$. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.
(Useful results for the unconditional case:
If $X \sim$ Binomial(N, p) where N, p are constants, then $\mathrm{E}(X) = Np$, $\mathrm{Var}(X) = Np(1 - p)$.
If $X \sim$ Uniform(a, b) where a, b are constants, then $\mathrm{E}(X) = (b + a)/2$, $\mathrm{Var}(X) = (b - a)^2/12$.)

Example 8. (Random sum) Let $X_1, X_2, \ldots$ be a series of independent and identically distributed Exponential random variables with parameter $\lambda = 0.5$ and N a Poisson(3) random variable. Define $S_N = X_1 + X_2 + \cdots + X_N$. Find $\mathrm{E}(S_N)$ and $\mathrm{Var}(S_N)$.
(Useful results for the unconditional case:
If $X \sim$ Exponential(λ) where λ is a constant, then $\mathrm{E}(X) = 1/\lambda$, $\mathrm{Var}(X) = 1/\lambda^2$.
if $X \sim$ Poisson(λ) where λ is a constant, then $\mathrm{E}(X) = \mathrm{Var}(X) = \lambda$.)

宋人有善為不龜手之藥者，世世以洴澼絖為事。客聞之，請買其方百金。聚族而謀曰：『我世世為洴澼絖，不過數金，今一朝而鬻技百金，請與之。』客得之，以說吳王。越有難，吳王使之將。冬，與越人水戰，大敗越人，裂地而封之。能不龜手，一也；或以封，或不免於洴澼絖，則所用之異也。

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