Fundamentals of Vibration Analysis and Vibroacoustics
Module 2 - Vibroacoustics of Musical Instruments
Assignment 1 - Axial vibration of undamped and damped bars

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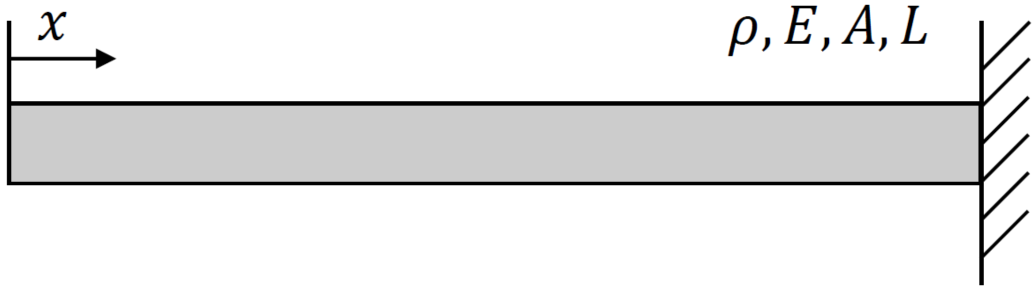
System schematic and parameters



$$\rho = 2700\,\mathrm{kg/m^3}\ ,\ E = 70\,\mathrm{GPa}\ ,\ L = 2\,\mathrm{m}\ ,\ b = h = 0.05\,\mathrm{m}$$

1 Natural frequencies and mode shapes in free-fixed configuration

Starting from the one-dimensional wave equation and applying it for the axial displacement $s(x,t)$ of point at position x at time t:

$$\frac{\partial^2 s(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 s(x,t)}{\partial t^2}$$

We already know that a solution to the equation is the standing wave expression, where space and time dependencies are separated by the mode shape function $\Phi(x)$ and the complex exponential $G(t)$:

$$s(x,t) = \Phi(x)\,G(t) = \big(A\,sin(kx) + B\,cos(kx)\big)\,e^{j\omega\,t}$$

The bar is in free-fixed condition, so the boundary conditions are

$$\begin{cases} N(0,t) = E\,S\,\frac{\partial s(x,t)}{\partial x}\Big|_{x=0} = 0 \;\Rightarrow\; E\,S\,A\,k\,e^{j\omega\,t} = 0 \;\Rightarrow\; A = 0 \text{ } (k = 0 \text{ is a trivial solution}) \\ s(L,t) = 0 \;\Rightarrow\; B\,cos(kL) = 0 \;\Rightarrow\; k^{\text{fr-fx}^{(i)}} = \frac{2i-1}{2L}\pi \;,\; i = 1,2,...,\infty \end{cases}$$

where S denotes the area of the bar's cross-section and N the normal axial load. From the second condition, the natural frequencies can be directly retrieved:

$$f_{\text{n}}^{\text{fr-fx}^{(i)}} = \frac{\omega_{\text{n}}^{\text{fr-fx}^{(i)}}}{2\pi} = \frac{c\,k^{\text{fr-fx}^{(i)}}}{2\pi} = \frac{2i-1}{4L}\sqrt{\frac{E}{\rho}} \;,\; i = 1,2,...,\infty$$

Our analysis is restricted to the frequency band $[0, f_{\text{max}}] = [0, 10k]$ Hz, so i assumes values within a finite set of indices:

$$\begin{aligned} f_{\text{max}} &= \frac{2\,i_{\text{max}}^{\text{fr-fx}} - 1}{4L}\sqrt{\frac{E}{\rho}} \\ \Rightarrow\; \lfloor i_{\text{max}}^{\text{fr-fx}} \rfloor &= \left\lfloor 2Lf_{\text{max}}\sqrt{\frac{\rho}{E}} + \frac{1}{2} \right\rfloor = \lfloor 8.35 \rfloor = 8 \end{aligned}$$

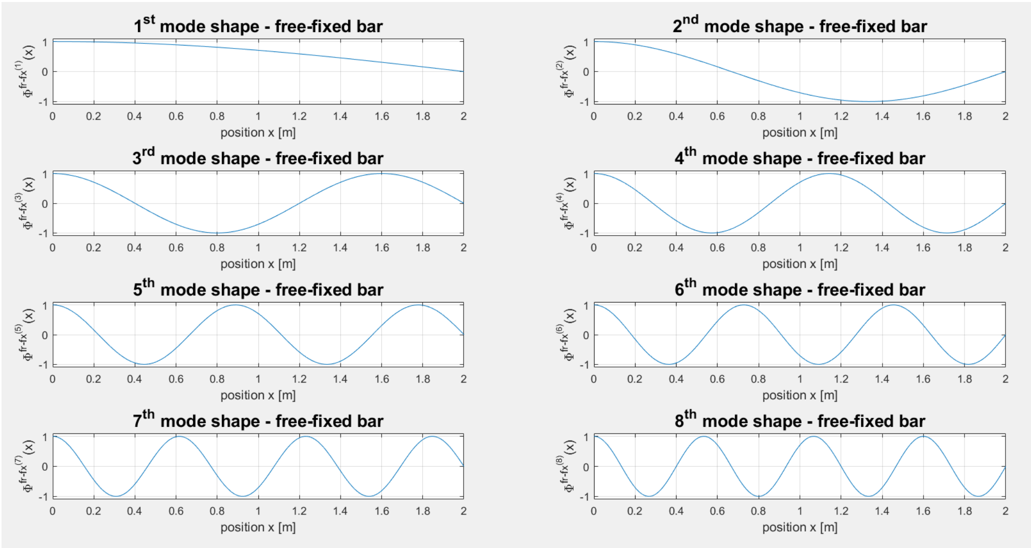
So $i = 1,2,...,8$ and the resulting natural frequencies are

$$\mathbf{f_n^{fr-fx}} = \begin{bmatrix} 636.47 & 1909.40 & 3182.30 & 4455.30 & 5728.20 & 7001.20 & 8274.10 & 9547.00 \end{bmatrix}$$

The mode shapes $\Phi^{\text{fr-fx}^{(i)}}$ are then, considering the constant coefficient $B = 1$,

$$\Phi^{\text{fr-fx}^{(i)}} = cos(k^{\text{fr-fx}^{(i)}}x) = cos\left(\frac{2i-1}{2L}\pi\,x\right) \;,\; i = 1,2,..,8$$

We hereby show them plotting the oscillation amplitude of each point of the bar versus the points position.



2 Natural frequencies and mode shapes in free-free configuration

Differently, in a free-free configuration of the bar the boundary condition on the normal force N is applied to the right end of the bar too:

$$\begin{cases} N(0,t) = E\,S\,\frac{\partial s(x,t)}{\partial x}\Big|_{x=0} = 0 \;\Rightarrow\; E\,S\,A\,k\,e^{j\omega\,t} = 0 \;\Rightarrow\; A = 0 \; (k = 0 \text{ is a trivial solution}) \\ N(L,t) = E\,S\,\frac{\partial s(x,t)}{\partial x}\Big|_{x=L} = 0 \;\Rightarrow\; -E\,S\,B\,k\,sin(kL) = 0 \;\Rightarrow\; sin(kL) = 0 \end{cases}$$

$$\Rightarrow\; k^{\text{fr-fx}^{(i)}} = \frac{i}{L}\pi \;,\; i = 0, 1, ..., \infty$$

So the natural frequencies are computed from $k^{\text{fr-fr}^{(i)}}$ as before:

$$f_{\text{n}}^{\text{fr-fr}^{(i)}} = \frac{\omega_{\text{n}}^{\text{fr-fr}^{(i)}}}{2\pi} = \frac{c\,k^{\text{fr-fr}^{(i)}}}{2\pi} = \frac{i}{2L}\sqrt{\frac{E}{\rho}} \;,\; i = 0, 1, ..., \infty$$

This time the maximum value for index i is

$$f_{\text{max}} = \frac{i_{\text{max}}^{\text{fr-fr}}}{2L}\sqrt{\frac{E}{\rho}}$$

$$\Rightarrow\; \lfloor i_{\text{max}}^{\text{fr-fr}} \rfloor = \left\lfloor 2L f_{\text{max}} \sqrt{\frac{\rho}{E}} \right\rfloor = \lfloor 7.86 \rfloor = 7$$

So $i = 0, 1, ..., 7$ and the corresponding natural frequencies are

$$\mathbf{f}_{\mathbf{n}}^{\mathbf{fr-fr}} = \begin{bmatrix} 0 & 1272.9 & 2545.9 & 3818.8 & 5091.8 & 6364.7 & 7637.6 & 8910.6 \end{bmatrix}$$

The number of resonances within f_{max} is the same, but the most important consideration is about the fact the system has now a resonance at $f = 0$, that is, a non-oscillating mode. The second consideration is that, while in the free-fixed case the resonance frequencies are odd integer multiples of the fundamental, in the free-free case these are all the integer multiples of the fundamental, both even and odd, which is twice that of the first case. In other words, naming f_0 the fundamental frequency of the free-fixed configuration, in the first case the resonances are

$$\mathbf{f_n^{fr-fx}} = \begin{bmatrix} f_0 & 3f_0 & 5f_0 & 7f_0 & 9f_0 & 11f_0 & 13f_0 & 15f_0 \end{bmatrix}$$

while in the second case

$$\mathbf{f_n^{fr-fr}} = \begin{bmatrix} 0 & 2f_0 & 4f_0 & 6f_0 & 8f_0 & 10f_0 & 12f_0 & 14f_0 \end{bmatrix}$$

Just like before, the resulting mode shapes $\Phi^{\text{fr-fr}^{(i)}}$ are computed and plotted considering $B = 1$.

$$\Phi^{\text{fr-fr}^{(i)}} = cos(k^{\text{fr-fr}^{(i)}} x) = cos\left(\frac{i}{L}\pi\, x\right) \text{ , } i = 0, 1, ..., 7$$