

Fundamentals of Vibration Analysis and Vibroacoustics

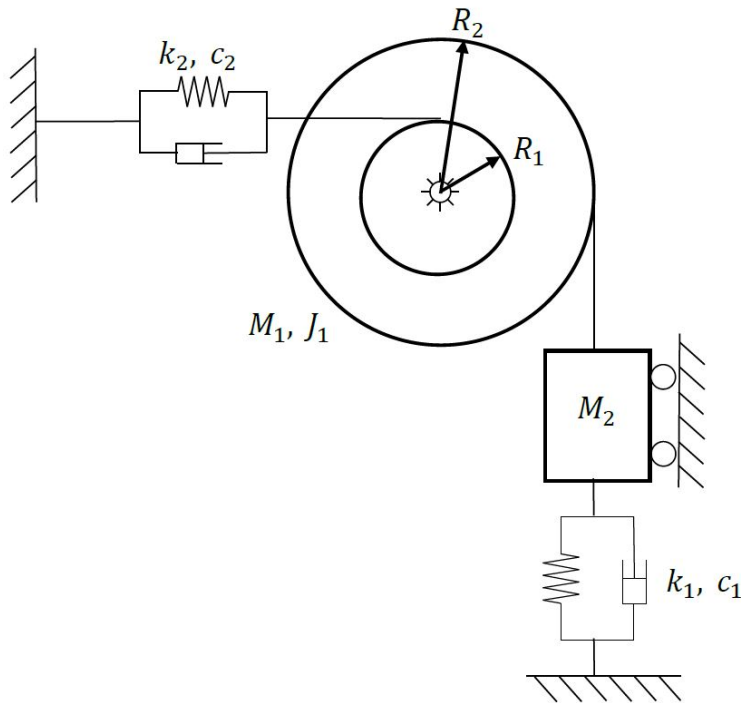
Module 1 - Fundamentals of Vibration Analysis

Assignment 1 - One-degree-of-freedom systems

Bombaci Nicola 10677942
 Fantin Jacopo 10591775
 Intagliata Emanuele 10544878

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System schematic and parameters



$$\begin{cases} M_1 = 1 \text{ kg} & , & J_1 = 0.005 \text{ kg m}^2 \\ M_2 = 0.35 \text{ kg} \end{cases}$$

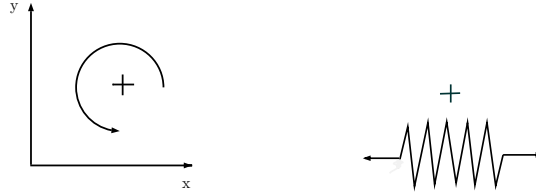
$$\begin{cases} R_1 = 0.1 \text{ m} \\ R_2 = 0.3 \text{ m} \end{cases}$$

$$\begin{cases} k_1 = 18 \text{ N/m} \\ k_2 = 25 \text{ N/m} \end{cases}$$

$$\begin{cases} c_1 = 0.7 \text{ Ns/m} \\ c_2 = 1.2 \text{ Ns/m} \end{cases}$$

1 Equation of motion

As a preliminary step, a reference system and sign conventions must be fixed. We chose to follow the commonly employed cartesian axes system, the counterclockwise rotation as the positive one and the spring elongation as the positive variation of the spring length:



1.a Equation derivation

Step 1: number of degrees of freedom identification

We can verify the system has one degree of freedom since

$$\begin{aligned}
 n_b \cdot 3 \text{ DOF} &= 6 \text{ DOF} - \\
 &2 \text{ DOF} - \quad (\text{hinge}) \\
 &2 \text{ DOF} - \quad (2 \text{ rollers}) \\
 &1 \text{ DOF} = \quad (\text{string}) \\
 &1 \text{ DOF}
 \end{aligned}$$

Where n_b is the number of bodies in the system, $n_b = 2$ in this case. We chose to solve the problem directly using Lagrange equation, so to have one equation only, as the system has one degree of freedom, depending on one independent variable, which we chose to be the disks M_1 rotation, θ .

Step 2: energy terms definition

$$E_c = \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} M_2 v_2^2$$

$$V_e = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2; \quad V_g = M_2 g h_2$$

$$\Rightarrow V = V_e + V_g = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 + M_2 g h_2$$

$$D = \frac{1}{2} c_1 \dot{\Delta l}_1^2 + \frac{1}{2} c_2 \dot{\Delta l}_2^2$$

Because the assignment's requests define external forces to compute the system's forced motion later on, we're assuming a vertical force $F(t)$, directed upward, applied on M_2 , so that we'll find a positive Lagrangian component.

$$\delta W = F(t) \delta y_2$$

Step 3: physical variables as functions of independent ones

The independent variable θ is chosen to be the one variable we need to describe the motion.

$$\omega_1 = \dot{\theta}$$

$$v_2 = \dot{y}_2 = \omega_1 R_2 = \dot{\theta} R_2$$

$$\begin{aligned} \Delta \dot{l}_1 &= \dot{\theta} R_2 \quad (\text{Rivals theorem}) \\ &\Rightarrow \Delta l_1 = \theta R_2 \end{aligned}$$

$$\begin{aligned} \Delta \dot{l}_2 &= -\dot{\theta} R_1 \quad (\text{Rivals theorem}) \\ &\Rightarrow \Delta l_2 = -\theta R_1 \end{aligned}$$

$$h_2 = y_2 = \theta R_2 \quad (\text{gravitational potential level} = 0 \text{ at equilibrium point})$$

$$\delta y_2 = \delta \theta R_2$$

Step 4: resulting equation

$$\delta W = Q_\theta \delta \theta = F(t) \delta y_2 = F(t) \delta \theta R_2 \Rightarrow Q_\theta = F(t) R_2$$

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{\partial E_c}{\partial \dot{\theta}} \right) - \frac{\partial E_c}{\partial \theta} + \frac{\partial V}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = Q_\theta \\ &\underbrace{(J_1 + M_2 R_2^2)}_{\text{generalized mass } m_g} \ddot{\theta} + \underbrace{(c_1 R_2^2 + c_2 R_1^2)}_{\text{generalized damping } c_g} \dot{\theta} + \underbrace{(k_1 R_2^2 + k_2 R_1^2)}_{\text{generalized stiffness } k_g} \theta = \underbrace{F(t) R_2 - M_2 g R_2}_{\substack{\text{active forces } C_g(t) \\ \text{(conservative/non conservative)}}} \end{aligned} \quad (1)$$

1.b Adimensional damping ratio

$$\xi = \frac{c_g}{c_{cr}} = \frac{c_g}{2 \sqrt{k_g m_g}} = \frac{c_1 R_2^2 + c_2 R_1^2}{2 \sqrt{(k_1 R_2^2 + k_2 R_1^2) (J_1 + M_2 R_2^2)}} \approx 0.134 \quad (2)$$

1.c Natural and damped frequency

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_g}{m_g}} = \sqrt{\frac{k_1 R_2^2 + k_2 R_1^2}{J_1 + M_2 R_2^2}} \approx 7.16 \text{ rad/s} \\ \alpha &= \xi \omega_n \approx 0.959 \text{ rad/s} \end{aligned} \quad (3)$$

$$\omega_d = \sqrt{\omega_n^2 - \alpha^2} \approx 7.09 \text{ rad/s} \quad (4)$$

2 Free motion of the system

The system's free motion is described by the general solution to the differential equation of motion, which corresponds to the solution to the homogeneous differential equation associated to (1):

$$m_g \ddot{\theta} + c_g \dot{\theta} + k_g \theta = 0$$

We solved this equation manually, finding the characteristic polynomial roots and summing the two resulting exponential functions, each with one of the two eigenvalues. The solution, plotted below, has the analytic form

$$\begin{aligned} \theta_{free}(t) &= X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t} = \\ &= X_1 e^{(-\alpha + j\omega_d)t} + X_2 e^{(-\alpha - j\omega_d)t} = \\ &= e^{-\alpha t} (X_1 e^{j\omega_d t} + X_2 e^{-j\omega_d t}) \end{aligned}$$

which represents an oscillatory behavior modulated by a decaying exponential, as expected, and where the coefficients X_1 and X_2 are computed starting from the system parameters and the initial conditions according to the following relations:

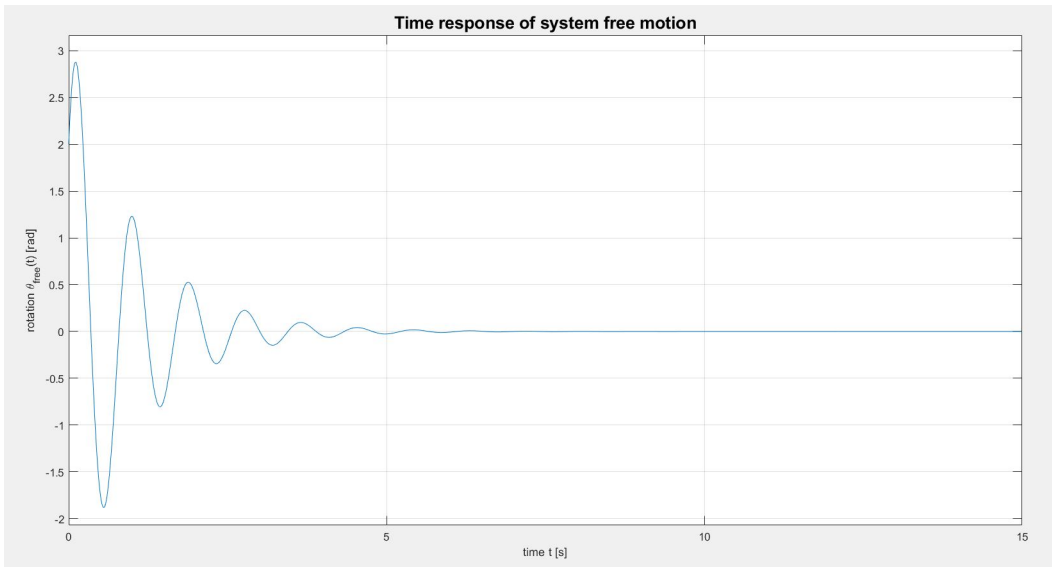
$$\begin{cases} \theta_0 = \theta_{free}(t=0) = X_1 + X_2 \\ \omega_0 = \dot{\theta}_{free}(t=0) = \lambda X_1 + \lambda X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \theta_0 - \frac{\omega_0 - \lambda_1 \theta_0}{\lambda_2 - \lambda_1} \\ X_2 = \frac{\omega_0 - \lambda_1 \theta_0}{\lambda_2 - \lambda_1} \end{cases}$$

2.a Generic initial conditions

In order for the equation to be computed and represented in a diagram, some initial conditions need to be settled. The shown time response is obtained with these arbitrary initial conditions:

$$\begin{cases} \theta_0 = 2 \text{ rad} \\ \omega_0 = 16 \text{ rad/s} \end{cases}$$

These values allow us for a clearer visualization of the motion, and an easier distinction between the transient and the steady-state behavior.



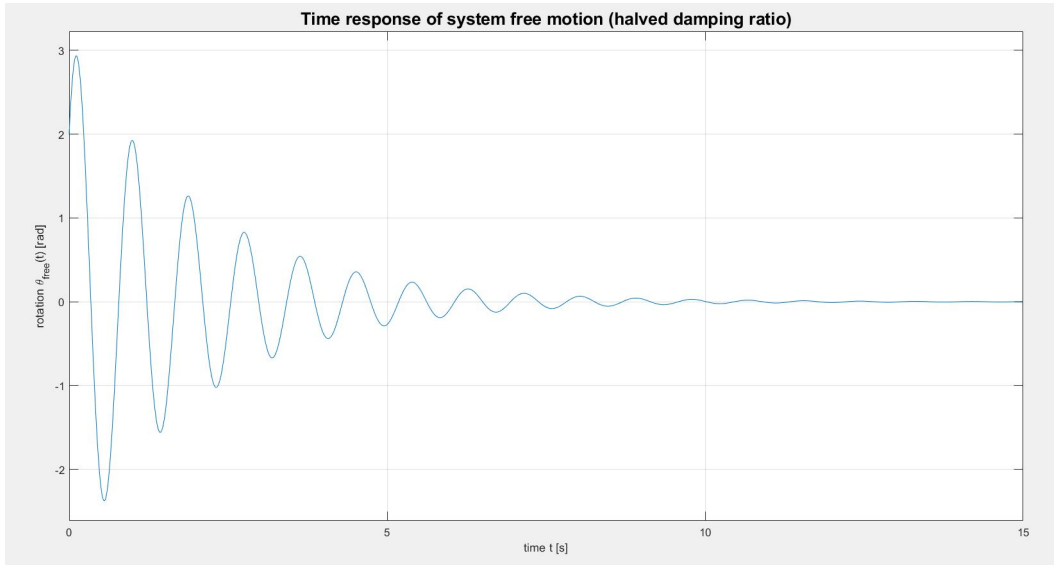
2.b Halved adimensional damping ratio

The system resulting from halving ξ is equivalent to that obtained by halving the damping coefficient c_g , according to (2). As a consequence, the system shows a much lighter damped free oscillation, and the time leading to an almost completely damped away motion is doubled. In fact, if t_0 and t'_0 are the earliest time instants for which the oscillation may be considered as over in the previous case and with halved ξ respectively (we assumed 10^{-6} times the initial amplitude), and remembering that, if ξ is halved, then α is halved too according to (3):

$$\begin{aligned} e^{-\alpha t_0} = 10^{-6} &\Rightarrow t_0 = \frac{\ln(1) - \ln(10^6)}{-\alpha} = \\ &= \frac{\ln(10^6)}{\alpha} \approx 14.41 \text{ s} \end{aligned}$$

$$\begin{aligned} e^{-\frac{\alpha}{2} t'_0} = 10^{-6} &\Rightarrow t'_0 = \frac{\ln(1) - \ln(10^6)}{-\frac{\alpha}{2}} = \\ &= 2 \frac{\ln(10^6)}{\alpha} = 2 t_0 \approx 28.81 \text{ s} \end{aligned}$$

Assuming the same initial conditions as before, the free time response is depicted in the following plot:

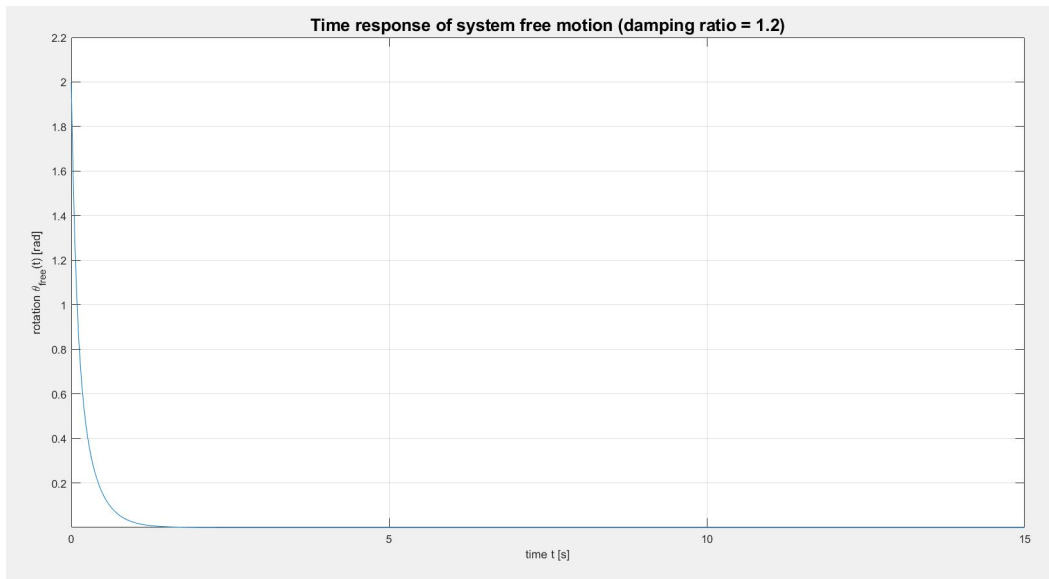


2.c Raised adimensional damping ratio

Instead, if the damping ratio is increased so that $\xi > 1$ holds, the damping coefficient becomes greater than the critical damping coefficient c_{cr} , which means that the system is overdamped. Choosing $\xi = 1.2$, we computed the corresponding c_g using (2) and we solved again the free motion equation, getting a totally analogous solution, except for the fact the exponential functions are not complex anymore, but real. This results in the sum of two decaying exponentials, which is in turn a decaying exponential.

$$\theta_{free}(t) = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t} = X_1 e^{-\alpha_1 t} + X_2 e^{-\alpha_2 t}$$

We can appreciate the absence of vibration due to the absence of a couple of complex conjugate roots in the characteristic equation solution.



3 Forced motion of the system

We now come to consider the system as excited by an external harmonic force in accordance with the force introduced in 1.a to evaluate its behavior in the steady-state and the complete response of the system.

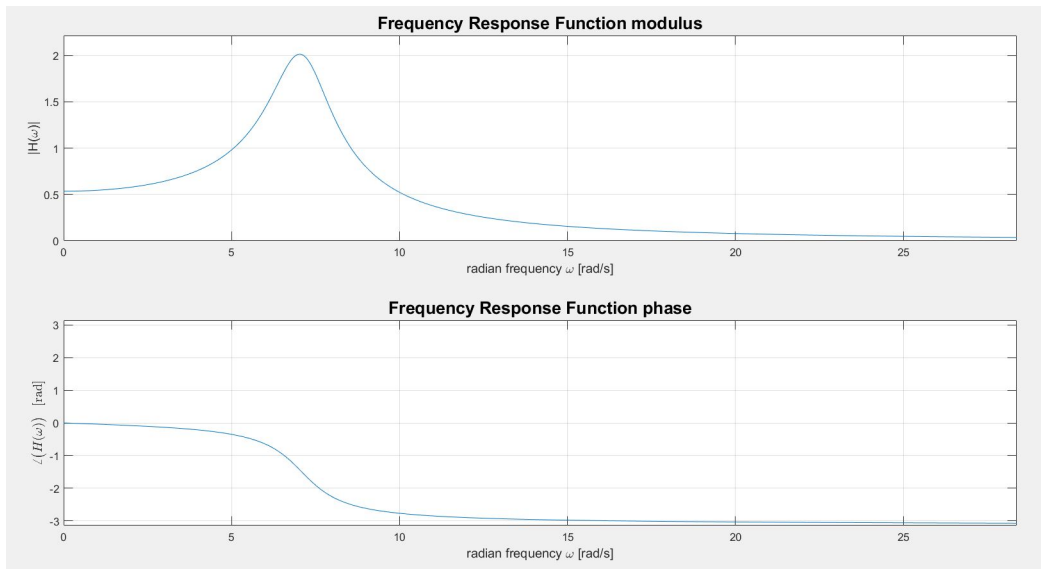
3.a Frequency Response Function

First, the Frequency Response Function (FRF) has been computed from the system parameters: it's defined as the ratio between the complex function embedding the forced vibration's amplitude and phase shift, which is a function of the forcing frequency, and the amplitude of the oscillation of the external harmonic excitation.

$$H(j\omega) = \frac{\tilde{\Theta}_0(\omega)}{F_0} = \frac{1}{m_g \omega^2 - j c_g \omega + k_g}$$

Case 1: generic initial conditions

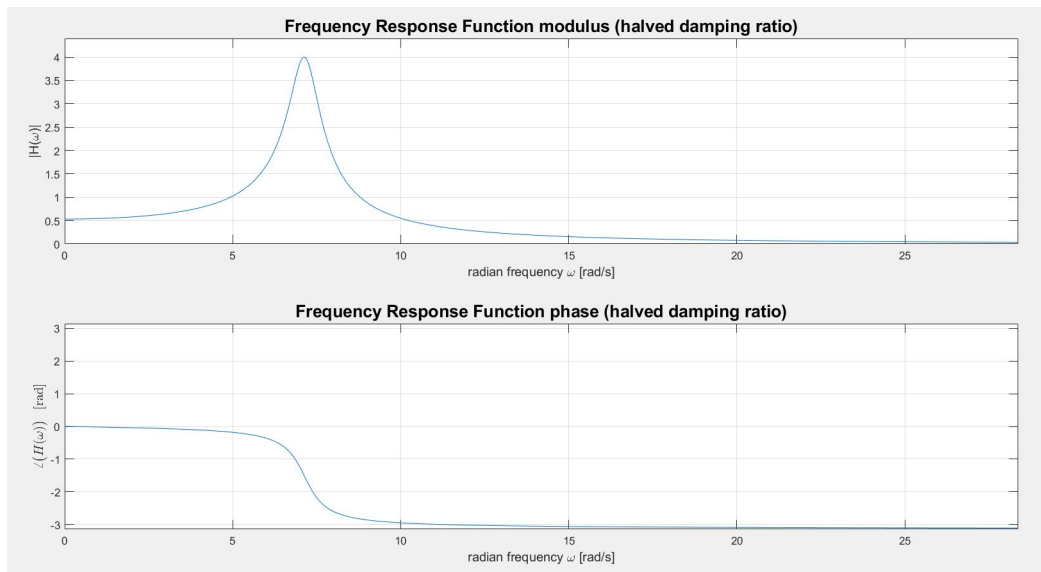
In this first case, we're plotting the FRF assuming the system parameters are the given ones, which we started with. Notably, a peak (absolute maximum) in the magnitude diagram



corresponds to the natural damped frequency ω_d of the system, computed at (4), which is reflected in the phase diagram in a $-\pi$ jump.

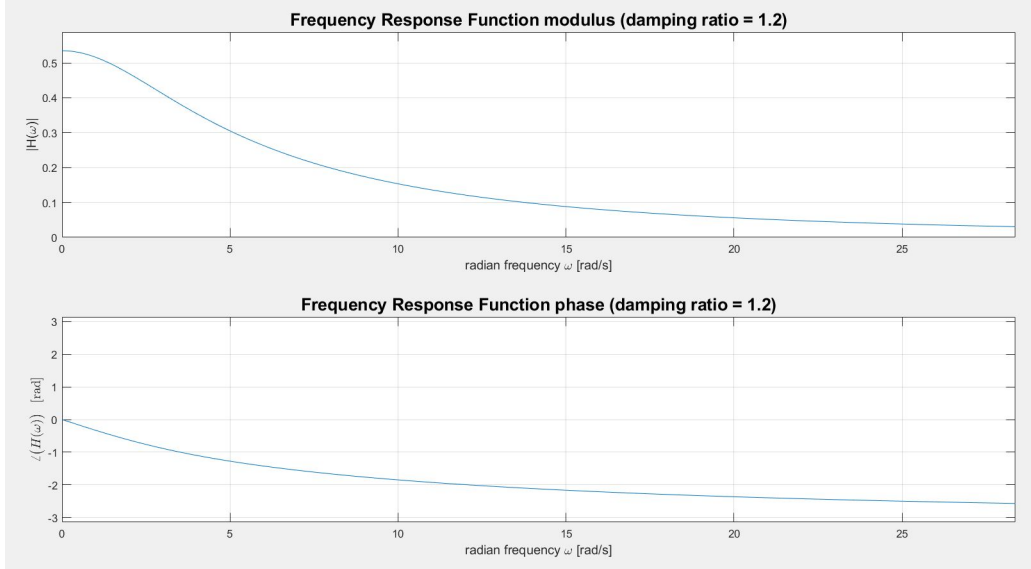
Case 2: halved adimensional damping ratio

Considering half the damping ratio, we expect the system to be closer to the ideal undamped case, which means that the peak in the modulus diagram is going to be sharper than in the previous case (getting closer to the asymptotic behavior typical of the undamped system) and the phase jump suddener in the phase diagram (getting closer to the ideal perfectly vertical jump of the undamped case). It is worthy to say also the natural damped frequency changes, in accordance with relation (4): in particular, its value decreases proportionally to the damping factor α , so the higher the damping, the lower the damped natural frequency, the more the peak moves to the left in the modulus plot.



Case 3: raised adimensional damping ratio

In this third case, the adimensional damping ratio is set to $\xi = 1.2$, so we find ourselves in the case of the overdamped system. We're expecting a smooth low-pass-like curve in the amplitude diagram as the damped natural frequency doesn't appear in the analytic formula of the time response, i.e. the system free motion does not oscillate, so the FRF shape is not depending on the natural frequency of the system. We might notice that the maximum steady-state oscillation amplitude is found for $\omega = 0$, i.e. for a constant external force.



3.b Temporal evolution of complete response of the system

The system's complete response is composed by its free response and its forced response, summed together. The former is the one obtained in 2.a. For the latter, the input force to the system is considered to be a harmonic force made up by a single sinusoidal wave:

$$F = A \cos(2\pi f + \phi)$$

where $A = 2.5 \text{ N}$ and $\phi = \frac{\pi}{3}$. We're considering two different cases for the frequency of this single harmonic component. In the first case

$$f = 0.15 \text{ Hz} \Rightarrow \omega = 2\pi f \approx 0.94 \text{ rad/s}$$

while in the second

$$f = 4.5 \text{ Hz} \Rightarrow \omega = 2\pi f \approx 28.27 \text{ rad/s}$$

In each case, the steady-state motion has been computed by applying the well-known theoretical result stating the output of a linear time-invariant system is equal to the system input modulated in amplitude and phase by the FRF computed for the particular frequency ω of the input:

$$x_{forced}(t) = |H(j\omega)| F_0 \cos(\omega t + \phi + \angle(H(j\omega))) \quad (5)$$

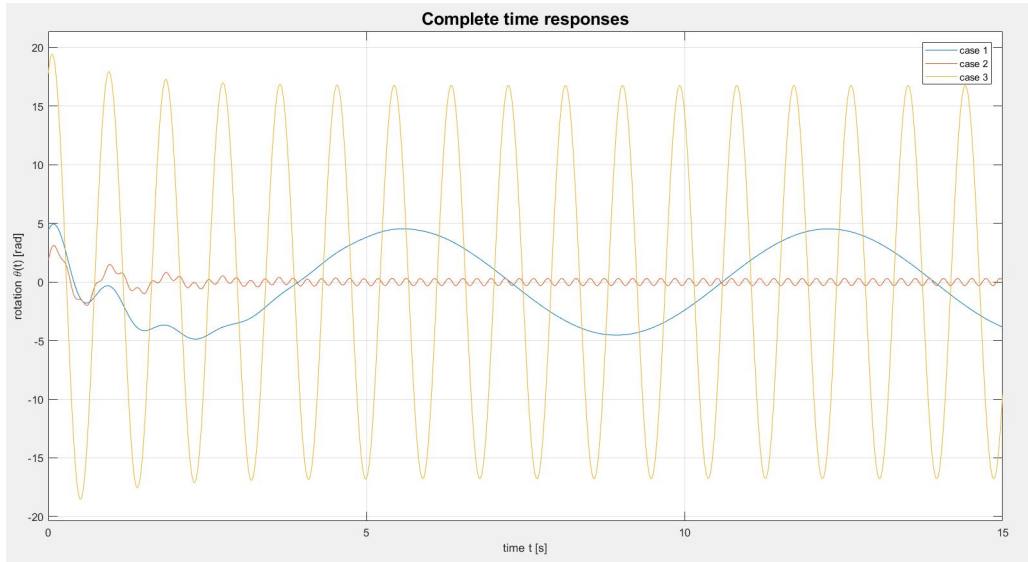
In our case, since the independent variable is an angle in radians, we need to divide this quantity by the radius R_2 that allows us to pass from the vertical displacement y_2 of body M_2 to the disks rotation θ , so the final formula becomes

$$\begin{aligned}\theta(t) &= \theta_{free} + \theta_{forced} = \\ &= e^{-\alpha t} (X_1 e^{j\omega_d t} + X_2 e^{-j\omega_d t}) + |H(j\omega)| \frac{A}{R_2} \cos(\omega t + \phi + \angle(H(j\omega)))\end{aligned}$$

We took the liberty of taking into account an additional case, where the external force frequency nearly matches the system's damped natural frequency, to check whether the amplitude of the resulting oscillation in steady-state is actually much larger than in the previous cases:

$$\omega = 7 \text{ rad/s}$$

The system's complete time responses for each of the three cases are visualized in the following diagram.

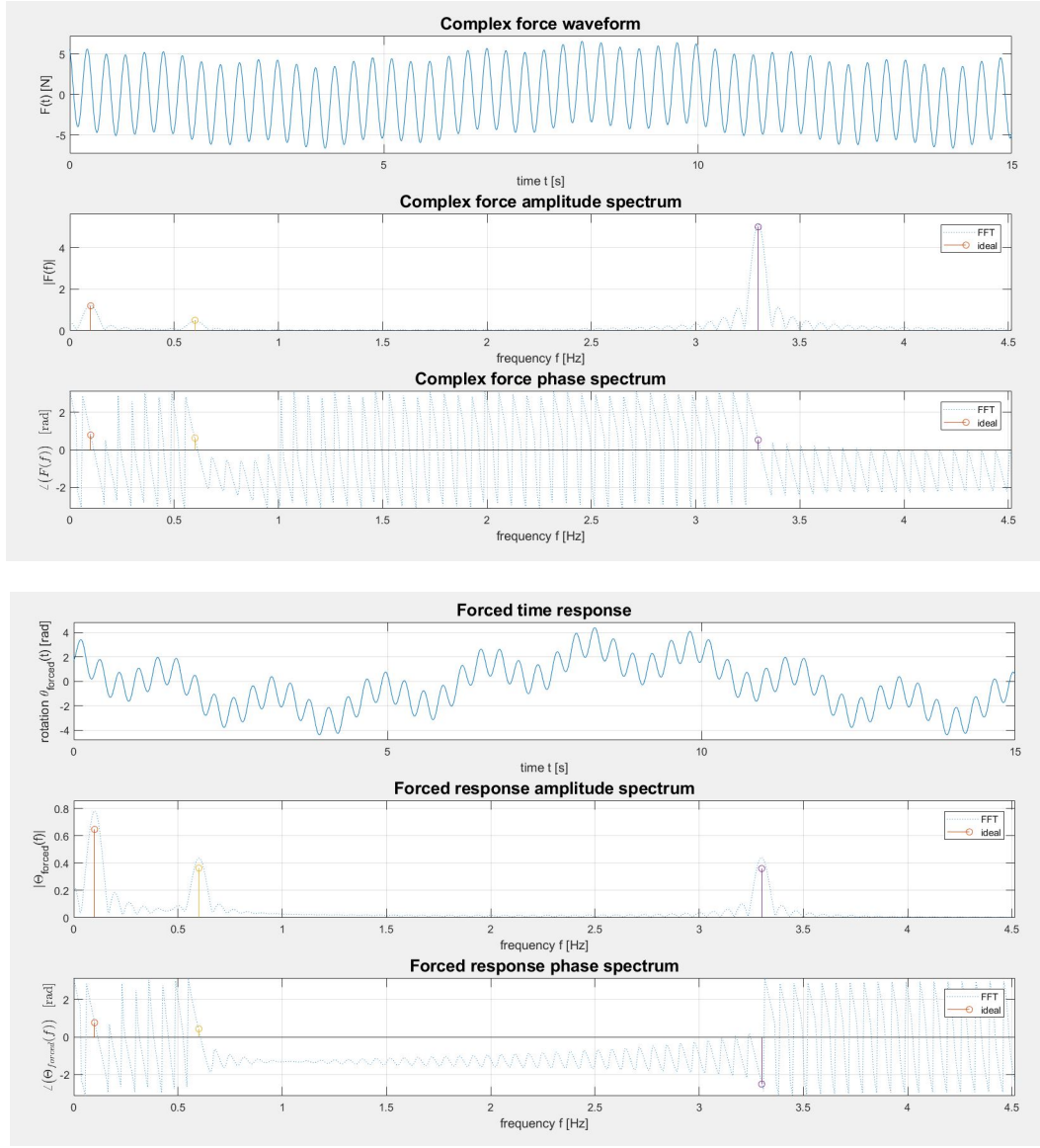


3.c Forced response of the system (steady-state)

Finally, let's consider this time a complex force composed by three harmonic components as system input:

$$F(t) = \sum_{k=1}^3 B_k \cos(2\pi f_k t + \phi_k)$$

with $B_1 = 1.2 \text{ N}$, $B_2 = 0.5 \text{ N}$, $B_3 = 5 \text{ N}$, $f_1 = 0.1 \text{ Hz}$ ($\omega_1 = 0.63 \text{ rad/s}$), $f_2 = 0.6 \text{ Hz}$ ($\omega_2 = 3.77 \text{ rad/s}$), $f_3 = 3.3 \text{ Hz}$ ($\omega_3 = 20.73 \text{ rad/s}$), $\phi_1 = \frac{\pi}{4}$, $\phi_2 = \frac{\pi}{5}$ and $\phi_3 = \frac{\pi}{6}$. The external force and the forced vibration of the system are analyzed both in time and in frequency domain. For analyzing the forced response, the superposition principle has been employed: using equation (5), we computed the system forced response for each of the individual harmonic components that constitute the oscillating input. We then summed together the three partial forced responses to get the total steady-state response of the system. To validate the result, we compared the spectrums with the Fast Fourier Transforms of the two time-domain



signals making use of the relative Matlab function. As we may see, the amplitude spectra computed by means of FFT, scaled by the FFT length, qualitatively coincide with the ones we constructed showing peaks in correspondence to the same frequencies.

As the external force is concerned, we can easily identify the three sinusoidal components that build it up in time domain, while the spectrum simply displays the given data in terms of amplitude, frequency and phase shift. As for the system's forced vibration, we are again able to tell the three frequency components apart in the time response. Furthermore, we can observe how different the weights of these three components are in the forced response, with respect to the input force: the most emphasized are the low frequencies, while the highest one is attenuated a big deal. This is due to the fact the original amplitudes for each cosine function in the external force are multiplied by the modulus of the system's Frequency Response Function computed for the frequency of the same cosine function, whereas the particular shape of the waveform is determined by the sum between the original phase shifts for each component with the phase value of the FRF evaluated for the corresponding frequency.