Fundamentals of Vibration Analysis and Vibroacoustics
Module 1 - Fundamentals of Vibration Analysis
Assignment 1 - One-degree-of-freedom systems

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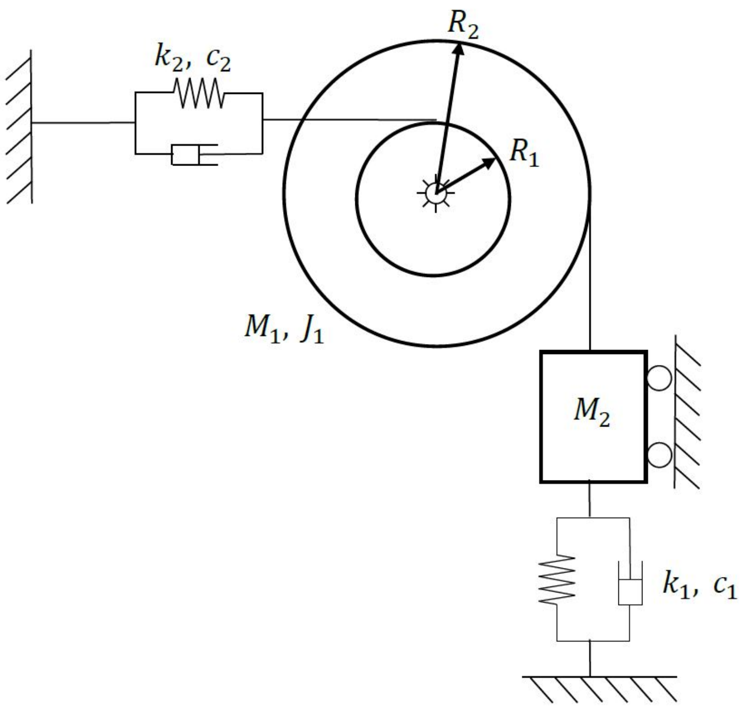
System schematic and parameters

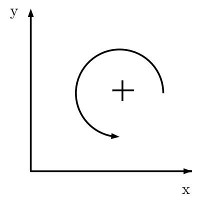


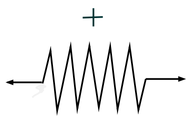
$$\begin{cases} M_1 = 1\,\mathrm{kg} \quad , \quad J_1 = 0.005\,\mathrm{kg\ m^2} \\ M_2 = 0.35\,\mathrm{kg} \end{cases}$$

$$\begin{cases} R_1 = 0.1\,\mathrm{m} \\ R_2 = 0.3\,\mathrm{m} \end{cases}$$

$$\begin{cases} k_1 = 18\,\mathrm{N/m} \\ k_2 = 25\,\mathrm{N/m} \end{cases}$$

$$\begin{cases} c_1 = 0.7\,\mathrm{Ns/m} \\ c_2 = 1.2\,\mathrm{Ns/m} \end{cases}$$





1 Equation of motion

As a preliminary step, a reference system and sign conventions must be fixed. We chose to follow the commonly employed cartesian axes system, the counterclockwise rotation as the positive one and the spring elongation as the positive variation of the spring length:

1.a Equation derivation

Step 1: number of degrees of freedom identification

We can verify the system has one degree of freedom since

$$\begin{aligned} n_b \cdot 3\ \text{DOF} = 6\ &\text{DOF}- \\ 2\ &\text{DOF}- \quad \text{(hinge)} \\ 2\ &\text{DOF}- \quad \text{(2 rollers)} \\ 1\ &\text{DOF}= \quad \text{(string)} \\ 1\ &\text{DOF} \end{aligned}$$

Where n_b is the number of bodies in the system, $n_\text{b} = 2$ in this case. We chose to solve the problem directly using Lagrange equation, so to have one equation only, as the system has one degree of freedom, depending on one independent variable, which we chose to be the disks M_1 rotation, θ.

Step 2: energy terms definition

$$E_\text{k} = \frac{1}{2}\,J_1\,\omega_1^2 + \frac{1}{2}\,M_2\,v_2^2$$

$$\begin{aligned} V_\text{e} &= \frac{1}{2}\,k_1\,\Delta l_1^2 + \frac{1}{2}\,k_2\,\Delta l_2^2\,; \quad V_\text{g} = M_2\,g\,h_2 \\ \Rightarrow\ V = V_\text{e} + V_\text{g} &= \frac{1}{2}\,k_1\,\Delta l_1^2 + \frac{1}{2}\,k_2\,\Delta l_2^2 + M_2\,g\,h_2 \end{aligned}$$

$$D = \frac{1}{2}\,c_1\,\dot{\Delta l_1}^2 + \frac{1}{2}\,c_2\,\dot{\Delta l_2}^2$$

Because the assignment's requests define external forces to compute the system's forced motion later on, we're assuming a vertical force $F(t)$, directed upward, applied on M_2, so that we'll find a positive Lagrangian component.

$$\delta W = F(t)\,\delta y_2$$

Step 3: physical variables as functions of independent ones

The independent variable θ is chosen to be the one variable we need to describe the motion.

$$
\begin{aligned}
&\omega_1 = \dot{\theta} \\
&v_2 = \dot{y}_2 = \omega_1 \, R_2 = \dot{\theta} \, R_2 \\
&\dot{\Delta l_1} = \dot{\theta} \, R_2 \quad \text{(Rivals theorem)} \;\Rightarrow\; \Delta l_1 = \theta \, R_2 \\
&\dot{\Delta l_2} = -\dot{\theta} \, R_1 \quad \text{(Rivals theorem)} \;\Rightarrow\; \Delta l_2 = -\theta \, R_1 \\
&h_2 = y_2 = \theta \, R_2 \quad \text{(gravitational potential level} = 0 \text{ at equilibrium point)} \\
&\delta y_2 = \delta\theta \, R_2
\end{aligned}
$$

Step 4: resulting equation

We first need to find the Lagrangian component Q_θ of the force $F(t)$:

$$\delta W = Q_\theta \, \delta\theta = F(t) \, \delta y_2 = F(t) \, \delta\theta \, R_2 \;\Rightarrow\; Q_\theta = F(t) \, R_2$$

Then, the equation of motion is

$$\frac{\partial}{\partial t}\Big(\frac{\partial E_c}{\partial \dot{\theta}}\Big) - \frac{\partial E_c}{\partial \theta} + \frac{\partial V}{\theta} + \frac{\partial D}{\partial \dot{\theta}} = Q_\theta$$

$$\Rightarrow\; (J_1 + M_2 \, R_2^2) \, \ddot{\theta} + (c_1 \, R_2^2 + c_2 \, R_1^2)\dot{\theta} + (k_1 \, R_2^2 + k_2 \, R_1^2) \, \theta = F(t) \, R_2 - M_2 \, g \, R_2$$

The right side of the equation shows a gravitational contribution in terms of torque. Since we know the system is in the static equilibrium position, this contribution doesn't do any work and, therefore, the static preload of the springs already compensates for it. We can include this static preload in the equation considering a new independent variable, $\bar{\theta}$, which is the independent variable θ in the static equilibrium point, called perturbed variable:

$$\bar{\theta} = \theta - \theta_{\text{eq}} \;\Rightarrow\; \dot{\bar{\theta}} = \dot{\theta} \;\Rightarrow\; \ddot{\bar{\theta}} = \ddot{\theta}$$

where θ_{eq} is a constant that accounts for the static preload, computed at the static ($\dot{\theta} = 0$) equilibrium ($\ddot{\theta} = 0$) position:

$$
\begin{aligned}
(k_1 \, R_2^2 + k_2 \, R_1^2) \, \theta_{\text{eq}} &= F(t) \, R_2 - M_2 \, g \, R_2 \\
\Rightarrow\; \theta_{\text{eq}} &= \frac{F(t) \, R_2 - M_2 \, g \, R_2}{(k_1 \, R_2^2 + k_2 \, R_1^2)}
\end{aligned}
$$

We consider the force as composed by a constant value and a variable one $F(t) = F_{\text{eq}} + F \, cos(\Omega \, t + \varphi)$. In the static equilibrium position, the variable part of the force is null, so we just have that $F(t) = F_{\text{eq}}$:

$$\theta_{\text{eq}} = \frac{F_{\text{eq}} \, R_2 - M_2 \, g \, R_2}{(k_1 \, R_2^2 + k_2 \, R_1^2)}$$

The final equation of motion in the equilibrium position is

$$\underbrace{(J_1 + M_2 \, R_2^2)}_{\substack{\text{generalized mass} \\ m_{\text{g}}}} \, \ddot{\bar{\theta}} + \underbrace{(c_1 \, R_2^2 + c_2 \, R_1^2)}_{\substack{\text{generalized damping} \\ c_{\text{g}}}} \, \dot{\bar{\theta}} + \underbrace{(k_1 \, R_2^2 + k_2 \, R_1^2)}_{\substack{\text{generalized stiffness} \\ k_{\text{g}}}} \, \bar{\theta} = F(t) \, R_2 = \underbrace{F \, R_2 \, cos(\Omega \, t + \varphi)}_{\substack{\text{generalized force} \\ Q_\theta}} \quad (1)$$

1.b Adimensional damping ratio

$$\xi = \frac{c_\mathrm{g}}{c_\mathrm{cr}} = \frac{c_\mathrm{g}}{2\sqrt{k_\mathrm{g}\, m_\mathrm{g}}} = \frac{c_1\, R_2^2 + c_2\, R_1^2}{2\sqrt{(k_1\, R_2^2 + k_2\, R_1^2)\,(J_1 + M_2\, R_2^2)}} \approx 0.134 \tag{2}$$

1.c Natural and damped frequency

$$\omega_\mathrm{n} = \sqrt{\frac{k_\mathrm{g}}{m_\mathrm{g}}} = \sqrt{\frac{k_1\, R_2^2 + k_2\, R_1^2}{J_1 + M_2\, R_2^2}} \approx 7.16\,\mathrm{rad/s}$$

$$\alpha = \xi\, \omega_\mathrm{n} \approx 0.959\,\mathrm{rad/s} \tag{3}$$

$$\omega_\mathrm{d} = \sqrt{\omega_n^2 - \alpha^2} \approx 7.09\,\mathrm{rad/s} \tag{4}$$

2 Free motion of the system

The system's free motion is described by the general solution to the differential equation of motion, which corresponds to the solution to the homogeneous differential equation associated to (1):

$$m_\mathrm{g}\, \ddot{\bar\theta} + c_\mathrm{g}\, \dot{\bar\theta} + k_\mathrm{g}\, \bar\theta = 0$$

We solved this equation manually, finding the characteristic polynomial roots and summing the two resulting exponential functions, each with one of the two eigenvalues. The solution, plotted below, has the analytic form

$$\begin{aligned}\bar\theta^\mathrm{free}(t) &= \bar X_1\, e^{\lambda_1\, t} + \bar X_2\, e^{\lambda_2\, t} = \\ &= \bar X_1\, e^{(-\alpha + j\omega_\mathrm{d})\, t} + \bar X_2\, e^{(-\alpha - j\omega_\mathrm{d})\, t} = \\ &= e^{-\alpha\, t}\,(\bar X_1\, e^{j\omega_\mathrm{d}\, t} + \bar X_2\, e^{-j\omega_\mathrm{d}\, t})\end{aligned}$$

which represents an oscillatory behavior modulated by a decaying exponential, as expected, and where the coefficients $\bar X_1$ and $\bar X_2$ are computed starting from the system parameters and the initial conditions according to the following relations:

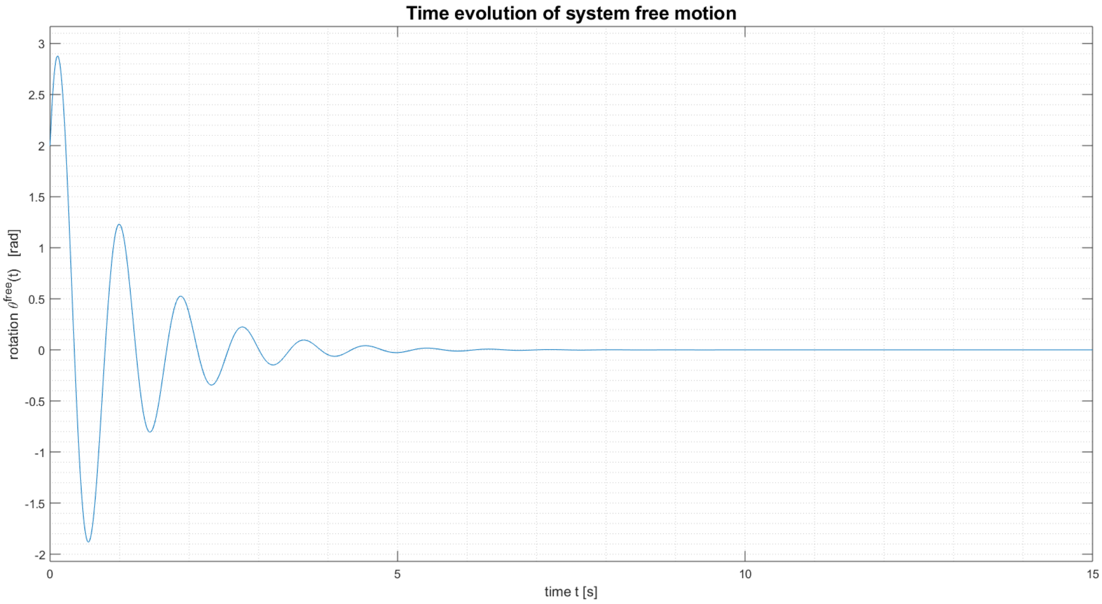
$$\begin{cases}\theta_0 = \bar\theta^\mathrm{free}(t=0) = \bar X_1 + \bar X_2 \\ \omega_0 = \dot{\bar\theta}^\mathrm{free}(t=0) = \lambda_1\, \bar X_1 + \lambda_2\, \bar X_2\end{cases} \quad \Rightarrow \quad \begin{cases}\bar X_1 = \theta_0 - \frac{\omega_0 - \lambda_1\, \theta_0}{\lambda_2 - \lambda_1} \\ \bar X_2 = \frac{\omega_0 - \lambda_1\, \theta_0}{\lambda_2 - \lambda_1}\end{cases}$$

2.a Generic initial conditions

In order for the equation to be computed and represented in a diagram, some initial conditions need to be settled. The shown response in time domain is obtained with these arbitrary initial conditions:

$$\begin{cases}\theta_0 = 2\,\mathrm{rad} \\ \omega_0 = 16\,\mathrm{rad/s}\end{cases}$$

These values allow us for a clearer visualization of the motion, and an easier distinction between the transient and the steady-state behavior.



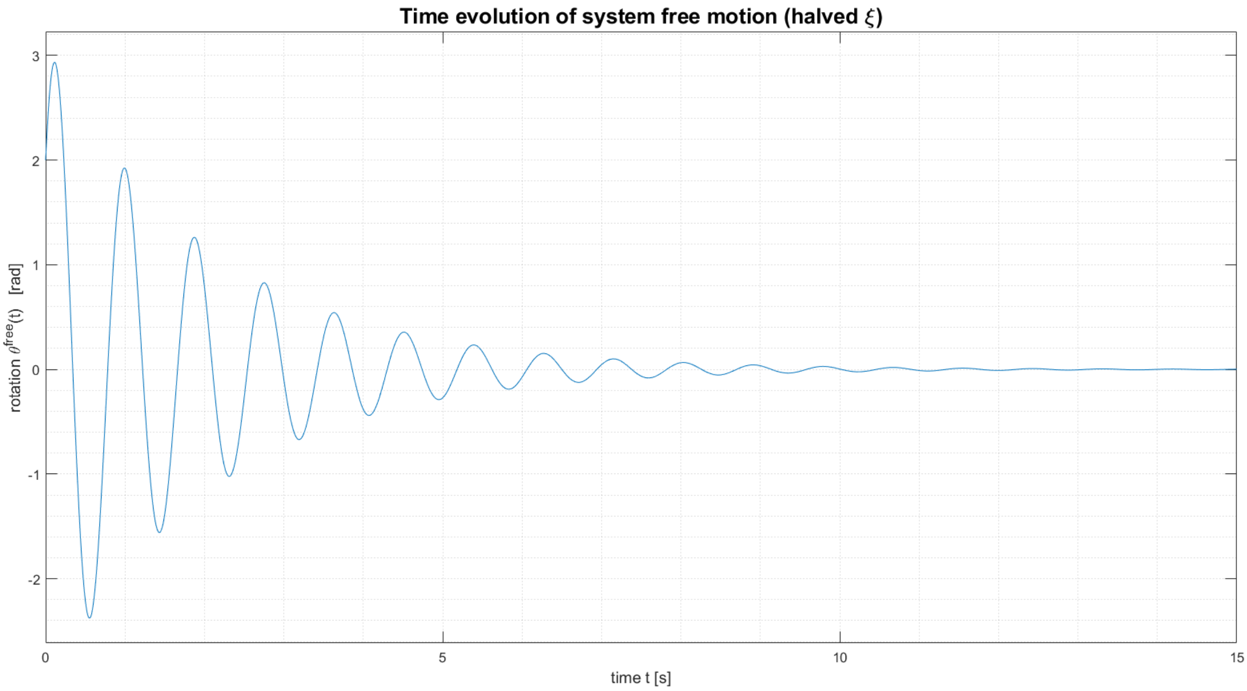
2.b Halved adimensional damping ratio

The system resulting from halving ξ is equivalent to that obtained by halving the damping coefficient c_g, in conformity with (2). As a consequence, the system shows a much lightlier damped free oscillation, and the time leading to an almost completely damped away motion is doubled. In fact, if t_0 and t_0' are the earliest time instants for which the oscillation may be considered as over in the previous case and with halved ξ respectively (we assumed 10^{-6} times the initial amplitude), and remembering that, if ξ is halved, then α is halved too according to (3):

$$\begin{aligned} e^{-\alpha\, t_0} = 10^{-6} \;\Rightarrow\; t_0 &= \frac{ln(1) - ln(10^6)}{-\alpha} = \\ &= \frac{ln(10^6)}{\alpha} \approx 14.41\,\mathrm{s} \end{aligned}$$

$$\begin{aligned} e^{-\frac{\alpha}{2}\, t_0'} = 10^{-6} \;\Rightarrow\; t_0' &= \frac{ln(1) - ln(10^6)}{-\alpha} = \\ &= 2\,\frac{ln(10^6)}{\alpha} = 2\,t_0 \approx 28.81\,\mathrm{s} \end{aligned}$$

Assuming the same initial conditions as before, the free response is depicted in the following plot:

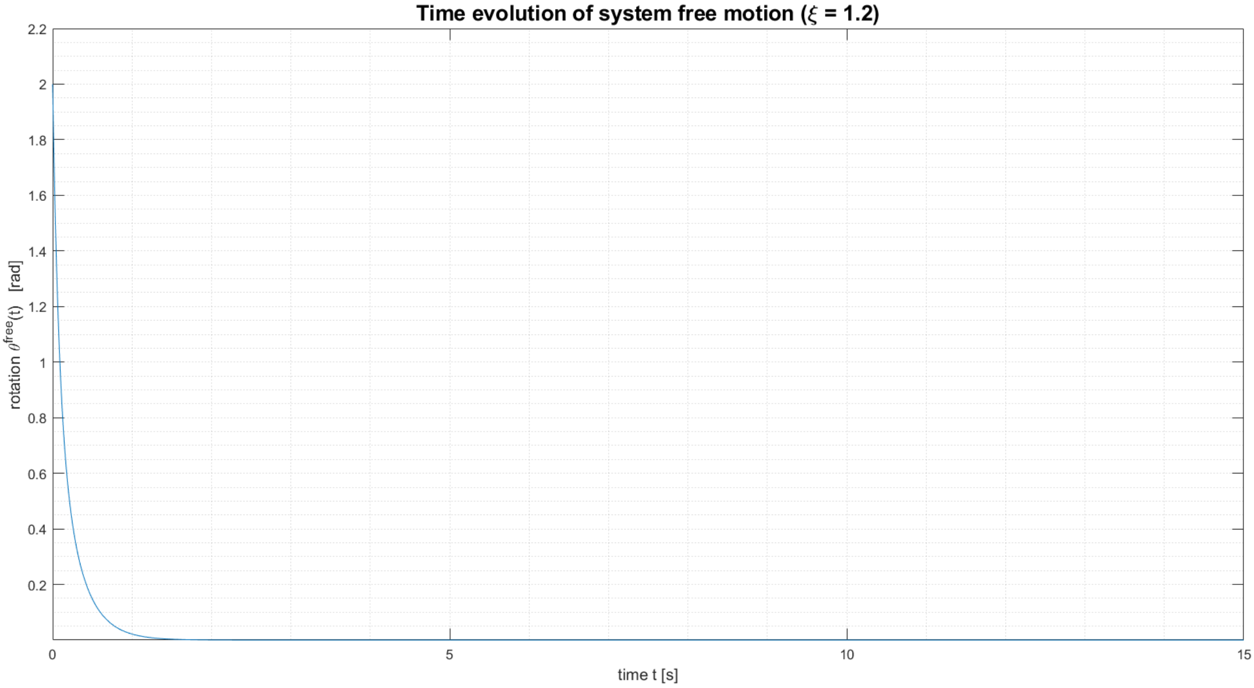


2.c Raised adimensional damping ratio

Instead, if the damping ratio is increased so that $\xi > 1$ holds, the damping coefficient becomes greater than the critical damping coefficient c_{cr}, which means that the system is overdamped. Choosing $\xi = 1.2$, we computed the corresponding c_{g} using (2) and we solved again the free motion equation, getting a totally analogous solution, except for the fact the exponential functions are not complex anymore, but real. This results in the sum of two decaying exponentials, which is in turn a decaying exponential.

$$\bar{\theta}^{\text{free}}(t) = \bar{X}_1\, e^{\lambda_1 t} + \bar{X}_2\, e^{\lambda_2 t} = \bar{X}_1\, e^{-\alpha_1 t} + \bar{X}_2\, e^{-\alpha_2 t}$$

We can appreciate the absence of vibration due to the absence of a couple of complex conjugate roots in the characteristic equation solution.



3 Forced motion of the system

3.a Frequency Response Function

We now come to consider the system as excited by an external torque $\tau(t) = T\, e^{j\Omega\, t}$, with unitary modulus and applied to the discs counterclockwise: the Frequency Response Function (FRF) of the system is, by definition, the response in terms of amplitude and phase shift (with respect to the applied generalized force itself) of the forced vibration of the independent variable we're considering to an external harmonic excitation whose virtual displacement has a 1x1 Jacobian equal to 1. This excitation is, in our case, a torque.

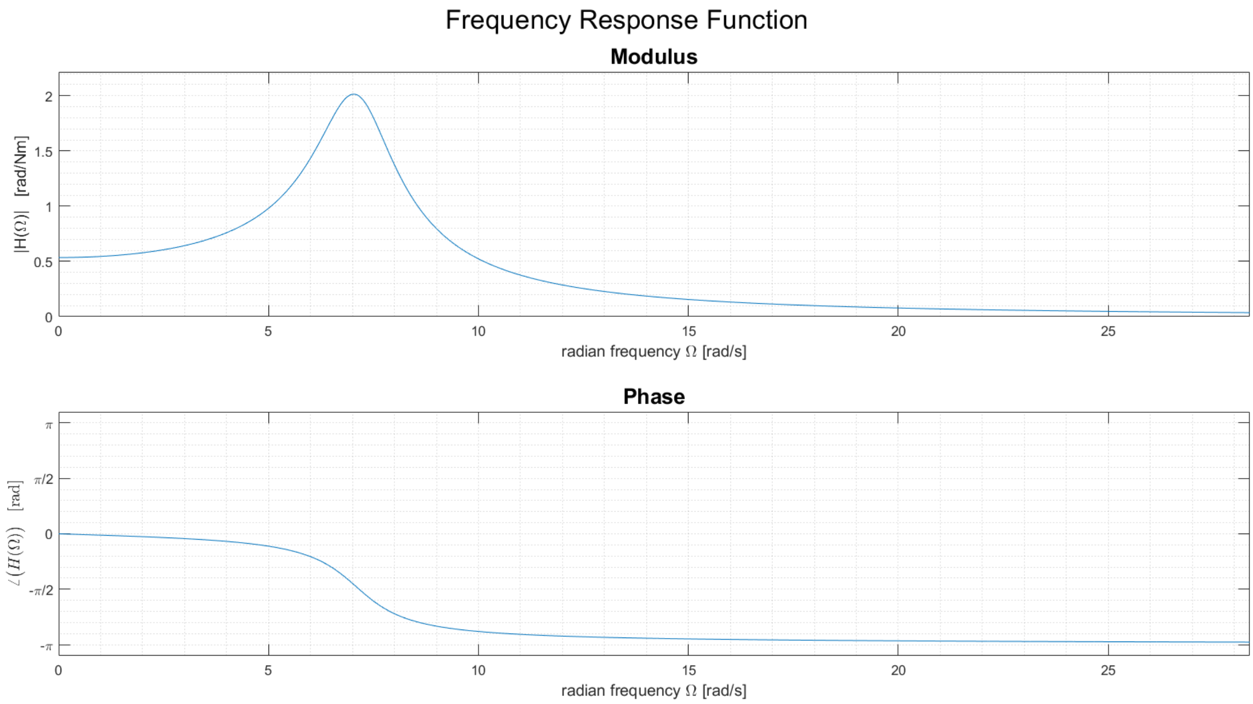
First, the FRF has been computed from the system parameters: specifically, in this case, it's defined as the ratio between the complex function embedding the steady-state discs rotation's amplitude and phase, which is a function of the forcing frequency, and the amplitude of the applied torque's oscillation.

$$H(\Omega) = \frac{\tilde{\bar{\Theta}}(\Omega)}{T} = \frac{1}{m_{\mathrm{g}}\, \Omega^2 - j c_{\mathrm{g}}\, \Omega + k_{\mathrm{g}}}$$

We're requested to apply it in three different cases.

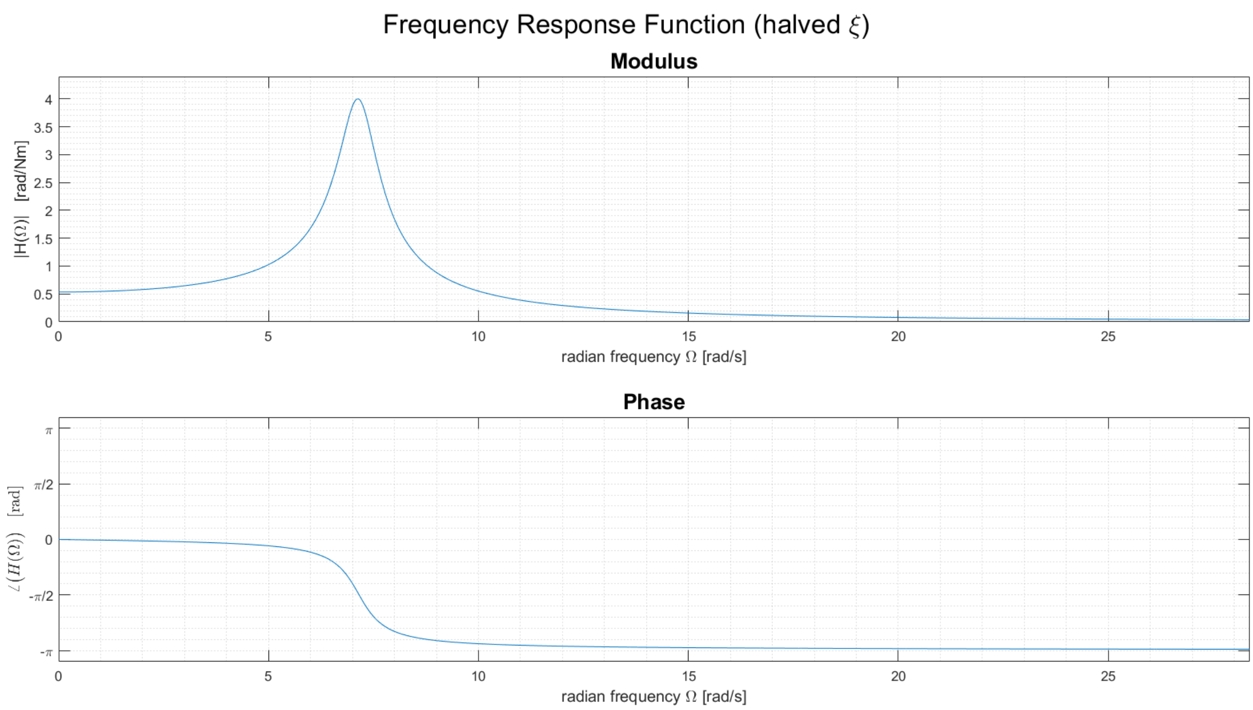
Case 1: generic initial conditions

In this first case, we're plotting the FRF assuming the system parameters are the given ones, which we started with. Notably, a peak (absolute maximum) in the magnitude diagram corresponds to the natural damped frequency ω_{d} of the system, computed at (4), which is reflected in the phase diagram in a $-\pi$ jump.



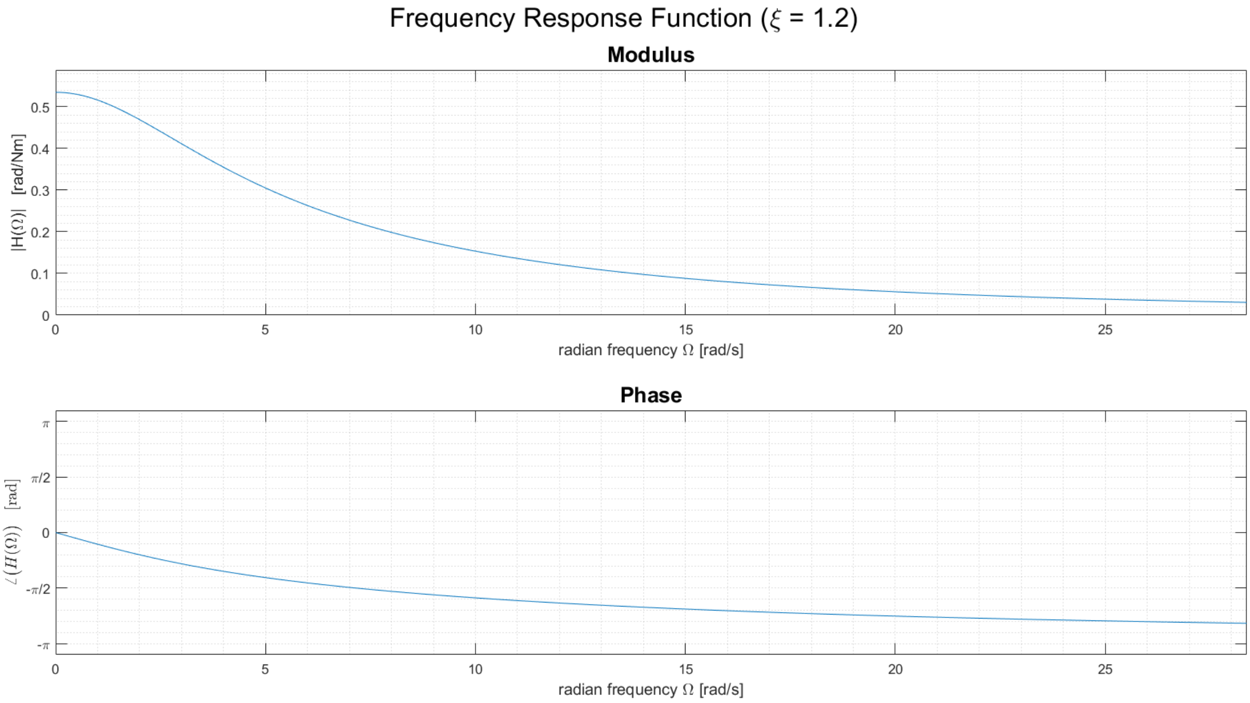
Case 2: halved adimensional damping ratio

Considering half the damping ratio, we expect the system to be closer to the ideal undamped case, which means that the peak in the modulus diagram is going to be sharper than in the previous case (getting closer to the asymptotic behavior typical of the undamped system) and the phase jump suddener in the phase diagram (getting closer to the ideal perfectly vertical jump of the undamped case). It is worthy to say also the natural damped frequency changes, in accordance with relation (4): in particular, its value decreases proportionally to the damping factor α, so the higher the damping, the lower the damped frequency, the more the peak moves to the left in the modulus plot.



Case 3: raised adimensional damping ratio

In this third case, the adimensional damping ratio is set to $\xi = 1.2$, so we find ourselves in the case of the overdamped system. We're expecting a smooth low-pass-like curve in the amplitude diagram as the damped frequency doesn't appear in the analytic formula of the response in time domain, i.e. the system's free motion does not oscillate, so the FRF shape is not depending on the natural frequency of the system. We might notice that the maximum steady-state oscillation amplitude is found for $\Omega = 0$, i.e. for a constant external force.



3.b Complete time response of the system

The system's complete response is composed by its free response and its forced response, summed together. The former is the one obtained in 2.a. For the latter, the input force to the system is considered to be a harmonic force applied to body 2, oriented in the vertical direction (we chose the verse to be upward, as it's not specified by the assignment) and made up by a single sinusoidal wave:

$$F(t) = A\,\cos(2\pi f_i\,t + \phi)$$

where $A = 2.5\,\mathrm{N}$ and $\phi = \frac{\pi}{3}$. Note that we chose the force for (1) according to the force we're using as the system's input at this point of the assignment. We're considering two different cases for the frequency of this single harmonic component. In the first case

$$f_i = f_1 = 0.15\,\mathrm{Hz} \;\Rightarrow\; \omega_i = \omega_1 = 2\pi f_1 \approx 0.94\,\mathrm{rad/s}$$

while in the second

$$f_i = f_2 = 4.5\,\mathrm{Hz} \;\Rightarrow\; \omega_i = \omega_2 = 2\pi f_2 \approx 28.27\,\mathrm{rad/s}$$

In each case, the steady-state motion has been computed by applying the well-known theoretical result stating the output of a linear time-invariant system is equal to the system input moduated in amplitude and phase by the FRF linking the independent variable with that input force computed for the particular frequency Ω of the input:

$$x_{\text{out}}(t) = |G(j\Omega)|\, F\, \cos\Big(\Omega t + \varphi + \angle\big(G(j\Omega)\big)\Big) \tag{5}$$

The FRF that links the discs rotation with this force is obtained by multiplying the FRF found at 3.a by the correspondent cinematic relationship (i.e. the force's 1x1 Jacobian), which is equal to R_2 in this case, so the final formula becomes

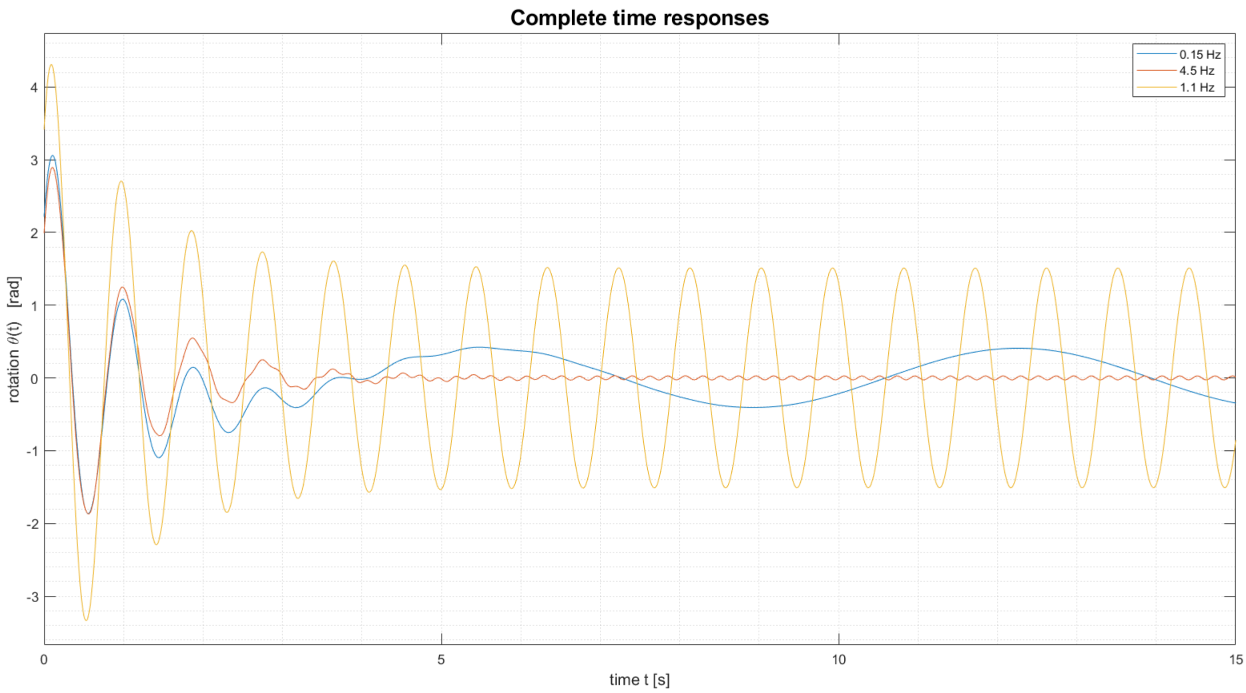
$$\begin{aligned}\bar{\theta}(t) &= \bar{\theta}^{\text{free}}(t) + \bar{\theta}^{\text{forced}}(t) = \\ &= e^{-\alpha t}\,(\bar{X}_1\, e^{j\omega_{\text{d}}\, t} + \bar{X}_2\, e^{-j\omega_{\text{d}}\, t}) + |H(\omega_i)|\, R_2\ A\ \cos\Big(\omega_i\, t + \phi + \angle\big(H(\omega_i)\big)\Big)\end{aligned}$$

Note that multiplying the FRF by a real positive constant number affects the amplitude only, since the resulting FRF has the amplitude of the system's FRF multiplied by that constant, and the phase of the system's FRF added to the constant's phase, which is zero.

We took the liberty of taking into account an additional case, where the external force frequency nearly matches the system's damped frequency, to check whether the amplitude of the resulting oscillation in steady-state is actually much larger than in the previous cases:

$$\omega_i = \omega_3 = 7\,\text{rad/s} \ \Rightarrow\ f_i = f_3 = \frac{\omega_3}{2\,\pi} \approx 1.11\,\text{Hz}$$

The system's complete responses over time for each of the three cases are visualized in the following diagram. Notably enough, in the first $4\,s$ of the motion, the transient greatly exhibits (much more than the respective forcing frequencies) the system's damped frequency even in the case the forcing frequency is not close to it.



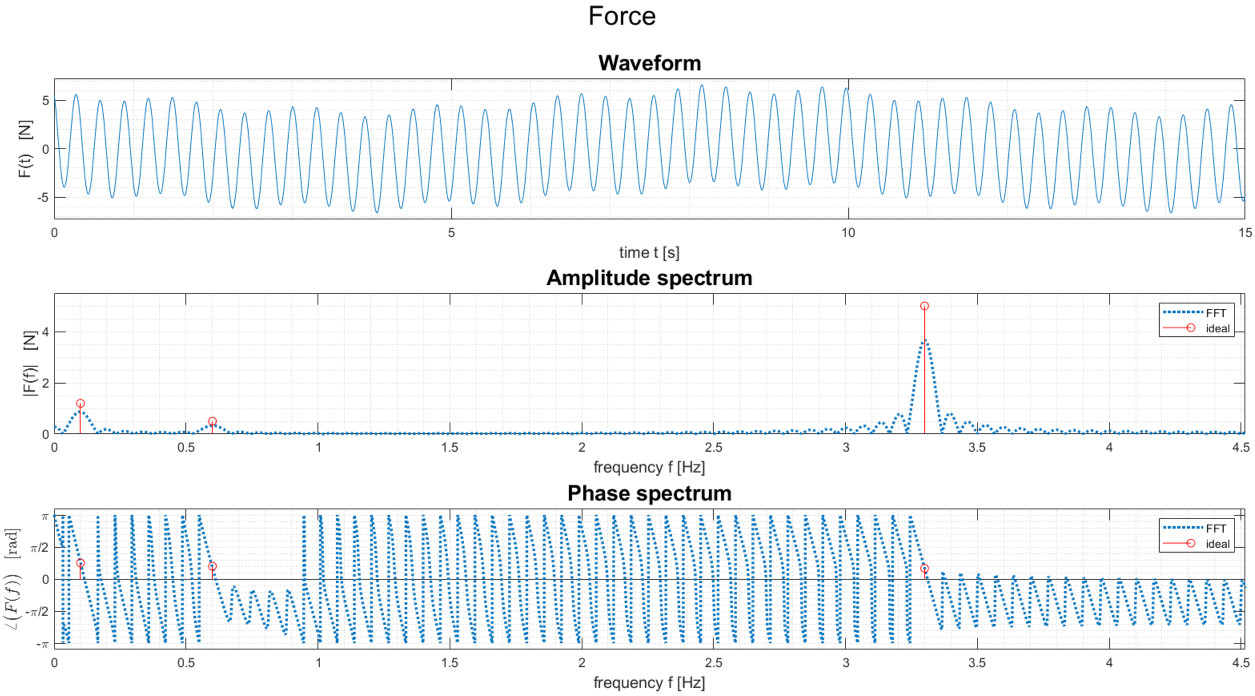
3.c Forced response of the system (steady-state)

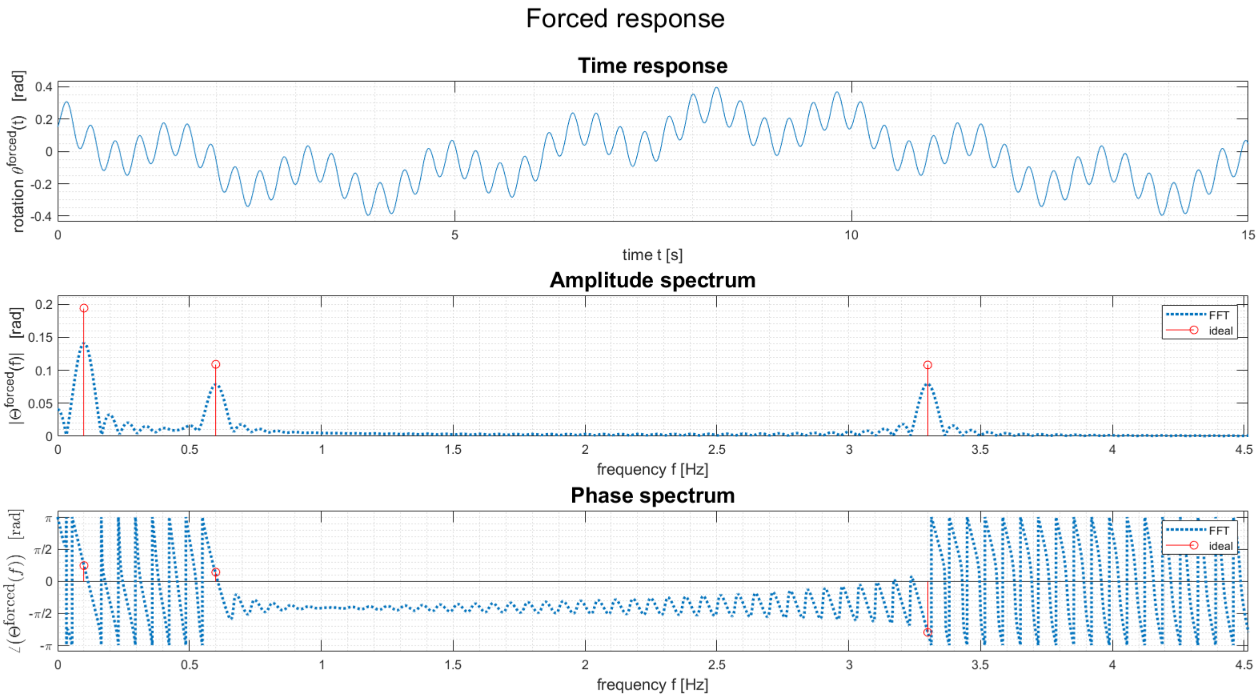
Finally, let's consider this time a complex force composed by three harmonic components as system input and applied as for the previous analysis case:

$$F(t) = \sum_{k=1}^{3} B_k \cos(2\pi f_k t + \phi_k)$$

with $B_1 = 1.2\,\mathrm{N}$, $B_2 = 0.5\,\mathrm{N}$, $B_3 = 5\,\mathrm{N}$, $f_1 = 0.1\,\mathrm{Hz}$ ($\omega_1 = 0.63\,\mathrm{rad/s}$), $f_2 = 0.6\,\mathrm{Hz}$ ($\omega_2 = 3.77\,\mathrm{rad/s}$), $f_3 = 3.3\,\mathrm{Hz}$ ($\omega_3 = 20.73\,\mathrm{rad/s}$), $\phi_1 = \frac{\pi}{4}$, $\phi_2 = \frac{\pi}{5}$ and $\phi_3 = \frac{\pi}{6}$. The external force and the forced vibration of the system are analyzed both in time and in frequency domain. For analyzing the forced response, the superposition principle has been employed: using equation (5), we computed the system forced response for each of the individual harmonic components that constitute the oscillating input. We then summed together the three partial forced responses to get the total steady-state response of the system. To validate the resut, we compared the spectrums with the Fast Fourier Transforms of the two time-domain signals making use of the relative Matlab function. As we may see, the amplitude spectra computed by means of FFT, scaled by the FFT length, qualitatively coincide with the ones we constructed showing peaks in correspondence to the same frequencies.

As the external force is concerned, we can easily identify the three sinusoidal components that build it up in time domain, while the spectrum simply displays the given data in terms of amplitude, frequency and phase shift. As for the system's forced vibration, we are again able to tell the three frequency components apart in the time-domain response. Furthermore, we can observe how different the weights of these three components are in the forced response, with respect to the input force: the low frequencies are the most emphasized, while the highest one is attenuated a big deal. This is due to the fact the original amplitudes for each cosine function in the external force are multiplied by the modulus of the corresponding Frequency Response Function computed for the frequency of the cosine functions itself, whereas the particular shape of the waveform is determined by the sum between the original phase shifts for each component with the phase value of the same FRF evaluated for the corresponding frequencies.


Force
Waveform
F(t) [N]
time t [s]
Amplitude spectrum
|F(f)| [N]
frequency f [Hz]
FFT
ideal
Phase spectrum
$\angle(F(f))$ [rad]
frequency f [Hz]


Forced response
Time response
rotation $\theta^{\text{forced}}(t)$ [rad]
time t [s]
Amplitude spectrum
$|\Theta^{\text{forced}}(f)|$ [rad]
frequency f [Hz]
FFT
ideal
Phase spectrum
$\angle(\Theta^{\text{forced}}(f))$ [rad]
frequency f [Hz]

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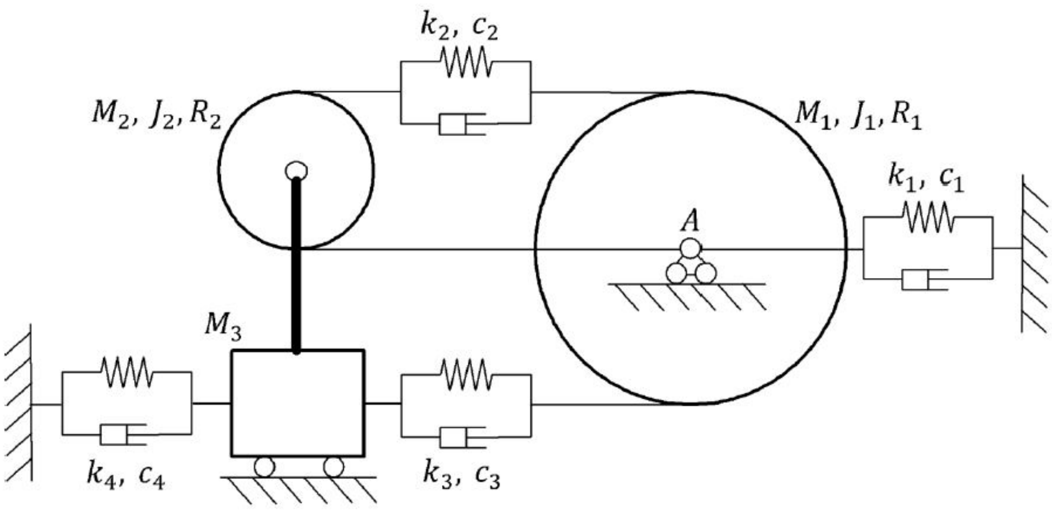
System schematic and parameters

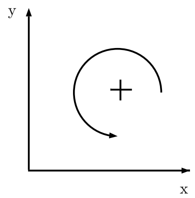


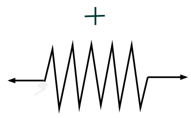
$$\begin{cases} M_1 = 5\,\text{kg} \quad , \quad J_1 = 2.5\,\text{kg m}^2 \\ M_2 = 1.25\,\text{kg} \quad , \quad J_2 = 0.16\,\text{kg m}^2 \\ M_3 = 10\,\text{kg} \end{cases} \qquad \begin{cases} R_1 = 1\,\text{m} \\ R_2 = 0.5\,\text{m} \end{cases}$$

$$\begin{cases} k_1 = 1000\,\text{N/m} \\ k_2 = 100\,\text{N/m} \\ k_3 = 560\,\text{N/m} \\ k_4 = 800\,\text{N/m} \end{cases} \qquad \begin{cases} c_1 = 0.5\,\text{Ns/m} \\ c_2 = 0.5\,\text{Ns/m} \\ c_3 = 1\,\text{Ns/m} \\ c_4 = 4\,\text{Ns/m} \end{cases}$$

1 Equation of motion

As a preliminary step, a reference system and sign conventions must be fixed. We chose to follow the commonly employed cartesian axes system, the counterclockwise rotation as the positive one and the spring elongation as the positive variation of the spring length:





1.a Equation derivation

Step 1: number of degrees of freedom identification

We proceeded computing the number of degrees of freedom N of the system using the same computation as for Assignment 1:

$$\begin{aligned} N = n_\mathrm{b} \cdot 3\ \mathrm{DOF} = 9\ \mathrm{DOF} - & \\ 1\ \mathrm{DOF} - &\quad \text{(roller on point A)} \\ 2\ \mathrm{DOF} - &\quad \text{(2 rollers for body 3)} \\ 1\ \mathrm{DOF} - &\quad \text{(string)} \\ 2\ \mathrm{DOF} = &\quad \text{(bar)} \\ 3\ \mathrm{DOF} & \end{aligned}$$

where n_b is the number of bodies in the system, $n_\mathrm{b} = 3$ in this case. We chose to solve the problem directly using Lagrange equation, resulting in one equation per independent variable, so three equations only. We chose the rotations θ_1 and θ_2 of bodies 1 and 2 respectively and the horizontal displacement x_3 of body 3 as the independent variables for our analysis. These are gathered together into the independent variables vector $\mathbf{u} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ x_3 \end{bmatrix}$.

Step 2: energy terms definition

$$E_\mathrm{k} = \frac{1}{2}\, J_1\, \omega_1^2 + \frac{1}{2}\, M_1\, v_1^2 + \frac{1}{2}\, J_2\, \omega_2^2 + \frac{1}{2}\, M_2\, v_2^2 + \frac{1}{2}\, M_3\, v_3^2$$

Gravitational contributions are not taken into account since none of the bodies can move in the vertical direction because of the system constraints:

$$V = V_\mathrm{e} = \frac{1}{2}\, k_1\, \Delta l_1^2 + \frac{1}{2}\, k_2\, \Delta l_2^2 + \frac{1}{2}\, k_3\, \Delta l_3^2 + \frac{1}{2}\, k_4\, \Delta l_4^2$$

$$D = \frac{1}{2}\, c_1\, \dot{\Delta l_1}^2 + \frac{1}{2}\, c_2\, \dot{\Delta l_2}^2 + \frac{1}{2}\, c_3\, \dot{\Delta l_3}^2 + \frac{1}{2}\, c_4\, \dot{\Delta l_4}^2$$

Because the assignment's requests define external forces to compute the system's forced motion later on, we're assuming a horizontal force $F(t)$, directed rightward, applied on point A:

$$\delta W = F(t)\, \delta x_\mathrm{F}$$

Step 3: physical variables as functions of independent ones

What we need now are the Jacobian matrices that allow to pass from physical coordinates to independent ones.

	$\dot{\theta}_1$	$\dot{\theta}_2$	$\dot{x}_3$
v_1	0	R_2	1
ω_1	1	0	0
v_2	0	0	1
ω_2	0	1	0
v_3	0	0	1

$$\Longrightarrow \qquad \mathbf{\Lambda_m} = \begin{bmatrix} 0 & R_2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

	θ_1	θ_2	x_3
Δl_1	0	$-R_2$	-1
Δl_2	$-R_1$	$2\,R_2$	0
Δl_3	R_1	R_2	0
Δl_4	0	0	1

$$\Longrightarrow \qquad \mathbf{\Lambda_k} = \begin{bmatrix} 0 & -R_2 & -1 \\ -R_1 & 2\,R_2 & 0 \\ R_1 & R_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

	θ_1	θ_2	x_3
δx_F	0	R_2	1

$$\Longrightarrow \qquad \mathbf{\Lambda_F} = \begin{bmatrix} 0 & R_2 & 1 \end{bmatrix}$$

Step 4: resulting equation

Remembering that the relationship between physical variables and independent ones is given by the Jacobians we found, we may write the Lagrangian component and the resulting motion equations (which are going to be three, as we have three independent variables) in matricial form as follows.

$$\delta W = \mathbf{Q_u}^{\mathrm{T}}\, \boldsymbol{\delta u} = F(t)\, \delta x_{\mathrm{F}} = F(t)\, \mathbf{\Lambda_F}\, \boldsymbol{\delta u} \;\Rightarrow\; \mathbf{Q_u}^{\mathrm{T}} = F(t)\, \mathbf{\Lambda_F} = \begin{bmatrix} 0 & F(t)\, R_2 & F(t) \end{bmatrix}$$
$$\Rightarrow\; \mathbf{Q_u} = \begin{bmatrix} 0 \\ F(t)\, R_2 \\ F(t) \end{bmatrix}$$

$$\left(\frac{\partial}{\partial t}\left(\frac{\partial E_{\mathrm{k}}}{\partial \dot{\mathbf{u}}}\right)\right)^{\mathrm{T}} - \left(\frac{\partial E_{\mathrm{k}}}{\partial \mathbf{u}}\right)^{\mathrm{T}} + \left(\frac{\partial V}{\mathbf{u}}\right)^{\mathrm{T}} + \left(\frac{\partial D}{\partial \dot{\mathbf{u}}}\right)^{\mathrm{T}} = \mathbf{Q_u}$$
$$\Rightarrow\; \underbrace{\mathbf{\Lambda_m}^{\mathrm{T}}\, \mathbf{M}\, \mathbf{\Lambda_m}}_{\substack{\text{generalized}\\ \text{mass matrix}\\ \mathbf{M}^*}}\, \ddot{\mathbf{u}} + \underbrace{\mathbf{\Lambda_k}^{\mathrm{T}}\, \mathbf{C}\, \mathbf{\Lambda_k}}_{\substack{\text{generalized}\\ \text{damping matrix}\\ \mathbf{C}^*}}\, \dot{\mathbf{u}} + \underbrace{\mathbf{\Lambda_k}^{\mathrm{T}}\, \mathbf{K}\, \mathbf{\Lambda_k}}_{\substack{\text{generalized}\\ \text{stiffness matrix}\\ \mathbf{K}^*}}\, \mathbf{u} = \underbrace{\mathbf{\Lambda_F}^{\mathrm{T}}\, F(t)}_{\substack{\text{generalized}\\ \text{forces vector}\\ \mathbf{Q_u}}} \tag{1}$$

1.b Eigenfrequencies and eigenvectors

We're asked to solve two eigenvalues-eigenvectors problem, one for the undamped system and one for the damped one. The to-be-found natural and damped eigenfrequencies of each mode of oscillation correspond, respectively, to the eigenvalues of the state matrices $\mathbf{A_{und}}$ and $\mathbf{A_d}$ of the undamped and damped system's characteristic equation, while the mode shapes to their eigenvectors, again both for undamped and damped case. The characteristic equation stems from the equation of motion where the external generalized forces term has been set equal to zero, i.e. analyzing the free motion of the system:

$$\mathbf{M}^*\,\ddot{\mathbf{u}} + \mathbf{C}^*\,\dot{\mathbf{u}} + \mathbf{K}^*\,\mathbf{u} = \mathbf{0}$$

For both the undamped and damped case, we made use of the Matlab function `eig()` passing $\mathbf{A_{und}}$ and $\mathbf{A_d}$ as parameters, which are, again, the state matrices of the undamped and damped system we're gonna see more in detail now.

Undamped case

In this case, the characteristic equation we obtain has no generalized damping matrix in it:

$$(\gamma^2\,\mathbf{M}^* + \mathbf{K}^*)\,\mathbf{U} = \mathbf{0}$$

where γ is the coefficient of time variable in the differential equation solution $\mathbf{u} = \mathbf{U}\,e^{\gamma\,t}$. The reference eigenvalues-eigenvectors problem is ascribable to the standard problem:

$$(\lambda\,\mathbf{I} - \mathbf{A_{und}})\,\mathbf{x} = \mathbf{0}$$

So the state matrix $\mathbf{A_{und}}$ in this case is

$$\mathbf{A_{und}} = -\mathbf{M}^{*^{-1}}\,\mathbf{K}^*$$

This yields the following natural eigenfrequencies and undamped mode shapes (normalized with respect to the first element), one for each of the three modes:

1st mode:

$$\omega_{\mathrm{n}}^{(1)} = 9.7840\,\mathrm{rad/s} \qquad \mathbf{U_{und}^{(1)}} = \begin{bmatrix} 1 \\ -2.3371 \\ 2.4924 \end{bmatrix}$$

2nd mode:

$$\omega_{\mathrm{n}}^{(2)} = 14.1841\,\mathrm{rad/s} \qquad \mathbf{U_{und}^{(2)}} = \begin{bmatrix} 1 \\ -0.8724 \\ 0.0018 \end{bmatrix}$$

3rd mode:

$$\omega_{\mathrm{n}}^{(3)} = 21.1479\,\mathrm{rad/s} \qquad \mathbf{U_{und}^{(3)}} = \begin{bmatrix} 1 \\ 2.5449 \\ -0.2877 \end{bmatrix}$$

As damping is being neglected, all the values in the eigenvectors are real.

Damped case

The construction of the state matrix for the damped case is not as trivial as for the undamped case: we resort to the state-space representation.

$$\mathbf{M}^*\,\ddot{\mathbf{u}} + \mathbf{C}^*\,\dot{\mathbf{u}} + \mathbf{K}^*\,\mathbf{u} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} \mathbf{M}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^* \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} -\mathbf{C}^* & -\mathbf{K}^* \\ \mathbf{M}^* & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \mathbf{u} \end{bmatrix}$$

$$\Rightarrow \mathbf{A_d} = \begin{bmatrix} \mathbf{M}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^* \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{C}^* & -\mathbf{K}^* \\ \mathbf{M}^* & \mathbf{0} \end{bmatrix}$$

This leads to the damping factor, damped frequency and normalized damped mode shape for each mode:

1st mode:

$$\alpha^{(1)} = 0.1910\,\mathrm{rad/s} \qquad \omega_{\mathrm{d}}^{(1)} = 9.7838\,\mathrm{rad/s} \qquad \mathbf{U_d^{(1)}} = \begin{bmatrix} 1 \\ -2.3361 - \jmath 0.0296 \\ 2.4856 + \jmath 0.1356 \end{bmatrix}$$

2nd mode:

$$\alpha^{(2)} = 0.3036\,\mathrm{rad/s} \qquad \omega_{\mathrm{d}}^{(2)} = 14.1792\,\mathrm{rad/s} \qquad \mathbf{U_d^{(2)}} = \begin{bmatrix} 1 \\ -0.8731 + \jmath 0.0014 \\ 0.0020 + \jmath 0.0106 \end{bmatrix}$$

3rd mode:

$$\alpha^{(3)} = 0.3850\,\mathrm{rad/s} \qquad \omega_{\mathrm{d}}^{(3)} = 21.1432\,\mathrm{rad/s} \qquad \mathbf{U_d^{(3)}} = \begin{bmatrix} 1 \\ 2.5433 + \jmath 0.0499 \\ -0.2874 - \jmath 0.0132 \end{bmatrix}$$

It's worth it to say the damped eigenfrequencies are very much close to the undamped ones, as the damping is light enough, and the eigenvectors are now complex, as damping has been introduced.

1.c Proportional damping matrix

In place of the generalized damping matrix $\mathbf{C}^*$, we may as well use a matrix $\mathbf{C_{ray}}$ that approximates it using the Rayleigh proportional damping, that is a linear combination of the generalized mass and stiffness matrices through coefficients α_{ray} and β_{ray}:

$$\mathbf{C_{ray}} = \alpha_{\mathrm{ray}}\,\mathbf{M}^* + \beta_{\mathrm{ray}}\,\mathbf{K}^*$$

We started computing the damping ratios ξ_i as $\frac{C^*_{ii}}{C^*_{\mathrm{cr}ii}} = \frac{C^*_{ii}}{2\,M^*_{ii}\,\omega_{\mathrm{n}}^{(i)}}$ for $i = 1, 2, 3$, stored in the vector $\boldsymbol{\xi}$, then exploited the relationship between the values of $\boldsymbol{\xi}$ and $\alpha_{\mathrm{ray}}, \beta_{\mathrm{ray}}$ obtained by substituting $C^*_{ii} = \alpha_{\mathrm{ray}}\,M^*_{ii} + \beta_{\mathrm{ray}}\,K^*_{ii}$ in the expression for ξ_i in order for us to find the values of α_{ray} and β_{ray}:

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega_{\mathrm{n}}^{(1)}} & \frac{\omega_{\mathrm{n}}^{(1)}}{2} \\ \frac{1}{\omega_{\mathrm{n}}^{(2)}} & \frac{\omega_{\mathrm{n}}^{(2)}}{2} \\ \frac{1}{\omega_{\mathrm{n}}^{(3)}} & \frac{\omega_{\mathrm{n}}^{(3)}}{2} \end{bmatrix} \begin{bmatrix} \alpha_{\mathrm{ray}} \\ \beta_{\mathrm{ray}} \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_{\mathrm{ray}} \\ \beta_{\mathrm{ray}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega_{\mathrm{n}}^{(1)}} & \frac{\omega_{\mathrm{n}}^{(1)}}{2} \\ \frac{1}{\omega_{\mathrm{n}}^{(2)}} & \frac{\omega_{\mathrm{n}}^{(2)}}{2} \\ \frac{1}{\omega_{\mathrm{n}}^{(3)}} & \frac{\omega_{\mathrm{n}}^{(3)}}{2} \end{bmatrix}^{-1} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0.7078 \\ -0.0008 \end{bmatrix}$$

$$\Rightarrow \mathbf{C_{ray}} = \begin{bmatrix} 1.2144 & -0.1514 & 0 \\ -0.1514 & 0.5858 & 1.3490 \\ 0 & 1.3490 & 9.9881 \end{bmatrix}$$

where as the inverse of the 3x2 matrix we used the Matlab function `pinv()` to compute its Moore-Penrose pseudoinverse. $\mathbf{C_{ray}}$ is a symmetric matrix, as expected, as it's a linear combination of symmetric matrices. From now on, this just-found proportional damping matrix is going to be employed as the damping matrix for the system.

2 Free motion of the system

Like for the single-degree-of-freedom systems, the equation describing the free motion of each independent variable comes from solving the motion equation when no external forces or torques are acting. We already solved this equation in the previous steps, which allowed us to obtain the damping coefficients vector $\boldsymbol{\alpha}$, the damped frequencies vector $\boldsymbol{\omega}_{\mathbf{d}}$ and the damped mode shapes $\mathbf{U_d}$:

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \end{bmatrix} = \begin{bmatrix} 0.1910 \\ 0.3036 \\ 0.3850 \end{bmatrix} \qquad \boldsymbol{\omega}_{\mathbf{d}} = \begin{bmatrix} \omega_{\mathrm{d}}^{(1)} \\ \omega_{\mathrm{d}}^{(2)} \\ \omega_{\mathrm{d}}^{(3)} \end{bmatrix} = \begin{bmatrix} 9.7838 \\ 14.1792 \\ 21.1432 \end{bmatrix}$$

$$\mathbf{U_d} = \begin{bmatrix} \mathbf{U_d^{(1)}} \mathbf{U_d^{(2)}} \mathbf{U_d^{(3)}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2.3361 - j0.0296 & -0.8731 + j0.0014 & 2.5433 + j0.0499 \\ 2.4856 + j0.1356 & 0.0020 + j0.0106 & -0.2874 - j0.0132 \end{bmatrix}$$

2.a Response to given initial conditions

We're now able to manually build the analytic expression for the free motion of each independent variable:

$$\mathbf{u}^{\mathbf{free}}(t) = \sum_{i=1}^{3} e^{-\alpha^{(i)}\, t}\, |\mathbf{U_d^{(i)}}| \Big(A^{(i)} \cos\big(\omega_{\mathrm{d}}^{(i)}\, t + \angle(\mathbf{U_d^{(i)}})\big) + B^{(i)} \sin\big(\omega_{\mathrm{d}}^{(i)}\, t + \angle(\mathbf{U_d^{(i)}})\big) \Big)$$

Prior to that, the relationship between the given initial conditions for both position and velocity of each independent variable and the coefficients of each sinusoidal term in the free response must be extracted: we manually computed the derivative with respect to time of the equation relative to the free motion for each variable to obtain the velocities, then evaluated the two expressions at initial time $t = 0$ and set them to be equal to the initial conditions, factoring out each of the cosine coefficient $A^{(i)}$, i=1,2,3 and each of the sine ones $B^{(i)}$ and, by doing so, writing each initial condition as a linear combination of those coefficients. The coefficients of the linear combinations, depending on system parameters only, are gathered in the matrix called $\mathbf{S}$:

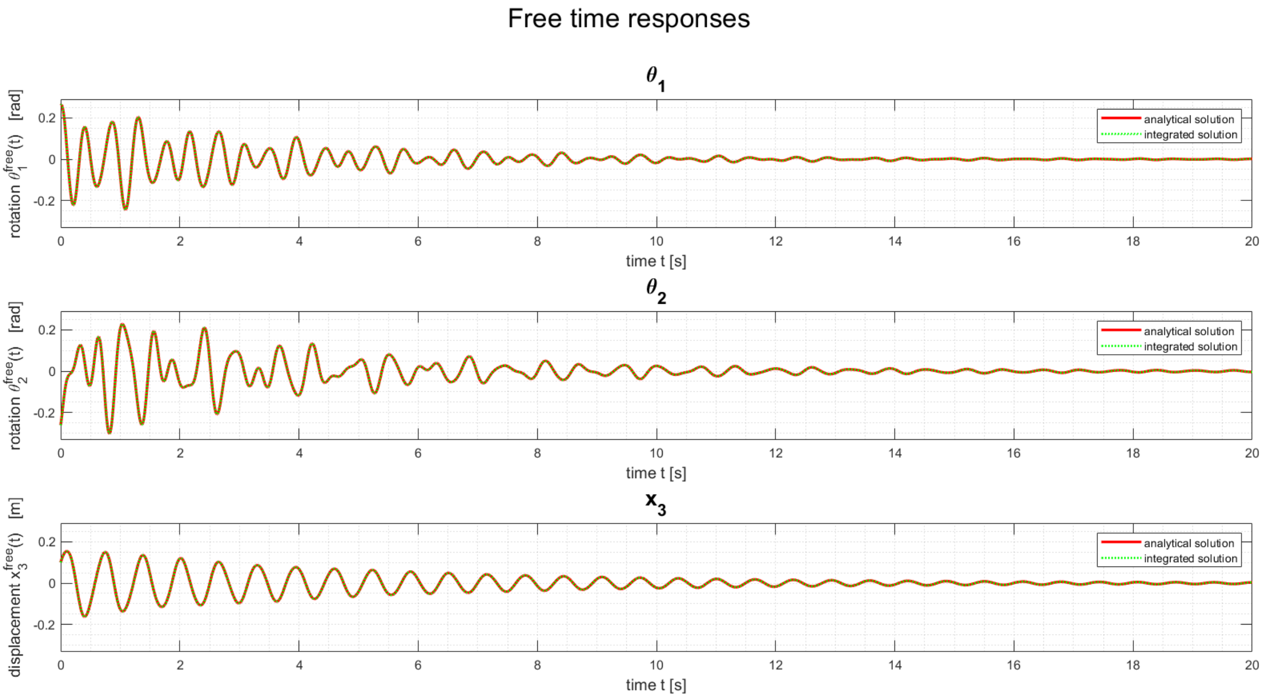
$$\begin{bmatrix} \mathbf{u_0} \\ \dot{\mathbf{u}}_0 \end{bmatrix} = \begin{bmatrix} \Re\{\mathbf{U_d}\} & \Im\{\mathbf{U_d}\} \\ -\boldsymbol{\alpha}^{\mathrm{T}} \circ \Re\{\mathbf{U_d}\} - \boldsymbol{\omega}_{\mathbf{d}}^{\mathrm{T}} \circ \Im\{\mathbf{U_d}\} & -\boldsymbol{\alpha}^{\mathrm{T}} \circ \Im\{\mathbf{U_d}\} + \boldsymbol{\omega}_{\mathbf{d}}^{\mathrm{T}} \circ \Re\{\mathbf{U_d}\} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{1_0} \\ \theta_{2_0} \\ x_{3_0} \\ \dot{\theta}_{1_0} \\ \dot{\theta}_{2_0} \\ \dot{x}_{3_0} \end{bmatrix} = \mathbf{S} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \\ B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{bmatrix} \Rightarrow \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \\ B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{bmatrix} = \mathbf{S}^{-1} \begin{bmatrix} \theta_{1_0} \\ \theta_{2_0} \\ x_{3_0} \\ \dot{\theta}_{1_0} \\ \dot{\theta}_{2_0} \\ \dot{x}_{3_0} \end{bmatrix} = \begin{bmatrix} 0.0065 \\ 0.2170 \\ 0.0382 \\ 0.0449 \\ -0.0654 \\ 0.0565 \end{bmatrix} \quad \text{with} \begin{cases} \theta_{1_0} = \frac{\pi}{12}\,\mathrm{rad} \\ \theta_{2_0} = -\frac{\pi}{12}\,\mathrm{rad} \\ x_{3_0} = 0.1\,\mathrm{m} \\ \dot{\theta}_{1_0} = 0.5\,\mathrm{rad/s} \\ \dot{\theta}_{2_0} = 2\,\mathrm{rad/s} \\ \dot{x}_{3_0} = 1\,\mathrm{m/s} \end{cases} \tag{2}$$

where the symbol $\circ$ is the row-wise product. These values for elements for vectors $\mathbf{A}$ and $\mathbf{B}$ shows that these initial conditions excite all vibration modes, as no value is null, and without particularly favoring one mode over the others. To be sure the matrix $\mathbf{S}$ has been correctly derived, the analytic solutions have been compared to the integrated ones obtained using the state matrix $\mathbf{A_d}$ with Matlab `ode45()` function:

```
odefun = @(t,y) A_damp*y;
[t_ode,sol] = ode45(odefun,[t(1) t(end)],[u_0_dot;u_0]);
```

In the following plot, we show the response of the three variables to the given initial conditions both by making use of the $\mathbf{S}$ matrix and by using the `ode45()` function. As we can see, the two perfectly coincide, validating our procedure. The most evident feature is that, while the two rotations are clearly a superposition of different modes and have a more chaotic motion, the displacement has a behavior very well approximated by the free motion of a single-degree-of-freedom system, meaning that, with these initial conditions that result in a quite uniform mixture of all three oscillation modes, one of these is prominent in terms of amplitude with respect to the other two, as for the third independent variable. From a very rough estimation, we may infer that this prominent mode is the first one, as the frequency is slightly higher than 3 cycles in the first 2 seconds, i.e. slightly higher than $1.5\,\mathrm{Hz} \approx 9.42\,\mathrm{rad/s}$, which corresponds to $\omega_{\mathrm{d}}^{(1)}$. This could have been predicted assessing the modulus of the elements of the mode shapes relative to the third independent varaible: the one for the first mode shape is two orders of magnitude greater than the one for the second mode shape and one order of magnitude greater than the one for mode 3.



2.b Exciting one mode only

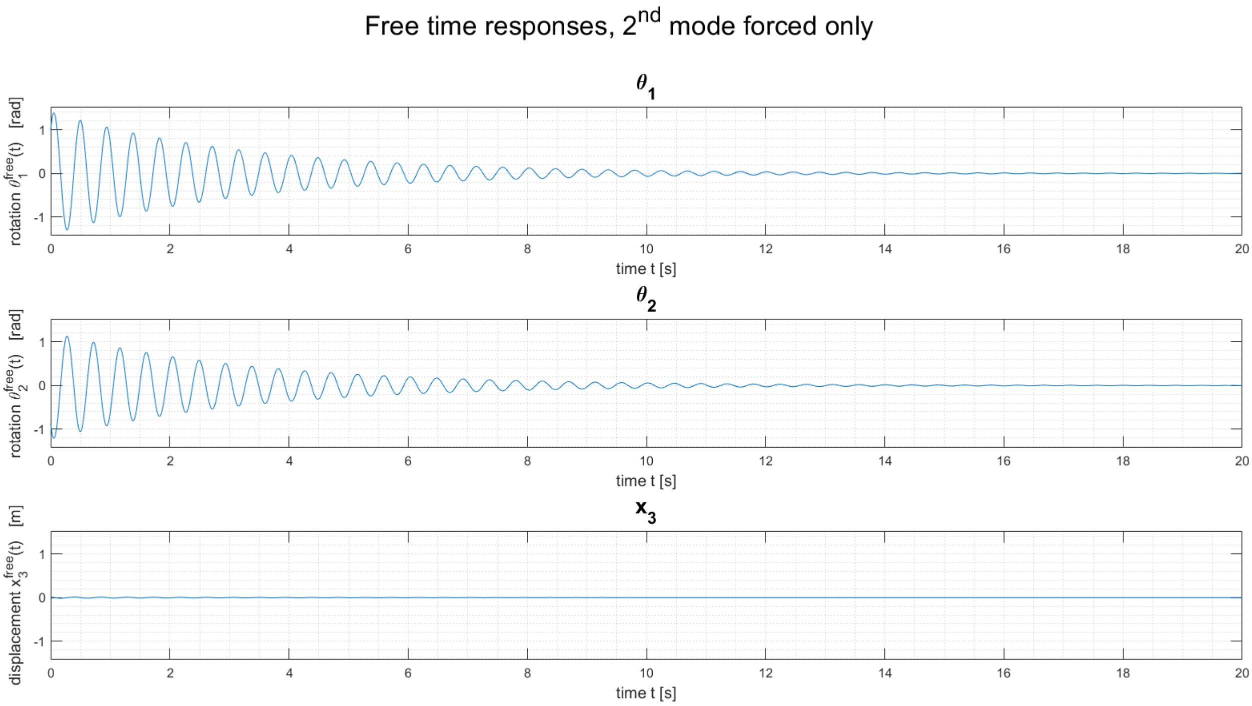
At this point, it's quite easy to set the initial conditions so that only one mode is going to show up in the free response: those are some initial conditions that make sure the $A^{(i)}$ and $B^{(i)}$ are different from zero for only one value of i, and equal to zero for all the others. For example, willing to force the second mode only and with a unitary value for both the coefficients, we obtain

$$\mathbf{A} = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{bmatrix} = \mathbf{B} = \begin{bmatrix} B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

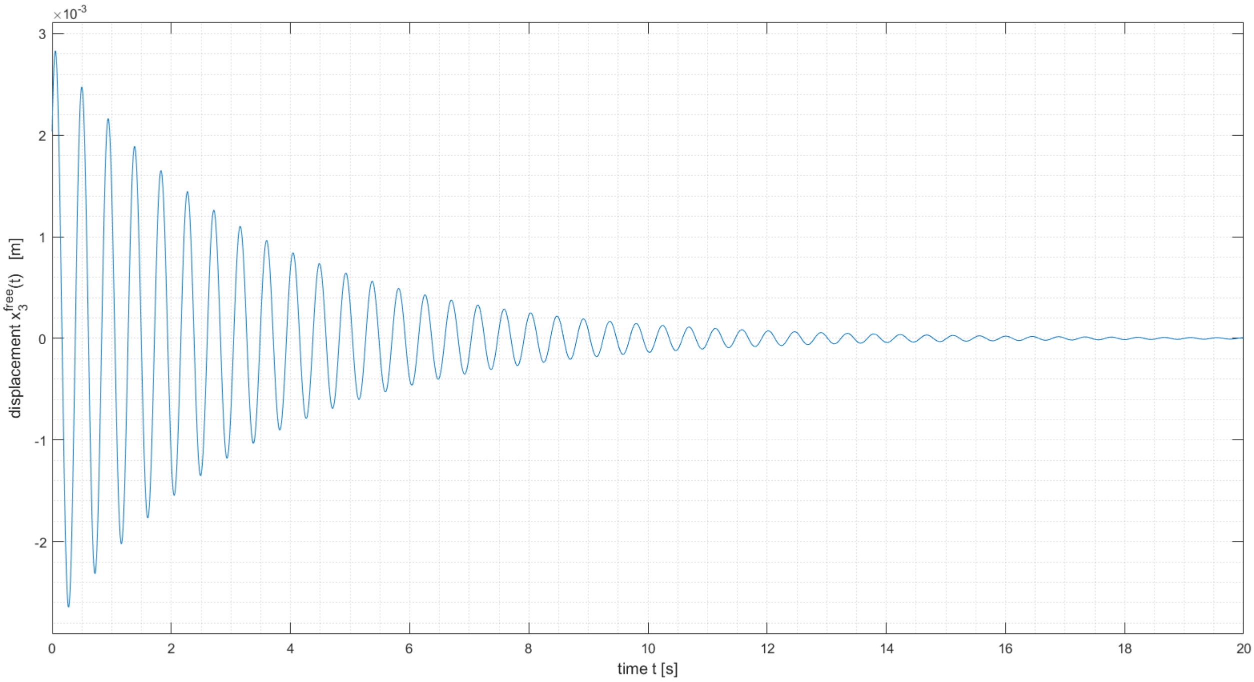
We may then apply the direct formula in (2) to result in the initial conditions to be applied:

$$\begin{bmatrix} \theta_{1_0} \\ \theta_{2_0} \\ x_{3_0} \\ \dot{\theta}_{1_0} \\ \dot{\theta}_{2_0} \\ \dot{x}_{3_0} \end{bmatrix} = \mathbf{S} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \\ B^{(1)} \\ B^{(2)} \\ B^{(3)} \end{bmatrix} = \mathbf{S} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0,8716 \\ 0.0126 \\ 13.8756 \\ -12.1349 \\ -0.1252 \end{bmatrix}$$

The free responses obtained with these initial conditions are depicted in time domain in the following. In accordance with the fact only one mode is present, each variable oscillates with different initial phases and amplitudes, but at just one frequency and at the same decay rate. Moreover, the third independent variable is almost a constant at zero value when comparing it with the other two, indicating the third body stays almost still during this free motion: this means the second mode contributes extremely little to the overall motion of that body, in comparison with the other modes.



A zoomed view of the graph for the third independent variable is showed here: we may again appreciate the fact a single mode is present in the motion.



3 Forced motion of the system

Considering the Rayleigh damping found in 1.c, we now study the forced motion of the system.

3.a Frequency Response Matrix

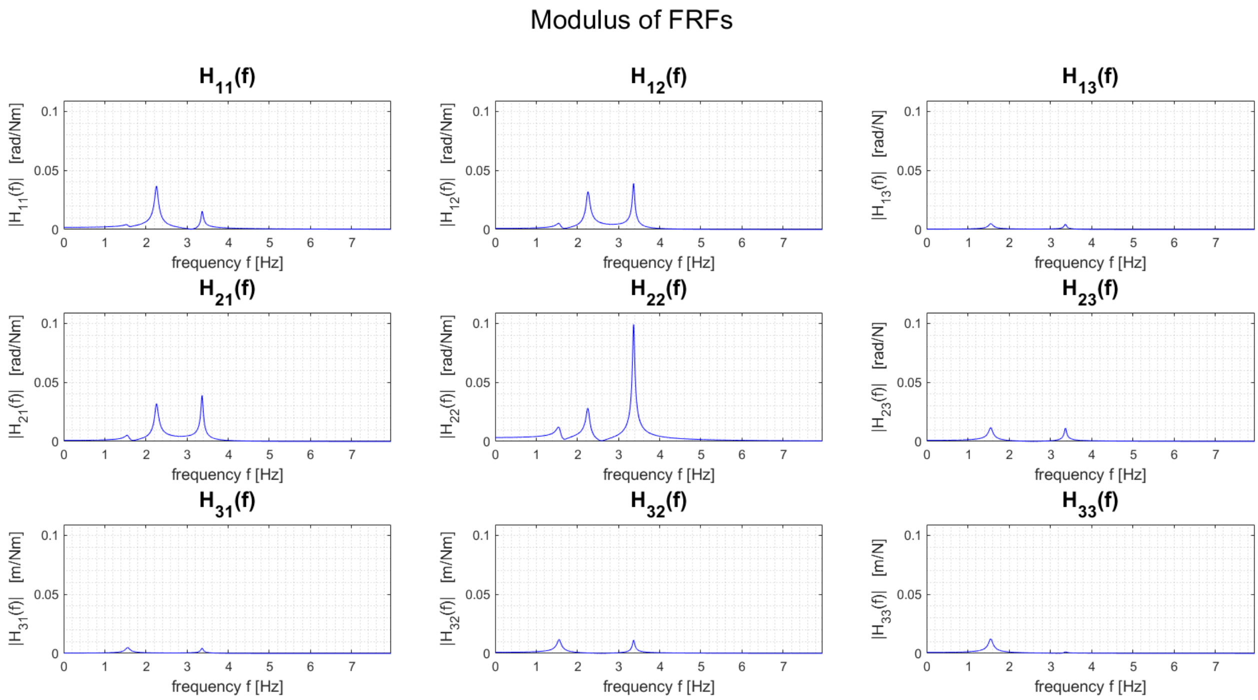
Analogously to the considerations we made for the one-degree-of-freedom system, for computing the system's Frequency Response Functions we are implicitly considering one generalized force for each variable, such that each of them is applied to the body corresponding to that independent variable and has a Jacobian matrix of zeros except for a 1 in correspondence with that variable. The FRFs that we find applying the formula

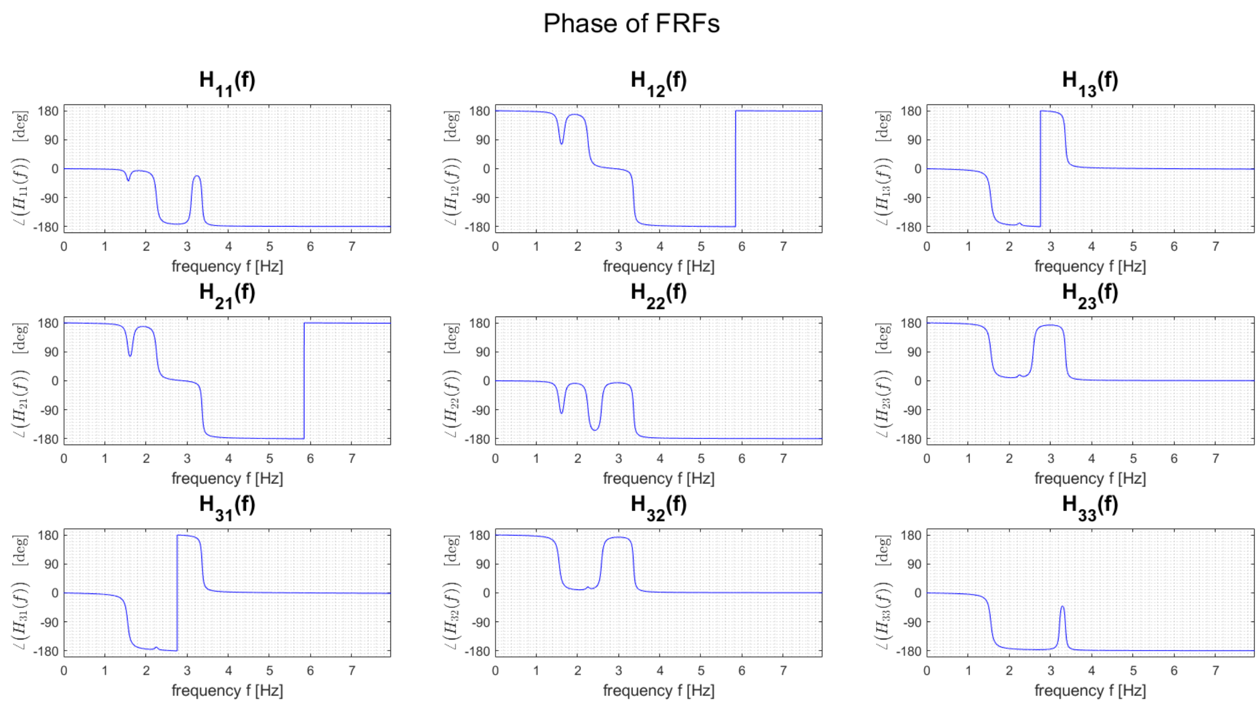
$$\mathbf{H}(\Omega) = \mathbf{D}(\Omega)^{-1} = (-\Omega^2\,\mathbf{M}^* + j\,\Omega\,\mathbf{C_{ray}} + \mathbf{K}^*)^{-1}$$

are, then, the transfer functions between each of the independent variables and each of these generalized forces. This way, we found a Frequency Response Matrix (FRM) where the element $H_{ij}(\Omega)$ is the transfer function between the i-th independent variable and the j-th generalized force. Note that the FRFs appearing on the diagonal of the matrix are the co-located FRFs, meaning that they relate an independent variable to the correspondent generalized force.

$$\mathbf{H}(\Omega) = \begin{bmatrix} H_{11}(\Omega) & H_{12}(\Omega) & H_{13}(\Omega) \\ H_{21}(\Omega) & H_{22}(\Omega) & H_{23}(\Omega) \\ H_{31}(\Omega) & H_{32}(\Omega) & H_{33}(\Omega) \end{bmatrix}$$

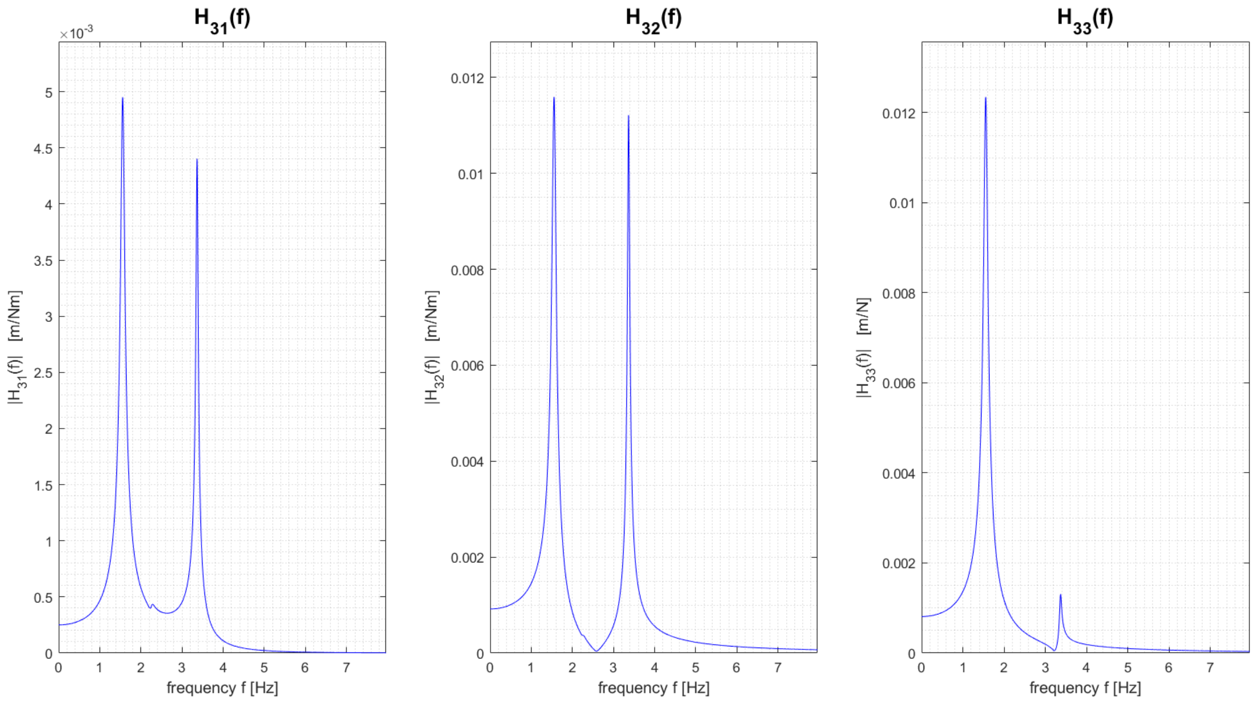
The corresponding plots in terms of magnitude and phase follow:





As expected, the matrix is symmetric, which also tells us that some sort of reciprocity principle holds: the same behavior is observed moving one body and observing another one or moving the latter and observing the former.

We immediately notice the displacement of the third body in resonance condition is by far smaller than the rotations of the two discs. We also see that the second mode is almost unperceivable in the displacement of body 3, indicating giving a hint for the presence of a node, which explains why x_3 has such a little oscillation with respect to θ_1 and θ_2 in 2.b. Here's a zoomed view of what happens in the FRF modulus matrix' last row: in correspondence with the second mode, it is barely possible to spot the small variations in the



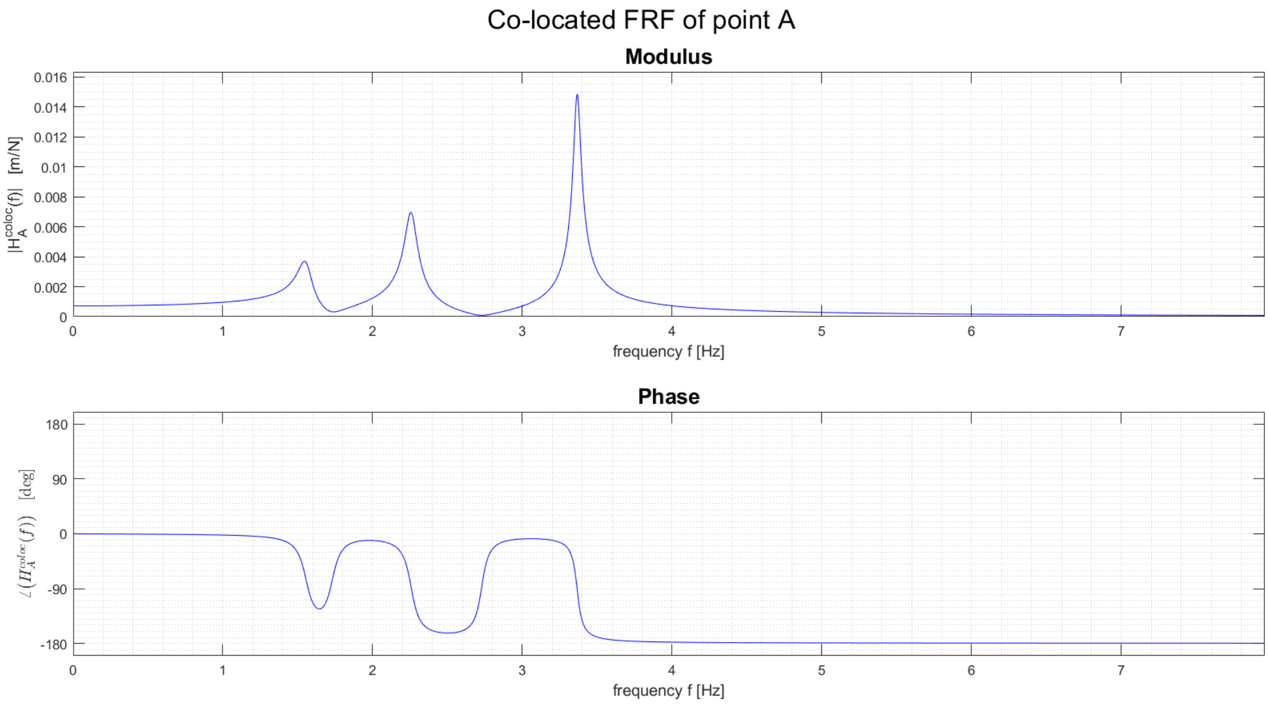
graphs. For $H_{33}(\Omega)$, the second mode's peak doesn't even correspond to a flat curve portion like in $H_{32}(\Omega)$, but only to a decrease in the graph's decreasing rate, i.e. to an increase in the function's first derivative or an inflection point.

3.b Co-located FRF of point A

Let's now consider the system as we did in (1), with the external force applied to point A, and let's retrieve the FRF between the displacement of point A itself and this force. In order to do so, the FRM must be multiplied by the Jacobian relative to the force computed at 1.a, which is already the force we're considering in this paragraph. So

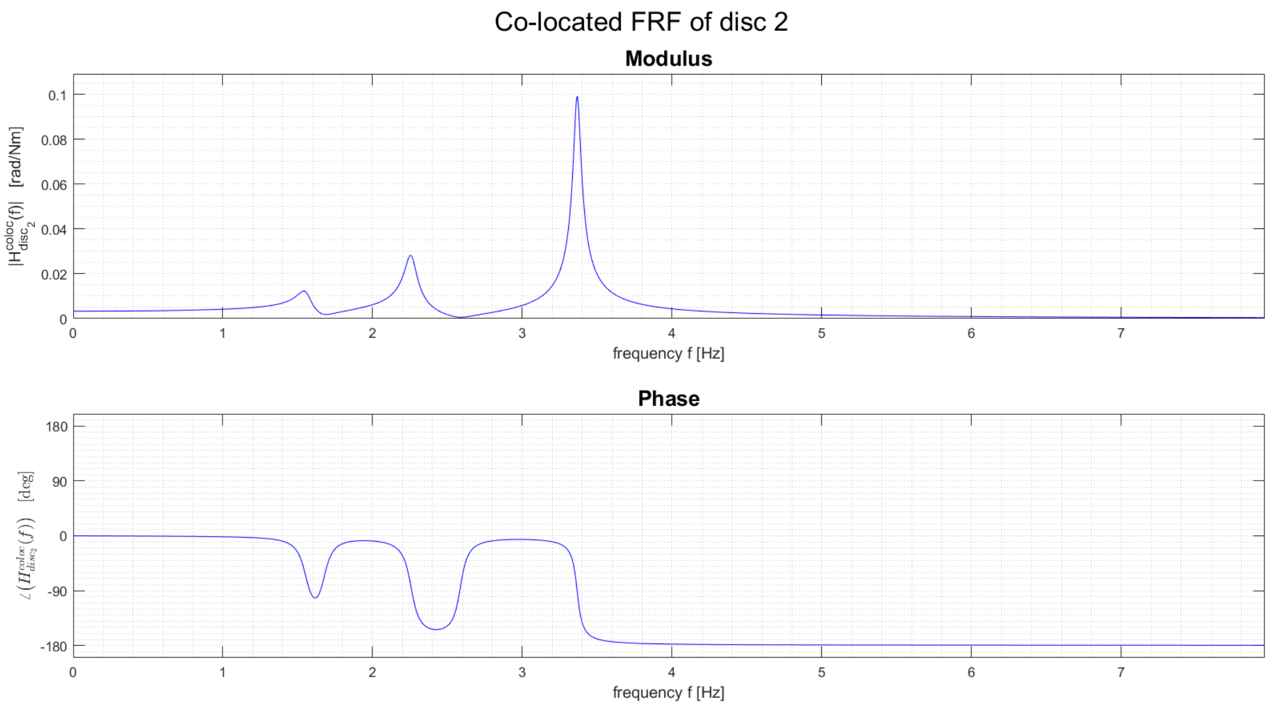
$$H_A^{\text{coloc}}(\Omega) = \mathbf{\Lambda_F}\, \mathbf{H}(\Omega)\, \mathbf{\Lambda_F}^{\mathrm{T}}$$

which gives the following amplitude and phase diagrams.



3.c Co-located FRF of disc 2

As a consequence of what we already said about the meaning of each element of the FRM, the requested co-located FRF is exactly the element H_{22} of the matrix:



3.d Complete time response of the system

The complete time response is obtained summing the free response to the forced one. The initial conditions we're considering are those specified at 2.a, so the free motion is the one we already computed $\mathbf{u}^{\mathbf{free}}(t)$. The forced motion is the response to the same horizontal force applied on point A that we considered so far, formed by two cosinusoidal components:

$$F(t) = A_1\,\cos(2\pi f_1\,t) + A_2\,\cos(2\pi f_2\,t)$$

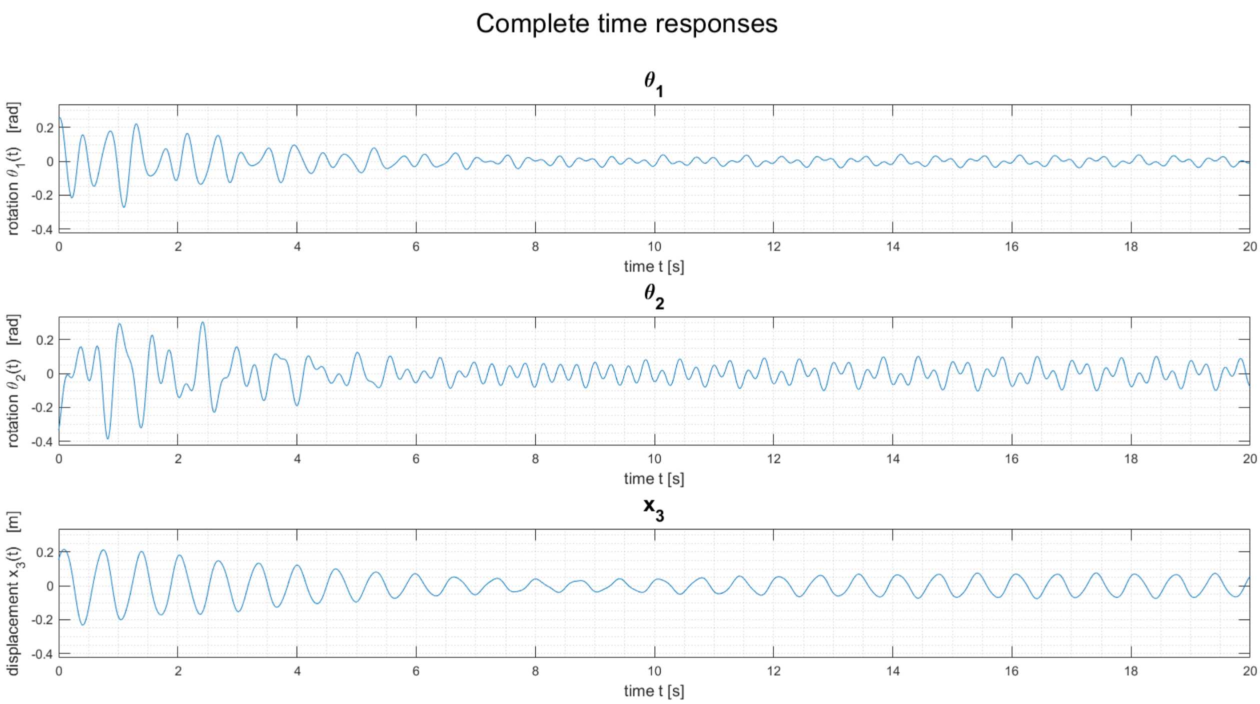
with $A_1 = 15\,\mathrm{N}$, $A_2 = 7\,\mathrm{N}$, $f_1 = 1.5\,\mathrm{Hz}$, $f_2 = 3.5\,\mathrm{Hz}$. The first step is to find the three FRFs for the case, so we're considering the case when the force is the generic $F(t) = F_0\,\cos(\Omega\,t)$. Starting from the equation of motion at 1 and proceeding as for the single-degree-of-freedom system, we have that

$$\mathbf{M}^*\,\ddot{\mathbf{u}} + \mathbf{C_{ray}}\,\dot{\mathbf{u}} + \mathbf{K}^* = \mathbf{Q_u} = \mathbf{\Lambda_F}^{\mathrm{T}}\,F(t) = \mathbf{\Lambda_F}^{\mathrm{T}}\,F_0\,\cos(\Omega\,t)$$
$$\Rightarrow\ \mathbf{D}(\Omega)\,\tilde{\mathbf{U}}\,e^{j\Omega\,t} = \mathbf{\Lambda_F}^{\mathrm{T}}\,F_0\,e^{j\Omega\,t}\ \Rightarrow\ \frac{\tilde{\mathbf{U}}}{F_0} = \mathbf{H}(\Omega)\,\mathbf{\Lambda_F}^{\mathrm{T}} = \mathbf{H_A}(\Omega)$$

Exploiting the superposition principle for the forced responses, the three time functions expressing the complete motion, in matricial form, are

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}^{\mathbf{free}}(t) + \mathbf{u}^{\mathbf{forced}}(t) = \\ &= \sum_{i=1}^{3} e^{-\alpha^{(i)}\,t}\,|\mathbf{U_d^{(i)}}|\Big(A^{(i)}\,\cos\big(\omega_{\mathrm{d}}^{(i)}\,t + \angle(\mathbf{U_d^{(i)}})\big) + B^{(i)}\,\sin\big(\omega_{\mathrm{d}}^{(i)}\,t + \angle(\mathbf{U_d^{(i)}})\big)\Big) + \\ &\quad + |\mathbf{H_A}(\boldsymbol{\omega_1})|\,A_1\,\cos\Big(\omega_1\,t + \angle\big(\mathbf{H_A}(\boldsymbol{\omega_1})\big)\Big) + |\mathbf{H_A}(\boldsymbol{\omega_2})|\,A_2\,\cos\Big(\omega_2\,t + \angle\big(\mathbf{H_A}(\boldsymbol{\omega_2})\big)\Big)\end{aligned}$$

where $\omega_1 = 2\,\pi\,f_1$ and $\omega_2 = 2\,\pi\,f_2$. The corresponding three plots are reported here.



3.e Forced response of the system (steady-state)

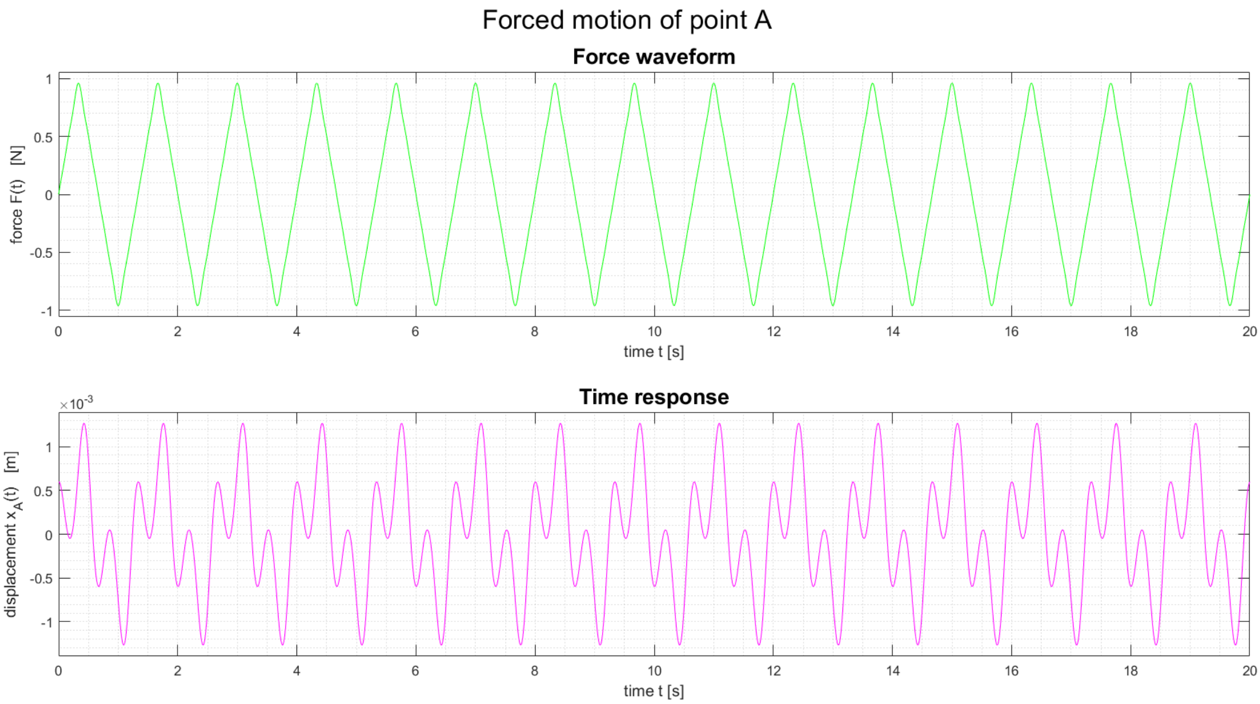
The force we're now considering is still the one we assumed in our equation of motion, but this time its waveform is a triangular wave approximated to the 5^{th} harmonic:

$$F(t) = \frac{8}{\pi^2} \sum_{k=0}^{4} (-1)^k \frac{\sin\big((2k+1)\, 2\pi f_0\, t\big)}{(2k+1)^2}$$

We need to find the forced response of point A, so we'll need to use the co-located FRF of paragraph 3.b and evaluate it for the frequency values $\omega_k^{\text{tr}} = (2k+1)\, 2\pi f_0$, $k = 0, ..., 5$, with $f_0 = 0.75\,\text{Hz}$, of the five harmonics of the force. The resulting horizontal displacement x_A for this forced motion is

$$x_A^{\text{forced}}(t) = \frac{8}{\pi^2} \sum_{k=0}^{4} (-1)^k \, |H_A^{\text{coloc}}(\omega_k^{\text{tr}})| \, \frac{\sin\Big(\omega_k^{\text{tr}}\, t + \angle\big(H_A^{\text{coloc}}(\omega_k^{\text{tr}})\big)\Big)}{(2k+1)^2}$$

and the plots representing the force waveform and the forced time response are



Notice that the horizontal motion of point A is very small compared to the oscillation amplitude of the external harmonic excitation.

4 Modal approach

Finally, the system is analyzed using a modal superposition method to find the FRM and the co-located FRFs, and to compare the results obtained with the two procedures. The system damping is again assumed to be the Rayleigh damping.

4.a Equation of motion and Frequency Response Matrix

The equation in modal coordinates is found thanks to the relation

$$\mathbf{u} = \boldsymbol{\phi}\,\mathbf{q}$$

where $\boldsymbol{\phi}$ is the modal matrix, equal to $\mathbf{U_{und}}$ obtained gathering together the modeshapes for the undamped system in 1.b. Applying this relation to the energy terms of 1.a, what we have is

$$E_\mathrm{k} = \frac{1}{2}\,\dot{\mathbf{u}}^\mathrm{T}\,\mathbf{M}^*\,\dot{\mathbf{u}} = \frac{1}{2}\,\dot{\mathbf{q}}^\mathrm{T}\,\boldsymbol{\phi}^\mathrm{T}\,\mathbf{M}^*\,\boldsymbol{\phi}\,\dot{\mathbf{q}}$$

$$V = V_\mathrm{e} = \frac{1}{2}\,\mathbf{u}^\mathrm{T}\,\mathbf{K}^*\,\mathbf{u} = \frac{1}{2}\,\mathbf{q}^\mathrm{T}\,\boldsymbol{\phi}^\mathrm{T}\,\mathbf{K}^*\,\boldsymbol{\phi}\,\mathbf{q}$$

$$D = \frac{1}{2}\,\dot{\mathbf{u}}^\mathrm{T}\,\mathbf{C_{ray}}\,\dot{\mathbf{u}} = \frac{1}{2}\,\dot{\mathbf{q}}^\mathrm{T}\,\boldsymbol{\phi}^\mathrm{T}\,\mathbf{C_{ray}}\,\boldsymbol{\phi}\,\dot{\mathbf{q}}$$

$$\delta W = \mathbf{Q_u}^\mathrm{T}\,\boldsymbol{\delta u} = \mathbf{Q_u}^\mathrm{T}\,\boldsymbol{\phi}\,\boldsymbol{\delta q}$$

The equation of motion becomes

$$\underbrace{\boldsymbol{\phi}^\mathrm{T}\,\mathbf{M}^*\,\boldsymbol{\phi}}_{\substack{\text{modal mass matrix}\\ \mathbf{M_q}}}\;\ddot{\mathbf{q}} + \underbrace{\boldsymbol{\phi}^\mathrm{T}\,\mathbf{C_{ray}}\,\boldsymbol{\phi}}_{\substack{\text{modal damping matrix}\\ \mathbf{C_q}}}\;\dot{\mathbf{q}} + \underbrace{\boldsymbol{\phi}^\mathrm{T}\,\mathbf{K}^*\,\boldsymbol{\phi}}_{\substack{\text{modal stiffness matrix}\\ \mathbf{K_q}}}\;\mathbf{q} = \underbrace{\boldsymbol{\phi}^\mathrm{T}\,\mathbf{Q_u}}_{\substack{\text{modal forces vector}\\ \mathbf{F_q}}}$$

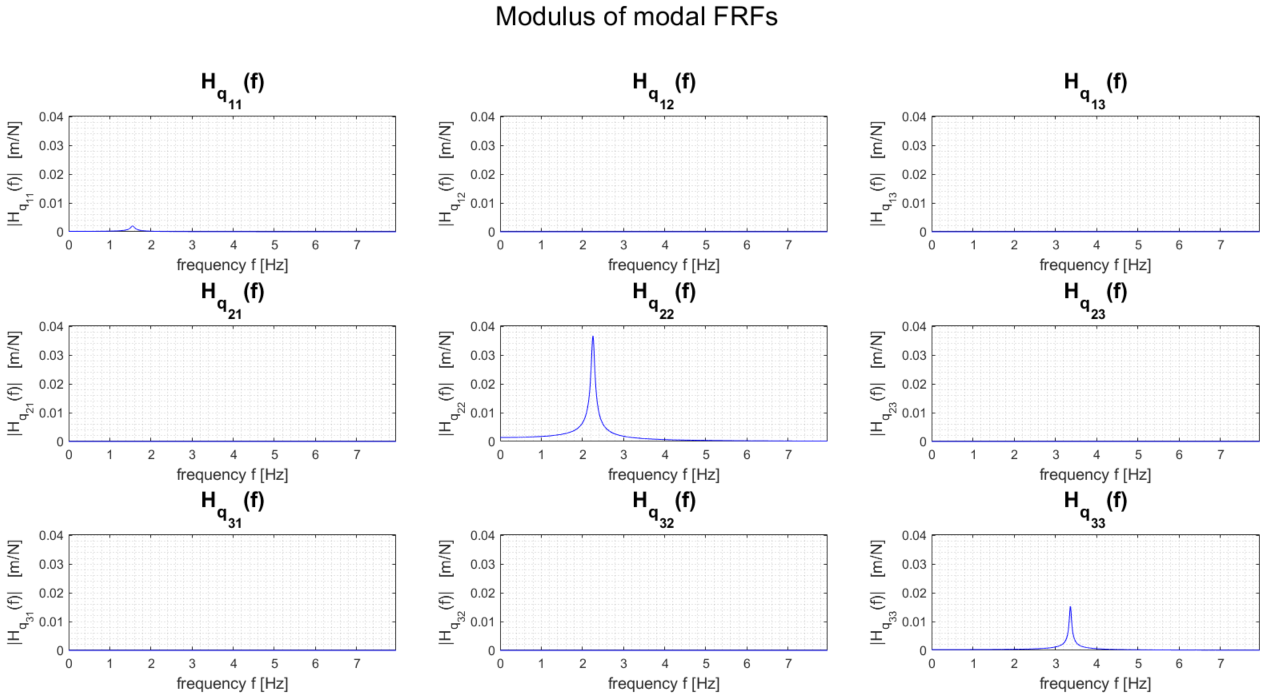
These form a system of N equations, each describing the motion of a single-degree-of-freedom system:

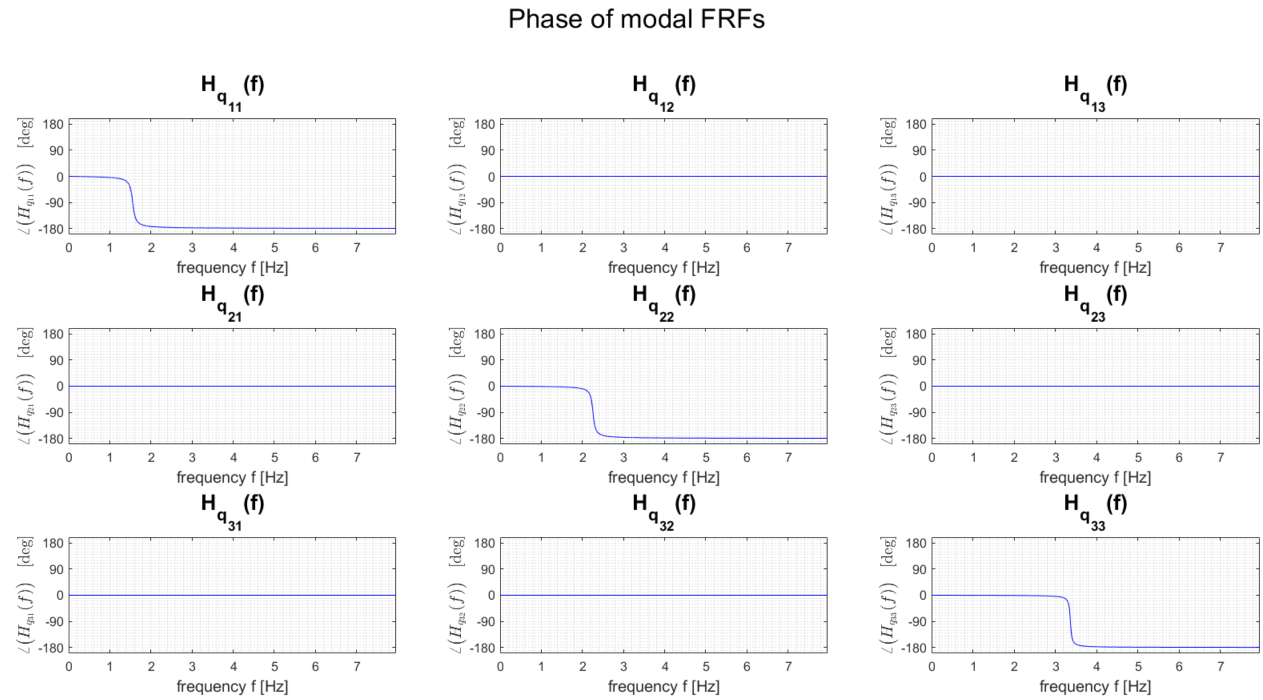
$$\begin{cases} m^{(1)}\,\ddot{q}^{(1)} + c^{(1)}\,\dot{q}^{(1)} + k^{(1)}\,q^{(1)} = F_\mathrm{q}^{(1)} \\ m^{(2)}\,\ddot{q}^{(2)} + c^{(2)}\,\dot{q}^{(2)} + k^{(2)}\,q^{(2)} = F_\mathrm{q}^{(2)} \\ m^{(3)}\,\ddot{q}^{(3)} + c^{(3)}\,\dot{q}^{(3)} + k^{(3)}\,q^{(3)} = F_\mathrm{q}^{(3)} \end{cases}$$

where $m^{(i)}$, $c^{(i)}$ and $k^{(i)}$, $i = 1, 2, 3$ are the element $M_{\mathrm{q}_{ii}}$, $C_{\mathrm{q}_{ii}}$ and $K_{\mathrm{q}_{ii}}$ of the modal mass, damping and stiffness matrix respectively. Note in particular that, since we assumed the system to be damped with Rayleigh damping, we don't run the risk of having a full modal damping matrix, but we're rather ending up with a diagonal $\mathbf{C_q}$. Since we decomposed the system into a superposition of N simple oscillatory systems, we expect the elements on the diagonal of the modal Frequency Response Matrix $\mathbf{H_q}(\Omega)$ to be the FRF for a single-degree-of-freedom system, with its own damping factor $\alpha^{(i)} = \frac{c^{(i)}}{2\,m^{(i)}}$, natural frequency $\omega_\mathrm{n}^{(i)} = \sqrt{\frac{k^{(i)}}{m^{(i)}}}$, damped frequency $\omega_\mathrm{d}^{(i)} = \sqrt{(\omega_\mathrm{n}^{(i)})^2 - (\alpha^{(i)})^2}$ and all the parameters that we already got in 1.b, exhibiting a single peak in the magnitude diagram and a single jump from 0 to $-\pi$ in the phase diagram, both in correspondence with the damped frequency. The N FRFs on the diagonal are in fact given by

$$H_q(\Omega) = \frac{\tilde{Q}^{(i)}(\Omega)}{F_{\mathrm{q}_0}^{(i)}} = \frac{1}{-m^{(i)}\,\Omega^2 + jc^{(i)}\,\Omega + k^{(i)}}$$

which are found solving the differential equations considering $F_\mathrm{q}^{(i)} = F_{\mathrm{q}_0}^{(i)}\,\cos(\Omega t)$. On the other hand, we expect the extra-diagonal elements to be equal to zero, as each mode is decoupled from the others. However, Matlab consider those terms as greater than zero, but since they're in the order of 10^{-17} we set them directly to zero, concluding they're due to numerical errors. The modulus and phase plots are the following:





Notice that the oscillation following the first mode is very small in terms of amplitude with respect to the one following the other two modes.

4.b Co-located FRF of point A

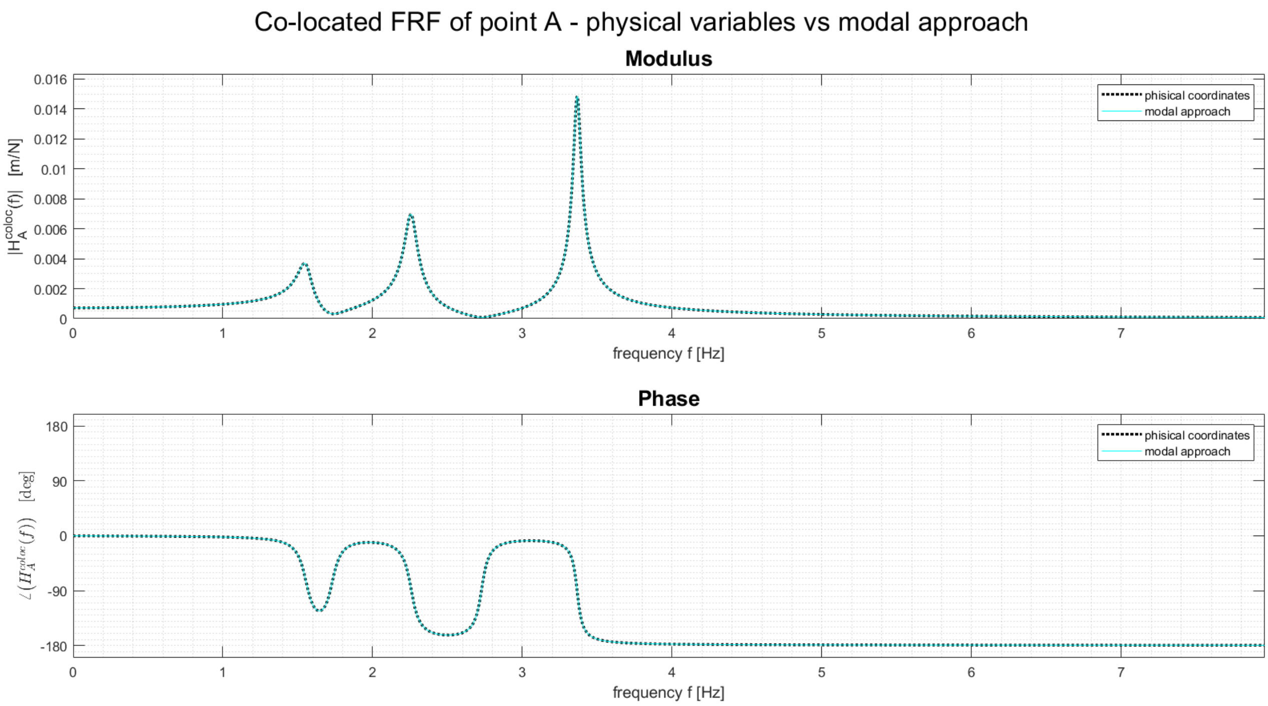
The co-located FRF of point A is now reconstructed using the modal approach, and compared to the one previously found. The reconstruction passes through finding a new Jacobian for the force:

$$\mathbf{\Lambda_{q_F}} = \mathbf{\Lambda_F}\,\boldsymbol{\phi}$$

Then the operations to find the FRF are the same as for 3.b:

$$H^{\mathrm{coloc}}_{\mathrm{q}_A}(\Omega) = \mathbf{\Lambda_{q_F}}\,\mathbf{H}(\Omega)\,\mathbf{\Lambda_{q_F}}^{\mathrm{T}}$$

As we may see in the following picture, the two approaches lead to the same plots:

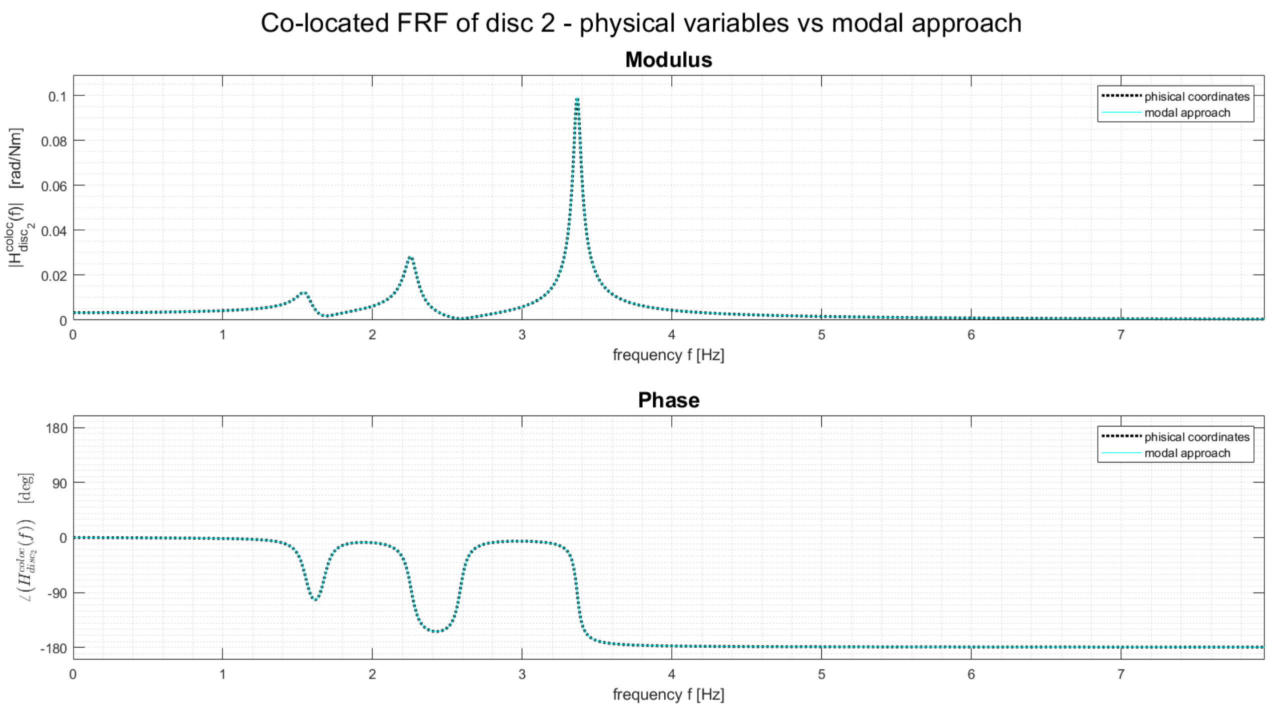


4.c Co-located FRF of disc 2

As a final step, we do the same for the co-located FRF of disc 2:

$$\mathbf{\Lambda}_{\mathbf{q}_\tau} = \mathbf{\Lambda}_\tau \, \boldsymbol{\phi} \;\Rightarrow\; H^{\text{coloc}}_{\text{q}_{disc_2}}(\Omega) = \mathbf{\Lambda}_{\mathbf{q}_\tau} \, \mathbf{H}(\Omega) \, \mathbf{\Lambda}_{\mathbf{q}_\tau}{}^{\text{T}}$$

where the Jacobian $\mathbf{\Lambda}_\tau$, since this FRF is the co-located FRF found in the FRM of 3.a for the second independent variable, is equal to [0 1 0]. Again, the FRFs found by means of physical coordinates and of modal ones perfectly coincide:



Fundamentals of Vibration Analysis and Vibroacoustics
Module 1 - Fundamentals of Vibration Analysis
Assignment 3 - Modal parameter identification

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In this problem, we're given a small set of measurements we have to extract information from. The virtual experiment consists in a hammer hitting impulsively a structure in one point, and measuring the displacements in four different positions, one of which is where the force is applied (the first one). The initial data are wherefore the time axis t used to represent the sampled measurements and the force, the impulse-like input force $F(t)$, and the four measurements:

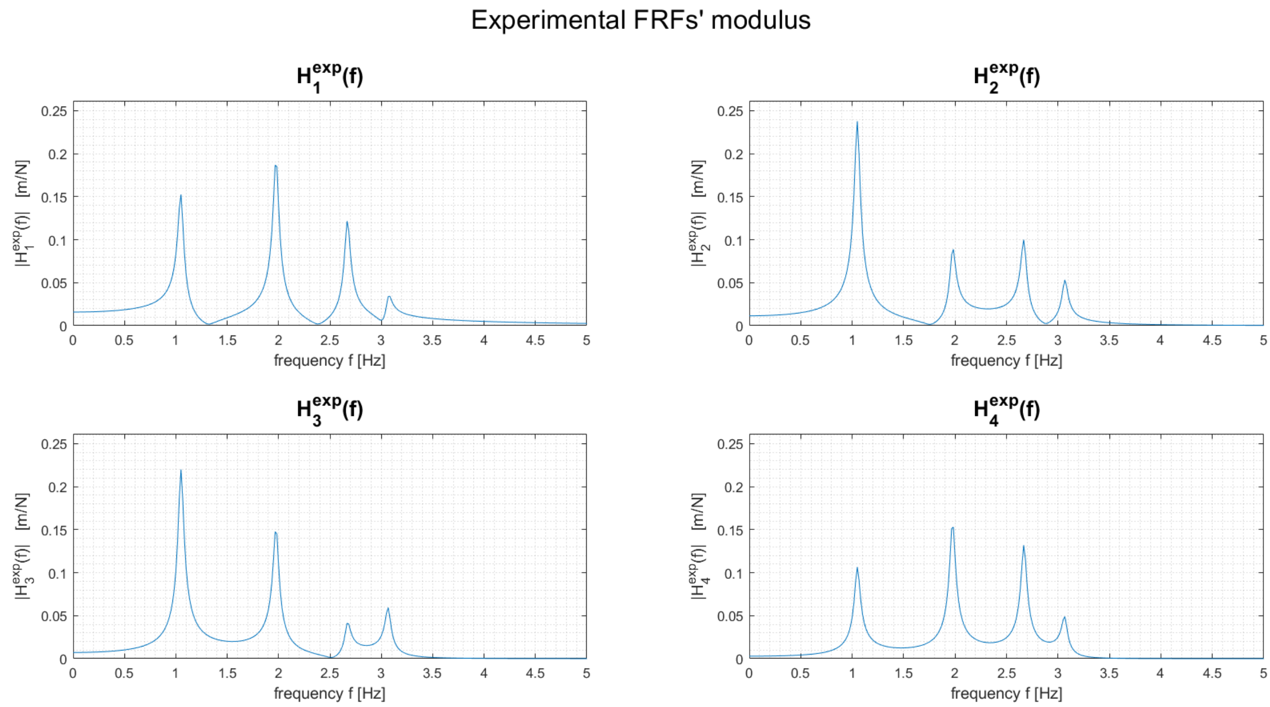
$$t \ , \ F(t) \ , \ x_1(t) \ , \ x_2(t) \ , \ x_3(t) \ , \ x_4(t)$$

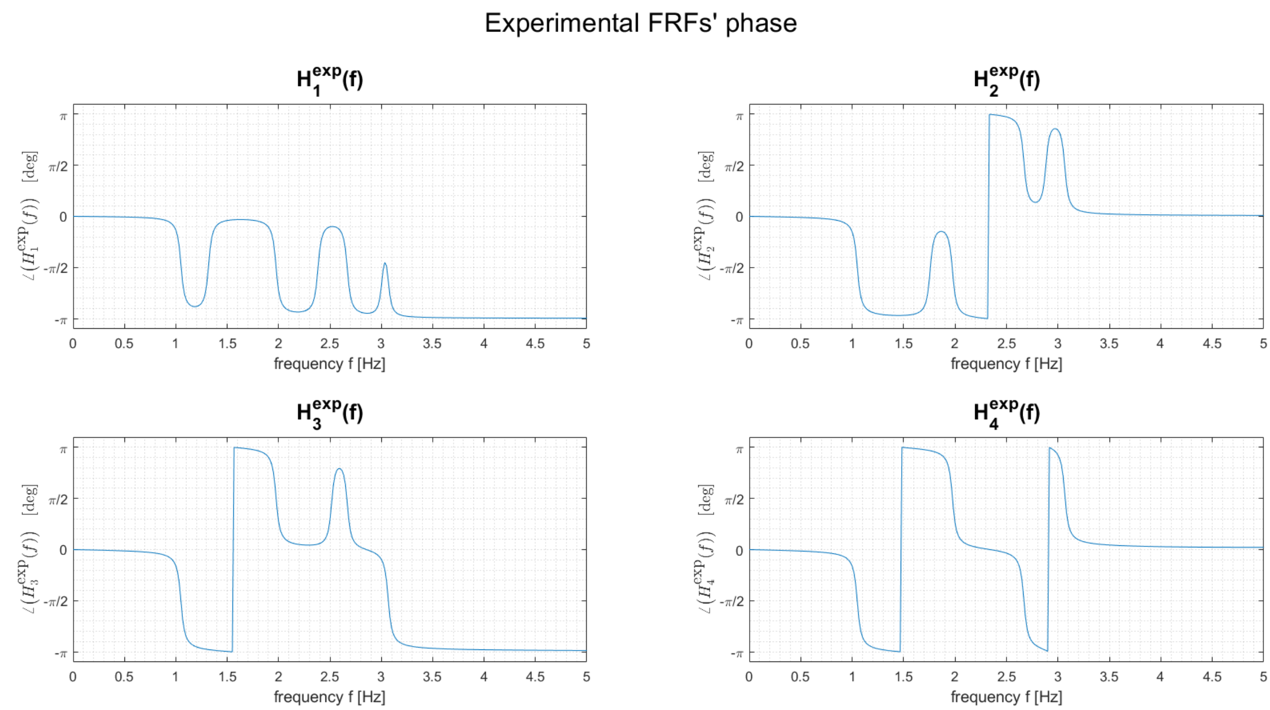
1 Experimental FRFs

The Frequency Response Functions linking the displacement of the four points to the external force can be retrieved computing the ratio between the Discrete Fourier Transform (DFT) of the former and of the latter. We used the Matlab function `fft` to find the DFT of the function input, here denoted with $\mathcal{F}[\cdot]$, through an optimized Fast Fourier Transform algorithm.

$$H_k^{\text{exp}}(f) = \frac{\mathcal{F}[x_k(t)]}{\mathcal{F}[F(t)]} \ , \ k = 1, 2, 3, 4$$

Note that the first element $H_1^{\text{exp}}(f)$ represents the co-located FRF. The corresponding plots are represented in magnitude and phase as follows:





2 Simplified methods for model identification

To estimate the system parameters, we first reached for simplified methods. The assumption we are working under are, as usual, low damping (order of magnitude for ξ: $10-2$), and well-distincted peaks in amplitude spectrum. As long as the damped frequencies are concerned, we found the relative maxima in each FRF setting one order of magnitude lower than the absolute maximum as threshold, and then found the frequencies corresponding to those values. The resulting matrix of damped frequencies is:

$$\mathbf{f_d} = \begin{bmatrix} f_{d_1}^{(1)} & f_{d_1}^{(2)} & f_{d_1}^{(3)} & f_{d_1}^{(4)} \\ f_{d_2}^{(1)} & f_{d_2}^{(2)} & f_{d_2}^{(3)} & f_{d_2}^{(4)} \\ f_{d_3}^{(1)} & f_{d_3}^{(2)} & f_{d_3}^{(3)} & f_{d_3}^{(4)} \\ f_{d_4}^{(1)} & f_{d_4}^{(2)} & f_{d_4}^{(3)} & f_{d_4}^{(4)} \end{bmatrix} = \begin{bmatrix} 1.0500 & 1.9667 & 2.6667 & 3.0833 \\ 1.0500 & 1.9833 & 2.6667 & 3.0667 \\ 1.0500 & 1.9667 & 2.6667 & 3.0667 \\ 1.0500 & 1.9833 & 2.6667 & 3.0667 \end{bmatrix}$$

where every row corresponds to one of the four different measurements, and each column to a vibration mode. The values are affected by small fluctuations as for the 2$^{\text{nd}}$ and 4$^{\text{th}}$ frequency, so we can reckon the values overall agree with each other. In order to have one value for each mode, we averaged the values for the four measurements and for each of the modes, obtaining

$$\mathbf{f_d} = \begin{bmatrix} 1.05 & 1.97 & 2.67 & 3.07 \end{bmatrix}$$

As a second step, we searched for the adimensional damping ratios, applying the half-power bandwidth method. This exploits the relationship between the width of a resonating mode bell-like peak and the damping ratio, which are dirctly proportional. Each ratio for each measurement is computed thanks to the formula

$$\xi_i = \frac{(f_2^{(i)})^2 - (f_1^{(i)})^2}{4\,(f_d^{(i)})^2}\ ,\ i = 1, 2, 3, 4$$

i being the mode number. The resulting matrix of damping ratios is:

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \xi_1^{(3)} & \xi_1^{(4)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \xi_2^{(3)} & \xi_2^{(4)} \\ \xi_3^{(1)} & \xi_3^{(2)} & \xi_3^{(3)} & \xi_3^{(4)} \\ \xi_4^{(1)} & \xi_4^{(2)} & \xi_4^{(3)} & \xi_4^{(4)} \end{bmatrix} = \begin{bmatrix} 0.0159 & 0.0128 & 0.0062 & 0.0081 \\ 0.0159 & 0.0084 & 0.0062 & 0.0054 \\ 0.0159 & 0.0128 & 0.0094 & 0.0054 \\ 0.0159 & 0.0126 & 0.0062 & 0.0081 \end{bmatrix}$$

organized like the damped frequency matrix. This time, the values vary a little more than before: this is due to non-negligible variations of the peaks width relative to the same resonance in distinct measurements, especially for mode 2 in measurement 2 and mode 3 in measurement 3. As for the 4$^{\text{th}}$ mode, conformity between measurement 1 and 4 and between measurement 2 and 3 may be observed. Values for the 1$^{\text{st}}$ mode are the most accurate both for the frequency value and for the damping ratio, in accordance to the fact the more the resonances are clear and their peaks are sharp for each measurement, the easier it is to make an estimation, because values will tend to be the same for all of the measurements. In fact, the first mode corresponds to a very clear peak for each of them. Analogously to what we did for the natural frequencies, we arrived to a single value of ξ for each mode computing the mean among the measurements. This time, though, we excluded values that more clearly differ from the others: the one for mode 2 and measurement 2 and the one for mode 3 and measurement 3. The final values are then

$$\boldsymbol{\xi} = \begin{bmatrix} 0.0159 & 0.0127 & 0.0062 & 0.0068 \end{bmatrix}$$

Finally, the mode shape of i-th mode is estimated equalling the experimental FRF for measurement $k = 1, 2, 3, 4$, computed in the previous section and evaluated at the i-th resonance frequency, $i = 1, 2, 3, 4$, to the approximated FRF $H_k^{\text{approx}^{(i)}}$ connecting displacement of point k and the force, evaluated at the resonant frequencies. The mode shape element relative to the force application position has been neglected, being it the same for all of the FRFs.

$$
\begin{aligned}
&H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)}) \approx H_k^{\mathrm{approx}^{(i)}}(\omega_{\mathrm{d}_k}^{(i)}) = \left.\frac{\phi_k^{(i)}}{-\omega^2 m_k^{(i)} + j\omega c_k^{(i)} + k_k^{(i)}}\right|_{\omega=\omega_{\mathrm{d}_k}^{(i)}} = \frac{\phi_k^{(i)}}{j\omega_{\mathrm{d}_k}^{(i)} c_k^{(i)}} = -j\frac{\phi_k^{(i)}}{\omega_{\mathrm{d}_k}^{(i)} c_k^{(i)}} \\
&\Rightarrow\ j\frac{\phi_k^{(i)}}{\omega_{\mathrm{d}_k}^{(i)} c_k^{(i)}} \approx -H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)}) \ \Rightarrow\ \frac{\phi_k^{(i)}}{\omega_{\mathrm{d}_k}^{(i)} c_k^{(i)}} \approx \Im\{-H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)})\} = -\Im\{H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)})\} \\
&\Rightarrow\ \phi_k^{(i)} \approx -\Im\{H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)})\}\, \omega_{\mathrm{d}_k}^{(i)} c_k^{(i)}\ ,\ i,k = 1,2,3,4
\end{aligned} \tag{1}
$$

where the contributions of other modes other than the i-th mode have been neglected in the computation of the i-th mode shape. The damping coefficients $c_k^{(i)}$ have been derived from the damping ratios following the definition of damping ratio:

$$
\xi_k^{(i)} = \frac{c_k^{(i)}}{2\, m_k^{(i)} \omega_{\mathrm{d}_k}^{(i)}} \ \Rightarrow\ c_k^{(i)} = 2\, \xi_k^{(i)} m_k^{(i)} \omega_{\mathrm{d}_k}^{(i)}
$$

and setting the elements of the modal mass matrix $m_k^{(i)}$ to 1. The modal matrix of all the elements $\phi_k^{(i)}$ is reported here:

$$
\boldsymbol{\phi} = \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} & \phi_1^{(3)} & \phi_1^{(4)} \\ \phi_2^{(1)} & \phi_2^{(2)} & \phi_2^{(3)} & \phi_2^{(4)} \\ \phi_3^{(1)} & \phi_3^{(2)} & \phi_3^{(3)} & \phi_3^{(4)} \\ \phi_4^{(1)} & \phi_4^{(2)} & \phi_4^{(3)} & \phi_4^{(4)} \end{bmatrix} = \begin{bmatrix} 0.1050 & 0.3438 & 0.2130 & 0.0683 \\ 0.1642 & 0.1065 & -0.1725 & -0.1071 \\ 0.1519 & -0.2704 & -0.1083 & 0.1195 \\ 0.0735 & -0.2841 & 0.2303 & -0.1465 \end{bmatrix}
$$

Normalizing the elements of each measure taking the first one as reference the result is

$$
\boldsymbol{\phi} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1.5632 & 0.3098 & -0.8098 & -1.5669 \\ 1.4459 & -0.7867 & -0.5087 & 1.7482 \\ 0.7000 & -0.8264 & 1.0811 & -2.1442 \end{bmatrix}
$$

These values are in agreement with the magnitude and phase plots of 1: the modulus of $\phi_k^{(i)}$ is approximatedly the relative value in the amplitude plots, while its sign follows the sign of the phase in the phase plots. The biggest divergence between the absolute value of the elements in the modal matrix and the plotted values is about measurement 3, 3$^{\text{rd}}$ mode, which should have a relative magnitude of $\frac{1}{3}$ with respect to the 1$^{\text{st}}$ measurement, but the corresponding element in $\boldsymbol{\phi}$ indicates $\frac{1}{2}$. This is certainly due to the approximated values of the modal matrix, obtained by means of a simplified method. In the next section, we go on with the analysis using a more accurate procedure, which is a modal parameter identification model.

3 Modal parameter identification

Modal identification uses the modal approach as analytical support, meaning that it reconstructs the analytical transfer function $H_k(\omega)$ of the system considering it as if it were made up of many one-DOF systems, since the various degrees of freedom are defined by the modal variables. $H_k(\omega)$ represents the harmonic transfer function for an output at the generic point k (recall that the input position is fixed) of the N-DOF system considered. So this transfer function will have modal parameters as unknown quantities:

$$H_k(\omega) = H_k(\omega, \omega_{\mathrm{d}_k}^{(i)}, m_k^{(i)}, k_k^{(i)}, \xi_k^{(i)}, \phi^{(i)}) \ , \ i, k = 1, 2, 3, 4$$

being:

- ω_{d_k} the natural frequencies;
- $m_k^{(i)}$ the modal masses;
- $k_k^{(i)}$ the modal stifness;
- $\xi_k^{(i)}$ the damping ratio;
- $\phi_k^{(i)}$ the element of the modal matrix $\boldsymbol{\phi}$.

With this approach, the dynamic response of the structure subjected to the known excitation is measured at several k points; this dynamic response, expressed in terms of experimental transfer functions $H_k^{\mathrm{exp}}(\omega)$, is then compared to the analytical response $H_k^{\mathrm{num}}(\omega)$ defined beforehand by minimising the difference between the analytical values and the experimental ones. It is thus possible to determine the set of modal parameters needed to characterize the static and dynamic behaviour of the system being analyzed.

From the modal approach theory, we know that, as far as a sufficient number of modes N are considered, the following gives the expression for $H_k^{\mathrm{exp}}(\omega)$:

$$H_k^{\mathrm{exp}}(\omega) = \sum_{i=1}^{N} \frac{\phi_k^{(i)}}{-\omega^2 m_k^{(i)} + j\omega c_k^{(i)} + k_k^{(i)}} \ , \ i, k = 1, 2, 3, 4$$

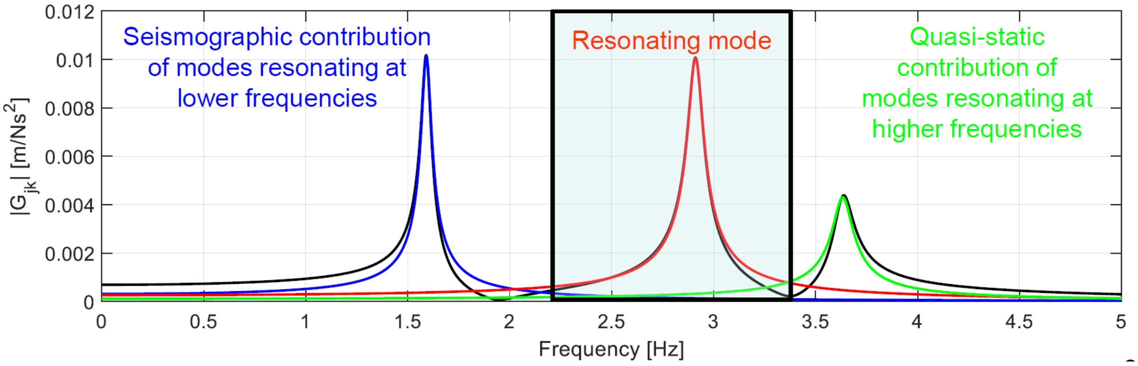
where, like in equation (1), k is the measurement index, and the element of the modal matrix relative to the forcing position has been neglected.

For well distinguished peaks and lightly damped structures, the experimental Frequency Response Function $H_k^{\mathrm{exp}}(\omega)$ can be approximated by the analytical Frequency Response Function $H_k^{\mathrm{num}}(\omega)$ around a certain $\omega_{\mathrm{d}_k}^{(i)}$ as:

$$H_k^{\mathrm{num}}(\omega) = \frac{A_k^{(i)} + jB_k^{(i)}}{-\omega^2 m_k^{(i)} + j\omega c_k^{(i)} + k_k^{(i)}} + C_k^{(i)} + jD_k^{(i)} + \frac{E_k^{(i)} + jF_k^{(i)}}{\omega^2}$$

In this form:

- the first term represent the contribution of the **resonating mode** in our frequency range of interest;
- the second term represent the contribution of the modes at **higher** frequency than $\omega_{\mathrm{d}_k}^{(i)}$; this contribution is approximately constant because our frequency range of interest falls in the **quasi-static region** of these modes;
- the third term represents the contribution of the modes at **lower** frequency than $\omega_{\mathrm{d}_k}^{(i)}$; this contribution has approximately a $\frac{1}{\omega^2}$ behavior because our frequency range of interest falls in the **seismographic region** of these modes.



The frequency ranges of interest are partitions of the frequency axis. Given the N well distinguished peaks we will have N ranges of interest. We chose to compute the bounds of these ranges as the local minima between one peak and the following one.

3.a Vibration modes identification procedure

For a given set of experimental FRFs $H_k^{\text{exp}}(\omega)$, obtained for a fixed excitation location and different measurement locations k (4 points), a least squares minimization procedure can be implemented for the estimation of the modal parameters.

The error function to be minimized (cost function) is:

$$\varepsilon = \sum_{s=s_{\text{inf}}}^{s_{\text{sup}}} \Re^2\{H_k^{\text{exp}}(\omega_s) - H_k^{\text{num}}(\omega_s)\} + \Im^2\{H_k^{\text{exp}}(\omega_s) - H_k^{\text{num}}(\omega_s)\}$$

Since the error function depends non-linearly on the unknown parameters, an iterative minimization procedure is used. This is implemented in MATLAB throught the function `fminsearch`. According to the documentation, `fminsearch` is a nonlinear programming solver. It searches for the minimum of a problem specified by $\min_x f(x)$. Its syntax is: `x = fminsearch(fun, x0, options)`.

`fun` is the function to minimize, specified as a function handle or function name. `fun` is a function that accepts a vector or array `x` and returns a real scalar `f` (the objective function evaluated at `x`).

An initial guess vector `x0` (`xpar0` in our code) is required, consisting of a preliminary estimate of:

1. $\omega_{\mathrm{d}_k}^{(i)}$ which is found from the maximum peak in the considered frequency range. We already discussed the procedure in section 2. It has been evaluated from each FRF and then averaged.

2. $\xi_k^{(i)}$ which is found through a simplified method (in our case the half power bandwidth method). We already discussed the procedure in section 2. It has been evaluated from each FRF and then averaged.

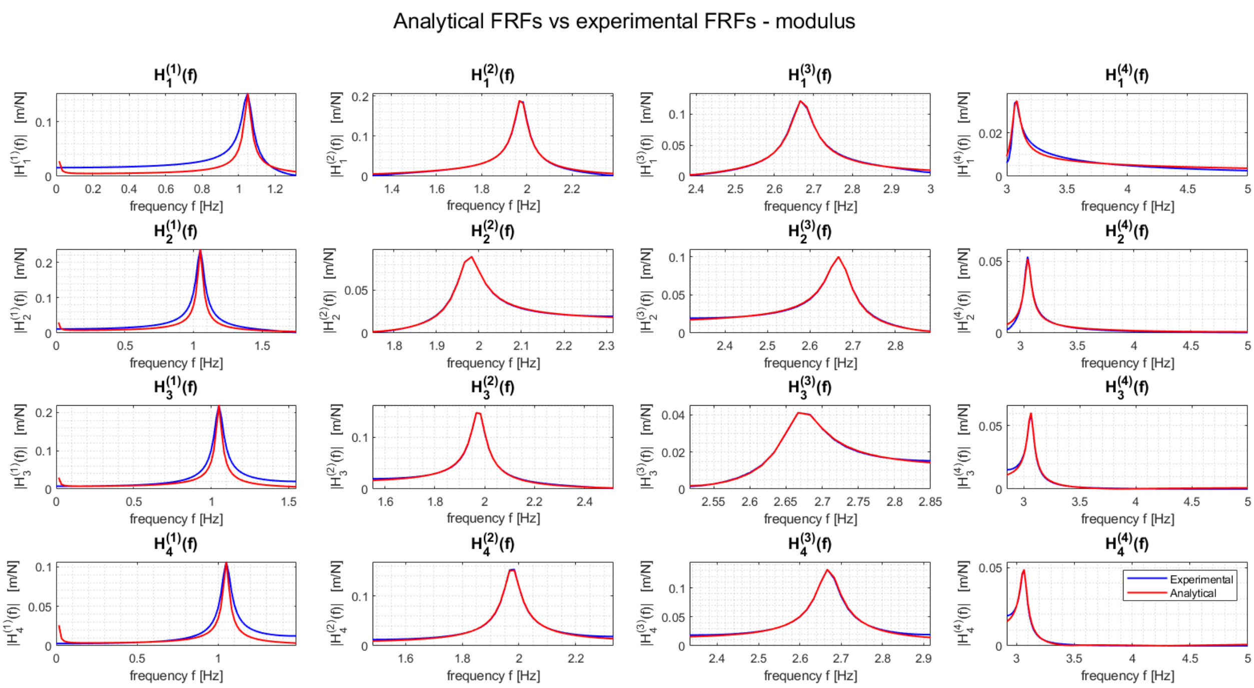
3. $A_k^{(i)}$ which is found considering each FRF at resonance and assuming real valued mode shapes (valid in resonance condition):

$$\phi_k^{(i)} = -\Im\{H_k^{\exp}(\omega_{\mathrm{d}_k}^{(i)})\}\, \omega_{\mathrm{d}_k}^{(i)} c_k^{(i)} = A_k^{(i)}$$

Again, we already discussed the procedure in section 2. It follows that $B_k^{(i)} = 0$.

4. $C_k^{(i)}, D_k^{(i)}, E_k^{(i)}, F_k^{(i)}$ are set to zero under the assumption of sufficiently distinguished peaks.

The non-linear minimization procedure elaborates separately the FRFs, leading to an estimate of the modal parameters $\omega_{\mathrm{d}_k}^{(i)}, \xi_k^{(i)}, A_k^{(i)}$ for every frequency range (i.e. for every peak) and for every measurement. Then the parameters are averaged. A further improvement of the algorithm could be to perform the minimization simultaneously on the whole set of FRFs, leading to a more precise estimate of the modal parameters. In this case the information given by the correlation of the FRFs is exploited. The quality of the estimates can be visually assessed comparing in a plot the identified FRFs $H_k^{\mathrm{num}}(\omega)$ with the experimental ones $H_k^{\exp}(\omega)$: we may conclude the algorithm worked very well for the 2$^{\text{nd}}$, 3$^{\text{rd}}$ and 4$^{\text{th}}$ mode, being the curves we started with very well approximated by the results, and fairly well for the 1$^{\text{st}}$ mode, for which the quasi-static contributions of the other modes, which we neglected for the initial guess of the minimization, evidently plays a greater role than for the other modes.



Analytical FRFs vs experimental FRFs - phase