# Fourier Series

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December 2023

## 1 7.4 Other Partial Differential Equations (PDE)

Fourier series is a method used to study periodic solutions of linear constant coefficient partial differential equations. This mean that Fourier series can be used to express functions that repeat over time or space in terms of sine and cosine functions, which themselves are inherently are periodic.
This will carry over to the following PDE's such as the wave equation and Laplace equation and the Fourier series will be very useful in solving these PDE's.
The 1D wave equation $u_{tt} = c^2 u_{xx}$ where $u$ is a function of both space and time, $c$ is the speed of the wave. This equation models the propagation of waves with a constant speed and can describe phenomena like waves on an elastic string, sound waves, or light waves in a vacuum. It is a second order equation in time two initial conditions are necessary to determine a unique solution. The following initial value problem for wave propagation on a circle if

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(0,x) &= f(x) \\ u_t(0,x) &= g(x) \text{where } f, g \in L^2(\mathbf{T}) \text{ are given functions} \end{aligned}$$

for $n \neq 0$ the separated solution of the wave equation that is proportional to $e^{inx}$ is $u(t,x) = (ae^{inct} + be^{inct})e^{inx}$
for $n = 0$ the other separated solution of the wave equation that is proportional to $e^{inx}$ is $u(t,x) = a + bt$. This signifies the wave like nature of the solution with n being the wave number. The general form for the solution is the following and we will explain how the constants in the solution are determined from the initial conditions of the problem.

(i) General solution for the wave equation:
$u(t,x) = a_0 + b_0 t + \sum_{n \neq 0} \{a_n e^{in(x-ct)} + b_n e^{in(x-ct)}\}$. This equation above represents a sum of waves traveling to the right and left with coefficients $a_n$ and $b_n$ respectively.

(ii)Determining the constants
Constants $a_0, b_0$ are determined from initial conditions $f_0$ and $g_0$ which are the initial displacement

and initial velocity of the wave. Constant $a_n$ and $b_n$ for $n \neq 0$ are calculated using the initial conditions through the formulas:
(1)$a_n = \frac{1}{2}(f_n - \frac{i}{nc}g_n)$
(2) $b_n = \frac{1}{2}(f_n - \frac{i}{nc}g_n)$
Here $f_n$ and $g_n$ are the Fourier coefficients of the initial displacement and velocity given by the integrals $f_n = \frac{1}{2\pi}\int_{\mathbf{T}} f(x)e^{-inx}$, $g_n = \frac{1}{2\pi}\int_{\mathbf{T}} f(x)e^{-inx}$.

Next, we'll show that the solution $u(x,t) = f(x+ct) + g(x-ct)$ is a solution of the 1-D wave equation,

$$u_t t - c^2 u_x x = 0. \tag{1}$$

We'll take the temporal Fourier transform.

$$\tilde{u}(x,w) = \int_{-\infty}^{\infty} u(x,t)e^{-iwt}dt, \tag{2}$$

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(x,w)e^{iwt}\frac{dw}{2\pi} \tag{3}$$

Equation (3) is the inverse Fourier transform. We begin solving this equation by multiplying the wave equation by $e^{-iwt}$ then integrate from $-\infty$ to $\infty$.

So we'll have

$$(iw)^2\tilde{u}(x,w) - c^2\frac{\partial^2}{\partial x^2}\tilde{u}(x,w) = 0. \tag{4}$$

The Fourier transform is very useful for the first term of the equation above because it turns the derivatives into multiplication. Since we took the temporal Fourier transform, or spatial derivative, we leave the second term as that.
We can rewrite our equation as

$$\tilde{u}'' + (\frac{w}{c})^2\tilde{u} = 0 \tag{5}$$

Let $k_w = \frac{w}{c}$. This can be recognized as a simple harmonic motion equation.

$$\tilde{u} = A_w e^{ik_w x} + B_w e^{-ik_w x} \tag{6}$$

where $A_w, B_w$ are arbitrary constants.

Using equation (3) and combining the complex exponentials we get,

$$u(x,t) = \int_{-\infty}^{\infty} A_w e^{i(wt+k_w x)} + B_w e^{i(wt-k_w x)}\frac{dw}{2\pi} = \int_{-\infty}^{\infty} A_w e^{iw(t+\frac{x}{c})}\frac{dw}{2\pi} + B_w e^{iw(t-\frac{x}{c})}\frac{dw}{2\pi} \tag{7}$$

since we let $k_w = \frac{w}{c}$

$$= f(t+\frac{x}{c}) + g(t-\frac{x}{c}) = f(\frac{1}{c}(x+ct)) + g(-\frac{1}{c}(x-ct)) = F(x+ct) + G(x-ct). \tag{8}$$

Above result is the general solution of the 1-D wave equation. As t increases we are translating the function F to the left of the graph by rate c and the function G to the right of the graph by rate c.

Comparison to the heat equation:
We can contrast the wave equation with the heat equation while the solution to the heat equation smooths out initial data over time (due to the diffusion process) the wave equation does not.The solution to the wave equation persist over time and does not converge to a stationary solution as $t \to \infty$. This reflects the difference in physical process: the heat equation models diffusion (which tends to even out differences over time), while the wave equation models propagation without loss of energy.

Next, we will focus on 2-dimensional Laplace equation. Laplace is often used for specific area or domain such as a circle. If we had a metal circle and knew the temperature at every point on the edge of the circle we would be able able to use the Laplace equation to figure out the temperature at any point inside the circle. In order to solve to solve this we need to use Fourier series in order to break down complex wave patterns. We can also use polar coordinates. A solution to the Laplace equation can include an infinite sum of terms but we only need a few to obtain a good approximation.

If we are looking at temperature across a flat surface, then there shouldn't be any change as we move around. We are particularly looking at a flat circle called unit disc. We are focusing on a heated circle where the heat doesn't focus on one particular spot.
KdV equation (Korteweg - de Vries) equation
This PDE is a mathematical formula used to describe waves on shallow water surfaces. The KdV equation is special because it describes waves that keep their shape while traveling. This is something that's known as solitary waves or solutions. Even though the equation is complex because it involves non-linearity it can still be solved with a technique called the inverse scattering method. The nonlinear PDE to describe water waves in shallow channels is $u_t = uu_x + u_{xxx}$. The component of the equation $uu_x$ is what makes the pde nonlinear. But if we assume that the wave height $u$ is very small, this nonlinear part doesn't have much effect so we can ignore it for simplicity. This will give us the linearized KdV equation: $u_t = u_{xxx}$ this is basically the simplified version of the fall KdV equation. The following is the General solution of the dimensional KdV equation: The general solution is a $2\pi$ periodic function of $x$ is $u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{in(x-n^2t)}$. The solution to this simplified equation is a sum of many waves, each represented by $e^{inx}$ where n is a number that represents the wave's characteristics and x, and t, are variables for space and time. Each wave has a speed that depends on its characteristics, and they all add up to give the overall wave motion. One key property of the linearized KdV equation is dispersion. Dispersion means that the waves speed out overtime. Different parts of the wave move at different speeds depending on their characteristics, so the shape of the wave changes as it travels. This is different from the nondispersive waves described by the normal wave equation where all parts of the wave more together at the same speed, keeping the shape constant overtime.

The two-dimensional *Laplace equation* is

$$u_{xx} + u_{yy} = 0. \tag{7.26}$$

We will use Fourier series to solve a boundary value problem for Laplace's equation in the unit disc

$$\Omega = \left\{(x, y) \mid x^2 + y^2 < 1\right\}.$$

The Dirichlet problem consists of (7.26) in $\Omega$ with the boundary condition

$$u = f \qquad \text{on } \partial\Omega, \tag{7.27}$$

where $f : \mathbb{T} \to \mathbb{R}$ is a given function. In polar coordinates $(r, \theta)$ we may write (7.26)–(7.27) for $u(r, \theta)$ as

$$\begin{aligned} &\frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \qquad \text{in } r < 1, \\ &u(1, \theta) = f(\theta). \end{aligned}$$

The Laplace equation in polar coordinates has the separated solutions

$$u(r, \theta) = \left(a r^n + b r^{-n}\right) e^{in\theta} \qquad \text{for } n \in \mathbb{Z}.$$

The general solution of Laplace's equation that is bounded inside the unit disc is therefore

$$u(r, \theta) = \sum^{\infty} a_n r^{|n|} e^{in\theta}. \tag{7.28}$$

The boundary condition implies that

$$a_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta}\, d\theta.$$

Using the convolution theorem, we may write (7.28) in $r < 1$ as

$$u(r, \theta) = (g^r * f)(\theta),$$

where $g^r : \mathbb{T} \to \mathbb{R}$ is the *Poisson kernel*,

$$g^r(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

The geometric series for $n > 0$ and $n < 0$ may be summed to give

$$g^r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

The series in (7.28) converges to an infinitely differentiable — in fact, analytic — function in $r < 1$ for any $f \in L^2(\mathbb{T})$, so the Laplace equation smoothes the boundary data.

## 2 7.5 More Applications of Fourier Series

The first is a solution of the isoperimetric problem, which states that of all closed curves of a given length of a circle encloses maximum area. This can be stated as an inequality: $4\pi A \leq L^2$ if and only if the curve is a circle. Without loss of generality we consider curves whose lengths normalize in $2\pi$ and is positively oriented in the counterclockwise direction with a smooth closed curve $\Gamma \in \mathbf{R}^2$ by $(x, y) = (f(s), g(s))$ where $f, g : \mathbf{T} \to \mathbf{R}$ are continuously differentiable such that $\dot{f}(s)^2 + \dot{g}(s)^2 = 1$ taking the derivative with respect to s.

**Theorem 1.** *Green's Theorem: $\Omega$ a region in the plane with a smooth, positively oriented boundary $\partial\Omega$ and $u, v\bar{\Omega} \to \mathbf{R}$ are continuously differentiable functions $\int_\Omega \{u_x + v_y\} dxdy = \int_{\partial\Omega} \{udy - vdx\}$*

**Theorem 2.** *Suppose that a curve $\Gamma$ is given by$x = f(s)$, $y = g(s)$ where $f, g \in H^1\mathbf{T}$ are real valued functions that satisfy $\dot{f}(s)^2 + \dot{g}(s)^2 = 1$ and the area A enclosed by $\Gamma$ is given $A = \frac{1}{2}\int_\mathbf{T}\{f(s)\dot{g}(s) + g(s)\dot{f}(s)\}ds$ then $A \leq \pi$ with inequality if and only if $\Gamma$ is a circle.*

*Proof.* We Fourier expand $f$ and $g$ as
$f(s) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\hat{f}_n e^{ins}$, $g(s) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\hat{g}_n e^{ins}$
Since $f$ and $g$ are real valued, we have $\hat{f}_{-n} = \bar{\hat{f}}_n$ and $\hat{g}_{-n} = \bar{\hat{g}}_n$ (these are our complex conjugate for our Fourier coefficients) for all n. Integration of $\dot{f}(s)^2 + \dot{g}(s)^2 = 1$ over $\mathbf{T}$ gives $2\pi = \int_\mathbf{T}\{\dot{f}(s)^2 + \dot{g}(s)^2\}ds$. From Parseval's Theorem, this equation implies that

$$
\begin{aligned}
f(s) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{ins} \quad 0 \leq 2\pi \\
f^{'}(s) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_n ine^{ins} \quad \text{differentiation term by term} \\
\hat{f}_n &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x) dx \text{then} \\
f_n^{'} &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f'(x) dx \quad \text{integration by parts} \\
&= \cancel{\frac{1}{\sqrt{2\pi}} e^{-inx} f(x)|_0^{2\pi}} - \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} -ine^{-inx} f(x) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} ine^{-inx} f(x) dx \\
&= in\hat{f}_n \\
g^{'}(s) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} in\hat{g}_n e^{ins} \\
\int_0^{2\pi} |f'(s)|^2 dx &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 n^2 2\pi \\
&= \sum_{n=-\infty}^{\infty} n^2 |\hat{f}_n|^2 \\
2\pi &=\rightarrow \sum_{n=-\infty}^{\infty} n^2 (|\hat{f}_n|^2 + |\hat{g}_n|^2) \\
\text{and equation} A &= \frac{1}{2} \int_{\mathbf{T}} \{f(s)\dot{(}s) - (s)g(s)\} ds \\
2A &= \sum_{n=-\infty}^{\infty} in\{\bar{\hat{f}}_n \hat{g}_n - \hat{f}_n \bar{\hat{g}}_n\}
\end{aligned}
$$

Subtracting these series and rearranging the result, we find that $2\pi - 2A = \frac{1}{2} \sum_{n \neq 0} \{|n\hat{f}_n - i\hat{g}_n|^2 + |n\hat{g}_n + i\hat{f}_n|^2 + (n^2 - 1)(|\hat{f}_n|^2 + |\hat{g}_n|^2)\}$ Since the terms in the series on the right hand side of this equation are nonnegative, it follows that $A \leq \pi$. Moreover, we have equality if and only if $\hat{f}_n = \hat{g}_n = 0$ for $n \geq 2$, and $\hat{f}_1 = i\hat{g}_1$. $2\pi = \sum_{n=-\infty}^{\infty} n^2(|\hat{f}_n|^2 + |\hat{g}_n|^2)$ implies that $|\hat{f}_1| = \sqrt{\frac{\pi}{2}}$

$$\hat{f}_1 = \sqrt{\tfrac{\pi}{2}} e^{i\delta}, \;\; \hat{g}_1 = -i\sqrt{\tfrac{\pi}{2}} e^{i\delta}$$

for some $\delta \in \mathbf{R}$. Writing $\hat{f}_0 = \sqrt{2\pi} x_0$ and $\hat{g}_0 = \sqrt{2\pi} y_0$, where $x_o, y_0 \in \mathbf{R}$ we find from

$$f(s) = \tfrac{1}{\sqrt{2\pi}} \textstyle\sum_{n=-\infty}^{\infty} \hat{f}_n e^{ins}, \;\; g(s) = \tfrac{1}{\sqrt{2\pi}} \textstyle\sum_{n=-\infty}^{\infty} \hat{g}_n e^{ins} \text{that}$$

$$f(s) = x_0 + cos(s + \delta), \quad g(s) = y_0 + sin(s + \delta).$$

Thus, $A = \pi$, the curve $x = f(s), y = g(s)$ is a circle. $\square$

If we have a curve represented by functions f and g meeting specific conditions related to their derivatives and a formula for the enclosed area A then this area will always be less than or equal to $\pi$. Area will only equal $\pi$ when the curve is a circle.

The final application is the Ergodic theorem for one of the simplest dynamical systems one can imagine, namely rotations of the circle. Let $\gamma \in \mathbb{R}$ Define map $F_\gamma : \mathbb{T} \to \mathbb{T}$ on a circle $\mathbb{T}$ by $F_\gamma(x) = x + 2\pi\gamma$.
Circle map (Rotation map): Imagine you have a circle, and you move around this circle by a certain amount each step. The amount you move is determine by a number we call $\gamma$. If $\gamma$ is a rational perhaps $\frac{1}{2}$ or $\frac{3}{4}$, you'll eventually end up at the start point after a few steps. If $\gamma$ is irrational (like $\pi$ or $\sqrt{2}$), you'll never exactly end up where you started, no matter how many steps you take.

Orbit or Trajectory: As you take steps around the circle, the path you follow is called an orbit or trajectory. If $\gamma$ is rational, your path will repeat after some time because you'll land on the same spots over and over. If $\gamma$ is irrational, your path will cover the circle without repeating.

Averages: Time average vs. Phase-space average.
If you take the average of a function (which is just a rule that gives that gives you a number for each spot on the circle) over your path, its called a time average. This is just like adding up all the numbers you get from the function at each step and then dividing by the number of steps. Phase-space average is another type of average where you consider the whole circle at once, not just the points you stepped on. This phase-space average may be regarded a a probabilistic average with respect to a uniform probability measure on $\mathbb{T}$. The following ergodic theorem states that time averages and phase-space average are equal to when $\gamma$ is irrational. Result is false when $\gamma$ is rational.

Ergodic Theorem

**Theorem 3.** *Suppose $\gamma$ is irrational, then*

$$\langle f \rangle_t(x_0) = \langle f \rangle_p h, \forall f \in C(\mathbb{T}) \tag{7.38}$$

Basically this theorem states that if $\gamma$ is irrational the time average will be the same as the phase-space average for any continuous function on the circle. This is a deep mathematical result because it connects the long-term behavior of moving around the circle with the overall properties of the circle itself.
The proof of the Weyl ergodic theorem is below.

***Proof.*** First, we show that (7.38) holds for the functions $e^{imx}$ for each $m \in \mathbb{Z}$. If $m = 0$, then both averages are equal to 1. If $m \neq 0$, then $\langle e^{imx} \rangle_{\mathrm{ph}} = 0$, and the time average may be explicitly computed as follows:

$$\begin{aligned}
\langle e^{imx} \rangle_{\mathrm{t}} &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} e^{im(x_0+2\pi n\gamma)} \\
&= \lim_{N\to\infty} \frac{1}{N+1} e^{imx_0} \sum_{n=0}^{N} [e^{2\pi i m\gamma}]^n \\
&= \lim_{N\to\infty} \frac{1}{N+1} e^{imx_0} \left( \frac{1-[e^{2\pi i m\gamma}]^{N+1}}{1-e^{2\pi i m\gamma}} \right) \\
&= 0,
\end{aligned}$$

where we use the fact that $e^{2\pi i m\gamma} \neq 1$ for irrational $\gamma$. Since both averages are linear in $f$, it follows that (7.38) holds for all trigonometric polynomials.

The trigonometric polynomials are dense in $C(\mathbb{T})$. Therefore, if $f \in C(\mathbb{T})$ and $\epsilon > 0$, then there is a trigonometric polynomial $p$ such that $\|f - p\|_\infty \leq \epsilon$, and

$$\left| \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) - \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx \right| \leq 2\epsilon + \left| \frac{1}{N+1} \sum_{n=0}^{N} p(x_n) - \frac{1}{2\pi} \int_0^{2\pi} p(x)\,dx \right|.$$

Taking the lim sup of this equation as $N \to \infty$, we obtain that

$$\limsup_{N\to\infty} \left| \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) - \langle f \rangle_{\mathrm{ph}} \right| \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves (7.38) for all $f \in C(\mathbb{T})$ and all $x_0 \in \mathbb{T}$. $\square$

A consequence of this theorem is that if $\gamma$ is irrational, the points you land on as you were to walk around the circle are spread out evenly. Those points do not bunch up in any part of the circle.

**Corollary 7.12** Suppose that $\gamma$ is irrational and $I$ is an interval in $\mathbb{T}$ of length $\lambda$. Then

$$\lim_{N\to\infty} \frac{\#\{n \mid 0 \le n \le N,\, x_n \in I\}}{N+1} = \frac{\lambda}{2\pi}, \tag{7.39}$$

where $\#S$ denotes the number of points in the set $S$.

***Proof.*** Let $\chi_I$ be the characteristic function of the interval $I$. Then (7.39) is equivalent to the statement that

$$\langle \chi_I \rangle_{\mathrm{t}} = \langle \chi_I \rangle_{\mathrm{ph}}. \tag{7.40}$$

This equation does not follow directly from Theorem 7.11 because $\chi_I$ is not continuous. We therefore approximate $\chi_I$ by continuous functions. We choose sequences $(f_k)$ and $(g_k)$ of nonnegative, continuous functions such that $f_k \le \chi_I \le g_k$ and

$$\int_{\mathbb{T}} f_k(x)\, dx \to \int_{\mathbb{T}} \chi_I(x)\, dx, \quad \int_{\mathbb{T}} g_k(x)\, dx \to \int_{\mathbb{T}} \chi_I(x)\, dx \qquad \text{as } k \to \infty.$$

We leave it to the reader to construct such sequences. Since $f_k \le \chi_I \le g_k$,

$$\frac{1}{N+1}\sum_{n=0}^{N} f_k(x_n) \le \frac{1}{N+1}\sum_{n=0}^{N} \chi_I(x_n) \le \frac{1}{N+1}\sum_{n=0}^{N} g_k(x_n).$$

Taking the limit as $N \to \infty$ of this equation, and applying Theorem 7.11 to the functions $f_k$ and $g_k$, we obtain that

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}} f_k(x)\, dx &\le \liminf_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} \chi_I(x_n) \\
&\le \limsup_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} \chi_I(x_n) \le \frac{1}{2\pi}\int_{\mathbb{T}} g_k(x)\, dx.
\end{aligned}$$

Letting $k \to \infty$, we find that

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}} \chi_I(x)\, dx &\le \liminf_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} \chi_I(x_n) \\
&\le \limsup_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} \chi_I(x_n) \\
&\le \frac{1}{2\pi}\int_{\mathbb{T}} \chi_I(x)\, dx.
\end{aligned}$$

It follows that the limit defining the time average of $\chi_I$ exists and satisfies equation (7.40) for all $x_0 \in \mathbb{T}$. $\square$

The proof of the corollary talks about a specific kind of function called the characteristic function which is a simple way of saying that some point is in a certain area of the circle. The ergodic theorem applies to this function as well, which is a bit tricky since the function is not continuous.
An example of a practical application is called the Monte Carlo method. This method introduces a way to solve complicated problems using random sampling (By taking random steps around the circle and observing where you end up at.) In essence the ergodic theorem is about the predictability of patterns when you follow certain rules. It tells us that for some certain number $\gamma \in \mathbb{R}$, the patterns we see over time are representative of the entire circle. This has powerful implications for understanding systems that appear randomly but actually have hidden uniform patterns.

# 3 7.6 Wavelets

Wavelets are mathematical functions that transform data into different scales or resolutions. They are useful for analyzing physical, functional or biological data unlike other transformations, like Fourier transforms which only analyze the frequency of signals, wavelets can also analyze the location in time.
Our $L^2([0,1])$ space which consists of functions whose squares are integrable, meaning we an calculate total area from 0 to 1. Our orthogonal basis for this space is a set of functions that can be used to represent any function in this space in a unique way. Imagine this as being able to build any structure using a specific set of building blocks.

The Haar wavelet is a simple example of a wavelet. It consists of a ”scaling function” and a ”wavelet function” shown below. It’s also used for analyzing time series data and great for detecting sudden changes in a signal.

We define the *Haar scaling function* $\varphi \in L^2(\mathbb{R})$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.41}$$

The function $\varphi$ is the characteristic function of the interval $[0,1)$, and is often referred to as a “box” function because of the shape of its graph. The basic *Haar wavelet*, or *mother wavelet*, $\psi \in L^2(\mathbb{R})$ is given by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.42}$$

These functions satisfy the scaling relations

$$\varphi(x) = \varphi(2x) + \varphi(2x-1), \tag{7.43}$$
$$\psi(x) = \varphi(2x) - \varphi(2x-1). \tag{7.44}$$

For $n, k \in \mathbb{Z}$, we define scaled translates $\varphi_{n,k}, \psi_{n,k} \in L^2(\mathbb{R})$ of $\varphi$, $\psi$ by

$$\varphi_{n,k}(x) = 2^{n/2}\varphi\left(2^n x - k\right), \quad \psi_{n,k}(x) = 2^{n/2}\psi\left(2^n x - k\right). \tag{7.45}$$

The "scaling relations" are essentially rules in which define how you can create wavelets of different sizes from a basic wavelet function.
The Haar wavelet can be used to create a complete and orthonormal basis for the space of functions $L^2([0,1])$. This means that any function in this space can be expressed exactly as an infinite sum of these wavelet functions. The scaling and wavelet functions, $\phi$ and $\psi$, are the basic building blocks of the wavelet theory. The scaling function $\phi$ helps to analyze the smooth parts of a signal, while the wavelet function $\psi$ captures the abrupt changes.

**Definition 7.15 (Multiresolution analysis)** A family $\{V_n \mid n \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R})$ and a function $\varphi \in L^2(\mathbb{R})$ are called a *multiresolution analysis* of $L^2(\mathbb{R})$ if the following properties hold:

(a) $f(x) \in V_n$ if and only if $f(2x) \in V_{n+1}$ for all $n \in \mathbb{Z}$ (scaling); (7.51)

(b) $V_n \subset V_{n+1}$ for all $n \in \mathbb{Z}$ (inclusion); (7.52)

(c) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$ (density); (7.53)

(d) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$ (maximality); (7.54)

(e) there is a function $\varphi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that
$\{\varphi(x-k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$ (basis). (7.55)

# 4 Homework Problem 7.11

Let $\Omega = \{(r,\theta)|r < 1\}$ be the unit disc in the plane, where $(r,\theta)$ are polar coordinates. The boundary of $\Omega$ is the unit circle $\mathbf{T}$. Let $u(r,\theta)$ be a solution of Laplace's equation in $\Omega$,

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad r < 1,$$

and define $f, g \in L^2\mathbf{T}$ by $f(\theta) = u_\theta(1,\theta)$, $g(\theta) = u_r(1,\theta)$. Show that $g = \mathbf{H}f$ where $\mathbf{H}$ is the periodic Hilbert transform.

*Proof.* WWTS $g = \mathbf{H}f$
We know $u(r,\theta)$ is a solution of Laplace equation
Our Fourier series in polar coordinates
$u(r,\theta) = \sum_{n=-\infty}^{\infty}(a_n(r)cosn\theta + b_n(r)sinn\theta)$
$a_n(r) = \frac{1}{\pi}\int_0^{2\pi} u(r,\theta)cos(n\theta)d\theta$
$b_n(r) = \frac{1}{\pi}\int_0^{2\pi} u(r,\theta)sin(n\theta)d\theta$
differentiating the Fourier series by terms we get:
$u_r(r,\theta) = \sum_{n=-\infty}^{\infty}[a_n^{'}(r)cos(n\theta) + b_n^{'}(r)sin(n\theta)]$
$u_\theta(r,\theta) = \sum_{n=-\infty}^{\infty}[-a_n(r)nsin(n\theta) + b_n(r)ncos(n\theta)]$
Evaluating the expression at $r = 1$
$f(\theta) = u_\theta(1,\theta) = \sum_{n=-\infty}^{\infty}(-a_n(1)nsin(n\theta) + b_n(1)ncos(n\theta))$

$g(\theta) = u_r(1,\theta) = \sum_{n=-\infty}^{\infty}[a_n^{'}(1)cos(n\theta) + b_n^{'}(1)sin(n\theta)]$
periodic Hilbert transform H is defined
$Hf(\theta) = \frac{1}{\pi}P\dot{V}\int_0^{2\pi}\frac{f(t)}{\theta-t}dt$
$Hf(\theta) = \frac{1}{\pi}P\dot{V}\int_0^{2\pi}\frac{\sum_{n=-\infty}^{\infty}[-a_n(1)nsin(n\theta)+b_n(1)ncos(n\theta)]}{\theta-t}dt$
changing the order of the summation and integration
$Hf(\theta) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}[-a_n(1)nsin(n\theta) + b_n(1)ncos(n\theta)]i$ imaginary unit
Comparing the expression for $g(\theta) and Hf(\theta)$ we observe they have the same Fourier coefficients for all n.
$\therefore g(\theta) = Hf(\theta)$

□