Problem Set #5

Econ 103

Lecture Progress

We made it to the end of the Chapter 4 (Updated) slides.

Homework Checklist

- ☐ **Book Problems (Chapter 4):** 7, 11, 15, 25, 27, 29
- ☐ **Book Problems (Chapter 5):** 1, 3, 5, 9, 11, 13, 17
- ☐ **Additional Problems:** See below
- ☐ **R Tutorial:** R Tutorial 4
- ☐ **Ask questions on Piazza**
- ☐ **Review slides**

Part II – Additional Problems

1. Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable *directly*, that is *without* using the shortcut formula.

2. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which $n = 1$. (Hint: compare pmfs.)

3. Suppose that X is a random variable with support $\{1, 2\}$ and Y is a random variable with support $\{0, 1\}$ where X and Y have the following joint distribution:

$$\begin{aligned} p_{XY}(1,0) &= 0.20, \qquad & p_{XY}(1,1) &= 0.30 \\ p_{XY}(2,0) &= 0.25, \qquad & p_{XY}(2,1) &= 0.25 \end{aligned}$$

   (a) Express the joint distribution in a 2×2 table.

(b) Using the table, calculate the marginal probability distributions of X and Y.

(c) Calculate the conditional probability distribution of $Y|X=1$ and $Y|X=2$.

(d) Calculate $E[Y|X]$.

(e) What is $E[E[Y|X]]$?

(f) Calculate the covariance between X and Y using the shortcut formula.

4. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:
   (a) $Cov(a+bX, c+dY) = bdCov(X, Y)$
   (b) $Corr(a+bX, c+dY) = Corr(X, Y)$

5. Fill in the missing steps from lecture to prove the shortcut formula for covariance:
$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

6. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1-\omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0,1]$, so that *negative* weights are not allowed. (This rules out short-selling.)
   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.
   (b) If $\omega \in [0,1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.
   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.
   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)
   (e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is
$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$
   In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio*.
   (f) If you want a challenge, check the second order condition from part (e).

7. Prove that if two random variables are independent, then their covariance is zero.

8. Prove that expectation of two random variables is linear. $E[aX+bY+c] = aE[X] + bE[Y] + c$