Economics 103 – Statistics for Economists

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Lecture # 6

Basic Probability – Part II

Recall: Axioms of Probability

Let S be the sample space. With each event $A \subseteq S$ we associate a real number $P(A)$ called the probability of A, satisfying the following conditions:

- Axiom 1 $0 \leq P(A) \leq 1$
- Axiom 2 $P(S) = 1$
- Axiom 3 If $A_1, A_2, A_3, \ldots$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

The Complement Rule: $P(A^c) = 1 - P(A)$

Since A, A^c are mutually exclusive and collectively exhaustive:

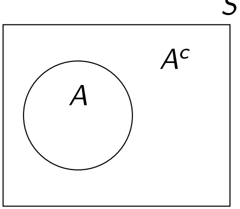

Figure: $A \cap A^c = \emptyset$, $A \cup A^c = S$

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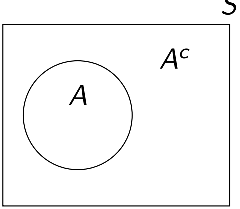

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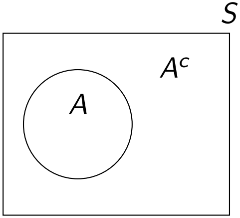

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Rearranging:

$$P(A^c) = 1 - P(A)$$

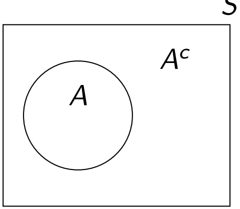

Figure: $A \cap A^c = \emptyset$, $A \cup A^c = S$

Another Important Rule – Equivalent Events

If A and B are Logically Equivalent, then $P(A) = P(B)$.

In other words, if A and B contain exactly the same basic outcomes, then $P(A) = P(B)$.

Although this seems obvious it's important to keep in mind, especially later in the course...

The Logical Consequence Rule

If B Logically Entails A, then $P(B) \leq P(A)$

In other words, $B \subseteq A \Rightarrow P(B) \leq P(A)$

Why is this so?

If $B \subseteq A$, then all the basic outcomes in B are also in A.

Proof of Logical Consequence Rule (Optional)

Proof won't be on a quiz or exam but is good practice with probability axioms.

Since $B \subseteq A$, we have $B = A \cap B$ and
$A = B \cup (A \cap B^c)$. Combining these,

$$A = (A \cap B) \cup (A \cap B^c)$$

Now since $(A \cap B) \cap (A \cap B^c) = \emptyset$,

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(B) + P(A \cap B^c) \\ &\geq P(B) \end{aligned}$$

because $0 \leq P(A \cap B^c) \leq 1$.

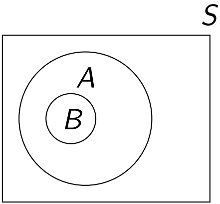

Figure:

$B = A \cap B$, and

$A = B \cup (A \cap B^c)$

“Odd Question” # 2

Pia is thirty-one years old, single, outspoken, and smart. She was a philosophy major. When a student, she was an ardent supporter of Native American rights, and she picketed a department store that had no facilities for nursing mothers. Rank the following statements in order from most probable to least probable.

(A) Pia is an active feminist.

(B) Pia is a bank teller.

(C) Pia works in a small bookstore.

(D) Pia is a bank teller and an active feminist.

(E) Pia is a bank teller and an active feminist who takes yoga classes.

(F) Pia works in a small bookstore and is an active feminist who takes yoga classes.

Using the Logical Consequence Rule...

(A) Pia is an active feminist.

(B) Pia is a bank teller.

(C) Pia works in a small bookstore.

(D) Pia is a bank teller and an active feminist.

(E) Pia is a bank teller and an active feminist who takes yoga classes.

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Any Correct Ranking Must Satisfy:

$$P(\mathrm{A}) \geq P(\mathrm{D}) \geq P(\mathrm{E})$$

$$P(\mathrm{B}) \geq P(\mathrm{D}) \geq P(\mathrm{E})$$

$$P(\mathrm{A}) \geq P(\mathrm{F})$$

$$P(\mathrm{C}) \geq P(\mathrm{F})$$

Throw a Fair Die Once

$E =$ roll an even number

What are the basic outcomes?

$\{1, 2, 3, 4, 5, 6\}$

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What is $P(E)$?

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What are the basic outcomes?

$\{1, 2, 3, 4, 5, 6\}$

What is $P(E)$?

$E = \{2, 4, 6\}$ and the basic outcomes are equally likely (and mutually exclusive), so

$$P(E) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$$

Throw a Fair Die Once

$E =$ roll an even number $\qquad M =$ roll a 1 or a prime number

What is $P(E \cup M)$?

Throw a Fair Die Once

$E =$ roll an even number $\qquad M =$ roll a 1 or a prime number

What is $P(E \cup M)$?

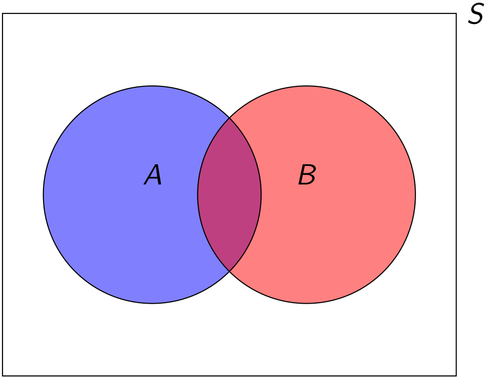
Key point: E and M are not mutually exclusive!

$$\begin{aligned} P(E \cup M) &= P(\{1,2,3,4,5,6\}) = 1 \\ P(E) &= P(\{2,4,6\}) = 1/2 \\ P(M) &= P(\{1,2,3,5\}) = 4/6 = 2/3 \end{aligned}$$

$$P(E) + P(M) = 1/2 + 2/3 = 7/6 \neq P(E \cup M) = 1$$

The Addition Rule – Don't Double-Count!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Construct a formal proof as an optional homework problem.

Who's on the other side?

Three Cards, Each with a Face on the Front and Back

1. Gaga/Gaga
2. Obama/Gaga
3. Obama/Obama

I draw a card at random and look at one side: it's Obama.

What is the probability that the other side is also Obama?

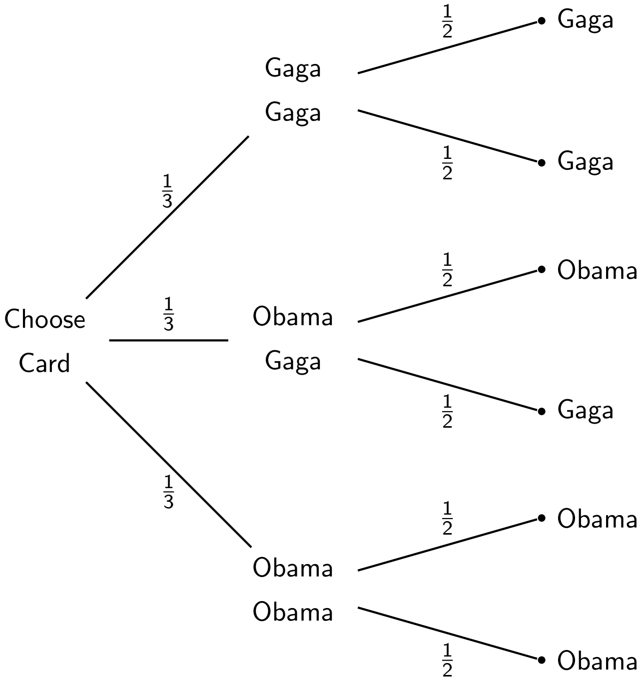
Let's Try The Method of Monte Carlo...

When you don't know how to calculate, simulate.

Procedure

1. Close your eyes and thoroughly shuffle your cards.
2. Keeping eyes closed, draw a card and place it on your desk.
3. Stand if Obama is face-up on your chosen card.
4. We'll count those standing and call the total N
5. Of those standing, sit down if Obama is *not* on the back of your chosen card.
6. We'll count those *still* standing and call the total m.

Monte Carlo Approximation of Desired Probability $= \dfrac{m}{N}$


Choose
Card
1/3
1/3
1/3
Gaga
Gaga
Obama
Gaga
Obama
Obama
1/2
1/2
1/2
1/2
1/2
1/2
Gaga
Gaga
Obama
Gaga
Obama
Obama

Conditional Probability – Reduced Sample Space

Set of relevant outcomes restricted by condition

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0$$

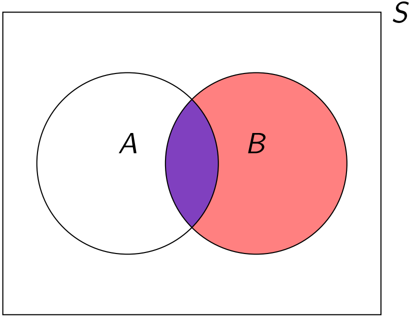


Figure: B becomes the "new sample space" so we need to re-scale by $P(B)$ to keep probabilities between zero and one.

Who's on the other side?

Let F be the event that Obama is on the front of the card of the card we draw and B be the event that he is on the back.

$$P(B|F) = \frac{P(B \cap F)}{P(F)} =$$

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Conditional Versions of Probability Axioms

1. $0 \leq P(A|B) \leq 1$
2. $P(B|B) = 1$
3. If $A_1, A_2, A_3, \ldots$ are mutually exclusive events, then
   $P(A_1 \cup A_2 \cup A_3 \cup \cdots | B) = P(A_1|B) + P(A_2|B) + P(A_3|B) \ldots$

Conditional Versions of Other Probability Rules

- $P(A|B) = 1 - P(A^c|B)$
- A_1 logically equivalent to $A_2 \iff P(A_1|B) = P(A_2|B)$
- $A_1 \subseteq A_2 \implies P(A_1|B) \leq P(A_2|B)$
- $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$

However: $P(A|B) \neq P(B|A)$ and $P(A|B^c) \neq 1 - P(A|B)$!

Independence and The Multiplication Rule

The Multiplication Rule

Rearrange the definition of conditional probability:

$P(A \cap B) = P(A|B)P(B)$

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By the Multiplication Rule

Independence $\iff P(A|B) = P(A)$

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Independence $\iff P(A|B) = P(A)$

Interpreting Independence

Knowledge that B has occurred tells nothing about whether A will.

Will Having 5 Children Guarantee a Boy?

A couple plans to have five children. Assuming that each birth is independent and male and female children are equally likely, what is the probability that they have at least one boy?

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By Independence and the Complement Rule,

$$\begin{aligned} P(\text{no boys}) &= P(5 \text{ girls}) \\ &= 1/2 \times 1/2 \times 1/2 \times 1/2 \times 1/2 \\ &= 1/32 \end{aligned}$$

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$$\begin{aligned} P(\text{at least 1 boy}) &= 1 - P(\text{no boys}) \\ &= 1 - 1/32 = 31/32 = 0.97 \end{aligned}$$

The Law of Total Probability

If $E_1, E_2, \ldots, E_k$ are mutually exclusive, collectively exhaustive events and A is another event, then

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_k)P(E_k)$$

Example of Law of Total Probability

Define the following events:

$$
\begin{aligned}
F &= \text{Obama on front of card} \\
A &= \text{Draw card with two Gagas} \\
B &= \text{Draw card with two Obamas} \\
C &= \text{Draw card with BOTH Obama and Gaga}
\end{aligned}
$$

$$P(F) = P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)$$

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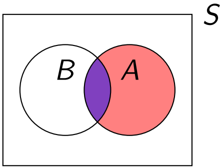
$$\begin{aligned} P(F) &= P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C) \\ &= 0 \times 1/3 + 1 \times 1/3 + 1/2 \times 1/3 \\ &= 1/2 \end{aligned}$$

Deriving the Law of Total Probability For $k = 2$

You are responsible for this proof on quizzes and exams.



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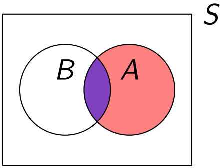
Since $A \cap B$ and $A \cap B^c$ are mutually exclusive and their union equals A,

$$P(A) = P(A \cap B) + P(A \cap B^c)$$



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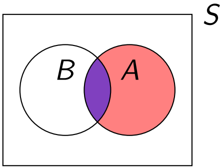
$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ P(A \cap B^c) &= P(A|B^c)P(B^c) \end{aligned}$$



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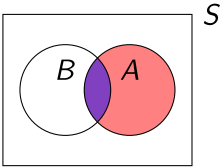
Combining,

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$



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How do prediction markets work?

To learn more, see Wolfers & Zitzewitz (2004)

THIS CERTIFICATE ENTITLES THE BEARER TO $1 IF TRUMP WINS THE 2016 US PRESIDENTIAL ELECTION.

Buyers – Purchase Right to Collect

Trump very likely to win $\Rightarrow$ buy for close to $1.

Trump very unlikely to win $\Rightarrow$ buy for close to $0.

Sellers – Sell Obligation to Pay

Trump very likely to win $\Rightarrow$ sell for close to $1.

Trump very unlikely to win $\Rightarrow$ sell for close to $0.

Probabilities from Beliefs

Market price of contract encodes market participants' beliefs in the form of probability:

$$\text{Price/Payout} \approx \text{Subjective Probability}$$

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Statistical Arbitrage

If the probabilities implied by prediction market prices violate *any* of our probability rules there is a *pure arbitrage opportunity*: a way make to make a guaranteed, risk-free profit.

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Last I checked a $1 contract on Trump winning was going for $0.29 on `predictit.org`

A Simple Example of Statistical Arbitrage

Courtesy of $\boxed{\text{Eric Crampton}}$

November 5th, 2012

- \$2.30 for contract paying \$10 if Romney wins on BetFair
- \$6.58 for contract paying \$10 if Obama wins on InTrade

Implied Probabilities

- BetFair: $P(Romney) \approx 0.23$
- InTrade: $P(Obama) \approx 0.66$

What's Wrong with This?

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What's Wrong with This?

Violates complement rule! $P(Obama) = 1 - P(Romney)$ but the implied probabilities here don't sum up to one!

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Arbitrage Strategy

Buy Equal Numbers of Each

- Cost = \$2.30 + \$6.58 =\$8.88 per pair
- Payout if Romney Wins: \$10
- Payout if Obama Wins: \$10
- Guaranteed Profit: \$10 - \$8.88 = \$1.12 per pair