

Econ 103 – Statistics for Economists

Chapter 3: Probability

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Odd Questions

“Odd Question” # 1

Is the following likely true or false and why?

*The counties with the **lowest** incidence of kidney cancer are mostly rural, sparsely populated, and located in traditionally Republican states in the Midwest, the South, and the East.*

- Less water or air pollution
- Lower stress
- Healthier food

“Odd Question” # 1

Is the following likely true or false and why?

*The counties with the **highest** incidence of kidney cancer are mostly rural, sparsely populated, and located in traditionally Republican states in the Midwest, the South, and the East.*

- Higher stress because of poverty
- Drink alcohol
- Higher tobacco use

“Odd Question” # 1

Which statement is right?

1. Lower rate in rural counties
2. Higher rate in rural counties
3. Both
4. Neither

How?

From *Thinking Fast and Slow* by Daniel Kahneman

Something is wrong, of course. The rural lifestyle cannot explain both very high and very low incidence of kidney cancer. The key factor is not that the counties were rural or predominately Republican. It is that rural counties have small populations.

“Odd Question” # 2

Pia is thirty-one years old, single, outspoken, and smart. She was a philosophy major. When a student, she was an ardent supporter of Native American rights, and she picketed a department store that had no facilities for nursing mothers.

Rank the following statements in order from most probable to least probable.

- (a) Pia is a bank teller.
- (b) Pia is a bank teller and an active feminist.

The Conjunction Fallacy

When it is assumed that specific conditions are more probable than a single general one.

Think Venn diagrams (we'll see this formally later in the lecture)

“Odd Question” # 3

Your aunt bought two lottery tickets and offers you one for free. Each ticket has six numbers, and if those numbers are drawn, you will win the jackpot. The two tickets are below

- I. Numbers 1, 2, 3, 4, 5, and 6
- II. Numbers 39, 36, 32, 21, 14, and 3

Which would you pick to maximize your chance of winning?

- (a) Ticket I
- (b) Ticket II
- (c) Indifferent

Probability

Remember!

Probability: Population \rightarrow Sample

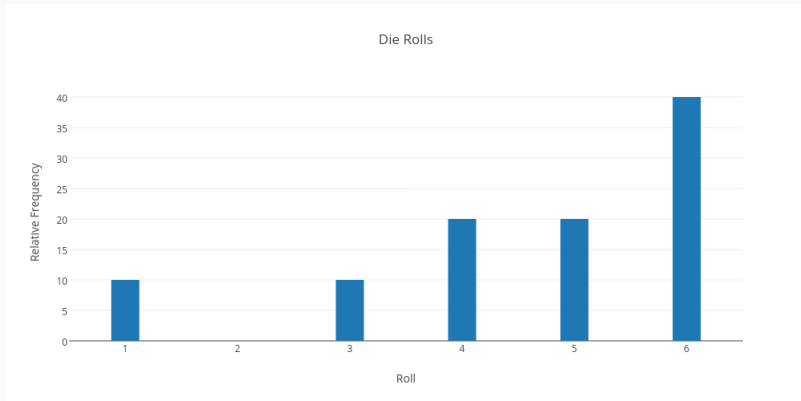
- Deductive: “safe” argument
 - All ducks waddle, swim, and quack. Donald is a duck.
Donald must waddle, swim, and quack.
- It turns out that we're really bad at this

Our Definition of Probability for this Course

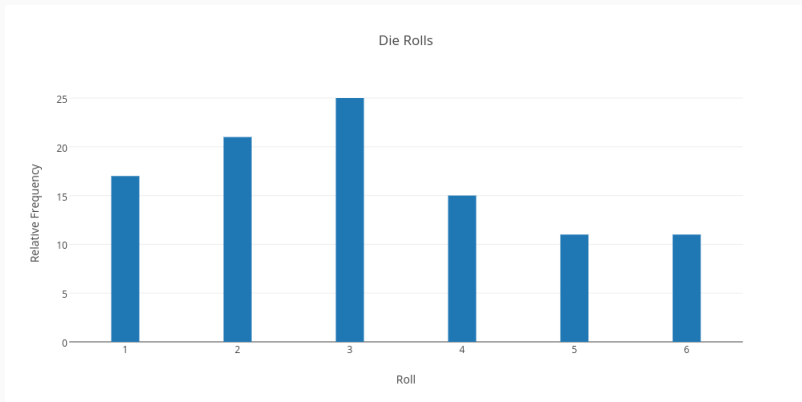
Probability: The long-run relative frequency

That is, relative frequencies settle down to probabilities if we carry out an experiment over, and over, and over...

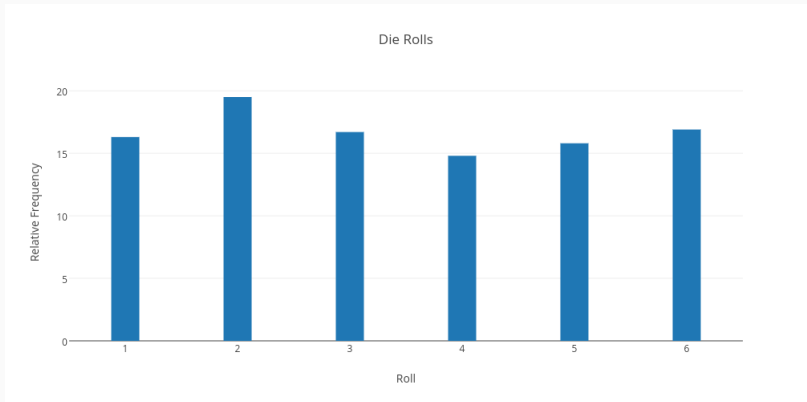
10 Die Rolls



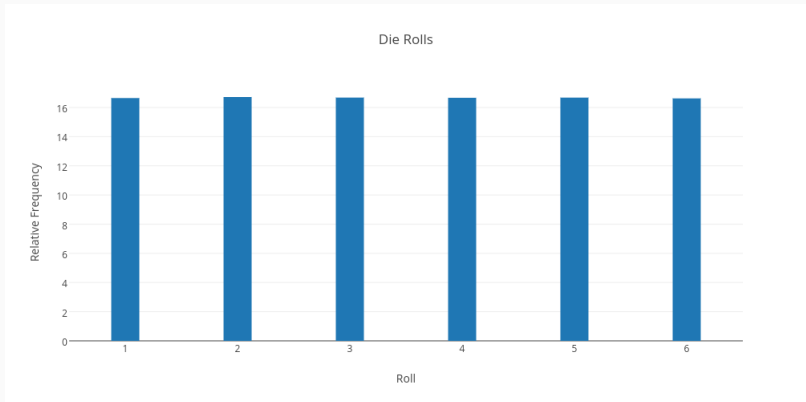
100 Die Rolls



1,000 Die Rolls



1,000,000 Die Rolls



What do you think of this argument?

- The probability of flipping heads is $1/2$: if we flip a coin many times, about half of the time it will come up heads.
- The last ten throws in a row the coin has come up heads.
- The coin is bound to come up tails next time – it would be very rare to get 11 heads in a row.

The Gambler's Fallacy

Relative frequencies settle down to probabilities, but this does not mean that the trials are dependent.

Dependent: “Memory” of previous trials

Independent: No “memory” of previous trials

Another Argument

Lucie visits Albert. As she enters, he rolls four dice and shouts “Hooray!” for he has just rolled four sixes. Lucie: “I bet you’ve been rolling the dice for a long time to get that result!” Now, Lucie may have many reasons for saying this – perhaps Albert is a lunatic dice-roller. But simply on the evidence that he has just rolled four sixes, is her conclusion reasonable?

The *Inverse* Gambler's Fallacy

This is true:

Albert is more likely to get four sixes if he rolls many times than if he rolls only once.

However:

Regardless of how long Albert has been rolling, the probability that he gets four sixes **on the particular roll that Lucie observes** is unchanged.

The outcome of that roll doesn't tell us anything about whether he has rolled the dice before, just like six heads in a row doesn't mean we're "due" for a tails.

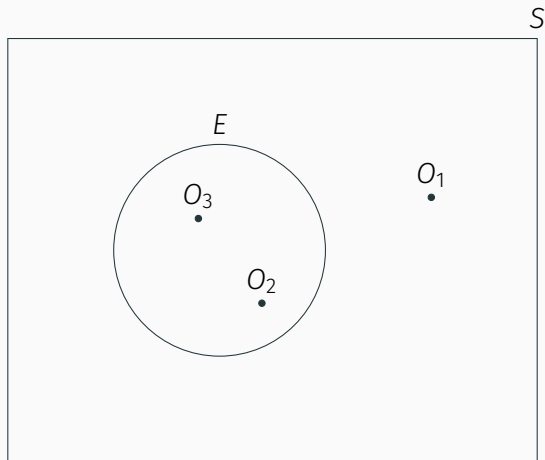
Definitions

- **Random Experiment:** An experiment whose outcomes are random.
- **Basic Outcomes:** Possible outcomes (mutually exclusive) of random experiment.
- **Sample Space (S):** Set of all basic outcomes of a random experiment.
- **Event (E):** A subset of the *sample space*. In set notation we write $E \subseteq S$.

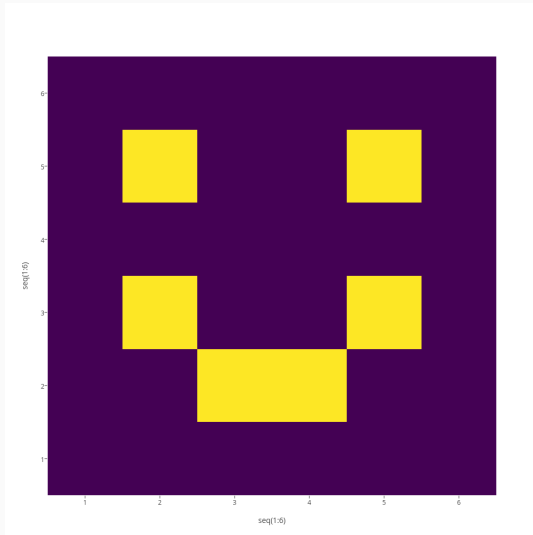
Examples

- **Experiment:** Tossing a pair of dice.
- **Basic Outcome:** An ordered pair (a, b) where $a, b \in \{1, 2, 3, 4, 5, 6\}$, e.g. $(2, 5)$
- **Sample Space:** $S = \text{All ordered pairs } (a, b) \text{ where } a, b \in \{1, 2, 3, 4, 5, 6\}$
- **Event:** $E = \{(2, 5), (5, 5), (2, 3), (3, 2), (4, 2), (5, 3)\}$

Visual Representation



Our Example



Probability is Defined on
Sets, and Events are Sets

Dice Problem

Set Theory

Complement of an Event: $A^c = \text{not } A$

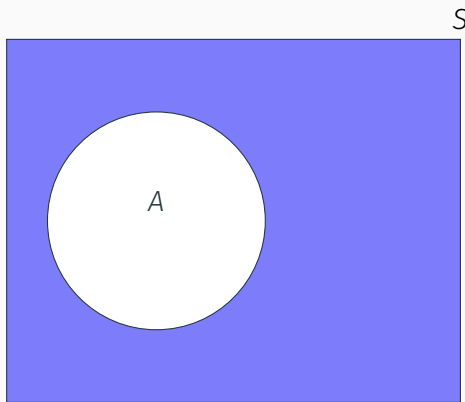


Figure 1: The complement A^c of an event $A \subseteq S$ is the collection of all basic outcomes from S not contained in A .

Intersection of Events: $A \cap B = A \text{ and } B$

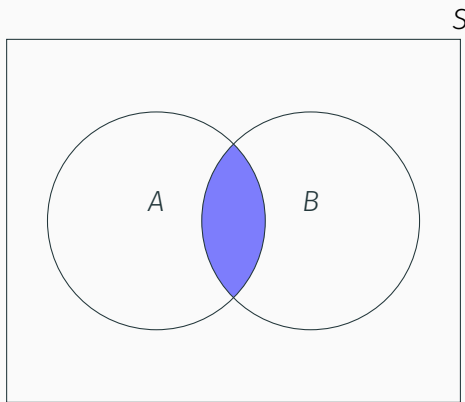


Figure 2: The intersection $A \cap B$ of two events $A, B \subseteq S$ is the collection of all basic outcomes from S contained in both A and B

Union of Events: $A \cup B = A \text{ or } B$

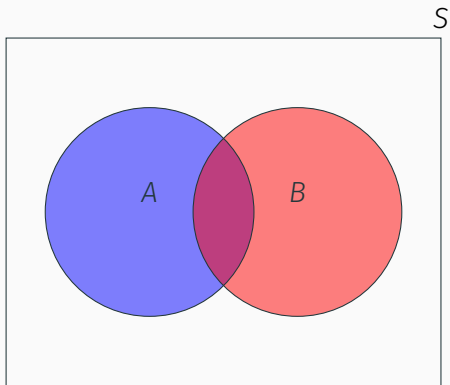


Figure 3: The union $A \cup B$ of two events $A, B \subseteq S$ is the collection of all basic outcomes from S contained in A , B or both.

Quick Set Theory Result

What is the formula for $A \cup B$?

$$A \cup B = A + B - (A \cap B)$$

Mutually Exclusive and Collectively Exhaustive

Mutually Exclusive Events

A collection of events E_1, E_2, E_3, \dots is *mutually exclusive* if $E_i \cap E_j$ of *any two different events* is empty (formally $E_i \cap E_j = \emptyset$ for any $i \neq j$).

Collectively Exhaustive Events

A collection of events E_1, E_2, E_3, \dots is *collectively exhaustive* if, taken together, they contain *all of the basic outcomes in S* (formally $E_1 \cup E_2 \cup \dots = S$).

Implications

Mutually Exclusive Events

If one of the events occurs, then none of the others did.

Collectively Exhaustive Events

One of these events *must* occur.

Can you come up with examples of

- Mutually exclusive events that are not collectively exhaustive?
- Collectively exhaustive events that are not mutually exclusive?

Mutually Exclusive but *not Collectively Exhaustive*

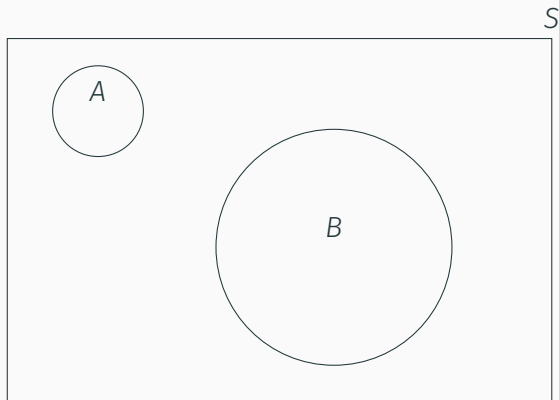


Figure 4: Although A and B don't overlap, they also don't cover S .

Collectively Exhaustive but *not Mutually Exclusive*

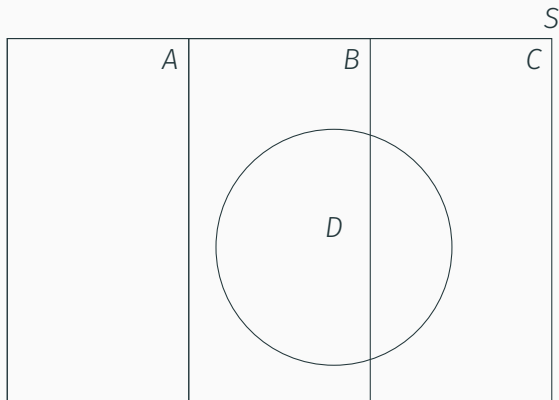


Figure 5: Together A , B , C and D cover S , but D overlaps with B and C .

Collectively Exhaustive *and* Mutually Exclusive

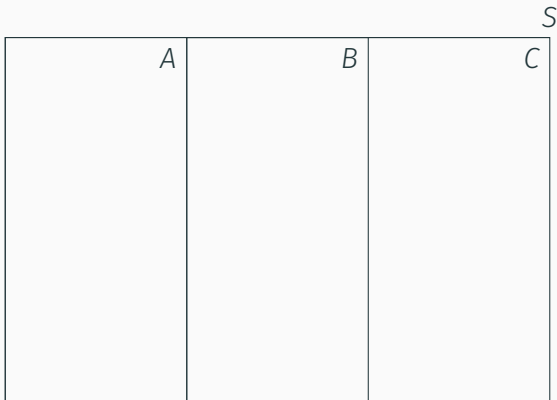


Figure 6: A , B , and C cover S and don't overlap.

The Complement Rule: $P(A^c) = 1 - P(A)$

Since A, A^c are *mutually exclusive* and *collectively exhaustive*:

$$P(A \cup A^c) = P(A) + P(A^c) = P(S) = 1$$

Rearranging:

$$P(A^c) = 1 - P(A)$$

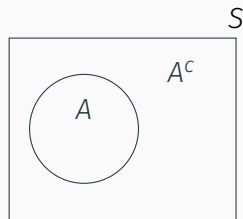


Figure 7: $A \cap A^c = \emptyset$,
 $A \cup A^c = S$

Will Having 5 Children Guarantee a Boy?

A couple plans to have five children. Assuming that each birth is independent and male and female children are equally likely, what is the probability that they have at least one boy?

By Independence and the Complement Rule,

$$\begin{aligned}P(\text{no boys}) &= P(5 \text{ girls}) \\&= 1/2 \times 1/2 \times 1/2 \times 1/2 \times 1/2 \\&= 1/32\end{aligned}$$

$$\begin{aligned}P(\text{at least 1 boy}) &= 1 - P(\text{no boys}) \\&= 1 - 1/32 = 31/32 = 0.97\end{aligned}$$

The Birthday Problem

What is the least number of persons required if the probability exceeds $1/2$ that two or more of them have the same birthday? (Year of birth need not match.)

[Hint: Use the Complement Rule.]

Coin Flipping

Axioms of Probability

We assign every event A in the sample space S a real number $P(A)$ called the **probability of A** such that:

Axiom 1 $0 \leq P(A) \leq 1$

Axiom 2 $P(S) = 1$

Axiom 3 If A_1, A_2, A_3, \dots are mutually exclusive events, then
$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$$

Equivalent Events

If A and B are Logically Equivalent, then $P(A) = P(B)$.

In other words, if A and B contain exactly the same basic outcomes, then $P(A) = P(B)$.

Although this seems obvious it's important to keep in mind, especially later in the course...

The Logical Consequence Rule

If B Logically Entails A , then $P(B) \leq P(A)$

In other words, $B \subseteq A \Rightarrow P(B) \leq P(A)$

Why is this so?

If $B \subseteq A$, then all the basic outcomes in B are also in A .

Proof of Logical Consequence Rule

Since $B \subseteq A$, we have $B = A \cap B$ and $A = B \cup (A \cap B^c)$. Combining these,

$$A = (A \cap B) \cup (A \cap B^c)$$

Now since $(A \cap B) \cap (A \cap B^c) = \emptyset$,

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(B) + P(A \cap B^c) \\ &\geq P(B) \end{aligned}$$

because $0 \leq P(A \cap B^c) \leq 1$.

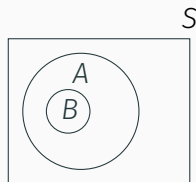


Figure 8:

$B = A \cap B$, and
 $A = B \cup (A \cap B^c)$

Permutations and Combinations

“Classical” Probability

When all of the basic outcomes are equally likely, the probability of an event is simply the total number of basic outcomes that make up the event divided by the total number of basic outcomes.

Recall from High School Math:

Multiplication Rule for Counting

n_1 ways to make first decision, n_2 ways to make second, \dots , n_k ways to make k th $\Rightarrow n_1 \times n_2 \times \dots \times n_k$ total ways to decide.

Corollary – Number of Possible Orderings

$$k \times (k-1) \times (k-2) \times \dots \times 2 \times 1 = k!$$

Permutations – Order n people in k slots

$$P_k^n = \frac{n!}{(n-k)!}$$

Combinations – Choose committee of k from group of n

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ where } 0! = 1$$

Cards, Tennis, and Obama

Conditional Probability

Conditional Probability – Reduced Sample Space

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

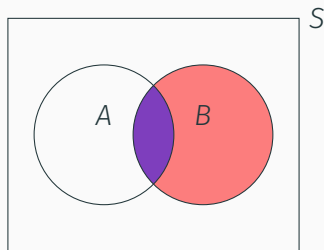


Figure 9: B becomes the “new sample space” so we need to re-scale by $P(B)$ to keep probabilities between zero and one.

Who's on the other side?

Let F be the event that Obama is on the front of the card of the card we draw and B be the event that he is on the back.

$$P(B|F) = \frac{P(B \cap F)}{P(F)} = \frac{1/3}{1/2} = 2/3$$

Conditional Versions of Probability Axioms

1. $0 \leq P(A|B) \leq 1$
2. $P(B|B) = 1$
3. If A_1, A_2, A_3, \dots are mutually exclusive events, then
$$P(A_1 \cup A_2 \cup A_3 \cup \dots | B) = P(A_1|B) + P(A_2|B) + P(A_3|B) \dots$$

Conditional Versions of Other Probability Rules

- $P(A|B) = 1 - P(A^c|B)$
- A_1 logically equivalent to $A_2 \iff P(A_1|B) = P(A_2|B)$
- $A_1 \subseteq A_2 \implies P(A_1|B) \leq P(A_2|B)$
- $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$

However: $P(A|B) \neq P(B|A)$ and $P(A|B^c) \neq 1 - P(A|B)$!

Independence and The Multiplication Rule

The Multiplication Rule

Rearrange the definition of conditional probability:

$$P(A \cap B) = P(A|B)P(B)$$

Statistical Independence

$$P(A \cap B) = P(A)P(B)$$

By the Multiplication Rule

$$\text{Independence} \iff P(A|B) = P(A)$$

Interpreting Independence

Knowing B has occurred tells nothing about whether A will.

The Law of Total Probability

The Law of Total Probability

If E_1, E_2, \dots, E_k are mutually exclusive, collectively exhaustive events and A is another event, then

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \dots + P(A|E_k)P(E_k)$$

Example of Law of Total Probability

Define the following events:

F = Obama on front of card

A = Draw card with two Gagas

B = Draw card with two Obamas

C = Draw card with BOTH Obama and Gaga

$$\begin{aligned}P(F) &= P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C) \\&= 0 \times 1/3 + 1 \times 1/3 + 1/2 \times 1/3 \\&= 1/2\end{aligned}$$

Deriving the Law of Total Probability For $k = 2$

Since $A \cap B$ and $A \cap B^c$ are mutually exclusive and their union equals A ,

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

But by the multiplication rule:

$$P(A \cap B) = P(A|B)P(B)$$

$$P(A \cap B^c) = P(A|B^c)P(B^c)$$

Combining,

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

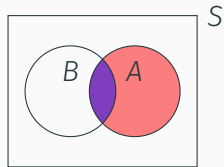


Figure 10:

$$A = (A \cap B) \cup (A \cap B^c), \\ (A \cap B) \cap (A \cap B^c) = \emptyset$$

Prediction Markets

How do prediction markets work?

THIS CERTIFICATE ENTITLES THE BEARER TO \$1 IF TRUMP WINS THE 2016 US PRESIDENTIAL ELECTION.

Buyers – Purchase Right to Collect

Trump very likely to win \Rightarrow buy for close to \$1.

Trump very unlikely to win \Rightarrow buy for close to \$0.

Sellers – Sell Obligation to Pay

Trump very likely to win \Rightarrow sell for close to \$1.

Trump very unlikely to win \Rightarrow sell for close to \$0.

Probabilities from Prediction Markets

Probabilities from Beliefs

Market price of contract encodes market participants' beliefs in the form of probability:

$$\text{Price/Payout} \approx \text{Subjective Probability}$$

Statistical Arbitrage

If the probabilities implied by prediction market prices violate any of our probability rules there is a *pure arbitrage opportunity*: a way make to make a guaranteed, risk-free profit.

A Simple Example of Statistical Arbitrage

November 5th, 2012

- \$2.30 for contract paying \$10 if Romney wins on BetFair
- \$6.58 for contract paying \$10 if Obama wins on InTrade

Implied Probabilities

- BetFair: $P(\text{Romney}) \approx 0.23$
- InTrade: $P(\text{Obama}) \approx 0.66$

What's Wrong with This?

Violates complement rule! $P(\text{Obama}) = 1 - P(\text{Romney})$ but the implied probabilities here don't sum up to one!

A Simple Example of Statistical Arbitrage

November 5th, 2012

- \$2.30 for contract paying \$10 if Romney wins on BetFair
- \$6.58 for contract paying \$10 if Obama wins on InTrade

Arbitrage Strategy

Buy Equal Numbers of Each

- Cost = $\$2.30 + \$6.58 = \$8.88$ per pair
- Payout if Romney Wins: \$10
- Payout if Obama Wins: \$10
- Guaranteed Profit: $\$10 - \$8.88 = \$1.12$ per pair

The Lie Detector

The Lie Detector Problem

From accounting records, we know that 10% of employees in the store are stealing merchandise.

The managers want to fire the thieves, but their only tool in distinguishing is a lie detector test that is 80% accurate:

Innocent \Rightarrow Pass test with 80% Probability

Thief \Rightarrow Fail test with 80% Probability

What is the probability that someone is a thief *given* that she has failed the lie detector test?

Monte Carlo Simulation

Managers will split up and visit employees. Employees roll the die twice **but keep the results secret!**

First Roll – Thief or not?

0 \Rightarrow Thief, 1 – 9 \Rightarrow Innocent

Second Roll – Lie Detector Test

0, 1 \Rightarrow Incorrect Test Result, 2 – 9 Correct Test Result

	0 or 1	2–9
Thief (0)	Pass	Fail
Innocent (> 0)	Fail	Pass

What percentage of those who failed the test are guilty?

Who Failed Lie Detector Test:

Of Thieves Among Those Who Failed:

Base Rate Fallacy – Failure to Consider Prior Information

Base Rate – Prior Information

Before the test we know that 10% of Employees are stealing.

People tend to focus on the fact that the test is 80% accurate and ignore the fact that only 10% of the employees are thieves.

Thief (Y/N), Lie Detector (P/F)

	0	1	2	3	4	5	6	7	8	9
0	YP	YP	YF	YF	YF	YF	YF	YF	YF	YF
1	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
2	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
3	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
4	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
5	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
6	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
7	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
8	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP
9	NF	NF	NP	NP	NP	NP	NP	NP	NP	NP

Table 1: Each outcome in the table is equally likely. The 26 given in red correspond to failing the test, but only 8 of these (YF) correspond to being a thief.

Base Rate of Thievery is 10%

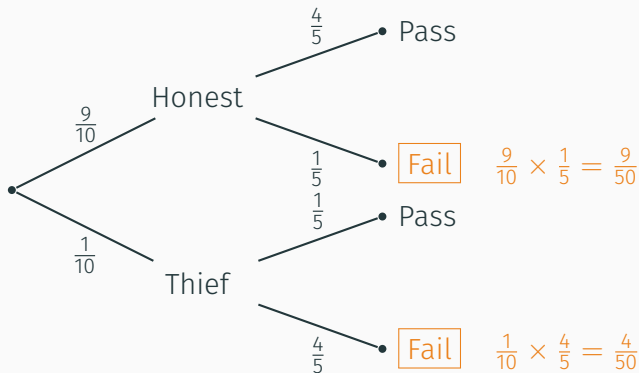


Figure 11: Although $\frac{9}{50} + \frac{4}{50} = \frac{13}{50}$ fail the test, only $\frac{4/50}{13/50} = \frac{4}{13} \approx 0.31$ are actually thieves!

Bayes' Rule

Deriving Bayes' Rule

Intersection is symmetric: $A \cap B = B \cap A$ so $P(A \cap B) = P(B \cap A)$

By the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

And by the multiplication rule:

$$P(B \cap A) = P(B|A)P(A)$$

Finally, combining these

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Understanding Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Reversing the Conditioning

Express $P(A|B)$ in terms of $P(B|A)$. *Relative magnitudes* of the two conditional probabilities determined by the ratio $P(A)/P(B)$.

Base Rate

$P(A)$ is called the “base rate” or the “prior probability.”

Denominator

Typically, we calculate $P(B)$ using the law of total probability

Solving the Lie Detector Problem with Bayes' Rule

T = Employee is a Thief, F = Employee Fails Lie Detector Test

$$P(T|F) = \frac{P(F|T)P(T)}{P(F)}$$

$$\begin{aligned} P(F) &= P(F|T)P(T) + P(F|T^c)P(T^c) \\ &= 0.8 \times 0.1 + 0.2 \times 0.9 \\ &= 0.08 + 0.18 = 0.26 \end{aligned}$$

$$P(T|F) = \frac{0.08}{0.26} = \frac{8}{26} = \frac{4}{13} \approx 0.31$$

“Odd” Question # 5

There are two kinds of taxis: green cabs and blue cabs. Of all the cabs on the road, 85% are green cabs. On a misty winter night a taxi sideswiped another car and drove off. A witness says it was a blue cab. The witness is tested under conditions like those on the night of the accident, and 80% of the time she correctly reports the color of the cab that is seen. That is, regardless of whether she is shown a blue or a green cab in misty evening light, she gets the color right 80% of the time.

Given that the witness said she saw a blue cab, what is the probability that a blue cab was the sideswiper?

Solving The Taxi Problem

G = Taxi is Green, $P(G) = 0.85$

B = Taxi is Blue, $P(B) = 0.15$

W_B = Witness says Taxi is Blue, $P(W_B|B) = 0.8$, $P(W_B|G) = 0.2$

$$P(B|W_B) = P(W_B|B)P(B)/P(W_B)$$

$$\begin{aligned}P(W_B) &= P(W_B|B)P(B) + P(W_B|G)P(G) \\&= 0.8 \times 0.15 + 0.2 \times 0.85 \\&= 0.12 + 0.17 = 0.29\end{aligned}$$

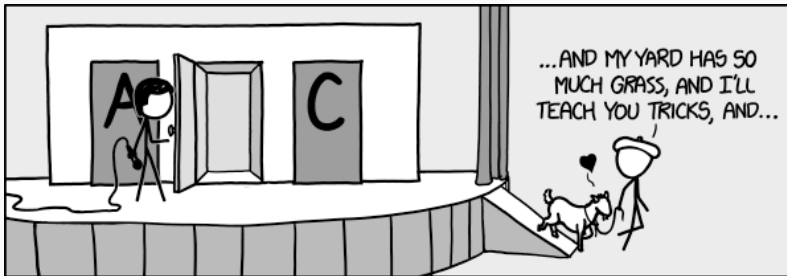
$$P(B|W_B) = 0.12/0.29 = 12/29 \approx 0.41$$

$$P(G|W_B) = 1 - (12/29) \approx 0.59$$

The Monty Hall Problem!



What is the probability that you win if you switch?



Key Point – Monte doesn't choose a door randomly: he *always* shows you a goat.

Without loss of generality, suppose you chose door #1

