Problem Set #6

Econ 103

Lecture Progress

We made it to the end of the Chapter 5 slides.

Homework Checklist

- ☐ **Book Problems (Chapter 4):** 19, 21, 25, 27, 29 (*When necessary, use R instead of the Normal tables in the textbook*)
- ☐ **Additional Problems:** See below
- ☐ **R Tutorial:** R Tutorial 5
- ☐ **Ask questions on Piazza**
- ☐ **Review slides**
- ☐ **Work on R Project!**

Additional Problems

1. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Graph the PDF of X.

Solution: It's an isosceles triangle with base from (0,0) to (2,0) and height 1.

(b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\, dx &= \int_0^1 x\, dx + \int_1^2 (2-x)\, dx = \frac{x^2}{2}\bigg|_0^1 + \left(2x - \frac{x^2}{2}\right)\bigg|_1^2 \\ &= 1/2 + (4-2) - (2-1/2) = 1\end{aligned}$$

(c) What is $P(0.5 < X < 1.5)$?

Solution:

$$\begin{aligned}P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x)\, dx = \int_{0.5}^1 x\, dx + \int_1^{1.5} (2-x)\, dx \\ &= \frac{x^2}{2}\bigg|_{0.5}^1 + \left(2x - \frac{x^2}{2}\right)\bigg|_1^{1.5} \\ &= (1/2 - 1/8) + (3 - 9/8) - (2 - 1/2) \\ &= 3/8 + 15/8 - 2 + 1/2 = 18/8 - 16/8 + 4/8 \\ &= 6/8 = 3/4 = 0.75\end{aligned}$$

2. Let X be a random variable with the following PDF

$$f(x) = \begin{cases} cx^2 & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is c?

Solution: Remember that $\int_{-\infty}^{\infty} f(x) = 1$. Hence, c must be such that

$$\int_{-\infty}^{\infty} f(x) = 1$$
$$\int_{-1}^{1} cx^2 = 1$$
$$\frac{cx^3}{3}\Big|_{-1}^{1} = 1$$
$$\frac{2c}{3} = 1$$
$$\Rightarrow c = \frac{3}{2}$$

(b) Find $E[X]$ and $Var(X)$

Solution: We can calculate the expected value as follows:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x) \\ &= \int_{-1}^{1} x * \frac{3}{2}x^2 \\ &= \frac{3x^4}{8}\Big|_{-1}^{1} \\ &= 0 \end{aligned}$$

To get variance, we will use the shortcut formula, so we need to find $E[X^2]$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x) \\ &= \int_{-1}^{1} x^2 * \frac{3}{2}x^2 \\ &= \frac{3x^5}{10}\Big|_{-1}^{1} \\ &= \frac{3}{5} \end{aligned}$$

Hence, since $Var(X) = E[X^2] - (E[X])^2 = \frac{3}{5} - 0^2 = \frac{3}{5}$

(c) Find $P(X \geq \frac{1}{2})$

Solution: We can rewrite this as

$$\begin{aligned}
P\left(X \geq \frac{1}{2}\right) &= 1 - P\left(X \leq \frac{1}{2}\right) \\
&= 1 - \int_{-1}^{\frac{1}{2}} \frac{3}{2}x^2 \\
&= 1 - \frac{x^3}{2}\bigg|_{-1}^{1/2} \\
&= \frac{7}{16}
\end{aligned}$$

Alternatively, you could do the following:

$$\begin{aligned}
P\left(X \geq \frac{1}{2}\right) &= \int_{1/2}^{1} \frac{3}{2}x^2 dx \\
&= \frac{7}{16}
\end{aligned}$$

3. A random variable is said to follow a Uniform(a, b) distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim \text{Uniform}(a, b)$.

(a) What is the support of X?

Solution: $[a, b]$

(b) Explain why the PDF of X is $f(x) = 1/(b - a)$ for $a \leq x \leq b$, zero elsewhere.

Solution: This simply generalizes the Uniform$(0, 1)$ random variable from class. To capture the idea that X is equally likely to take on any value in the range $[a, b]$, the PDF must be constant. To ensure that it integrates to 1, the denominator must be $b - a$.

(c) Using the PDF from part (b), calculate the CDF of X.

Solution:

$$F(x_0) \quad = \quad \int_{-\infty}^{x_0} f(x)\, dx = \int_{a}^{x_0} \frac{dx}{b-a} = \frac{x}{b-a}\bigg|_{a}^{x_0} = \frac{x_0 - a}{b - a}$$

(d) Verify that $f(x) = F'(x)$ for the present example.

Solution:

$$F'(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a} = f(x)$$

(e) Calculate $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

(f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2+a^2+ab)$.

Solution:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a} = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3-a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3} \end{aligned}$$

(g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

Solution:

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 = \frac{b^2+a^2+ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2+a^2+ab}{3} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12} \\ &= \frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

4. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.

   For each of the following, use the transformation to a standard normal in your computations, but also include the R commands you would use for an arbitrary normal variable.

   (a) Calculate $P(-8 \leq X \leq 8)$.

   Solution:

   $$P(-8 \leq X \leq 8) = P(-8/4 \leq X/4 \leq 8/4) = P(-2 \leq Z \leq 2) \approx 0.95$$

   where Z is a standard normal random variable. Using R, you would enter `pnorm(8, mean = 0, sd = 4) - pnorm(-8, mean = 0, sd = 4)` ≈ 0.95

   (b) Calculate $P(0 \leq Y \leq 4)$.

   Solution:

   $$P(0 \leq Y \leq 4) = P\left(\frac{0-2}{2} \leq \frac{Y-2}{2} \leq \frac{4-2}{2}\right) = P(-1 \leq Z \leq 1) \approx 0.68$$

   where Z is a standard normal random variable. Using R, you would enter `pnorm(4, mean = 2, sd = 2) - pnorm(0, mean = 2, sd = 2)` ≈ 0.68

   (c) Calculate $P(-1 \leq Y \leq 6)$.

   Solution:

   $$\begin{aligned} P(-1 \leq Y \leq 6) &= P\left(\frac{-1-2}{2} \leq \frac{Y-2}{2} \leq \frac{6-2}{2}\right) \\ &= P(-1.5 \leq Z \leq 2) \\ &= \Phi(2) - \Phi(-1.5) \\ &= \texttt{pnorm(2) - pnorm(-1.5)} \\ &\approx 0.91 \end{aligned}$$

   where Z is a standard normal random variable. Using R, you would enter `pnorm(6, mean = 2, sd = 2) - pnorm(-1, mean = 2, sd = 2)` ≈ 0.91

(d) Calculate $P(X \geq 10)$.

Solution:

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) = 1 - P(X/4 \leq 10/4) = 1 - P(Z \leq 2.5) \\ &= 1 - \Phi(2.5) = 1 - \texttt{pnorm(2.5)} \\ &\approx 0.006 \end{aligned}$$

Using R, you would enter `1 - pnorm(10, mean = 0, sd = 4)` ≈ 0.006

Note: *In the following five questions* $X_1, X_2 \sim$ *iid* $N(\mu, \sigma^2)$, $Y = (X_1 - \mu)/\sigma$, $Z = (X_2 - \mu)/\sigma$.

5. (a) What is the distribution of $X_1 + X_2$?

Solution: $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$

(b) Use R to calculate $P(X_1 + X_2 > 5)$ if $\mu = 5$ and $\sigma^2 = 50$.

Solution: In this case, $X_1 + X_2 \sim N(10, 100)$, hence

$$\begin{aligned} P(X_1 + X_2 > 5) &= 1 - P(X_1 + X_2 \leq 5) \\ &= 1 - P\left(\frac{X_1 + X_2 - 10}{10} \leq \frac{5 - 10}{10}\right) \\ &= 1 - \texttt{pnorm(-0.5)} \\ &\approx 0.6914625 \end{aligned}$$

Alternatively, we could use `1- pnorm(5, mean = 10, sd = 10)`, which gives the same result.

(c) Use R to calculate the 10th percentile of the distribution of $X_1 + X_2$.

Solution: `qnorm(p = 0.1, mean = 10, sd = 10)` gives -2.815516.

6. (a) What is the distribution of Y^2?

Solution: As the sum of squares of one standard normal RV, $Y^2 \sim \chi^2(1)$.

(b) Use R to calculate $P(Y^2 \geq 1)$.

Solution:

$$P(Y^2 \geq 1) = 1 - P(Y^2 \leq 1) = 1 - \texttt{pchisq(1, df = 1)} \approx 0.3173105$$

7. (a) What is the distribution of $Y^2 + Z^2$?

Solution: Since this is the sum of squares of two independent standard normal random variables, $Y^2 + Z^2 \sim \chi^2(2)$.

(b) Use R to calculate the 95th percentile of the distribution of $Y^2 + Z^2$.

Solution: `qchisq(p = 0.95, df = 2)` gives 5.991465

8. (a) What is the distribution of $Z/\sqrt{Y^2}$?

Solution: Since it is the ratio of a standard normal to the square root of an independent χ^2 random variable divided by its degrees of freedom (in this case one), $Z/\sqrt{Y^2} \sim t(1)$.

(b) What value of c satisfies $P(-c \leq Z/\sqrt{Y^2} \leq c) = 0.95$?

Solution: By the symmetry of the t-distribution, it suffices to find the 97.5th percentile (this allocates 2.5% probability to the upper and lower tails). The command `qt(p = 0.975, df = 1)` gives 12.7062, so $c \approx 12.7$. Alternatively, we could have calculated the 2.5th percentile: `qt(p = 0.025, df = 1)` gives -12.7062.

(c) How does the interval in part (b) compare to the corresponding interval for Z?

Solution: Since Z is a standard normal RV, $P(-2 \leq Z \leq 2) \approx 0.95$. We see that the interval for a $t(1)$ RV is *much wider* than the corresponding interval for a standard normal. In other words, extreme outcomes are much more likely under the $t(1)$ distribution.

9. (a) What is the distribution of Y^2/Z^2?

Solution: This is the ratio of two independent χ^2 random variables, each divided by its degrees of freedom (in this case, one). Hence $Y^2/Z^2 \sim F(1, 1)$.

(b) Use R to calculate the 95th percentile of the distribution of Y^2/Z^2.

Solution: `qf(p = 0.95, df1 = 1, df2 = 1)` gives 161.4476