Econ 103 – Statistics for Economists

Chapter 4: Probability Distributions

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Random Variables

Random Variables

A random variable is neither random nor a variable.

Random Variable (RV): X

A *fixed* function that assigns a *number* to each basic outcome of a random experiment.

Realization: x

A particular numeric value that an RV could take on. We write $\{X = x\}$ to refer to the *event* that the RV X took on the value x.

Support Set (aka Support)

The set of all possible realizations of a RV.

Random Variables (continued)

Notation

Capital letters for RVs, e.g. X, Y, Z, and the corresponding lowercase letters for their realizations, e.g. x, y, z.

Intuition

You can think of an RV as a machine that spits out random numbers: although the machine is deterministic, its inputs, the outcomes of a random experiment, are not.

Example: Coin Flip Random Variable

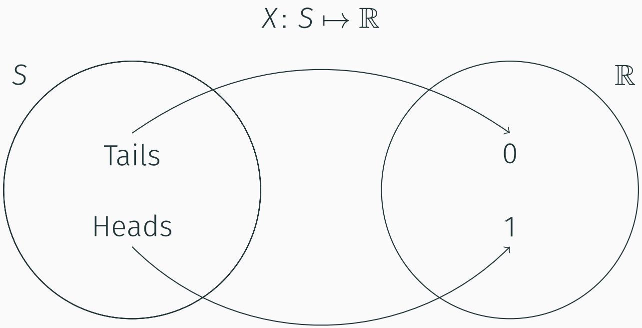


Figure 1: This random variable assigns numeric values to the random experiment of flipping a fair coin once: Heads is assigned 1 and Tails 0.

Which of these is a realization of the Coin Flip RV?

(a) Tails
(b) 2
(c) 0
(d) Heads
(e) 1/2

What is the support set of the Coin Flip RV?

(a) $\{\text{Heads}, \text{Tails}\}$

(b) $1/2$

(c) 0

(d) $\{0, 1\}$

(e) 1

Let X denote the Coin Flip RV

What is $P(X = 1)$?

(a) 0
(b) 1
(c) 1/2
(d) Not enough information to determine

Two Kinds of RVs: Discrete and Continuous

Discrete support set is discrete, e.g. $\{0, 1, 2\}$, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Continuous support set is continuous, e.g. $[-1, 1]$, $\mathbb{R}$.

We will start with the discrete case since it's easier, but most of the ideas we learn will carry over to the continuous case.

Discrete Random Variables

Probability Mass Function (pmf)

A function that gives $P(X = x)$ for any realization x in the support set of a discrete RV X. We use the following notation for the pmf:

$$p(x) = P(X = x)$$

Plug in a realization x, get out a probability $p(x)$.

Probability Mass Function for Coin Flip RV

$$X = \begin{cases} 0, \text{Tails} \\ 1, \text{Heads} \end{cases}$$

$$\begin{aligned} p(0) &= 1/2 \\ p(1) &= 1/2 \end{aligned}$$

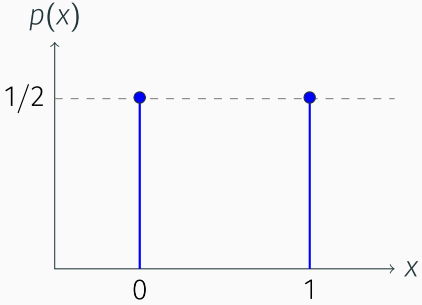


Figure 2: Plot of pmf for Coin Flip Random Variable

Important Note about Support Sets

Whenever you write down the pmf of a RV, it is crucial to also write down its Support Set. Recall that this is the set of *all possible realizations for a RV*. Outside of the support set, all probabilities are zero. In other words, the pmf is only defined on the support.

Properties of Probability Mass Functions

If $p(x)$ is the pmf of a random variable X, then

(i) $0 \leq p(x) \leq 1$ for all x

(ii) $\sum_{\text{all } x} p(x) = 1$

where "all x" is shorthand for "all x in the support of X."

Cumulative Distribution Function (CDF)

The CDF gives the probability that a RV X does not exceed a specified threshold x_0, as a function of x_0

$$F(x_0) = P(X \leq x_0)$$

Important!

The threshold x_0 is allowed to be *any real number*. In particular, it doesn't have to be in the support of X!

Discrete RVs: Sum the pmf to get the CDF

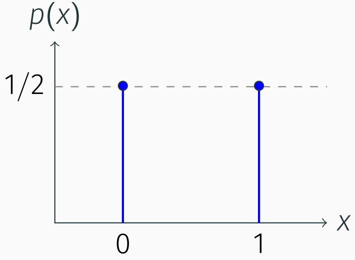
$$\boxed{F(x_0) = \sum_{x \leq x_0} p(x)}$$

Why?

The events $\{X = x\}$ are mutually exclusive, so we sum to get the probability of their union for all $x \leq x_0$:

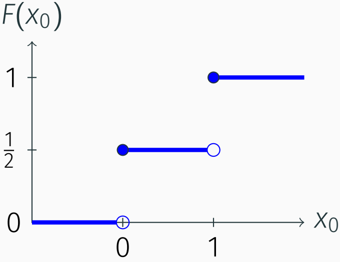
$$F(x_0) = P(X \leq x_0) = P\left(\bigcup_{x \leq x_0} \{X = x\}\right) = \sum_{x \leq x_0} P(X = x) = \sum_{x \leq x_0} p(x)$$

Probability Mass Function



$$
\begin{aligned}
p(0) &= 1/2 \\
p(1) &= 1/2
\end{aligned}
$$

Cumulative Dist. Function



$$
F(x_0) = \begin{cases} 0, & x_0 < 0 \\ \frac{1}{2}, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}
$$

Properties of CDFs

1. $\lim_{x_0 \to \infty} F(x_0) = 1$
2. $\lim_{x_0 \to -\infty} F(x_0) = 0$
3. Non-decreasing: $x_0 < x_1 \Rightarrow F(x_0) \leq F(x_1)$
4. Right-continuous ("open" versus "closed" on prev. slide)

Since $F(x_0) = P(X \leq x_0)$, we have $0 \leq F(x_0) \leq 1$ for all x_0

Bernoulli Random Variable – Generalization of Coin Flip

Support Set

$\{0, 1\}$ – 1 traditionally called "success," 0 "failure"

Probability Mass Function

$$
\begin{aligned}
p(0) &= 1 - p \\
p(1) &= p
\end{aligned}
$$

Cumulative Distribution Function

$$
F(x_0) = \begin{cases} 0, & x_0 < 0 \\ 1 - p, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}
$$

Average Winnings Per Trial

If the realizations of the coin-flip RV were payoffs, how much would you expect to win per play *on average* in a long sequence of plays?

$$X = \begin{cases} \$0, \text{Tails} \\ \$1, \text{Heads} \end{cases}$$

Expected Value (aka Expectation)

The expected value of a discrete RV X is given by

$$E[X] = \sum_{\text{all } x} x \cdot p(x)$$

In other words, the expected value of a discrete RV is the *probability-weighted average of its realizations.*

Notation

We sometimes write μ as shorthand for $E[X]$.

Expected Value of Bernoulli RV

$$X = \begin{cases} 0, \text{Failure: } 1 - p \\ 1, \text{Success: } p \end{cases}$$

$$\sum_{\text{all } x} x \cdot p(x) = 0 \cdot (1 - p) + 1 \cdot p = p$$

Your Turn to Caculate an Expected Value

Let X be a random variable with support set $\{1, 2, 3\}$ where $p(1) = p(2) = 1/3$. Calculate $E[X]$.

$$E[X] = \sum_{\text{all } x} x \cdot p(x) = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$$

Random Variables and Parameters

Notation: $X \sim \text{Bernoulli}(p)$

Means X is a Bernoulli RV with $P(X = 1) = p$ and $P(X = 0) = 1 - p$. The tilde is read "distributes as."

Parameter

Any constant that appears in the definition of a RV, here p.

Constants Versus Random Variables

This is a crucial distinction that students sometimes miss:

Random Variables

- Suppose X is a RV – the values it takes on are random
- A function $g(X)$ of a RV is itself a RV as we'll learn today.

Constants

- $E[X]$ is a constant (you should convince yourself of this)
- Realizations x are constants. What is random is *which* realization the RV takes on.
- Parameters are constants (e.g. p for Bernoulli RV)
- Sample size n is a constant

The St. Petersburg Game

How much would you be willing to pay for the right to play the following game?

> *Imagine a fair coin. The coin is tossed once. If it falls heads, you receive a prize of \$2 and the game stops. If not, it is tossed again. If it falls heads on the second toss, you get \$4 and the game stops. If not, it is tossed again. If it falls heads on the third toss, you get \$8 and the game stops, and so on. The game stops after the first head is thrown. If the first head is thrown on the* x^{th} *toss, the prize is* $\$2^x$

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \ldots = \infty$$

Functions of Random Variables are Themselves Random Variables

Example: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Support of Y

$\{e^0, e^1\} = \{1, e\}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation of $Y = e^X$

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Expectation of the Function

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

Function of the Expectation

$$e^{E[X]} = e^p$$

$$E[g(X)] \neq g(E[X])$$

(Expected value of Function $\neq$ Function of Expected Value)

Expectation of a Function of a Discrete RV

Let X be a random variable and g be a function. Then:

$$\boxed{E[g(X)] = \sum_{\text{all } x} g(x)p(x)}$$

This is how we proceeded in the St. Petersburg Game Example

Your Turn: Calculate $E[X^2]$

X has support $\{-1, 0, 1\}$, $p(-1) = p(0) = p(1) = 1/3$.

$$
\begin{aligned}
E[X^2] &= \sum_{\text{all } x} x^2 p(x) = \sum_{x \in \{-1,0,1\}} x^2 p(x) \\
&= (-1)^2 \cdot (1/3) + (0)^2 \cdot (1/3) + (1)^2 \cdot (1/3) \\
&= 1/3 + 1/3 \\
&= 2/3 \approx 0.67
\end{aligned}
$$

Linearity of Expectation

Let X be a RV and a, b be constants. Then:

$$E[a + bX] = a + bE[X]$$

This is one of the most important facts in the course: the special case in which $E[g(X)] = g(E[X])$ is $g = a + bX$.

Example: Linearity of Expectation

Let $X \sim \text{Bernoulli}(1/3)$ and define $Y = 3X + 2$

1. What is $E[X]$? $E[X] = 0 \times 2/3 + 1 \times 1/3 = 1/3$
2. What is $E[Y]$? $E[Y] = E[3X + 2] = 3E[X] + 2 = 3$

Proof: Linearity of Expectation For Discrete RV

$$\begin{aligned}
E[a+bX] &= \sum_{\text{all } x}(a+bx)p(x) \\
&= \sum_{\text{all } x} p(x)\cdot a + \sum_{\text{all } x} p(x)\cdot bx \\
&= a\sum_{\text{all } x} p(x) + b\sum_{\text{all } x} x\cdot p(x) \\
&= a + bE[X]
\end{aligned}$$