Econ 103 – Statistics for Economists

Chapter 9: Regression

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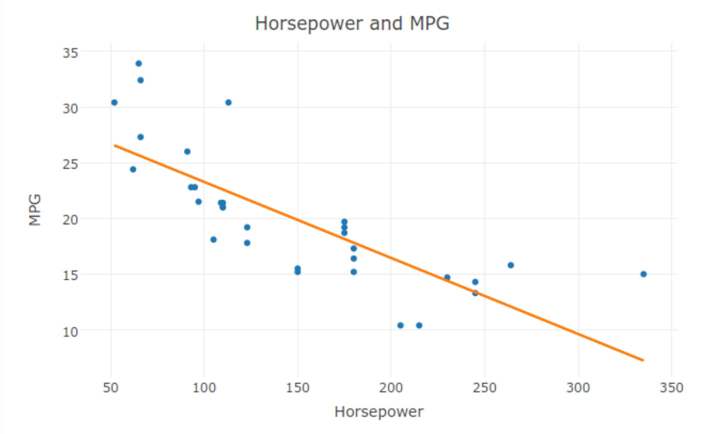
Recall: “Best Fitting” Line Through Cloud of Points



Recall: Regression as a Data Summary

Linear Model

$\hat{y} = a + bx$

Choose a, b to Minimize Sum of Squared Vertical Deviations

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

The Prediction

Predict a car's MPG given its horsepower

Recall: Regression as a Data Summary

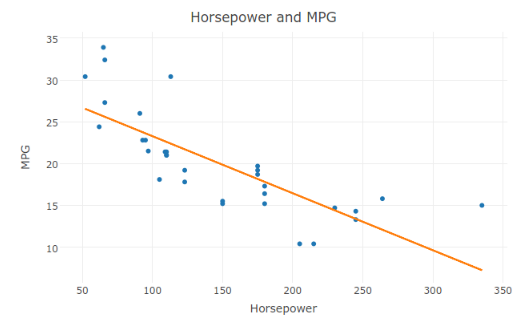
Problem

$$\min_{a,b} \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

Solution

$$\begin{aligned} b &= \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2} = r\frac{s_y}{s_x} \\ a &= \bar{y} - b\bar{x} \end{aligned}$$

Beyond Regression as a Data Summary



The estimated slope was about -0.07 hp/mpg and the estimated intercept was about 30 hp.

What if anything does this tell us about the relationship between horsepower and mpg *in the population*?

How is Y (mpg) related to X (horsepower) in the population?

Assumption I: Linearity

The random variable Y is linearly related to X according to

$$Y = \beta_0 + \beta_1 X + \epsilon$$

β_0, β_1 are two unknown population parameters (constants).

Assumption II: Error Term ϵ

$E[\epsilon] = 0$, $Var(\epsilon) = \sigma^2$ and ϵ is independent of X. The error term ϵ measures the unpredictability of Y *after controlling for* X

Predictive Interpretation of Regression

Under Assumptions I and II

$$E[Y|X] = \beta_0 + \beta_1 X$$

- "Best guess" of Y having observed $X = x$ is $\beta_0 + \beta_1 x$
- If $X = 0$, we predict $Y = \beta_0$
- If two people differ by one unit in X, we predict that they will differ by β_1 units in Y.

The only problem is, we don't know β_0, β_1...

Estimating β_0, β_1

Suppose we observe an iid sample $(Y_1, X_1), \ldots, (Y_n, X_n)$ from the population: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$. Then we can *estimate* β_0, β_1:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

$$\widehat{\beta}_0 = \bar{Y}_n - \widehat{\beta}_1 \bar{X}_n$$

Once we have estimators, we can think about sampling uncertainty...

Sampling Uncertainty: Pretend the Cars in Our Data is our Population

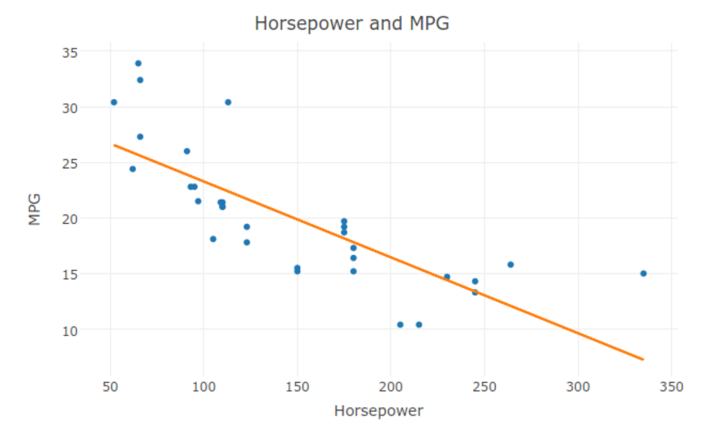
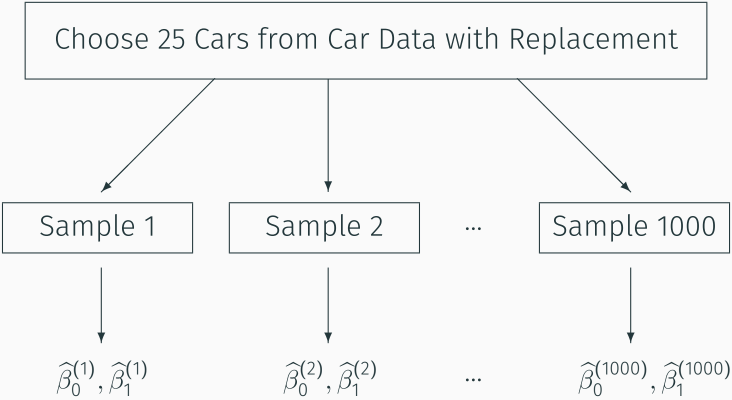


Figure 1: Estimated Slope = -0.07, Estimated Intercept = 30

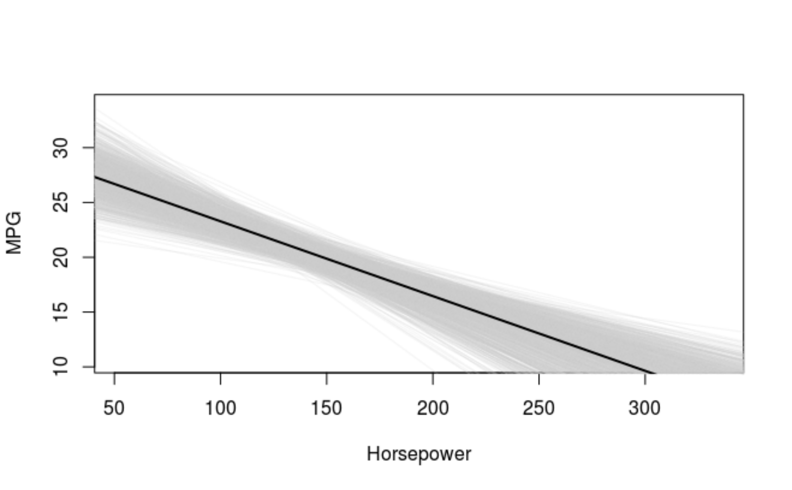
Sampling Distribution of Regression Coefficients $\widehat{\beta}_0$ and $\widehat{\beta}_1$



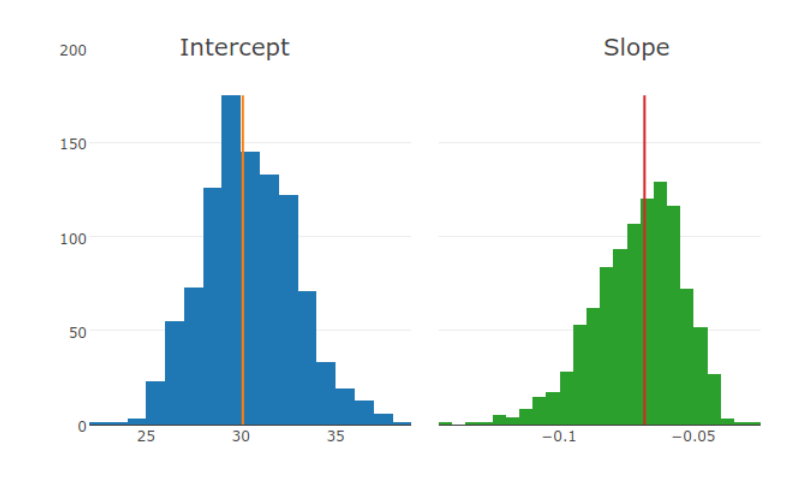
Repeat 1000 times $\rightarrow$ get 1000 different pairs of estimates

Sampling Distribution: long-run relative frequencies

1000 Replications, $n = 25$



Population: Intercept = 30, Slope = -0.07



Based on 1000 Replications, $n = 25$

Inference for Linear Regression

Central Limit Theorem

$$\frac{\widehat{\beta} - \beta}{\widehat{SE}(\widehat{\beta})} \approx N(0, 1)$$

How to calculate $\widehat{SE}$?

- Complicated
  - Depends on variance of errors ϵ and all predictors in regression.
  - We'll look at a few simple examples
  - R does this calculation for us
- Requires assumptions about population errors ϵ_i
  - Simplest (and R default) is to assume $\epsilon_i \sim iid(0, \sigma^2)$
  - Weaker assumptions in Econ 104

Intuition for What Affects $SE(\widehat{\beta}_1)$ for Simple Regression

$$SE(\widehat{\beta}_1) \approx \frac{\sigma}{\sqrt{n}} \cdot \frac{1}{s_X}$$

- $\sigma = SD(\epsilon)$ – inherent variability of the Y, even after controlling for X
- n is the sample size
- s_X is the sampling variability of the X observations.

Cars Data

MPG = $\beta_0 + \epsilon$

```
lm(formula = mpg ~ 1, data = mtcars)
            coef.est coef.se
(Intercept) 20.09     1.07
---
n = 32, k = 1
```

MPG = $\beta_0 + \epsilon$

```
lm(formula = mpg ~ 1, data = mtcars)
            coef.est coef.se
(Intercept) 20.09    1.07
---
n = 32, k = 1
```

```
> mean(mtcars$mpg)
[1] 20.09062
```

MPG = $\beta_0 + \epsilon$

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lm(formula = mpg ~ 1, data = mtcars)
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(Intercept) 20.09     1.07
---
n = 32, k = 1

> mean(mtcars$mpg)
[1] 20.09062

> sd(mtcars$mpg)/sqrt(length(mtcars$mpg))
[1] 1.065424
```

Dummy Variable (aka Binary Variable)

A predictor variable that takes on only two values: 0 or 1. Used to represent two categories, e.g. Automatic/Manual.

MPG = $\beta_0 + \beta_1$ Manual $+\epsilon$

```
lm(formula = mpg ~ am, data = mtcars)
            coef.est coef.se
(Intercept) 17.15     1.12
am           7.24     1.76
---
n = 32, k = 2
residual sd = 4.90, R-Squared = 0.36
```

MPG = $\beta_0 + \beta_1$ Manual $+\epsilon$

```
lm(formula = mpg ~ am, data = mtcars)
            coef.est coef.se
(Intercept) 17.15     1.12
am           7.24     1.76
---
n = 32, k = 2
residual sd = 4.90, R-Squared = 0.36

> mean(manual$mpg) - mean(automatic$mpg)
[1] 7.244939
```

MPG = $\beta_0 + \beta_1$ Manual $+\epsilon$

```
lm(formula = mpg ~ am, data = mtcars)
            coef.est coef.se
(Intercept) 17.15     1.12
am           7.24     1.76
---
n = 32, k = 2
residual sd = 4.90, R-Squared = 0.36

> mean(manual$mpg) - mean(automatic$mpg)
[1] 7.244939

> sqrt(var(manual$mpg)/length(manual$mpg) +
     var(automatic$mpg)/length(automatic$mpg))
[1] 1.923202
```

MPG = $\beta_0 + \beta_1$ Manual $+\epsilon$

What is the ME for an approximate 95% confidence interval for the difference of population means of transmission: (automatic - manual)?

```
lm(formula = mpg ~ am, data = mtcars)
            coef.est coef.se
(Intercept) 17.15     1.12
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n = 32, k = 2
residual sd = 4.90, R-Squared = 0.36
```

$$7.24 \pm 2 * 1.76 = [3.72, 10.76]$$

MPG = $\beta_0 + \beta_1$ Horsepower $+\epsilon$

```
lm(formula = mpg ~ hp, data = mtcars)
            coef.est coef.se
(Intercept) 30.10     1.63
hp          -0.07     0.01
---
n = 32, k = 2
residual sd = 3.86, R-Squared = 0.60
```

MPG = $\beta_0 + \beta_1$ Horsepower $+\epsilon$

What is the ME for an approximate 95% CI for β_1?

```
lm(formula = mpg ~ hp, data = mtcars)
            coef.est coef.se
(Intercept) 30.10     1.63
hp          -0.07     0.01
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n = 32, k = 2
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MPG = $\beta_0 + \beta_1$ Horsepower $+\epsilon$

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lm(formula = mpg ~ hp, data = mtcars)
            coef.est coef.se
(Intercept) 30.10     1.63
hp          -0.07     0.01
---
n = 32, k = 2
residual sd = 3.86, R-Squared = 0.60
```

$$-0.07 \pm 2 * 0.01 = [-0.09, -0.05]$$

Terminology

Y is the "outcome" and X is the "predictor."

Simple Regression

One predictor variable: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$

Multiple Regression

More than one predictor variable:

$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$

- In both cases $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \sim \text{iid}(0, \sigma^2)$
- Multiple regression coefficient estimates $\widehat{\beta}_1, \widehat{\beta}_1, \ldots, \widehat{\beta}_k$ calculated by minimizing sum of squared vertical deviations, but formula requires linear algebra so we won't cover it.

Predictive Interpretation

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$$

β_j is the difference in Y that we would predict between two individuals who differed by one unit in predictor X_j *but who had the same values for the other X variables.*

What About an Example?

In a few minutes, we'll work through an extended example of multiple regression using real data.

In addition to estimating the coefficients $\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_k$ for us, R will calculate the corresponding standard errors. It turns out that

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}(\widehat{\beta})} \approx N(0,1)$$

for *each* of the $\widehat{\beta}_j$ by the CLT provided that the sample size is large.

MPG = $\beta_0 + \beta_1$ Horsepower $+\epsilon$

What are `residual sd` and `R-squared`?

```
lm(formula = mpg ~ hp, data = mtcars)
            coef.est coef.se
(Intercept) 30.10    1.63
hp          -0.07    0.01
---
n = 32, k = 2
residual sd = 3.86, R-Squared = 0.60
```

Fitted Values and Residuals

Fitted Value $\widehat{y}_i$

Predicted y-value for a car i given its x-variables using estimated regression coefficients: $\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_k x_{ik}$

Residual $\widehat{\epsilon}_i$

Car i's *vertical deviation* from regression line: $\widehat{\epsilon}_i = y_i - \widehat{y}_i$.

The residuals are *stand-ins* for the unobserved errors ϵ_i.

Residual Standard Deviation: $\widehat{\sigma}$

- Idea: use residuals $\widehat{\epsilon}_i$ to estimate σ

$$\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} \widehat{\epsilon}_i^2}{n-k}}$$

- Measures avg. distance of y_i from regression line.
    - E.g. if Y is points scored on a test and $\widehat{\sigma} = 16$, the regression predicts to an accuracy of about 16 points.
- Same units as Y (Exam practice: verify this)
- Denominator $(n - k)$ = (# Datapoints - # of X variables)

Proportion of Variance Explained: R^2

$$R^2 \approx 1 - \frac{\widehat{\sigma^2}}{s_y^2}$$

- R^2 = proportion of $Var(Y)$ "explained" by the regression.
  - Higher value $\implies$ greater proportion explained
- Unitless, between 0 and 1
- Generally harder to interpret than $\widehat{\sigma}$, but...
- For simple linear regression $R^2 = (r_{xy})^2$ and this where its name comes from!

MPG = $\beta_0 + \beta_1$ Horsepower $+\epsilon$

```
lm(formula = mpg ~ hp, data = mtcars)
            coef.est coef.se
(Intercept) 30.10     1.63
hp          -0.07     0.01
---
n = 32, k = 2
residual sd = 3.86, R-Squared = 0.60
> cor(mtcars$mpg, mtcars$hp)^2
[1] 0.6024373
```

Which Gives Better Predictions: (a) Transmission or (b) Horsepower?

```
lm(formula = mpg ~ am, data = mtcars)
            coef.est coef.se
(Intercept) 17.15     1.12
am           7.24     1.76
---
n = 32, k = 2
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