Problem Set #6

Econ 103

Part I – Problems from the Textbook

Chapter 4: 19, 21, 23 (*When necessary, use R rather than the Normal tables in the front of the textbook.*) Chapter 4: 7, 9, 11, 13, 15, 25, 27, 29
Chapter 5: 1, 3, 5, 9, 11, 13, 17

Part II – Additional Problems

1. Suppose that X is a random variable with the following PDF

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Graph the PDF of X.

   (b) Show that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

   (c) What is $P(0.5 < X < 1.5)$?

2. A random variable is said to follow a Uniform(a, b) distribution if it is equally likely to take on any value in the range $[a, b]$ and never takes a value outside this range. Suppose that X is such a random variable, i.e. $X \sim$ Uniform(a, b).

   (a) What is the support of X?

   (b) Explain why the PDF of X is $f(x) = 1/(b - a)$ for $a \leq x \leq b$, zero elsewhere.

   (c) Using the PDF from part (b), calculate the CDF of X.

   (d) Verify that $f(x) = F'(x)$ for the present example.

   (e) Calculate $E[X]$.

   (f) Calculate $E[X^2]$. *Hint:* recall that $b^3 - a^3$ can be factorized as $(b-a)(b^2 + a^2 + ab)$.

   (g) Using the shortcut formula and parts (e) and (f), show that $Var(X) = (b-a)^2/12$.

3. Suppose that $X \sim N(0, 16)$ independent of $Y \sim N(2, 4)$. Recall that our convention is to express the normal distribution in terms of its mean and variance, i.e. $N(\mu, \sigma^2)$. Hence, X has a mean of zero and variance of 16, while Y has a mean of 2 and a variance of 4. In completing some parts of this question you will need to use the R function `pnorm` described in class. In this case, please write down the command you used as well as the numeric result.
   (a) Calculate $P(-8 \leq X \leq 8)$.
   (b) Calculate $P(0 \leq Y \leq 4)$.
   (c) Calculate $P(-1 \leq Y \leq 6)$.
   (d) Calculate $P(X \geq 10)$.

Note: *In the following five questions* $X_1, X_2 \sim$ *iid* $N(\mu, \sigma^2)$, $Y = (X_1 - \mu)/\sigma$, $Z = (X_2 - \mu)/\sigma$.

4. (a) What is the distribution of $X_1 + X_2$?
   (b) Use R to calculate $P(X_1 + X_2 > 5)$ if $\mu = 5$ and $\sigma^2 = 50$.
   (c) Use R to calculate the 10th percentile of the distribution of $X_1 + X_2$.
5. (a) What is the distribution of Y^2?
   (b) Use R to calculate $P(Y^2 \geq 1)$.
6. (a) What is the distribution of $Y^2 + Z^2$?
   (b) Use R to calculate the 95th percentile of the distribution of $Y^2 + Z^2$.
7. (a) What is the distribution of $Z/\sqrt{Y^2}$?
   (b) What value of c satisfies $P(-c \leq Z/\sqrt{Y^2} \leq c) = 0.95$?
   (c) How does the interval in part (b) compare to the corresponding interval for Z?
8. (a) What is the distribution of Y^2/Z^2?
   (b) Use R to calculate the 95th percentile of the distribution of Y^2/Z^2.
9. Fill in the missing details from class to calculate the variance of a Bernoulli Random Variable *directly*, that is *without* using the shortcut formula.
10. Prove that the Bernoulli Random Variable is a special case of the Binomial Random variable for which $n = 1$. (Hint: compare pmfs.)

11. Suppose that X is a random variable with support $\{1,2\}$ and Y is a random variable with support $\{0,1\}$ where X and Y have the following joint distribution:

$$p_{XY}(1,0) = 0.20, \qquad p_{XY}(1,1) = 0.30$$
$$p_{XY}(2,0) = 0.25, \qquad p_{XY}(2,1) = 0.25$$

   (a) Express the joint distribution in a 2×2 table.
   (b) Using the table, calculate the marginal probability distributions of X and Y.
   (c) Calculate the conditional probability distribution of $Y|X = 1$ and $Y|X = 2$.
   (d) Calculate $E[Y|X]$.
   (e) What is $E[E[Y|X]]$?
   (f) Calculate the covariance between X and Y using the shortcut formula.

12. Let X and Y be discrete random variables and a, b, c, d be constants. Prove the following:
   (a) $Cov(a + bX, c + dY) = bdCov(X,Y)$
   (b) $Corr(a + bX, c + dY) = Corr(X,Y)$

13. Fill in the missing steps from lecture to prove the shortcut formula for covariance:

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

14. Let X_1 be a random variable denoting the returns of stock 1, and X_2 be a random variable denoting the returns of stock 2. Accordingly let $\mu_1 = E[X_1]$, $\mu_2 = E[X_2]$, $\sigma_1^2 = Var(X_1)$, $\sigma_2^2 = Var(X_2)$ and $\rho = Corr(X_1, X_2)$. A *portfolio*, Π, is a linear combination of X_1 and X_2 with weights that sum to one, that is $\Pi(\omega) = \omega X_1 + (1-\omega)X_2$, indicating the proportions of stock 1 and stock 2 that an investor holds. In this example, we require $\omega \in [0,1]$, so that *negative* weights are not allowed. (This rules out short-selling.)
   (a) Calculate $E[\Pi(\omega)]$ in terms of ω, μ_1 and μ_2.
   (b) If $\omega \in [0,1]$ is it possible to have $E[\Pi(\omega)] > \mu_1$ *and* $E[\Pi(\omega)] > \mu_2$? What about $E[\Pi(\omega)] < \mu_1$ and $E[\Pi(\omega)] < \mu_2$? Explain.
   (c) Express $Cov(X_1, X_2)$ in terms of ρ and σ_1, σ_2.
   (d) What is $Var[\Pi(\omega)]$? (Your answer should be in terms of ρ, σ_1^2 and σ_2^2.)
   (e) Using part (d) show that the value of ω that minimizes $Var[\Pi(\omega)]$ is

$$\omega^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

   In other words, $\Pi(\omega^*)$ is the *minimum variance portfolio.*
   (f) If you want a challenge, check the second order condition from part (e).