Econ 103 – Statistics for Economists

Chapter 4 and 5: Discrete Random Variables

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Motivation

Why Should I Care?

- These next few lectures will be a crash course in random variables and probability
- This will help build your toolbox of how to think about real-world problems using statistics.
- Economists are always concerned about the “distribution” of things. This will help you understand why this is such a critical concept.

Random Variables

Definitions

A random variable is neither random nor a variable.

- **Random Variable (RV):** A random variable X is a *fixed* function that assigns a *number* to each basic outcome of a random experiment.
- **Realization:** A realization x of a RV X is a particular numeric value that the RV could take. We write $\{X = x\}$ to denote the event that X took on the value x.
- **Support Set:** The support is the set of all possible realizations of a RV.

Note: RVs are CAPITAL LETTERS while their realizations are lowercase letters.

Definitions

- **Discrete RV:** A random variable X is *discrete* if its support set is discrete (e.g. $\{0, 1, 2\}$).
- **Continuous RV:** A random variable X is *continuous* if its support set is continuous (e.g. $[0, 1], \mathbb{R}$).

What are examples of each kind of random variable?

Example: Coin Flip Random Variable

$$X\colon S \mapsto \mathbb{R}$$

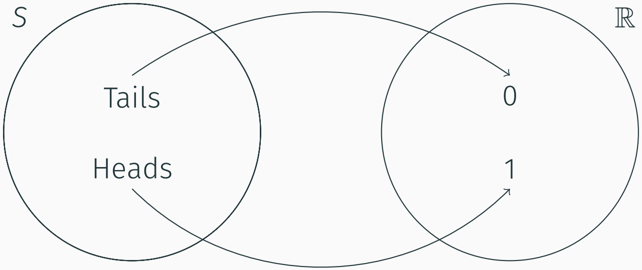


Figure 1: This random variable assigns numeric values to the random experiment of flipping a fair coin once: Heads is assigned 1 and Tails 0.

Which of these is a realization of the Coin Flip RV?

(a) Tails
(b) 2
(c) 0
(d) Heads
(e) 1/2

What is the support set of the Coin Flip RV?

(a) $\{\text{Heads}, \text{Tails}\}$

(b) $1/2$

(c) 0

(d) $\{0, 1\}$

(e) 1

Let X denote the Coin Flip RV

What is $P(X = 1)$?

(a) 0

(b) 1

(c) 1/2

(d) π

(e) Not enough information to determine

CDFs and PMFs

Probability Mass Function (PMF)

A function that gives $P(X = x)$ for any x in the support set of a discrete RV X. We use the following notation for the PMF:

$$p(x) = P(X = x)$$

Plug in a realization x, get out a probability $p(x)$.

Probability Mass Function for Coin Flip RV

$$X = \begin{cases} 0, \text{Tails} \\ 1, \text{Heads} \end{cases}$$

$$\begin{aligned} p(0) &= 1/2 \\ p(1) &= 1/2 \end{aligned}$$

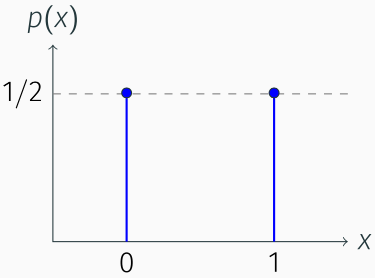


Figure 2: Plot of Coin Flip RV PMF

Important Note about Support Sets

Whenever you write down the PMF of a RV, it is crucial to also write down its support (*all possible realizations for a RV*). Outside of the support set, all probabilities are zero. In other words, the PMF is only defined on its support.

Properties of Probability Mass Functions

If $p(x)$ is the PMF of a random variable X, then

(i) $0 \leq p(x) \leq 1$ for all x

(ii) $\displaystyle\sum_{\text{all } x} p(x) = 1$

where "all x" is shorthand for "all x in the support of X."

Cumulative Distribution Function (CDF)

The CDF gives the probability that a RV X is less than or equal to some threshold x_0, as a function of x_0

$$F(x_0) = P(X \leq x_0)$$

Important!

The threshold x_0 is allowed to be *any real number*. In particular, it doesn't have to be in the support of X!

Discrete RVs: Sum the PMF to get the CDF

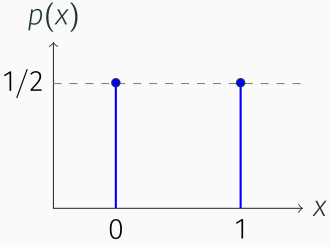
$$\boxed{F(x_0) = \sum_{x \leq x_0} p(x)}$$

Why?

The events $\{X = x\}$ are mutually exclusive, so we sum to get the probability of their union for all $x \leq x_0$:

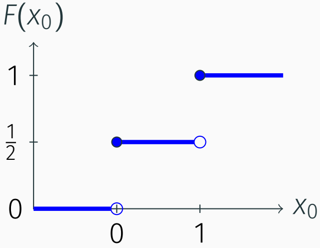
$$F(x_0) = P(X \leq x_0) = P\left(\bigcup_{x \leq x_0} \{X = x\}\right) = \sum_{x \leq x_0} P(X = x) = \sum_{x \leq x_0} p(x)$$

Probability Mass Function



$$\begin{aligned} p(0) &= 1/2 \\ p(1) &= 1/2 \end{aligned}$$

Cumulative Dist. Function



$$F(x_0) = \begin{cases} 0, & x_0 < 0 \\ \frac{1}{2}, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}$$

Properties of CDFs

1. $\lim_{x_0 \to \infty} F(x_0) = 1$
2. $\lim_{x_0 \to -\infty} F(x_0) = 0$
3. Non-decreasing: $x_0 < x_1 \Rightarrow F(x_0) \leq F(x_1)$
4. Right-continuous (“open” versus “closed” on prev. slide)

Since $F(x_0) = P(X \leq x_0)$, we have $0 \leq F(x_0) \leq 1$ for all x_0

Lotteries

Easy Lottery

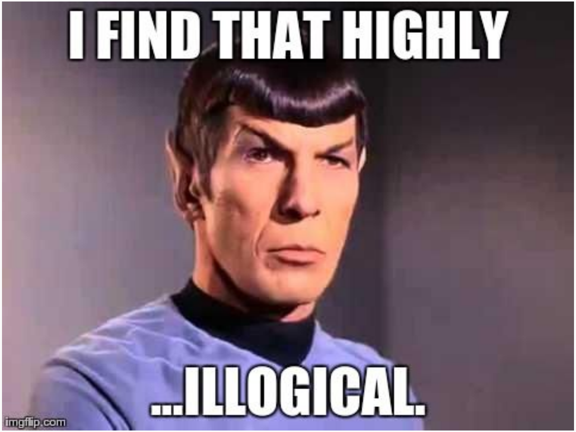
Choose between the following two lotteries:

- Lottery A:
  - You get \$1 million for sure
- Lottery B:
  - 10% chance of \$5 million
  - 89% chance of \$1 million
  - 1% chance of nothing

Hard Lottery

Choose between the following two lotteries:

- Lottery C:
    - 11% chance of $1 million
    - 89% chance of nothing
- Lottery D:
    - 10% chance of $5 million
    - 90% chance of nothing



If you chose $\{A, D\}$ or $\{B, C\}$, you are highly illogical.

Expectation

Expected Value (aka Expectation)

The expected value of a discrete RV X is given by

$$E[X] = \sum_{\text{all } x} x \cdot p(x)$$

In other words, the expected value of a discrete RV is the *probability-weighted average of its realizations.*

Notation

We sometimes write μ as shorthand for $E[X]$.

How Can I Think About Expected Value?

- The long-run average of a random variable
- How much you expect to win from a gamble where the payoffs are the values of the RV

If the realizations of the coin-flip RV were payoffs, how much would you expect to win per play *on average* in a long sequence of plays?

$$X = \begin{cases} \$0, \text{Tails} \\ \$1, \text{Heads} \end{cases}$$

Your Turn to Calculate an Expected Value

Let X be a random variable with support set $\{1, 2, 3\}$ where $p(1) = p(2) = 1/3$. Calculate $E[X]$.

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Let X be a random variable with support set $\{1, 2, 3\}$ where $p(1) = p(2) = 1/3$. Calculate $E[X]$.

$$E[X] = \sum_{\text{all } x} x \cdot p(x) = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$$

Expectation of a Function of a Discrete RV

Let X be a random variable and g be a function. Then:

$$\boxed{E[g(X)] = \sum_{\text{all } x} g(x)p(x)}$$

Is $E[g(X)] = g(E[X])$? What if $g(x) = x^2$?

IMPORTANT

$$E[g(X)] \neq g(E[X])$$

(Expected Value of Function $\neq$ Function of Expected Value)

Linearity of Expectation

$$\boxed{E[a + bX] = a + bE[X]}$$

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$$\boxed{E[a + bX] = a + bE[X]}$$

Proof:

$$\begin{aligned}
E[a + bX] &= \sum_{\text{all } x} (a + bx)p(x) \\
&= \sum_{\text{all } x} p(x) \cdot a + \sum_{\text{all } x} p(x) \cdot bx \\
&= a \sum_{\text{all } x} p(x) + b \sum_{\text{all } x} x \cdot p(x) \\
&= a + bE[X]
\end{aligned}$$

One of the few instances where $E[g(X)] = g(E[X])$

This is known as the Allais paradox.

Let $U(x)$ be the utility you get from money (notice, we are not assuming that every dollar is worth the same to you).

If you chose A, then we know that

$$U(1) > 0.1U(5) + 0.89U(1) + 0.01U(0)$$

Why Was I Highly Illogical?

This is known as the Allais paradox.

Let $U(x)$ be the utility you get from money (notice, we are not assuming that every dollar is worth the same to you).

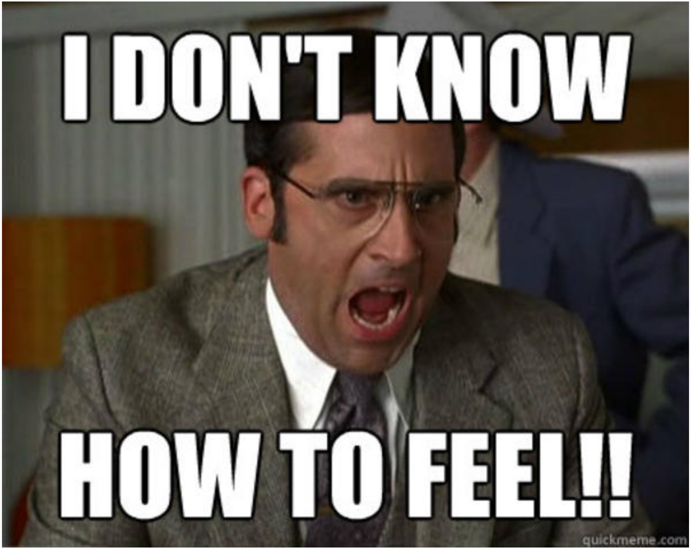
If you chose A, then we know that

$$U(1) > 0.1U(5) + 0.89U(1) + 0.01U(0)$$

If you also chose D, then we know that

$$0.1U(5) + 0.9U(0) > 0.11U(1) + 0.89U(0)$$
$$0.1U(5) + 0.89U(1) + 0.01U(0) > U(1)$$

Which Is It?



Variance

Variance (Var)

$$\sigma^2 = Var(X) = E\left[(X-\mu)^2\right] = E\left[(X-E[X])^2\right]$$

Standard Deviation (SD)

$$\sigma = \sqrt{\sigma^2} = SD(X)$$

These look similar to their definitions from last chapter

Variance (Var)

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Standard Deviation (SD)

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These look similar to their definitions from last chapter

Variance and std. dev. are *expectations of functions of a RV*

How To Calculate Variance for Discrete RV?

$$
\begin{aligned}
Var(X) = E\left[(X-\mu)^2\right] &= \sum_{\text{all } x}(x-\mu)^2 p(x) \\
&= \sum_{\text{all } x}(x^2 - 2x\mu + \mu^2)p(x) \\
&= \sum_{\text{all } x} x^2 p(x) - 2\mu \sum_{\text{all } x} xp(x) + \mu^2 \sum_{\text{all } x} p(x) \\
&= E[X^2] - 2\mu E[X] + \mu^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}
$$

Shortcut Formula For Variance

This is *not* the definition, it's a shortcut for doing calculations:

$$Var(X) = E\left[(X-\mu)^2\right] = E[X^2] - (E[X])^2$$

Variance of a Linear Function

Suppose X is a random variable with $Var(X) = \sigma^2$ and a, b are constants. What is $Var(a + bX)$?

(a) σ^2
(b) $a + \sigma^2$
(c) $b\sigma^2$
(d) $a + b\sigma^2$
(e) $b^2\sigma^2$

Variance and SD are *NOT* Linear

$$
\begin{aligned}
Var(a + bX) &= b^2\sigma^2 \\
SD(a + bX) &= |b|\sigma
\end{aligned}
$$

These should look familiar from the related results for sample variance and std. dev. that you worked out on an earlier problem set.

Variance of a Linear Transformation

$$
\begin{aligned}
Var(a+bX) &= E\left[\{(a+bX)-E(a+bX)\}^2\right] \\
&= E\left[\{(a+bX)-(a+bE[X])\}^2\right] \\
&= E\left[(bX-bE[X])^2\right] \\
&= E[b^2(X-E[X])^2] \\
&= b^2E[(X-E[X])^2] \\
&= b^2Var(X) = b^2\sigma^2
\end{aligned}
$$

The key point here is that variance is defined in terms of expectation and expectation is linear.

Constants Versus Random Variables

This is a crucial distinction that students sometimes miss:

- Random Variables
    - Suppose X is a RV – the values it takes on are random
    - A function $g(X)$ of a RV is itself a RV
- Constants
    - $E[X]$ and $Var(X)$ are constants
    - Realizations x are constants, but *which* realization the RV takes on is random
    - Parameters are constants
    - Sample size n is a constant

Bernoulli Random Variable

Bernoulli Random Variable – Generalization of Coin Flip

Support Set

$\{0, 1\}$ – 1 traditionally called “success,” 0 “failure”

Probability Mass Function

$$
\begin{aligned}
p(0) &= 1 - p \\
p(1) &= p
\end{aligned}
$$

Cumulative Distribution Function

$$
F(x_0) = \begin{cases} 0, & x_0 < 0 \\ 1 - p, & 0 \leq x_0 < 1 \\ 1, & x_0 \geq 1 \end{cases}
$$

Expected Value of Bernoulli RV

$$X = \begin{cases} 0, \text{Failure: } 1-p \\ 1, \text{Success: } p \end{cases}$$

$$\sum_{\text{all } x} x \cdot p(x) = 0 \cdot (1-p) + 1 \cdot p = p$$

Variance of Bernoulli RV – via the Shortcut Formula

Step 1 – $E[X]$

$$\mu = E[X] = \sum_{x \in \{0,1\}} p(x) \cdot x = (1-p) \cdot 0 + p \cdot 1 = p$$

Step 2 – $E[X^2]$

$$E[X^2] = \sum_{x \in \{0,1\}} x^2 p(x) = 0^2(1-p) + 1^2 p = p$$

Step 3 – Combine with Shortcut Formula

$$\sigma^2 = Var[X] = E[X^2] - (E[X])^2 = \ p - p^2 = p(1-p)$$

Variance of Bernoulli RV – Without Shortcut

You will fill in the missing steps on Problem Set 5.

$$
\begin{aligned}
\sigma^2 &= Var(X) = \sum_{x \in \{0,1\}} (x-\mu)^2 p(x) \\
&= \sum_{x \in \{0,1\}} (x-p)^2 p(x) \\
&\vdots \\
&= p(1-p)
\end{aligned}
$$

Notation: $X \sim$ **Bernoulli**(p)

Means X is a Bernoulli RV with $P(X = 1) = p$ and $P(X = 0) = 1 - p$. The tilde is read "distributes as."

Parameter

Any constant that appears in the definition of a RV, here p.

The St. Petersburg Game

How Much Would You Pay?

How much would you be willing to pay for the right to play the following game?

> *Imagine a fair coin. The coin is tossed once. If it falls heads, you receive a prize of \$2 and the game stops. If not, it is tossed again. If it falls heads on the second toss, you get \$4 and the game stops. If not, it is tossed again. If it falls heads on the third toss, you get \$8 and the game stops, and so on. The game stops after the first head is thrown. If the first head is thrown on the x^{th} toss, the prize is \$$2^x$*

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) =$$

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \dots$$

$X =$ Trial Number of First Head

x	2^x	$p(x)$	$2^x \cdot p(x)$
1	2	1/2	1
2	4	1/4	1
3	8	1/8	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$1/2^n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E[Y] = \sum_{\text{all } x} 2^x \cdot p(x) = 1 + 1 + 1 + \ldots = \infty$$

Functions of Random Variables

Example: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Support of Y

Example: Function of Bernoulli RV

$$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$$

Support of Y

$\{e^0, e^1\} = \{1, e\}$

Probability Mass Function for Y

Example: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Support of Y

$\{e^0, e^1\} = \{1, e\}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation of $Y = e^X$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation of $Y = e^X$

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) =$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Probability Mass Function for Y

$$p_Y(y) = \begin{cases} p & y = e \\ 1 - p & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expectation of $Y = e^X$

$$\sum_{y \in \{1,e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

Expectation: Function of Bernoulli RV

$\boxed{\text{Let } Y = e^X \text{ where } X \sim \text{Bernoulli}(p)}$

Expectation of the Function

$$\sum_{y \in \{1, e\}} y \cdot p_Y(y) = (1 - p) \cdot 1 + p \cdot e = 1 + p(e - 1)$$

Function of the Expectation

$$e^{E[X]} = e^p$$

Binomial Random Variable

Binomial Random Variable

Let $X =$ the sum of n independent Bernoulli trials, each with probability of success p. Then we say that: $X \sim \text{Binomial}(n, p)$

Parameters

$p =$ probability of "success," $n =$ # of trials

Support

$\{0, 1, 2, \ldots, n\}$

Probability Mass Function (PMF)

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Where does the Binomial PMF come from?

Question

Let's flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

Where does the Binomial PMF come from?

Question

Let's flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

Answer

Three basic outcomes make up this event: $\{HHT, HTH, THH\}$, each has probability $1/8 = 1/2 \times 1/2 \times 1/2$. Basic outcomes are mutually exclusive, so sum to get $3/8 = 0.375$

Where does the Binomial PMF come from?

Question

Let's flip an *unfair* coin 3 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

Answer

All basic outcomes are not equally likely, but those with exactly two heads *still are*

$$\begin{aligned} P(HHT) &= (1/3)^2(1-1/3) = 2/27 \\ P(THH) &= 2/27 \\ P(HTH) &= 2/27 \end{aligned}$$

Summing gives $2/9 \approx 0.22$

Where does the Binomial PMF come from?

Suppose we flip an unfair coin 4 times, where the probability of heads is 1/3. What is the probability that we get exactly 2 heads?

HHTT	TTHH
HTHT	THTH
HTTH	THHT

Six equally likely, mutually exclusive basic outcomes make up this event:

$$\binom{4}{2}(1/3)^2(2/3)^2$$

Joint Distributions

Definition

Sometimes we are interested in how two random variables behave together (aka jointly). Because of this, we need a way to characterize their joint behavior.

Joint probability mass function ($p_{XY}(x, y)$): Let X and Y be discrete random variables. The joint probability mass function $p_{XY}(x, y)$ gives the probability of each pair of realizations (x, y) in the support:

$$\boxed{p_{XY}(x, y) = P(X = x \cap Y = y)}$$

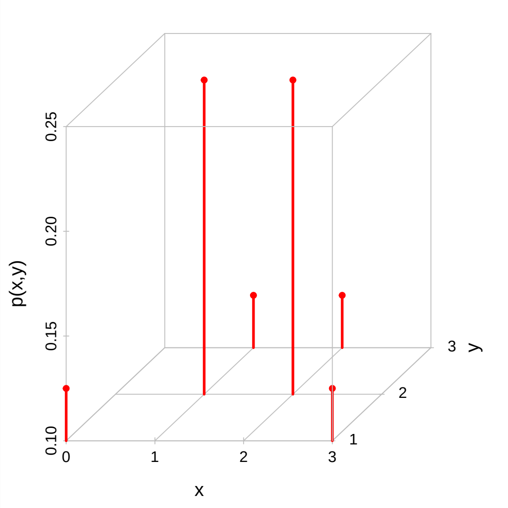
Example: Joint PMF in Tabular Form

For discrete RVs, we often use tables

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

One example might be that X is one's home region (0 is Northeast, 1 is Northwest, 2 is Southeast, 3 is Southwest) and Y is the car they own (1 is sedan, 2 is SUV, 3 is truck).

Plot of Joint PMF



What is $p_{XY}(1, 2)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

What is $p_{XY}(1,2)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1,2) = P(X = 1 \cap Y = 2) = 1/4$$

What is $p_{XY}(2, 1)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

What is $p_{XY}(2, 1)$?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(2, 1) = P(X = 2 \cap Y = 1) = 0$$

Properties of Joint PMF

1. $0 \leq p_{XY}(x, y) \leq 1$ for any pair (x, y)
2. The sum of $p_{XY}(x, y)$ over all pairs (x, y) in the support is 1:

$$\sum_{x} \sum_{y} p(x, y) = 1$$

Does this satisfy the properties of a joint PMF?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

Does this satisfy the properties of a joint PMF?

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

1. $p(x, y) \geq 0$ for all pairs (x, y)
2. $\sum_x \sum_y p(x, y) = 1/8 + 1/4 + 1/8 + 1/4 + 1/8 + 1/8 = 1$

Joint versus Marginal PMFs

What if we just care about one RV? Then, use the marginal PMF

Marginal PMFs

$p_X(x) = P(X = x)$
$p_Y(y) = P(Y = y)$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

Marginals from Joint

$$\boxed{p_X(x) = \sum_{\text{all } y} p_{XY}(x,y)} \qquad \boxed{p_Y(y) = \sum_{\text{all } x} p_{XY}(x,y)}$$

Why?

$$\begin{aligned} p_Y(y) &= P(Y=y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right) \\ &= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x,y) \end{aligned}$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
	0	1/8	0	0	1/8
X	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	

$$p_X(0) \quad = \quad 1/8 + 0 + 0 = 1/8$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
	0	1/8	0	0	1/8
X	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	
	3	1/8	0	0	

$$\begin{aligned} p_X(0) &= 1/8 + 0 + 0 = 1/8 \\ p_X(1) &= 0 + 1/4 + 1/8 = 3/8 \end{aligned}$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	

$$\begin{aligned} p_X(0) &= 1/8 + 0 + 0 = 1/8 \\ p_X(1) &= 0 + 1/4 + 1/8 = 3/8 \\ p_X(2) &= 0 + 1/4 + 1/8 = 3/8 \end{aligned}$$

To get the marginals sum "into the margins" of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$
\begin{aligned}
p_X(0) &= 1/8 + 0 + 0 = 1/8 \\
p_X(1) &= 0 + 1/4 + 1/8 = 3/8 \\
p_X(2) &= 0 + 1/4 + 1/8 = 3/8 \\
p_X(3) &= 1/8 + 0 + 0 = 1/8
\end{aligned}
$$

Marginal Probabilities

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	

Marginal Probabilities

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4			

$$p_Y(1) \quad = \quad 1/8 + 0 + 0 + 1/8 = 1/4$$

Marginal Probabilities

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2		

$$\begin{aligned} p_Y(1) &= 1/8 + 0 + 0 + 1/8 = 1/4 \\ p_Y(2) &= 0 + 1/4 + 1/4 + 0 = 1/2 \end{aligned}$$

Marginal Probabilities

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$
\begin{aligned}
p_Y(1) &= 1/8 + 0 + 0 + 1/8 = 1/4 \\
p_Y(2) &= 0 + 1/4 + 1/4 + 0 = 1/2 \\
p_Y(3) &= 0 + 1/8 + 1/8 + 0 = 1/4
\end{aligned}
$$

Conditional Distributions

Definition of Conditional PMF

How does the distribution of y change with x? In other words, if we know x, does that tell us anything about y?

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y \cap X = x)}{P(X = x)} = \frac{p_{XY}(x, y)}{p_X(x)}$$

Conditional PMF of Y given $X = 2$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$p_{Y|X}(2|2) = \frac{p_{XY}(2,2)}{p_X(2)} = \frac{1/4}{3/8} = 2/3$$

$$p_{Y|X}(3|2) = \frac{p_{XY}(2,3)}{p_X(2)} = \frac{1/8}{3/8} = 1/3$$

What is $p_{X|Y}(1|2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

What is $p_{X|Y}(1|2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2 \tag{1}$$

What is $p_{X|Y}(1|2)$?

		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2 \tag{1}$$

Similarly:

$$p_{X|Y}(0|2) = 0, \quad p_{X|Y}(2|2) = 1/2, \quad p_{X|Y}(3|2) = 0$$

Independent RVs: Joint Equals Product of Marginals

Definition

Two discrete RVs are independent if and only if

$$p_{XY}(x,y) = p_X(x)p_Y(y)$$

for all pairs (x,y) in the support.

Equivalent Definition

$$p_{Y|X}(y|x) = p_Y(y) \text{ and } p_{X|Y}(x|y) = p_X(x)$$

for all pairs (x,y) in the support.

Are X and Y Independent?

		Y			
		1	2	3	
	0	1/8	0	0	1/8
X	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

Are X and Y Independent?

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$\begin{aligned} p_{XY}(2,1) &= 0 \\ p_X(2) \times p_Y(1) &= (3/8) \times (1/4) \neq 0 \end{aligned}$$

Therefore X and Y are *not* independent.

Conditional Expectation

Conditional Expectation

Intuition
$E[Y|X]$ = "best guess" of realization that Y after observing realization of X.

$E[Y|X]$ is a Random Variable
While $E[Y]$ is a constant, $E[Y|X]$ is a function of X, hence a Random Variable.

$E[Y|X=x]$ is a Constant
The constant $E[Y|X=x]$ is the "guess" of Y if we see $X=x$.

Calculating $E[Y|X=x]$
Take the mean of the conditional pmf of Y given $X=x$.

Conditional Expectation: $E[Y|X=2]$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

We showed above that the conditional pmf of $Y|X=2$ is:

$$\boxed{p_{Y|X}(1|2) = 0 \quad p_{Y|X}(2|2) = 2/3 \quad p_{Y|X}(3|2) = 1/3}$$

Hence

$$E[Y|X=2] = 2 \times 2/3 + 3 \times 1/3 = 7/3$$

Conditional Expectation: $E[Y|X=0]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X=0$ is

$$\boxed{p_{Y|X}(1|0) = 1 \quad p_{Y|X}(2|0) = 0 \quad p_{Y|X}(3|0) = 0}$$

Hence $E[Y|X=0] = 1$

Calculate $E[Y|X=3]$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X=3$ is

$$\boxed{p_{Y|X}(1|3) = 1 \quad p_{Y|X}(2|3) = 0 \quad p_{Y|X}(3|3) = 0}$$

Hence $E[Y|X=3] = 1$

Calculate $E[Y|X = 1]$

		Y			
		1	2	3	
	0	1/8	0	0	1/8
X	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

Calculate $E[Y|X=1]$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

The conditional pmf of $Y|X=1$ is

$$\boxed{p_{Y|X}(1|1)=0 \quad p_{Y|X}(2|1)=2/3 \quad p_{Y|X}(3|1)=1/3}$$

Hence

$$E[Y|X=1]=2\times 2/3+3\times 1/3=7/3$$

$E[Y|X]$ is a Random Variable

For this example:

$$E[Y|X] = \begin{cases} 1 & X = 0 \\ 7/3 & X = 1 \\ 7/3 & X = 2 \\ 1 & X = 3 \end{cases}$$

From above the marginal distribution of X is:

$$\begin{aligned} P(X=0) &= 1/8 \qquad & P(X=1) &= 3/8 \\ P(X=2) &= 3/8 \qquad & P(X=3) &= 1/8 \end{aligned}$$

$E[Y|X]$ takes the value 1 with prob. 1/4 and 7/3 with prob. 3/4.

Law of Iterated Expectations

The Law of Iterated Expectations

Remember that $E[Y|X]$ is a random variable, so it makes sense to ask what its expectation is. It turns out that there's a nice solution:

$$\boxed{E\left[E\left[Y|X\right]\right] = E[Y]}$$

Proof of The Law of Iterated Expectations (Helpful for 104)

$$\begin{aligned}
E_X\left[E_{Y|X}\left[Y|X\right]\right] &= E_X\left[\sum_y y\; p_{Y|X}(y|x)\right] = \sum_x \left(\sum_y y\; p_{Y|X}(y|x)\right) p_X(x) \\
&= \sum_x \sum_y y\; p_X(x)\; p_{Y|X}(y|x) = \sum_x \sum_y y\; p_{XY}(x,y) \\
&= \sum_y y \sum_x p_{XY}(x,y) = \sum_y y\; p_Y(y) = E[Y]
\end{aligned}$$

Law of Iterated Expectations for Our Example

Marginal pmf of Y

$$\begin{aligned} P(Y=1) &= 1/4 \\ P(Y=2) &= 1/2 \\ P(Y=3) &= 1/4 \end{aligned}$$

$$\begin{aligned} E[Y] &= 1 \times 1/4 + 2 \times 1/2 + 3 \times 1/4 \\ &= 2 \end{aligned}$$

$E[Y|X]$

$$E[Y|X] = \begin{cases} 1 & \text{w/ prob. } 1/4 \\ 7/3 & \text{w/ prob. } 3/4 \end{cases}$$

$$\begin{aligned} E\left[E\left[Y|X\right]\right] &= 1 \times 1/4 + 7/3 \times 3/4 \\ &= 2 \end{aligned}$$

Expectation of Function of Two Discrete RVs

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y)p_{XY}(x,y)}$$

Let $\mu_X = E[X], \mu_Y = E[Y]$

Covariance

$$\sigma_{XY} = Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Correlation

$$\rho_{XY} = Corr(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Shortcut Formula for Covariance

Much easier for calculating:

$$\boxed{Cov(X, Y) = E[XY] - E[X]E[Y]}$$

I'll mention this again in a few slides...

Calculating $Cov(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

Calculating $Cov(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

Calculating $Cov(X, Y)$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

Calculating $Cov(X, Y)$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= 3 - 3/2 \times 2 = 0 \end{aligned}$$

Calculating $Cov(X, Y)$

		Y 1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= 3 - 3/2 \times 2 = 0 \end{aligned}$$

$$Corr(X, Y) = Cov(X, Y) / [SD(X)SD(Y)] = 0$$

Zero Covariance versus Independence

- From this example we learn that zero covariance (correlation) *does not* imply independence.
- However, it turns out that independence *does* imply zero covariance (correlation).

You will prove this on the homework

In general, $E[g(X,Y)] \neq g(E[X],E[Y])$. The key exception is when g is a linear function:

$$\boxed{E[aX + bY + c] = aE[X] + bE[Y] + c}$$

You will also show this on the homework

Another Application: Shortcut Formula for Covariance

Similar to Shortcut for Variance: in fact $Var(X) = Cov(X, X)$

$$
\begin{aligned}
Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
&= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
&\vdots \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

You'll fill in the details for homework...

Functions of Independent RVs are Independent

If X and Y are independent random variables and g and h are functions, then the random variables $g(X)$ and $h(Y)$ are also independent.

Revisiting Binomial RVs

Expected Value of Sum = Sum of Expected Values

Repeatedly applying the linearity of expectation,

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

whether or not they are dependent or independent.

Example

$X_1, X_2, \ldots X_n \sim$ iid Bernoulli(p)

Independent

Realization of one of the RVs gives no information about the others.

Identically Distributed

Each X_i is the same kind of RV, with the same values for any parameters. (Hence same pmf, cdf, mean, variance, etc.)

Binomial(n, p) Random Variable

Definition

Sum of n independent Bernoulli RVs, each with probability of "success," i.e. 1, equal to p

Parameters

$p =$ probability of "success," $n =$ # of trials

Support

$\{0, 1, 2, \ldots, n\}$

Using Our New Notation

Let $X_1, X_2, \ldots, X_n \sim$ iid Bernoulli(p), $Y = X_1 + X_2 + \ldots + X_n$. Then $Y \sim$ Binomial(n, p).

Which of these is the PMF of a Binomial(n, p) RV?

(a) $p(x) = p^x(1-p)^{n-x}$

(b) $p(x) = \binom{n}{x} p^x(1-p)^{n-x}$

(c) $p(x) = \binom{x}{n} p^x$

(d) $p(x) = \binom{n}{x} p^{n-x}(1-p)^x$

(e) $p(x) = p^n(1-p)^x$

Which of these is the PMF of a Binomial(n, p) RV?

(a) $p(x) = p^x(1-p)^{n-x}$

(b) $p(x) = \binom{n}{x} p^x(1-p)^{n-x}$

(c) $p(x) = \binom{x}{n} p^x$

(d) $p(x) = \binom{n}{x} p^{n-x}(1-p)^x$

(e) $p(x) = p^n(1-p)^x$

$$p(x) = \binom{n}{x} p^x(1-p)^{n-x}$$

Expected Value of Binomial RV

Use the fact that a Binomial(n, p) RV is defined as the sum of n iid Bernoulli(p) Random Variables and the Linearity of Expectation:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n] \\ &= p + p + \ldots + p \\ &= np \end{aligned}$$

Variance of a Sum $\neq$ Sum of Variances!

$$Var(aX + bY) \quad = \quad E\left[\{(aX + bY) - E[aX + bY]\}^2\right]$$

Variance of a Sum $\neq$ Sum of Variances!

$$
\begin{aligned}
Var(aX+bY) &= E\left[\{(aX+bY)-E[aX+bY]\}^2\right] \\
&= E\left[\{a(X-\mu_X)+b(Y-\mu_Y)\}^2\right]
\end{aligned}
$$

Variance of a Sum $\neq$ Sum of Variances!

$$
\begin{aligned}
Var(aX + bY) &= E\left[\{(aX + bY) - E[aX + bY]\}^2\right] \\
&= E\left[\{a(X - \mu_X) + b(Y - \mu_Y)\}^2\right] \\
&= E\left[a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)\right]
\end{aligned}
$$

Variance of a Sum $\neq$ Sum of Variances!

$$\begin{aligned}
Var(aX + bY) &= E\left[\{(aX + bY) - E[aX + bY]\}^2\right] \\
&= E\left[\{a(X - \mu_X) + b(Y - \mu_Y)\}^2\right] \\
&= E\left[a^2(X - \mu_X)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)\right] \\
&= a^2E[(X - \mu_X)^2] + b^2E[(Y - \mu_Y)^2] + 2abE[(X - \mu_X)(Y - \mu_Y)]
\end{aligned}$$

Variance of a Sum $\neq$ Sum of Variances!

$$
\begin{aligned}
Var(aX+bY) &= E\left[\{(aX+bY)-E[aX+bY]\}^2\right] \\
&= E\left[\{a(X-\mu_X)+b(Y-\mu_Y)\}^2\right] \\
&= E\left[a^2(X-\mu_X)^2+b^2(Y-\mu_Y)^2+2ab(X-\mu_X)(Y-\mu_Y)\right] \\
&= a^2E[(X-\mu_X)^2]+b^2E[(Y-\mu_Y)^2]+2abE[(X-\mu_X)(Y-\mu_Y)] \\
&= a^2Var(X)+b^2Var(Y)+2abCov(X,Y)
\end{aligned}
$$

Since $\sigma_{XY} = \rho\sigma_X\sigma_Y$, this is sometimes written as:

$$\boxed{Var(aX+bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y}$$

Independence $\Rightarrow Var(X+Y) = Var(X) + Var(Y)$

X and Y independent $\implies Cov(X,Y) = 0.$ Hence, independence implies

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) + 2Cov(X,Y) \\ &= Var(X) + Var(Y) \end{aligned}$$

Also true for three or more RVs

If $X_1, X_2, \ldots, X_n$ are independent, then

$$Var(X_1 + X_2 + \ldots X_n) = Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

Crucial Distinction

Expected Value

Always true that

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

Variance

Not true in general that

$$Var[X_1 + X_2 + \ldots + X_n] = Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$

except in the special case where $X_1, \ldots X_n$ are independent (or at least uncorrelated).

Variance of Binomial Random Variable

Definition from Sequence of Bernoulli Trials

If $X_1, X_2, \ldots, X_n \sim$ iid Bernoulli(p) then

$$Y = X_1 + X_2 + \ldots + X_n \sim \text{Binomial}(n, p)$$

Using Independence

$$\begin{aligned} Var[Y] &= Var[X_1 + X_2 + \ldots + X_n] \\ &= Var[X_1] + Var[X_2] + \ldots + Var[X_n] \\ &= \end{aligned}$$

Variance of Binomial Random Variable

Definition from Sequence of Bernoulli Trials

If $X_1, X_2, \ldots, X_n \sim$ iid Bernoulli(p) then

$$Y = X_1 + X_2 + \ldots + X_n \sim \text{Binomial}(n, p)$$

Using Independence

$$\begin{aligned} Var[Y] &= Var[X_1 + X_2 + \ldots + X_n] \\ &= Var[X_1] + Var[X_2] + \ldots + Var[X_n] \\ &= p(1-p) + p(1-p) + \ldots + p(1-p) \\ &= \end{aligned}$$

Variance of Binomial Random Variable

Definition from Sequence of Bernoulli Trials

If $X_1, X_2, \ldots, X_n \sim$ iid Bernoulli(p) then

$$Y = X_1 + X_2 + \ldots + X_n \sim \text{Binomial}(n, p)$$

Using Independence

$$\begin{aligned}
Var[Y] &= Var[X_1 + X_2 + \ldots + X_n] \\
&= Var[X_1] + Var[X_2] + \ldots + Var[X_n] \\
&= p(1-p) + p(1-p) + \ldots + p(1-p) \\
&= np(1-p)
\end{aligned}$$

Supplemental Reading

- Chapter 4: Sections 4-1 to 4-3 and 4-6
- Chapter 5: Sections 5-1 to 5-4