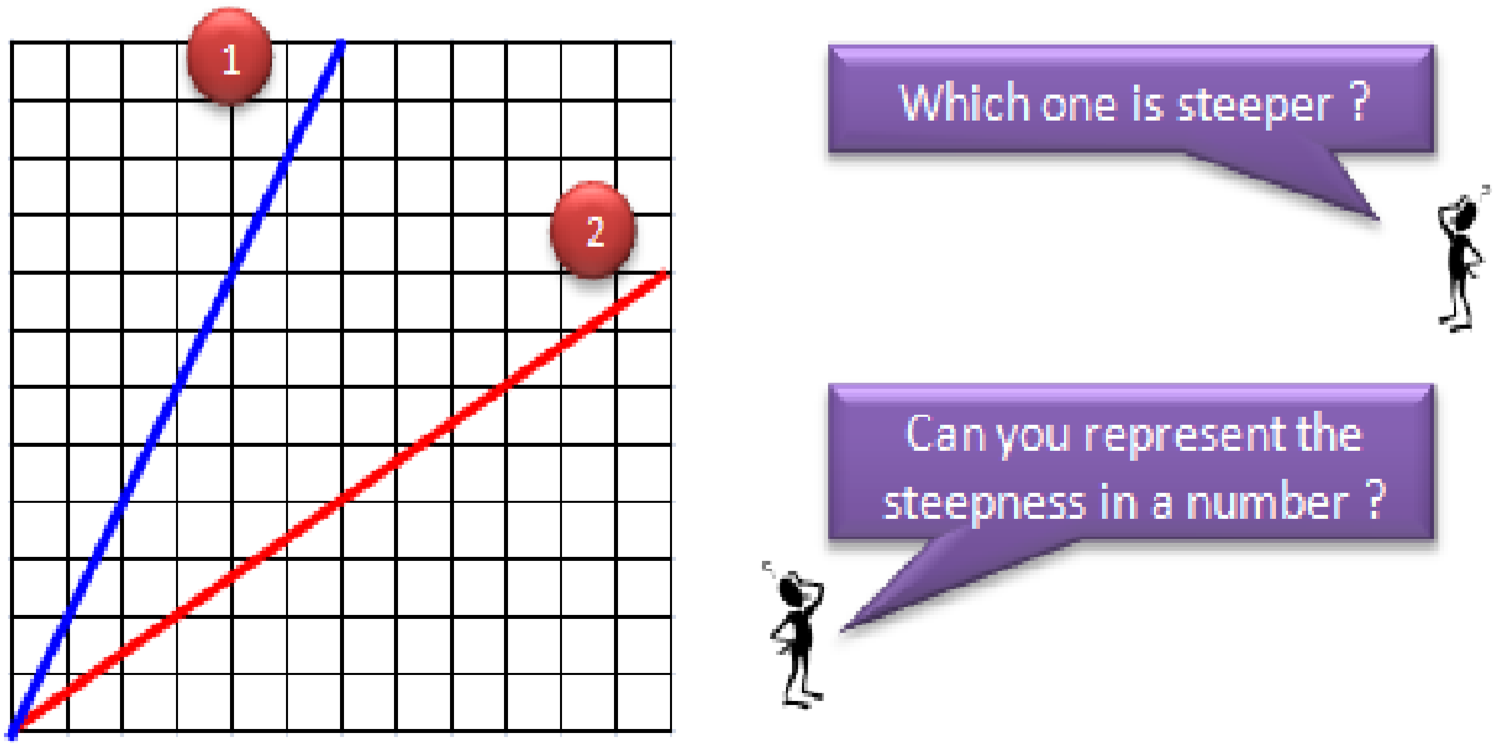
Derivatives, Partial Derivatives & Gradients

Why do we need calculus?

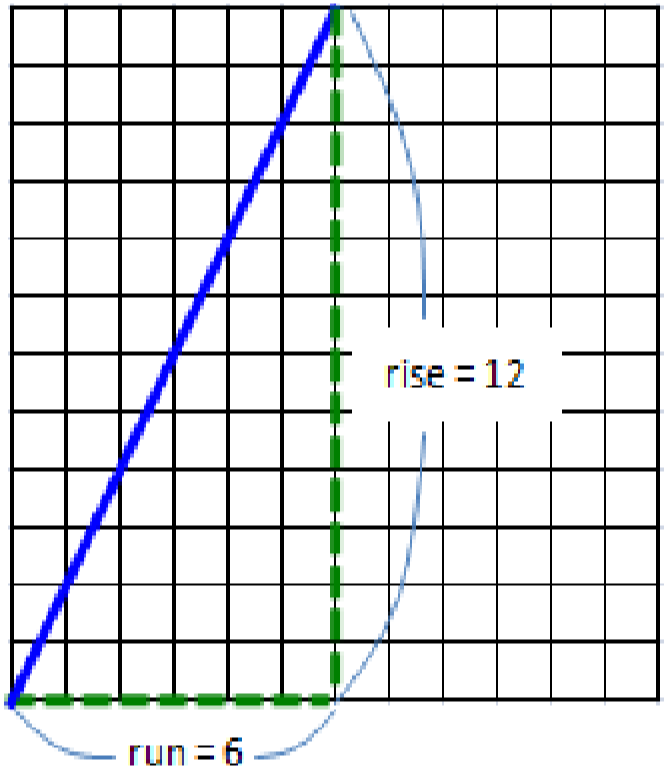
- To study rate of change(Physics context)
- To solve optimization problems(Data science context)

The rate of change in a graph is represented as 'Slope'. What does it mean by 'Slope' ?

Steepness

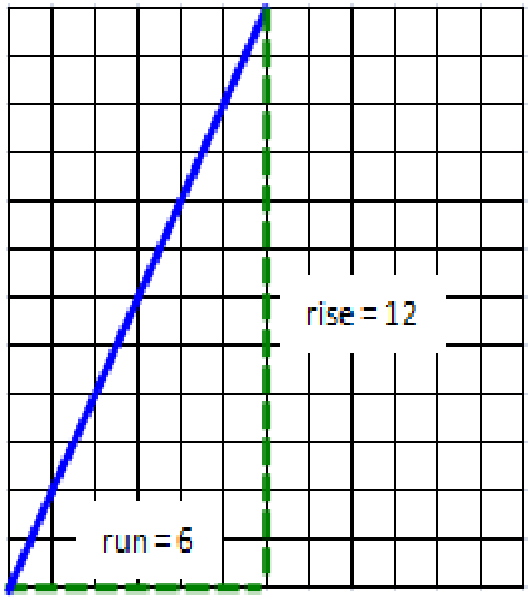


How do you represent the steepness in a number ?



$$\text{steepness} = \frac{\text{rise}}{\text{run}} = \frac{12}{6} = 2$$

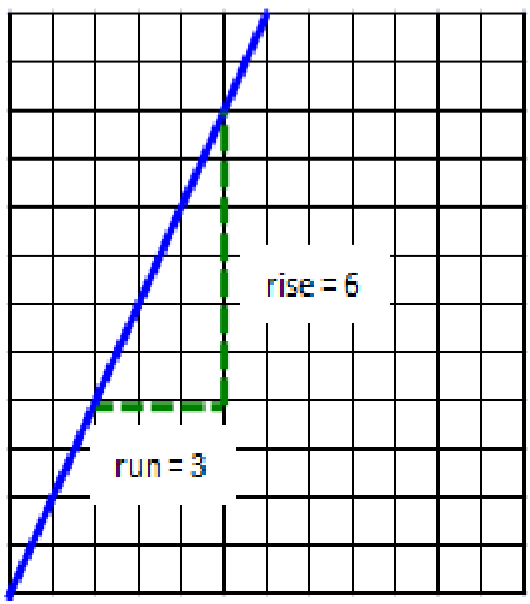
This is called 'Slope'



$$\text{steepness(slope)} = \frac{\text{rise}}{\text{run}} = \frac{12}{6} = 2$$

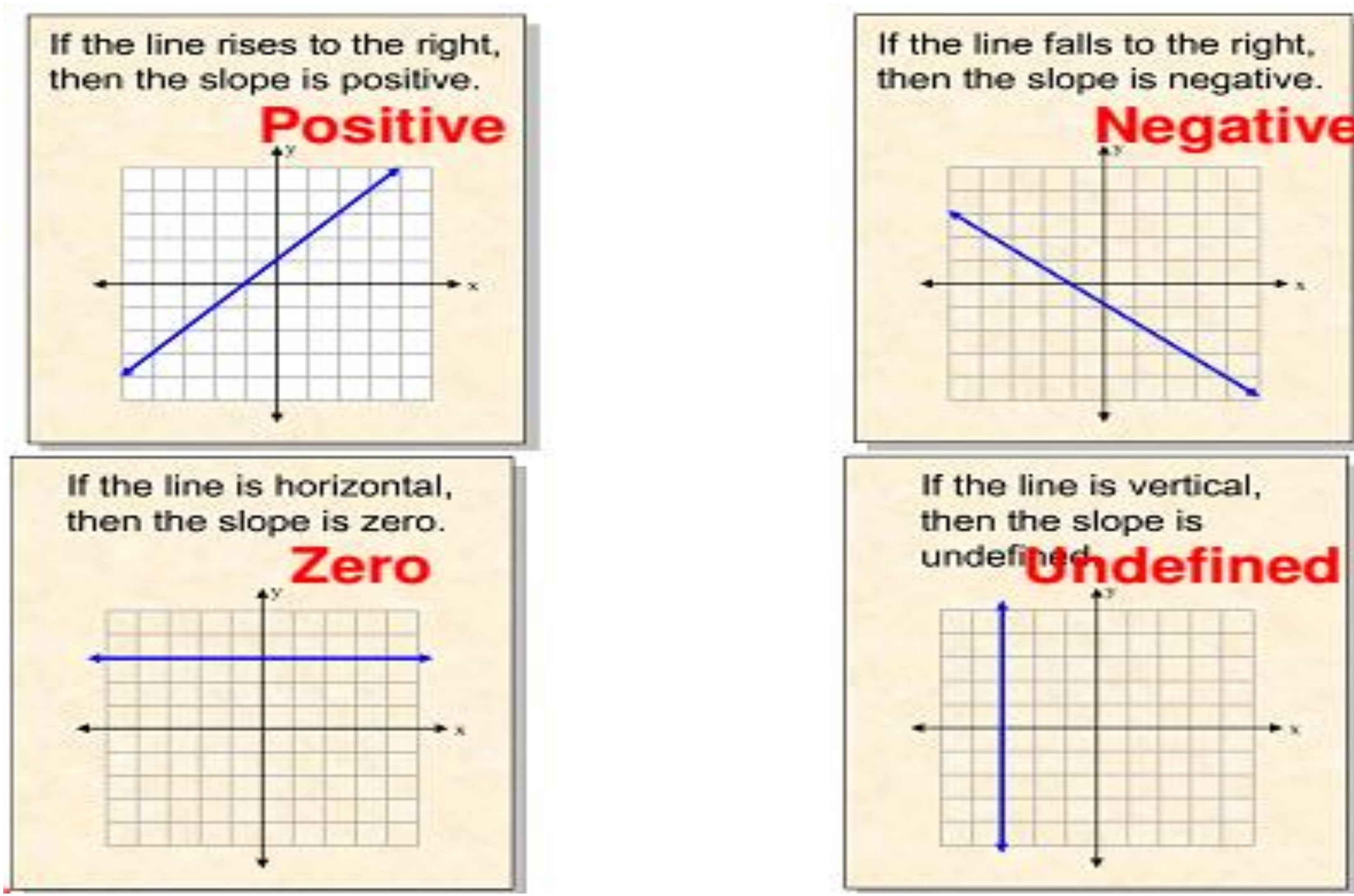
Run and Rise values are different between the two but the ratio of sun and rise (Slope) is same.

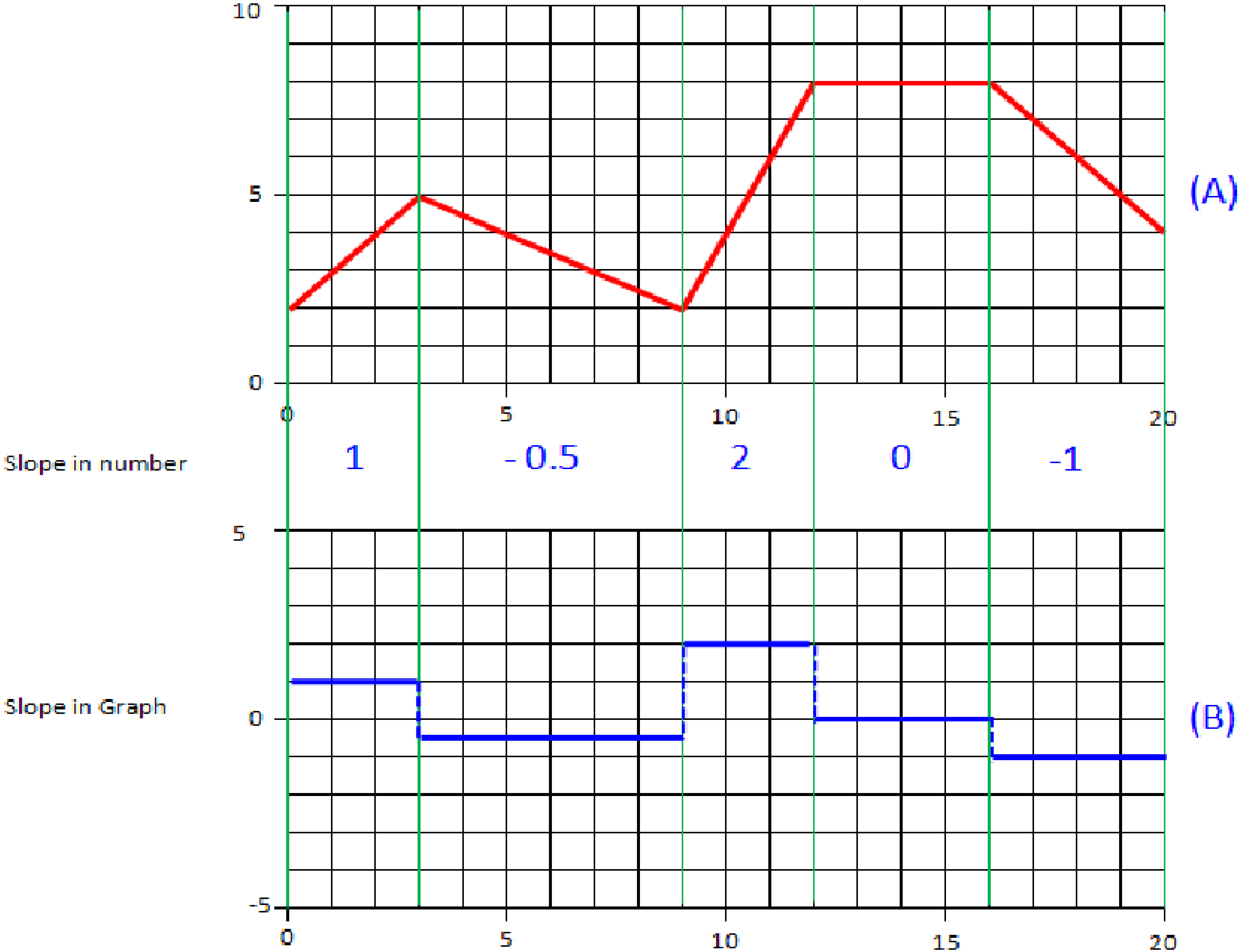
Same



$$\text{steepness(slope)} = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = 2$$

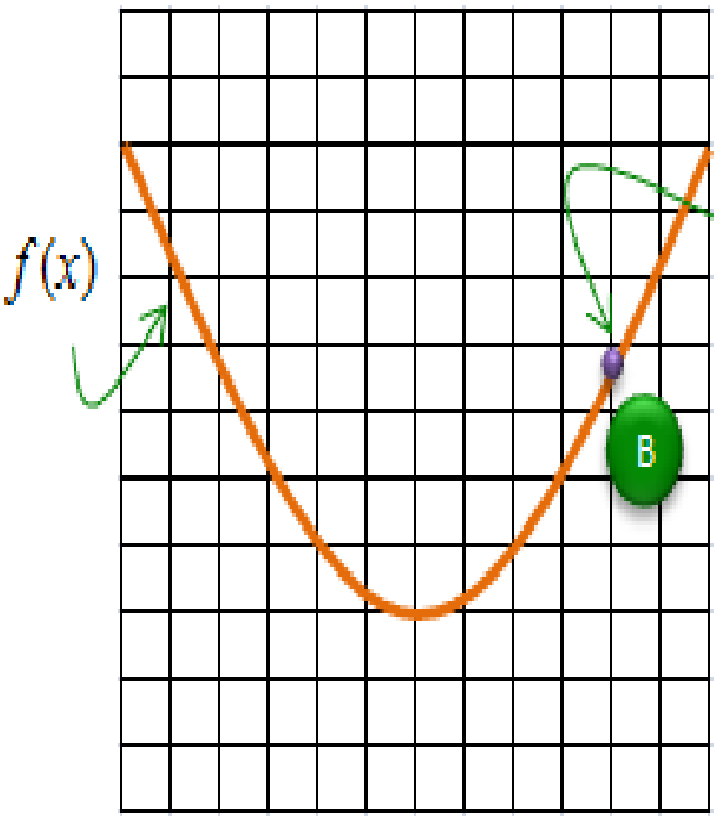
Slope


If the line rises to the right, then the slope is positive.
Positive
If the line falls to the right, then the slope is negative.
Negative
If the line is horizontal, then the slope is zero.
Zero
If the line is vertical, then the slope is
Undefined

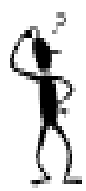


10
5
0
0
5
10
15
20
(A)
Slope in number
1
- 0.5
2
0
-1
5
0
-5
Slope in Graph
(B)
0
5
10
15
20

Slope on Curve



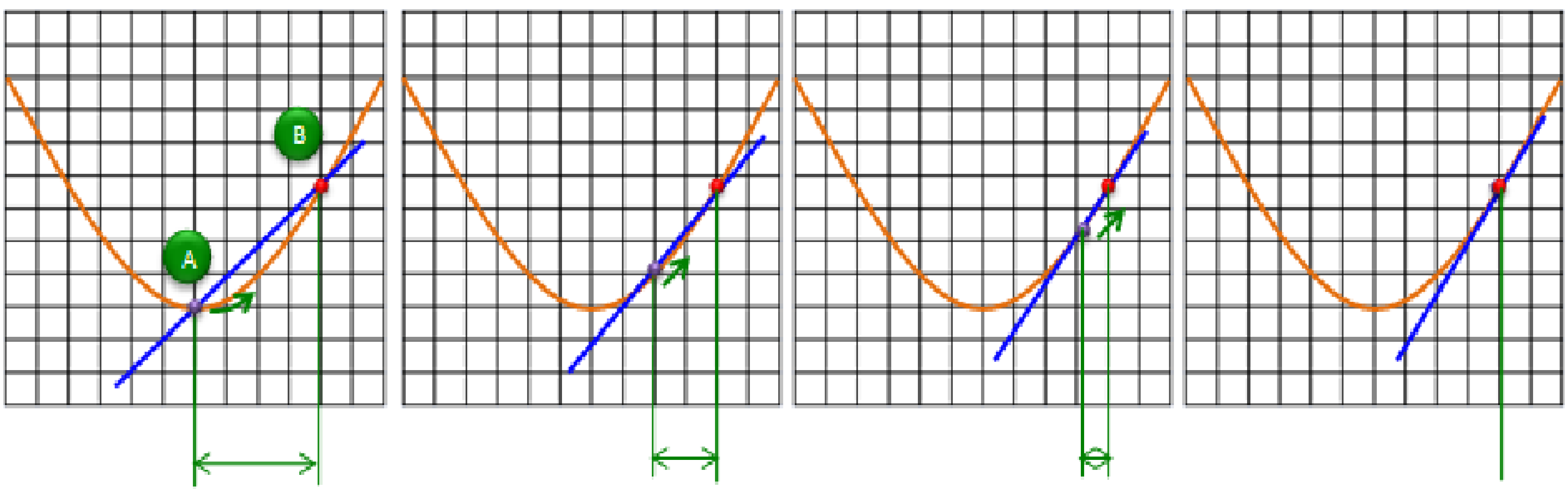
What is the slope at this point (a point on a curve) ?



How can I define the slope on a curve ?

What is the meaning of it ?

Slope finding with Tangent Line



(1) (2) (3) (4)

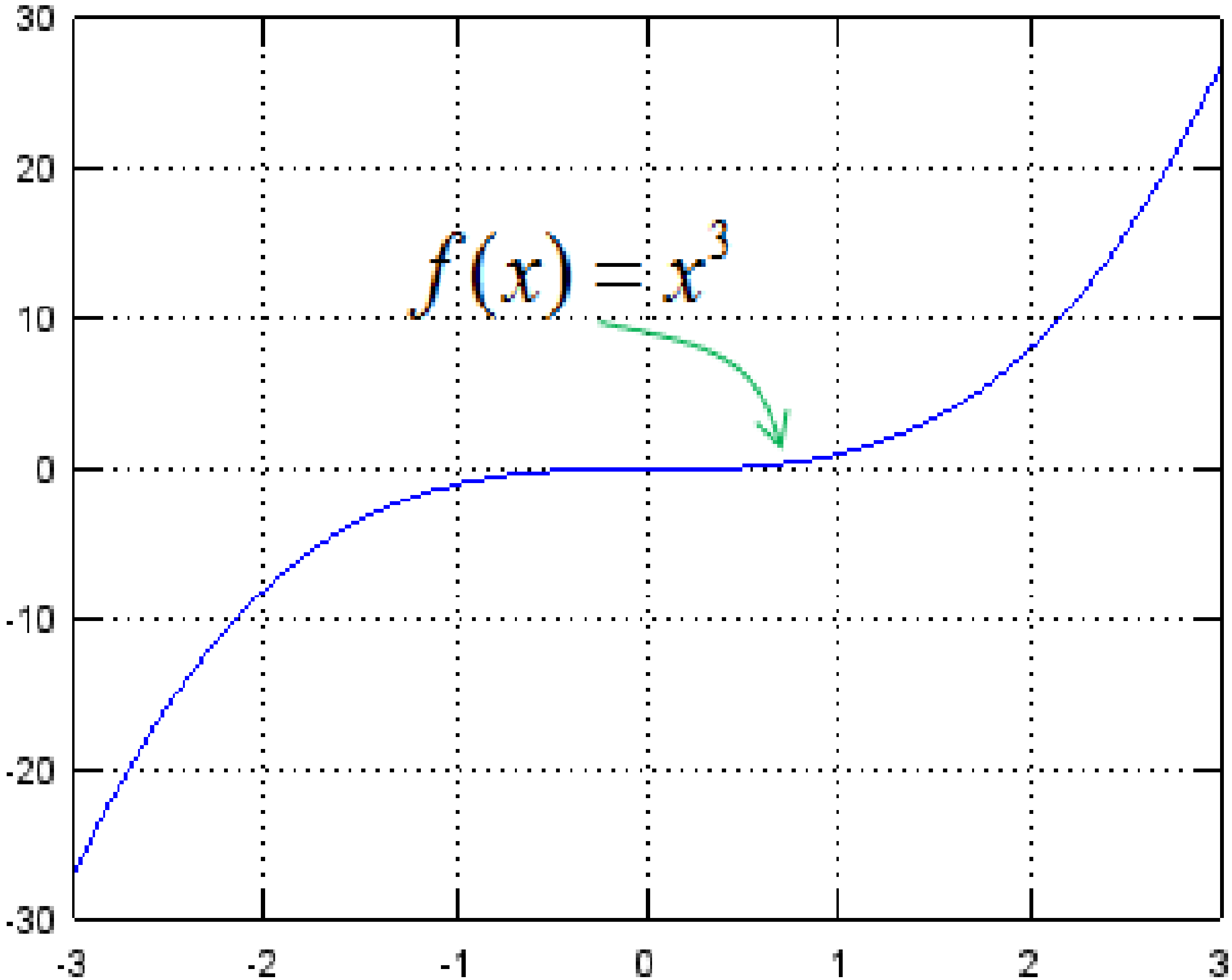
- Limit helps us to find the slope of a tangent line at any point

Limit

- Value of function when its variables are having specific value

- Value of function when its variables are approaching some value

Example1



Limit

$$\lim_{x \to 2} f(x) = ?$$

When x approaches to 2,

To which value f(x) approaches ?

When x approaches to 2,

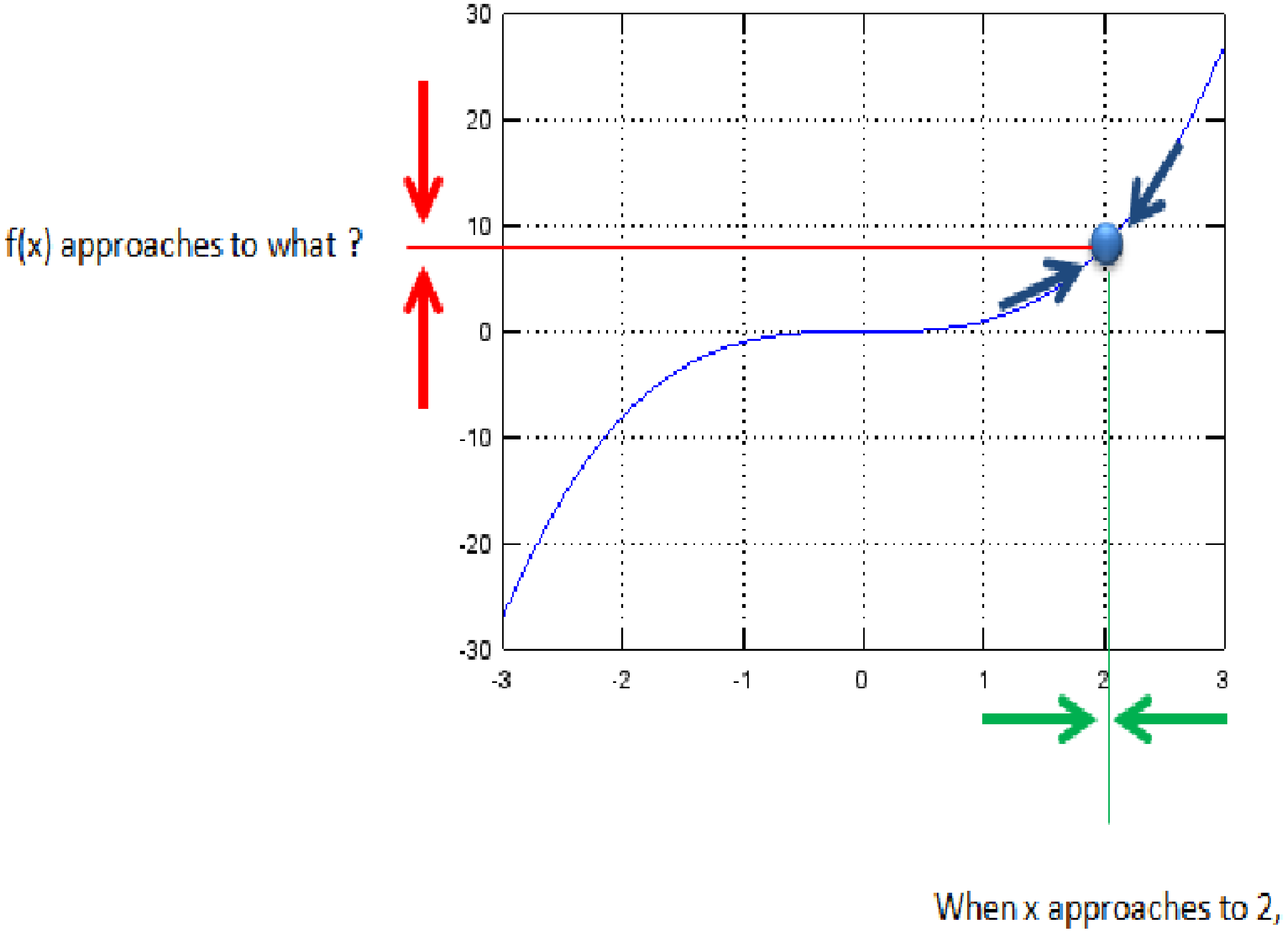
x	f(x)
1.900000	6.859000
1.950000	7.414875
1.975000	7.703734
1.987500	7.850936
1.993750	7.925234
1.996875	7.962559
1.998438	7.981265
1.999219	7.990629
1.999609	7.995313
1.999805	7.997656
1.999902	7.998828
1.999951	7.999414
1.999976	7.999707
1.999988	7.999854
1.999994	7.999927
1.999997	7.999963

f(x) approaches to what ?

When x approaches to 2,

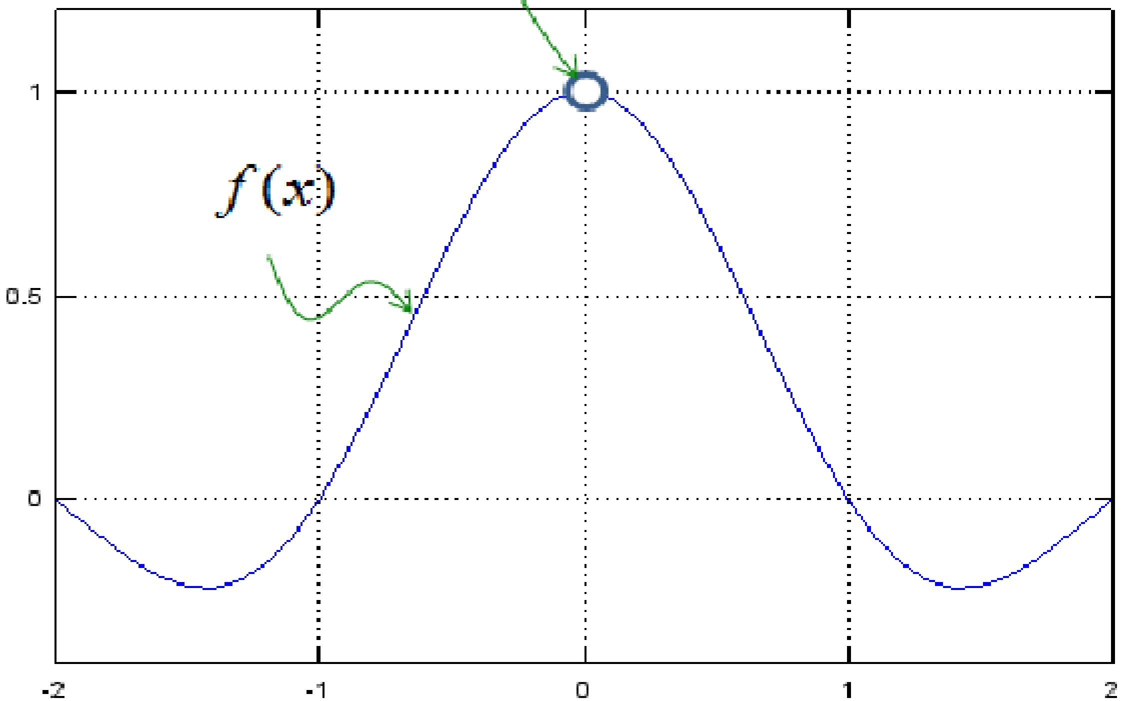
x	f(x)
2.100000	9.261000
2.050000	8.615125
2.025000	8.303766
2.012500	8.150939
2.006250	8.075235
2.003125	8.037559
2.001563	8.018765
2.000781	8.009379
2.000391	8.004688
2.000195	8.002344
2.000098	8.001172
2.000049	8.000586
2.000024	8.000293
2.000012	8.000146
2.000006	8.000073
2.000003	8.000037

f(x) approaches to what ?


f(x) approaches to what ?
30
20
10
0
-10
-20
-30
-3
-2
-1
0
1
2
3
When x approaches to 2,

Example2

f(x) value is not defined (not exist) at x = 0



$$f(0) = ?$$

$$\lim_{x \to 0} f(x) = ?$$

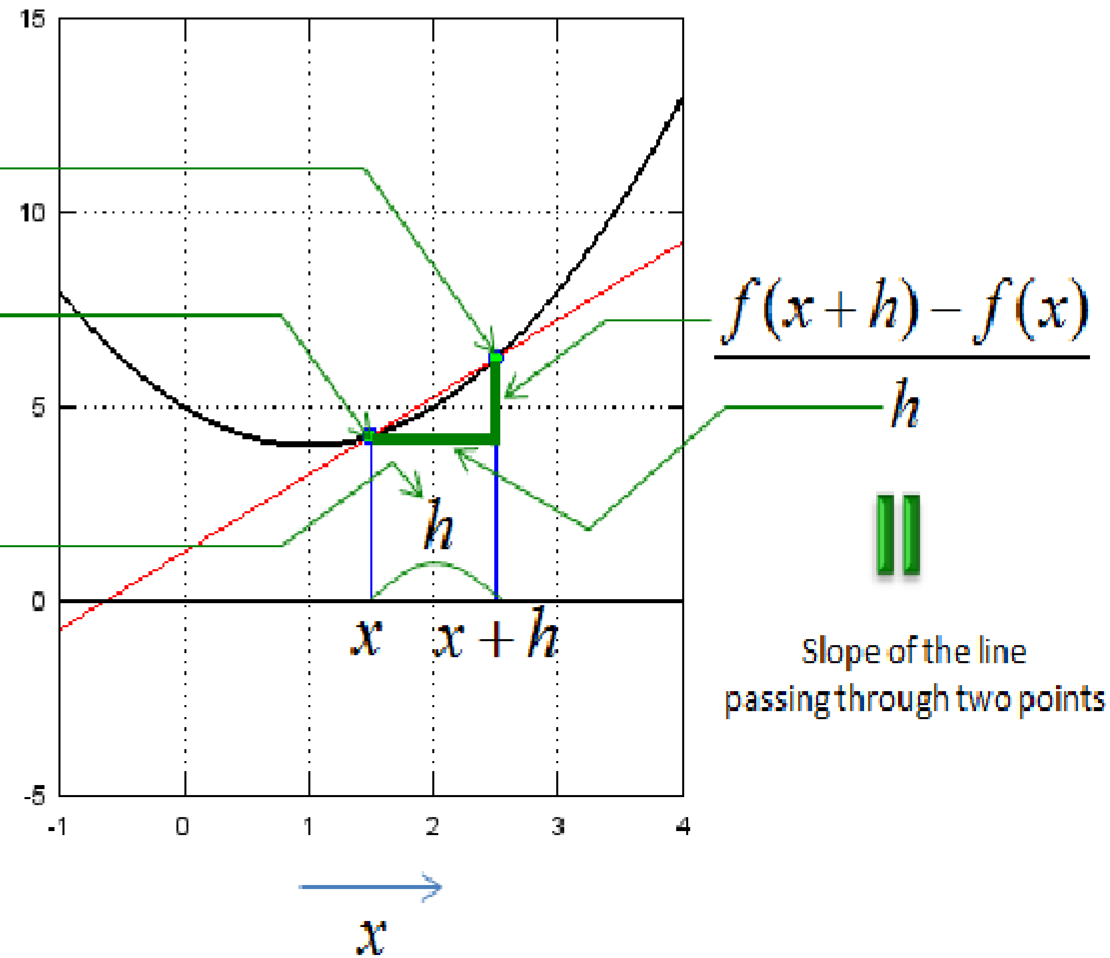
When x approaches to 0,

To which value f(x) approaches ?

Just by looking at the graph, you will notice that as x approaches to 0, f(x) approaches to 1.

Using Limit to find the slope of tangent line

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



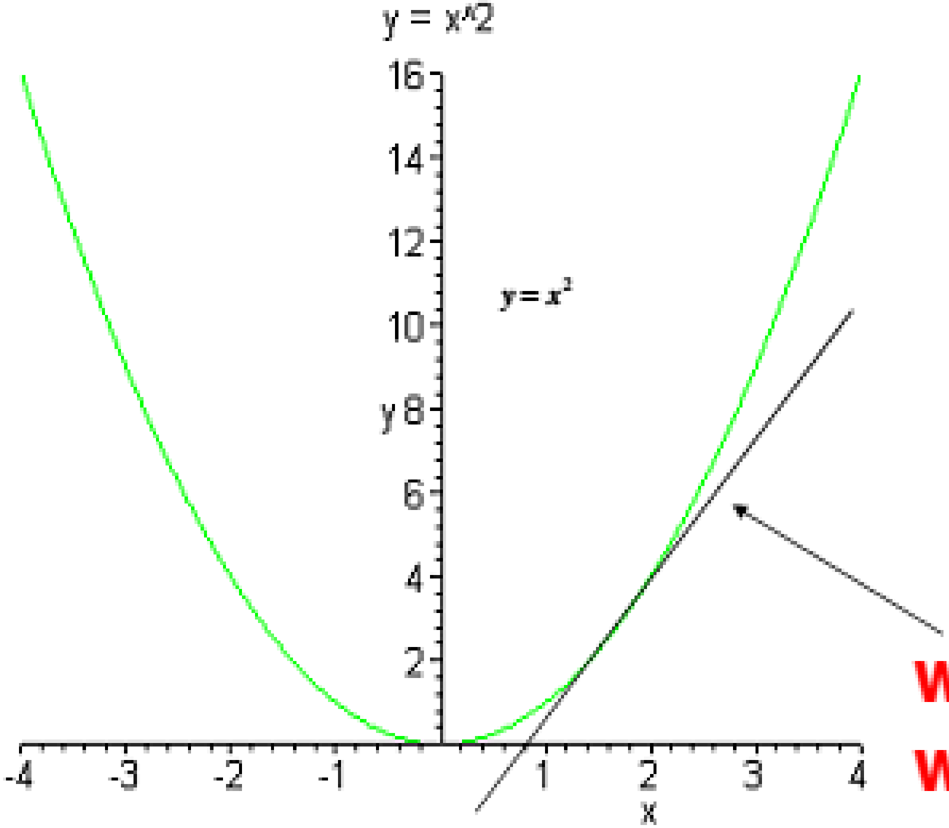
Derivative

Derivative of a function provides the rate of change in one variable at any point w.r.t. the changes in other variable.

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{change\ in\ f(x)}{change\ in\ x}$$

A VERY simple example...

$$y = x^2$$



want the slope where a=2

$$\lim \frac{f(x+h)-f(x)}{h} = \lim \frac{(x+h)^2 - x^2}{h}$$

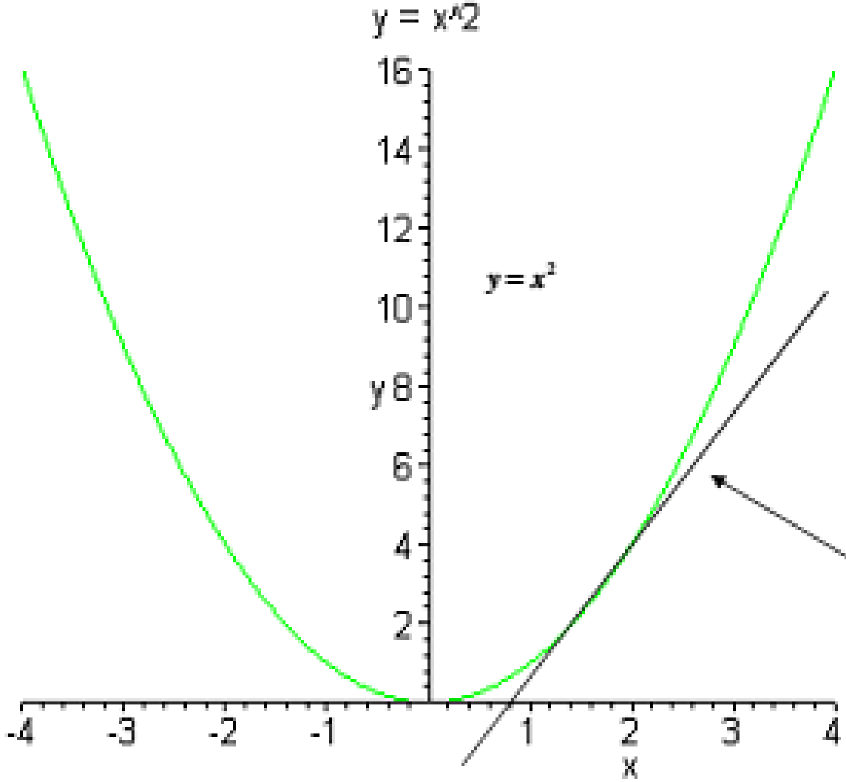
$$= \lim \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim \frac{h(2x+h)}{h}$$

For X=2

$$\lim (2x+h) = 4$$

As h $\rightarrow$ 0

$$y = x^2 \qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

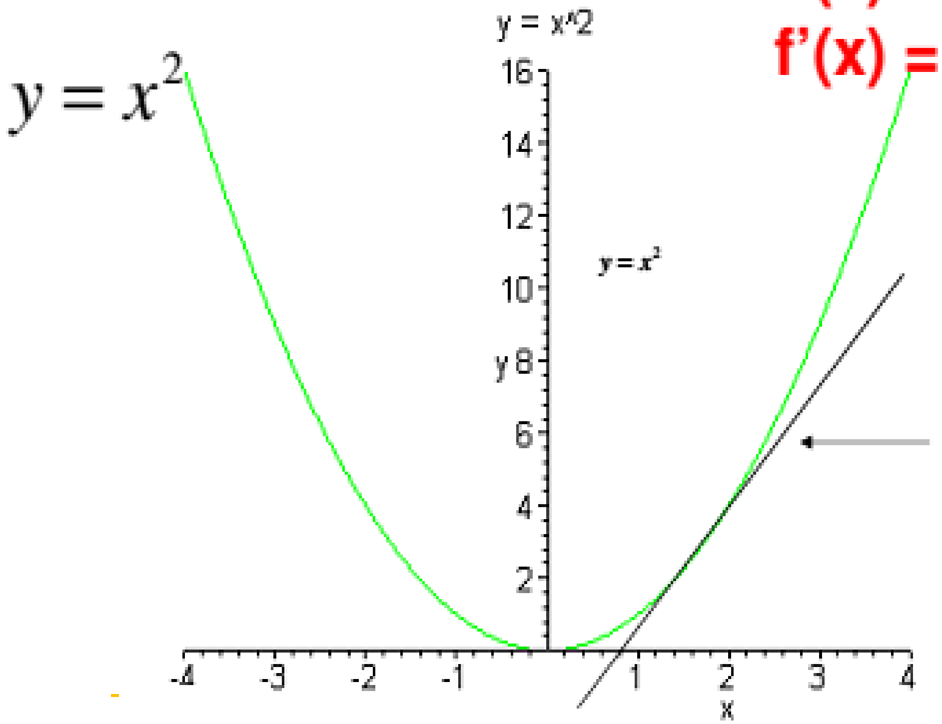


When a=2, the slope is 4

Using derivatives….

f’(x) = 2x when a =2

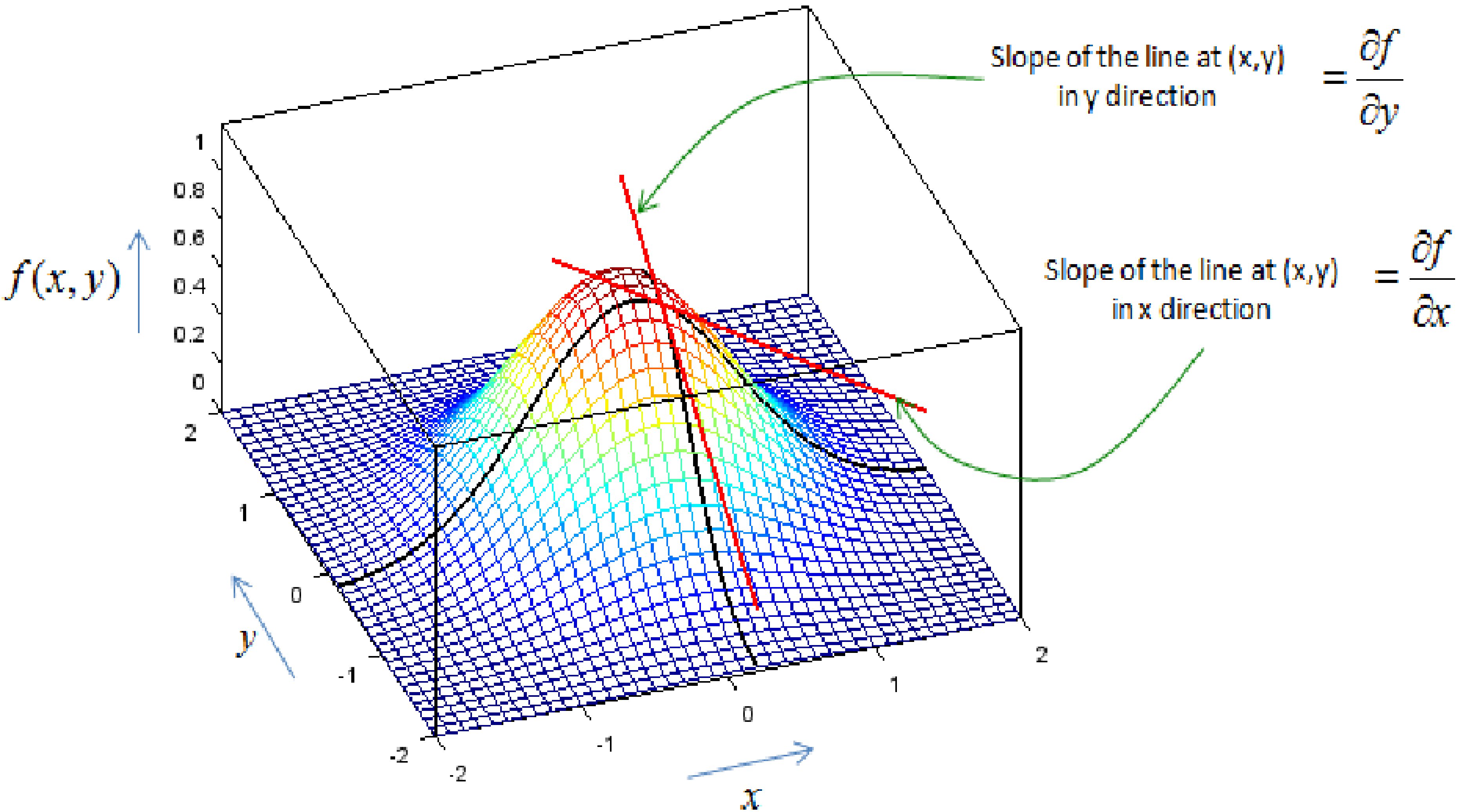
f’(x) = 2*2=4



When a=2, f’(2)=

the slope is 4

Meaning of Partial Derivative



Multivariable Differentiation

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

What is Gradient?

A vector representing the direction of the steepest downward path at specified point

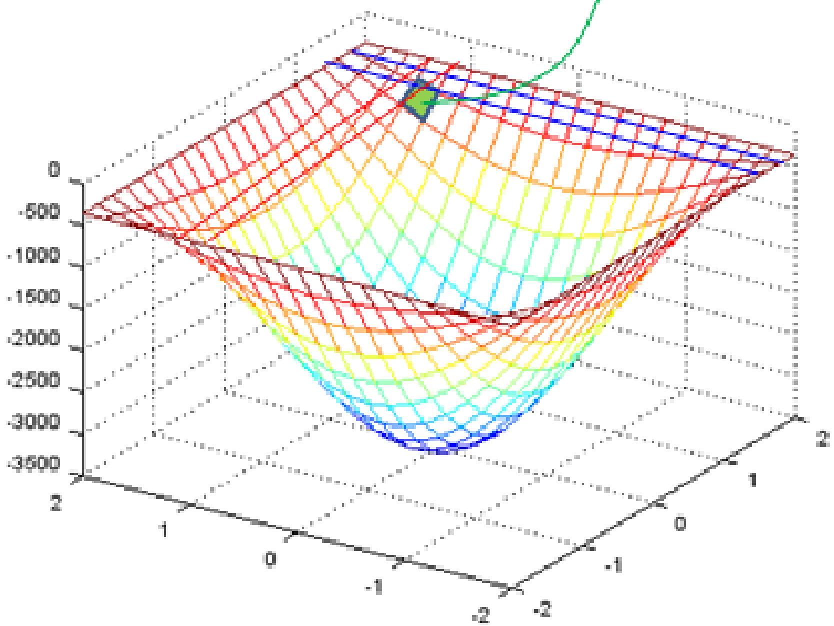
Symbol of Gradient

$$\nabla f = \overbrace{\frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}}^{\text{Definition of Gradient}}$$

Scalar　　Vector

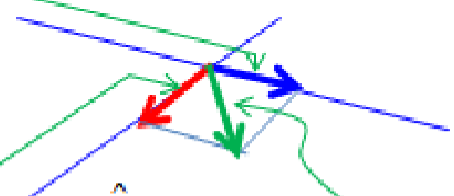
The result is a Vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$



Direction of this vector is determined by $\hat{j}$

Size of this vector is determined by $\dfrac{\partial f}{\partial y}$



∇f

Direction of this vector is determined by $\hat{i}$

Size of this vector is determined by $\dfrac{\partial f}{\partial x}$

Applying Calculus: Optimization problem solving

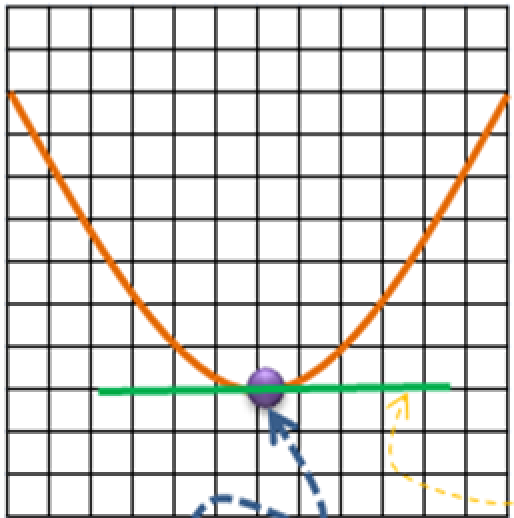
Numerical example

- Given the following relationship between x and y,

$$y = x^3 + 3x^2 - 9x - 4$$

find the values of x that minimize/maximize y.

Finding Min/Max



Tangent Line

Slope of this line = 0

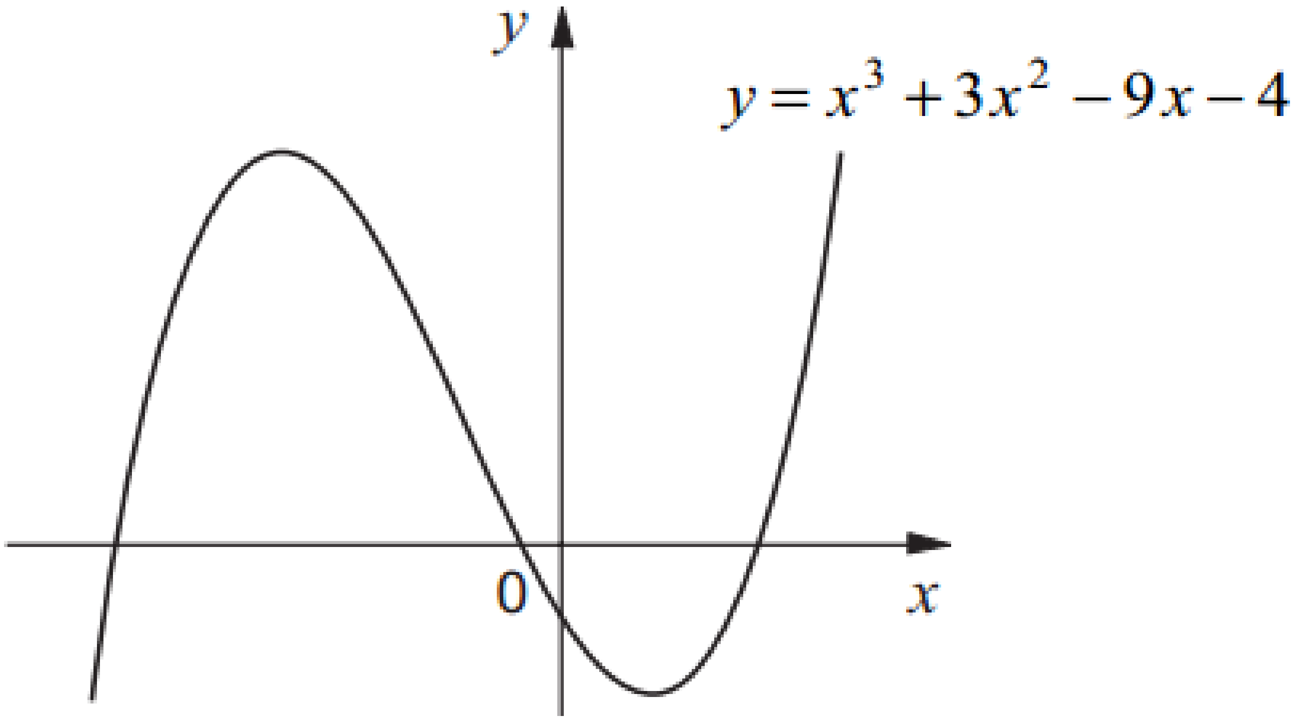
How do you explain about this point ?

It means $\frac{dy}{dx} = 0$ at this point

This is the minimum value point

This is the point where the slope of tangent line become '0'

Finding Min/Max



For any stationary point, dy/dx = 0

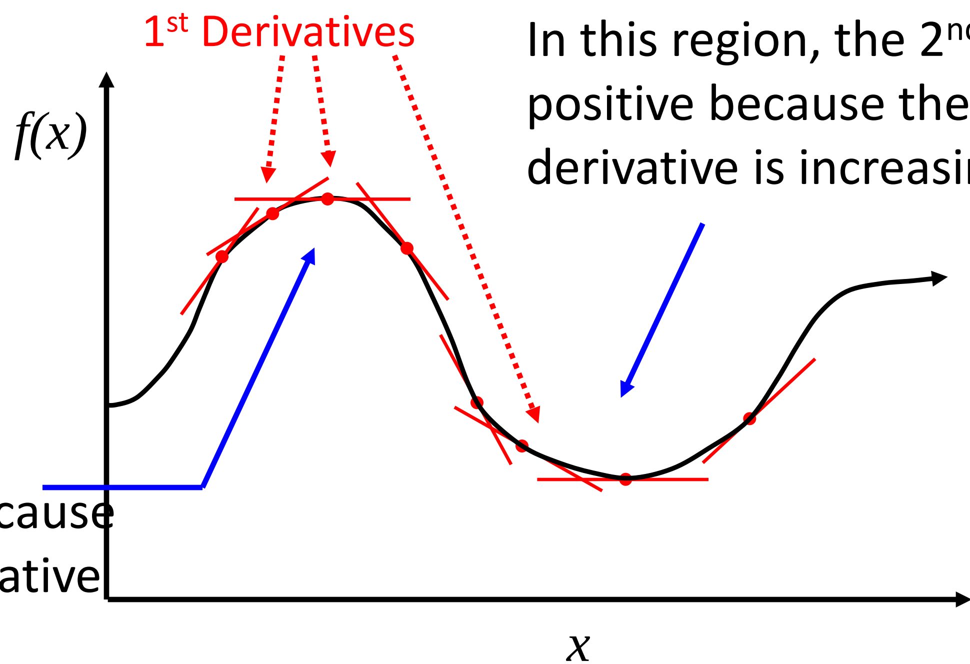
The values of x that satisfies above equation are x = 1 and x = -3.

Finding Min/Max

- How do we know which value of x will minimize/maximize the given function?

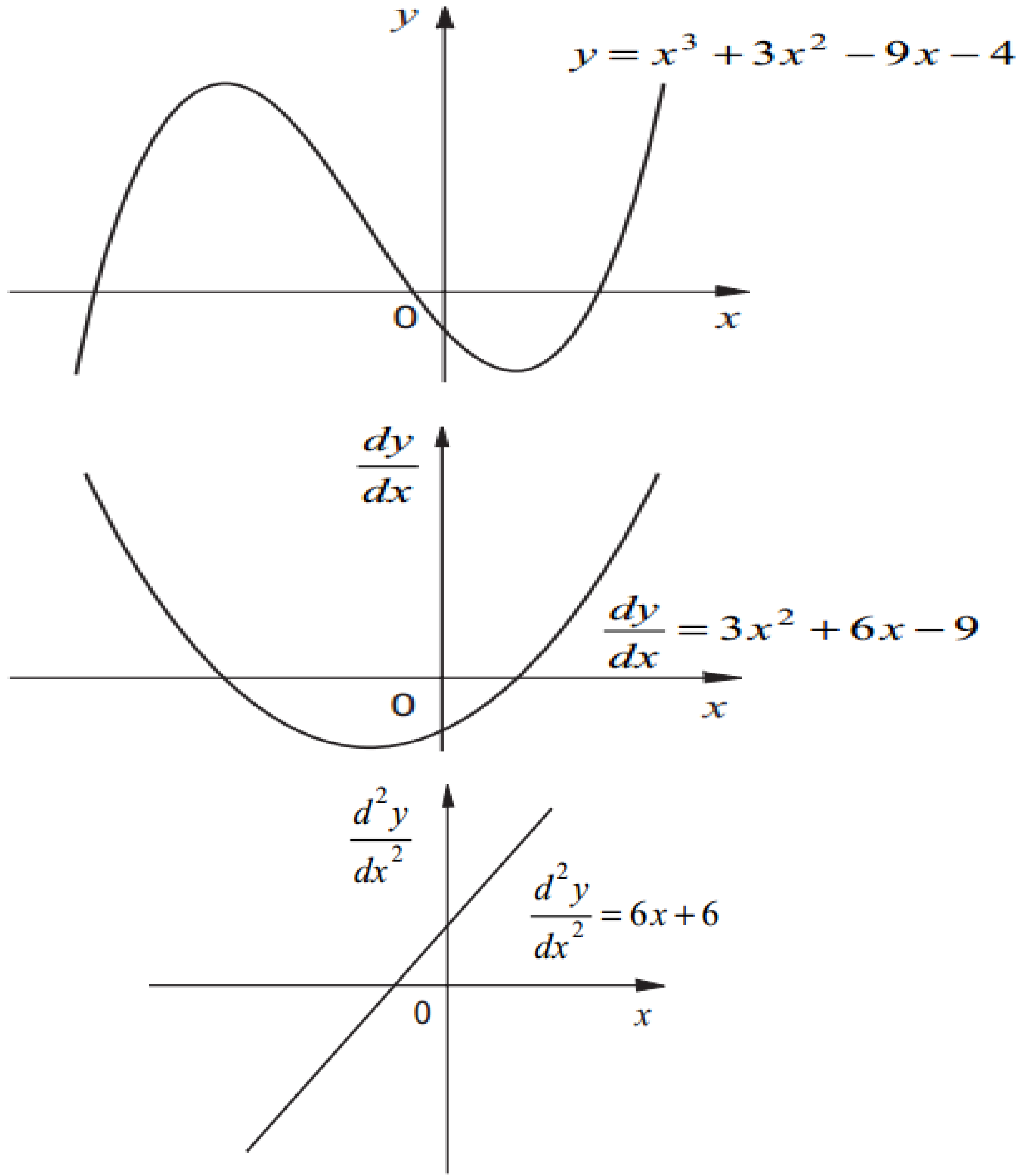
Calculus Review

- 1st Derivative: Rate of change of the function. Also, tangent line.
- 2nd Derivative: Rate of change of the 1st derivative



In this region, the 2nd derivative is positive because the slope of the 1st derivative is increasing

In this region, the 2nd derivative is negative because the slope of the 1st derivative is decreasing



y
$y = x^3 + 3x^2 - 9x - 4$
O
x
$\frac{dy}{dx}$
$\frac{dy}{dx} = 3x^2 + 6x - 9$
O
x
$\frac{d^2y}{dx^2}$
$\frac{d^2y}{dx^2} = 6x + 6$
0
x

For **stationary points** of a function $y(x)$

$$\frac{dy}{dx} = 0 \, .$$

If $\dfrac{d^2y}{dx^2} < 0$ at a stationary point, it corresponds to a **maximum** value of y.

If $\dfrac{d^2y}{dx^2} > 0$ at a stationary point, it corresponds to a **minimum** value of y.