The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\text{(8)} \qquad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely.
Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} \overline{(z_1 + z_2)} &= \bar{z}_1 + \bar{z}_2, & \overline{(z_1 - z_2)} &= \bar{z}_1 - \bar{z}_2, \\ \overline{(z_1 z_2)} &= \bar{z}_1 \bar{z}_2, & \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\begin{aligned} \overline{(z_1 z_2)} &= \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i, \\ \bar{z}_1 \bar{z}_2 &= (4 - 3i)(2 - 5i) = -7 - 26i. \end{aligned}$$

PROBLEM SET 13.1

1. **(Powers of i)** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **(Rotation)** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.
3. **(Division)** Verify the calculation in (7).
4. **(Multiplication)** If the product of two complex numbers is zero, show that at least one factor must be zero.
5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **(Laws for conjugates)** Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15 COMPLEX ARITHMETIC

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$
8. $\bar{z}_1\bar{z}_2$
9. $\operatorname{Re}(1/z_1^{\,2})$
10. $\operatorname{Re}(z_2^{\,2})$, $(\operatorname{Re} z_2)^2$
11. z_2/z_1
12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
13. $(4z_1 - z_2)^2$
14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$
15. $(z_1 + z_2)/(z_1 - z_2)$

16–19 Let $z = x + iy$. Find:

16. $\operatorname{Im} z^3$, $(\operatorname{Im} z)^3$
17. $\operatorname{Re}(1/\bar{z})$
18. $\operatorname{Im}[(1 + i)^8 z^2]$
19. $\operatorname{Re}(1/\bar{z}^2)$

20. **(Laws of addition and multiplication)** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \quad \textit{(Commutative laws)}$$

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3), \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3) \end{aligned} \quad \textit{(Associative laws)}$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad \textit{(Distributive law)}$$

$$0 + z = z + 0 = z,$$

$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n) \ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the n distinct values of the nth root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and z^c has only finitely many distinct values. However, if c is real irrational or genuinely complex, then z^c is infinitely many-valued.

EXAMPLE 3 General Power

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2} i \pm 2n\pi i\right)\right] = e^{-(\pi/2) \mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$(1+i)^{2-i} = \exp[(2-i)\ln(1+i)] = \exp\left[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}\right]$$
$$= 2e^{\pi/4 \pm 2n\pi}\left[\sin\left(\tfrac{1}{2}\ln 2\right) + i\cos\left(\tfrac{1}{2}\ln 2\right)\right].$$

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\operatorname{Ln} z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–9 Principal Value Ln z. Find $\operatorname{Ln} z$ when z equals:

1. -10
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. -100
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

10–16 All Values of ln z. Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln(-1)$
12. $\ln e$
13. $\ln(-6)$
14. $\ln(4 + 3i)$
15. $\ln(-e^{-i})$
16. $\ln(e^{3i})$

17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z:

18. $\ln z = (2 - \frac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \frac{1}{4}\pi i$

22–28 **General Powers.** Showing the details of your work, find the principal value of:

22. i^{2i}, $(2i)^i$
23. 4^{3+i}
24. $(1 - i)^{1+i}$
25. $(1 + i)^{1-i}$
26. $(-1)^{1-2i}$
27. $i^{1/2}$
28. $(3 - 4i)^{1/3}$

29. How can you find the answer to Prob. 24 from the answer to Prob. 25?

30. TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions. By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued.***) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln (z + \sqrt{z^2 - 1})$

(b) $\arcsin z = -i \ln (iz + \sqrt{1 - z^2})$

(c) $\operatorname{arccosh} z = \ln (z + \sqrt{z^2 - 1})$

(d) $\operatorname{arcsinh} z = \ln (z + \sqrt{z^2 + 1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Add, subtract, multiply, and divide $26 - 7i$ and $3 + 4i$ as well as their complex conjugates.

2. Write the two given numbers in Prob. 1 in polar form. Find the principal value of their arguments.

3. What is the triangle inequality? Its geometric meaning? Its significance?

4. If you know the values of $\sqrt[6]{1}$, how do you get from them the values of $\sqrt[6]{z}$ for any z?

5. State the definition of the derivative from memory. It looks similar to that in calculus. But what is the big difference?

6. What is an analytic function? How would you test for analyticity?

7. Can a function be differentiable at a point without being analytic there? If yes, give an example.

8. Are $|z|$, $\bar{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$ analytic? Give reason.

9. State the definitions of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ and the relations between these functions. Do these relations have analogs in real?

10. What properties of e^z are similar to those of e^x ? Which one is different?

11. What is the fundamental region of e^z ? Its significance?

12. What is an entire function? Give examples.

13. Why is $\ln z$ much more complicated than $\ln x$? Explain from memory.

14. What is the principal value of $\ln z$?

15. How is the general power z^c defined? Give examples.

16–21 **Complex Numbers.** Find, in the form $x + iy$, showing the details:

16. $(1 + i)^{12}$
17. $(-2 + 6i)^2$
18. $1/(3 - 7i)$
19. $(1 - i)/(1 + i)^2$
20. $\sqrt{-5 - 12i}$
21. $(43 - 19i)/(8 + i)$

22–26 **Polar Form.** Represent in polar form, with the principal argument:

22. $1 - 3i$
23. $-6 + 6i$
24. $\sqrt{20}/(4 + 2i)$
25. $-12i$
26. $2 + 2i$

27–30 **Roots.** Find and graph all values of

27. $\sqrt{8i}$
28. $\sqrt[4]{256}$
29. $\sqrt[4]{-1}$
30. $\sqrt{32 - 24i}$

31–35 **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check for analyticity.

31. $u = x/(x^2 + y^2)$
32. $v = e^{-3x} \sin 3y$
33. $u = x^2 - 2xy - y^2$
34. $u = \cos 2x \cosh 2y$
35. $v = e^{x^2-y^2} \sin 2xy$

36–39 **Harmonic Functions.** Are the following functions harmonic? If so, find a harmonic conjugate.

36. x^2y^2
37. xy
38. $e^{-x/2} \cos \frac{1}{2}y$
39. $x^2 + y^2$

40–45 **Special Function Values.** Find the values of

40. $\sin (3 + 4\pi i)$
41. $\sinh 4\pi i$
42. $\cos (5\pi + 2i)$
43. $\operatorname{Ln} (0.8 + 0.6i)$
44. $\tan (1 + i)$
45. $\cosh (1 + \pi i)$

SUMMARY OF CHAPTER 13
Complex Numbers and Functions

For arithmetic operations with **complex numbers**

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta), \tag{1}$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{2}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0 \tag{4}$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The complex **exponential function** (Sec. 13.5)

$$e^z = \exp z = e^x(\cos y + i\sin y) \tag{5}$$

reduces to e^x if $z = x$ $(y = 0)$. It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 13.6)

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned} \tag{6}$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \quad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

(7) $\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}) = -i \sin iz$

etc. The functions (5)–(7) are **entire,** that is, analytic everywhere in the complex plane.

The **natural logarithm** is (Sec. 13.7)

(8) $$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z \pm 2n\pi i$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln z of ln z; thus $\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$.

General powers are defined by (Sec. 13.7)

(9) $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$
(13)
$$\operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Complex Trigonometric and Hyperbolic Functions Are Related. If in (11), we replace z by iz and then use (1), we obtain

$$\text{(14)} \qquad \cosh iz = \cos z, \qquad \sinh iz = i \sin z.$$

Similarly, if in (1) we replace z by iz and then use (11), we obtain conversely

$$\text{(15)} \qquad \cos iz = \cosh z, \qquad \sin iz = i \sinh z.$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

PROBLEM SET 13.6

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.

2. Verify by differentiation that $\text{Re} \cos z$ and $\text{Im} \sin z$ are harmonic.

3–6 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

3. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$\sinh z = \sinh x \cos y + i \cosh x \sin y.$

4. $\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

5. $\cosh^2 z - \sinh^2 z = 1$

6. $\cosh^2 z + \sinh^2 z = \cosh 2z$

7–15 Function Values. Compute (in the form $u + iv$)

7. $\cos (1 + i)$

8. $\sin (1 + i)$

9. $\sin 5i$, $\cos 5i$

10. $\cos 3\pi i$

11. $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$

12. $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$

13. $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$

14. $\sinh (4 - 3i)$

15. $\cosh (4 - 6\pi i)$

16. (Real and imaginary parts) Show that

$$\text{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\text{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

17–21 Equations. Find all solutions of the following equations.

17. $\cosh z = 0$

18. $\sin z = 100$

19. $\cos z = 2i$

20. $\cosh z = -1$

21. $\sinh z = 0$

22. Find all z for which (a) $\cos z$, (b) $\sin z$ has real values.

23–25 Equations and Inequalities. Using the definitions, prove:

23. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

24. $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

25. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

PROBLEM SET 13.5

1. Using the Cauchy–Riemann equations, show that e^z is entire.

2–8 Values of e^z. Compute e^z in the form $u + iv$ and $|e^z|$, where z equals:

2. $3 + \pi i$
3. $1 + 2i$
4. $\sqrt{2} - \frac{1}{2}\pi i$
5. $7\pi i/2$
6. $(1 + i)\pi$
7. $0.8 - 5i$
8. $9\pi i/2$

9–12 Real and Imaginary Parts. Find Re and Im of:

9. e^{-2z}
10. e^{z^3}
11. e^{z^2}
12. $e^{1/z}$

13–17 Polar Form. Write in polar form:

13. $\sqrt{i}$
14. $1 + i$
15. $\sqrt[n]{z}$
16. $3 + 4i$
17. -9

18–21 Equations. Find all solutions and graph some of them in the complex plane.

18. $e^{3z} = 4$
19. $e^z = -2$
20. $e^z = 0$
21. $e^z = 4 - 3i$

22. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

13.6 Trigonometric and Hyperbolic Functions

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions.* We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 18.

PROBLEM SET 13.4

1–10 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? [Use (1) or (7).]

1. $f(z) = z^4$ **2.** $f(z) = \text{Im}\,(z^2)$

3. $e^{2x}(\cos y + i \sin y)$ **4.** $f(z) = 1/(1 - z^4)$

5. $e^{-x}(\cos y - i \sin y)$ **6.** $f(z) = \text{Arg}\; \pi z$

7. $f(z) = \text{Re}\, z + \text{Im}\, z$ **8.** $f(z) = \ln |z| + i \,\text{Arg}\, z$

9. $f(z) = i/z^8$ **10.** $f(z) = z^2 + 1/z^2$

11. (Cauchy–Riemann equations in polar form) Derive (7) from (1).

12–21 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = xy$ **13.** $v = xy$

14. $v = -y/(x^2 + y^2)$ **15.** $u = \ln |z|$

16. $v = \ln |z|$ **17.** $u = x^3 - 3xy^2$

18. $u = 1/(x^2 + y^2)$ **19.** $v = (x^2 - y^2)^2$

20. $u = \cos x \cosh y$ **21.** $u = e^{-x} \sin 2y$

22–24 Determine a, b, c such that the given functions are harmonic and find a harmonic conjugate.

22. $u = e^{3x} \cos ay$ **23.** $u = \sin x \cosh cy$

24. $u = ax^3 + by^3$

25. (Harmonic conjugate) Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

26. TEAM PROJECT. Conditions for $f(z) = const.$ Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = const.$

(a) $\text{Re}\, f(z) = const$

(b) $\text{Im}\, f(z) = const$

(c) $f'(z) = 0$

(d) $|f(z)| = const$ (see Example 3)

27. (Two further formulas for the derivative). Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

28. CAS PROJECT. Equipotential Lines. Write a program for graphing equipotential lines $u = const$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2xy$, (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, (c) $u = e^x \cos y$, $v = e^x \sin y$.

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$(1) \qquad e^z = e^x(\cos y + i \sin y).$$

Surprising as Example 4 may be, it merely illustrates that differentiability of a *complex* function is a rather severe requirement.

The idea of proof (approach of z from different directions) is basic and will be used again as the crucial argument in the next section.

Analytic Functions

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Hence analyticity of $f(z)$ at z_0 means that $f(z)$ has a derivative at every point in some neighborhood of z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighborhoods). This concept is *motivated* by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but not throughout some neighborhood of z_0. Team Project 26 gives an example.

A more modern term for *analytic in D* is *holomorphic in D.*

EXAMPLE 5 **Polynomials, Rational Functions**

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials,** that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function.** This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–10 CURVES AND REGIONS OF PRACTICAL INTEREST

Find and sketch or graph the sets in the complex plane given by

1. $|z - 3 - 2i| = \frac{4}{3}$

2. $1 \leqq |z - 1 + 4i| \leqq 5$

3. $0 < |z - 1| < 1$

4. $-\pi < \text{Re } z < \pi$

5. $\text{Im } z^2 = 2$

6. $\text{Re } z > -1$

7. $|z + 1| = |z - 1|$

8. $|\text{Arg } z| \leqq \frac{1}{4}\pi$

9. $\text{Re } z \leqq \text{Im } z$

10. $\text{Re } (1/z) < 1$

11. WRITING PROJECT. Sets in the Complex Plane. Extend the part of the text on sets in the complex plane by formulating that part in your own words and including examples of your own and comparing with calculus when applicable.

COMPLEX FUNCTIONS AND DERIVATIVES

12–15 **Function Values.** Find Re f and Im f. Also find their values at the given point z.

12. $f = 3z^2 - 6z + 3i,\ z = 2 + i$

13. $f = z/(z + 1),\ z = 4 - 5i$

14. $f = 1/(1 - z),\ z = \frac{1}{2} + \frac{1}{4}i$

15. $f = 1/z^2,\ z = 1 + i$

16–19 **Continuity.** Find out (and give reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

16. $[\text{Re}\,(z^2)]/|z|^2$

17. $[\text{Im}\,(z^2)]/|z|$

18. $|z|^2\ \text{Re}\,(1/z)$

19. $(\text{Im}\, z)/(1 - |z|)$

20–24 **Derivative.** Differentiate

20. $(z^2 - 9)/(z^2 + 1)$

21. $(z^3 + i)^2$

22. $(3z + 4i)/(1.5iz - 2)$

23. $i/(1 - z)^2$

24. $z^2/(z + i)^2$

25. CAS PROJECT. Graphing Functions. Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves Re $f(z) = const$ and Im $f(z) = const$ in the same figure, and the curves $|f(z)| = const$ in another figure, where (a) $f(z) = z^2$, (b) $f(z) = 1/z$, (c) $f(z) = z^4$.

26. TEAM PROJECT. Limit, Continuity, Derivative

(a) Limit. Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

(b) Limit. If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

(c) Continuity. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

(d) Continuity. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

(e) Differentiability. Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z. Can you find other such functions?

(f) Differentiability. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

13.4 Cauchy–Riemann Equations. Laplace's Equation

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GR9] in App. 1.

hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\text{(15)} \qquad \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta + 2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n - 1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\text{(16)} \qquad \sqrt[n]{1} = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1.$$

These n values are called the ***n*th roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 324–326 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

$$\text{(17)} \qquad w_1, \qquad w_1\omega, \qquad w_1\omega^2, \qquad \cdots, \qquad w_1\omega^{n-1}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

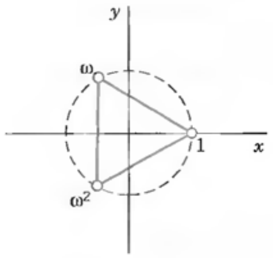

Fig. 324. $\sqrt[3]{1}$

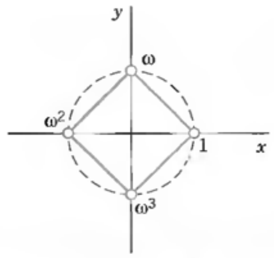

Fig. 325. $\sqrt[4]{1}$

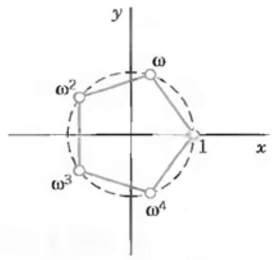

Fig. 326. $\sqrt[5]{1}$

PROBLEM SET 13.2

1–8 POLAR FORM

Do these problems very carefully since polar forms will be needed frequently. Represent in polar form and graph in the complex plane as in Fig. 322 on p. 608. (Show the details of your work.)

1. $3 - 3i$

2. $2i, -2i$

3. -5

4. $\frac{1}{2} + \frac{1}{4}\pi i$

5. $\dfrac{1 + i}{1 - i}$

6. $\dfrac{3\sqrt{2} + 2i}{-\sqrt{2} - (2/3)i}$

7. $\dfrac{-6 + 5i}{3i}$

8. $\dfrac{2 + 3i}{5 + 4i}$

9–15 PRINCIPAL ARGUMENT

Determine the principal value of the argument.

9. $-1 - i$
10. $-20 + i, -20 - i$
11. $4 \pm 3i$
12. $-\pi^2$
13. $7 \pm 7i$
14. $(1 + i)^{12}$
15. $(9 + 9i)^3$

16–20 CONVERSION TO $x + iy$

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos \frac{1}{2}\pi + i \sin (\pm\frac{1}{2}\pi)$
17. $3(\cos 0.2 + i \sin 0.2)$
18. $4(\cos \frac{1}{3}\pi \pm i \sin \frac{1}{3}\pi)$
19. $\cos (-1) + i \sin (-1)$
20. $12(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$

21–25 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$
22. $\sqrt[8]{1}$
23. $\sqrt[4]{-1}$
24. $\sqrt[3]{3 + 4i}$
25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right],$$

$$(18) \quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] = -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19) \quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)i\sqrt{\tfrac{1}{2}(|z| - x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

27–30 EQUATIONS

Solve and graph all solutions, showing the details:

27. $z^2 - (8 - 5i)z + 40 - 20i = 0$ (Use (19).)
28. $z^4 + (5 - 14i)z^2 - (24 + 10i) = 0$
29. $8z^2 - (36 - 6i)z + 42 - 11i = 0$
30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

32–35 INEQUALITIES AND AN EQUATION

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re } z| \leqq |z|$, $|\text{Im } z| \leqq |z|$.
33. **(Parallelogram equality)** Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.
35. **(Triangle inequality)** Prove (6).

13.3 Derivative. Analytic Function

Our study of complex functions will involve point sets in the complex plane. Most important will be the following ones.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 327) has already occurred in Sec. 13.2. Figure 328 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

PROBLEM SET 14.1

1–9 PARAMETRIC REPRESENTATIONS

Find and sketch the path and its orientation given by:

1. $z(t) = (1 + 3i)t \ (1 \leqq t \leqq 4)$
2. $z(t) = 5 - 2it \ (-3 \leqq t \leqq 3)$
3. $z(t) = 4 + i + 3e^{it} \ (0 \leqq t \leqq 2\pi)$
4. $z(t) = 1 + i + e^{-\pi it} \ (0 \leqq t \leqq 2)$
5. $z(t) = e^{it} \ (0 \leqq t \leqq \pi)$
6. $z(t) = 3 + 4i + 5e^{it} \ (\pi \leqq t \leqq 2\pi)$
7. $z(t) = 6 \cos 2t + 5i \sin 2t \ (0 \leqq t \leqq \pi)$
8. $z(t) = 1 + 2t + 8it^2 \ (-1 \leqq t \leqq 1)$
9. $z(t) = t + \frac{1}{2}it^3 \ (-1 \leqq t \leqq 2)$

10–18 PARAMETRIC REPRESENTATIONS

Sketch and represent parametrically:

10. Segment from $1 + i$ to $4 - 2i$
11. Unit circle (clockwise)
12. Segment from $a + ib$ to $c + id$
13. Hyperbola $xy = 1$ from $1 + i$ to $4 + \frac{1}{4}i$
14. Semi-ellipse $x^2/a^2 + y^2/b^2 = 1, \ y \geqq 0$
15. Parabola $y = 4 - 4x^2 \ (-1 \leqq x \leqq 1)$
16. $|z - 2 + 3i| = 4$ (counterclockwise)
17. $|z + a + ib| = r$ (clockwise)
18. Ellipse $4(x - 1)^2 + 9(y + 2)^2 = 36$

19–29 INTEGRATION

Integrate by the first method or state why it does not apply and then use the second method. (Show the details of your work.)

19. $\int_C \text{Re}\, z \, dz$, C the shortest path from 0 to $1 + i$
20. $\int_C \text{Re}\, z \, dz$, C the parabola $y = x^2$ from 0 to $1 + i$
21. $\int_C e^{2z} \, dz$, C the shortest path from πi to $2\pi i$
22. $\int_C \sin z \, dz$, C any path from 0 to $2i$
23. $\int_C \cos^2 z \, dz$ from $-\pi i$ along $|z| = \pi$ to πi in the right half-plane
24. $\int_C (z + z^{-1}) \, dz$, C the unit circle (counterclockwise)
25. $\int_C \cosh 4z \, dz$, C any path from $-\pi i/8$ to $\pi i/8$
26. $\int_C \bar{z} \, dz$, C from $-1 + i$ along the parabola $y = x^2$ to $1 + i$
27. $\int_C \sec^2 z \, dz$, C any path from $\pi/4$ to $\pi i/4$
28. $\int_C \text{Im}\, z^2 \, dz$ counterclockwise around the triangle with vertices $z = 0, 1, i$
29. $\int_C ze^{z^2/2} \, dz$, C from i along the axes to 1

30. **(Sense reversal)** Verify (5) for $f(z) = z^2$, where C is the segment from $-1 - i$ to $1 + i$.
31. **(Path partitioning)** Verify (6) for $f(z) = 1/z$ and C_1 and C_2 the upper and lower halfs of the unit circle
32. **(*ML*-inequality)** Find an upper bound of the absolute value of the integral in Prob. 19.
33. **(Linearity)** Illustrate (4) with an example of your own. Prove (4).
34. **TEAM PROJECT. Integration. (a) Comparison.** Write a short report comparing the essential points of the two integration methods.

    (b) Comparison. Evaluate $\int_C f(z)\, dz$ by Theorem 1 and check the result by Theorem 2, where:

    (i) $f(z) = z^4$ and C is the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane,

    (ii) $f(z) = e^{2z}$ and C is the shortest path from 0 to $1 + 2i$.

    (c) Continuous deformation of path. Experiment with a family of paths with common endpoints, say, $z(t) = t + ia \sin t$, $0 \leqq t \leqq \pi$, with real parameter a. Integrate nonanalytic functions (Re z, Re (z^2), etc.) and explore how the result depends on a. Then take analytic functions of your choice. (Show the details of your work.) Compare and comment.

    (d) Continuous deformation of path. Choose another family, for example, semi-ellipses $z(t) = a \cos t + i \sin t$, $-\pi/2 \leqq t \leqq \pi/2$, and experiment as in (c).
35. **CAS PROJECT. Integration.** Write programs for the two integration methods. Apply them to problems of your choice. Could you make them into a joint program that also decides which of the two methods to use in a given case?