TRANSFORMATIONS OF BOUNDARIES IN PARAMETRIC FORM

Suppose that in the z plane a curve C [Fig. 8-9], which may or may not be closed, has parametric equations given by

$$x = F(t), \quad y = G(t) \tag{11}$$

where we assume that F and G are continuously differentiable. Then the transformation

$$z = F(w) + i\,G(w) \tag{12}$$

maps curve C on to the real axis C' of the w plane [Fig. 8-10].

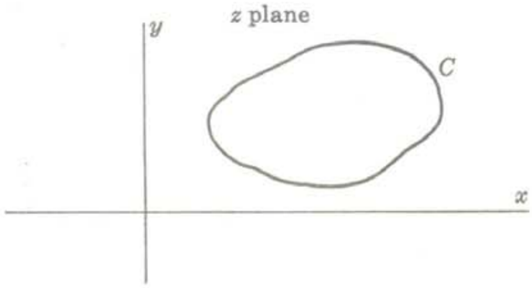


Fig. 8-9

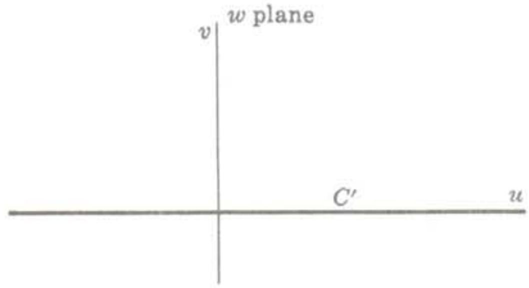


Fig. 8-10

SOME SPECIAL MAPPINGS

For reference purposes we list here some special mappings which are useful in practice. For convenience we have listed separately the mapping functions which map the given region $\mathcal{R}$ of the w or z plane on to the upper half of the z or w plane or the unit circle in the z or w plane, depending on which mapping function is simpler. As we have already seen there exists a transformation [equation (8)] which maps the upper half plane on to the unit circle.

A. Mappings on the Upper Half Plane

A–1 Infinite sector of angle π/m — $w = z^m, \quad m \geqq 1/2$

Fig. 8-11 — z plane

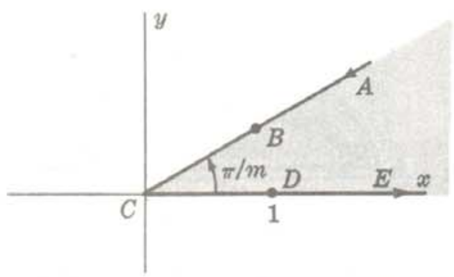
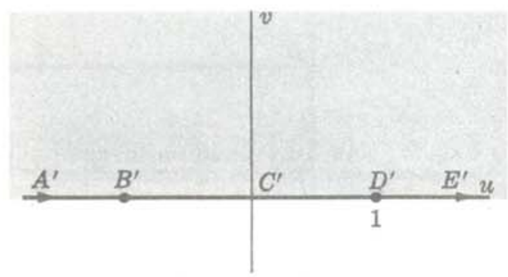


Fig. 8-12 — w plane



A–2 Infinite strip of width a — $w = e^{\pi z/a}$

Fig. 8-13 — z plane

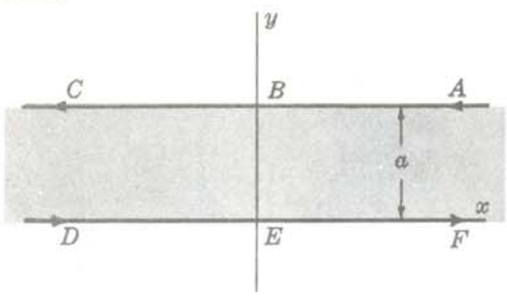
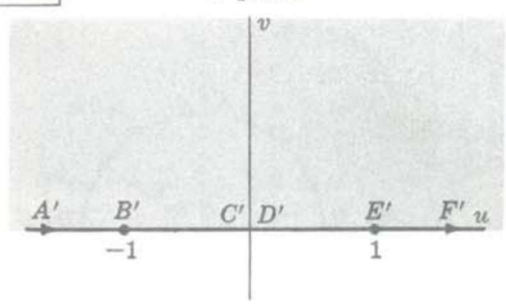


Fig. 8-14 — w plane



A–3 Semi-Infinite strip of width a

(*a*) $w = \sin\frac{\pi z}{a}$

Fig. 8-15 z plane

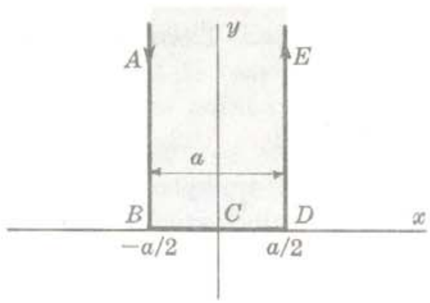
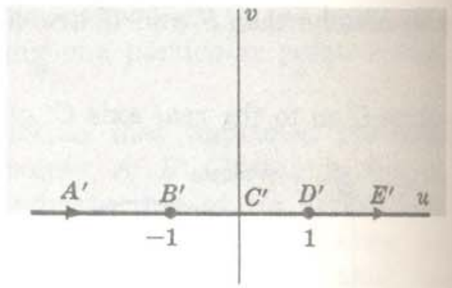


Fig. 8-16 w plane



(*b*) $w = \cos\frac{\pi z}{a}$

Fig. 8-17 z plane

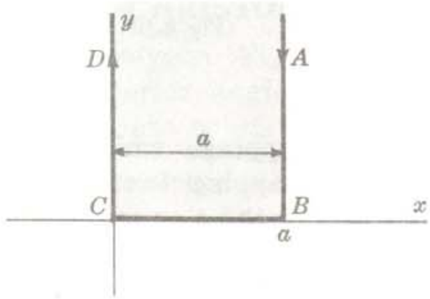
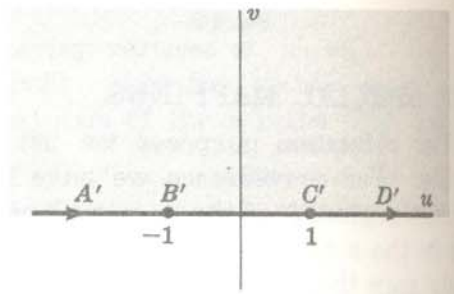


Fig. 8-18 w plane



(*c*) $w = \cosh\frac{\pi z}{a}$

Fig. 8-19 z plane

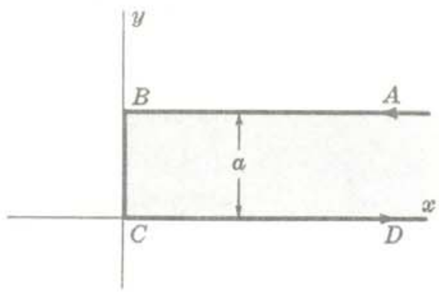
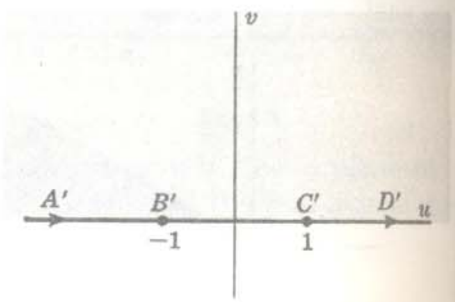


Fig. 8-20 w plane



A–4 Half plane with semicircle removed $w = \frac{a}{2}\left(z + \frac{1}{z}\right)$

Fig. 8-21 z plane

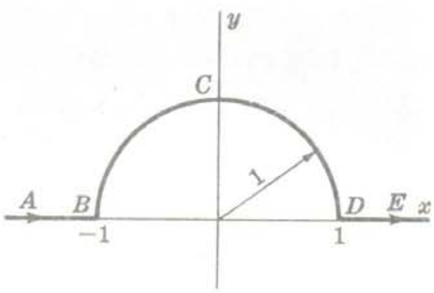
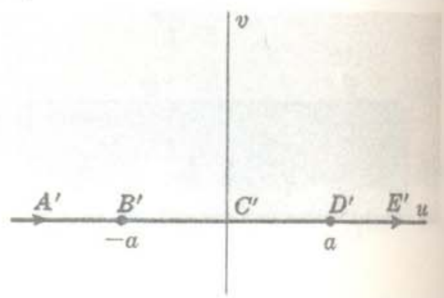


Fig. 8-22 w plane



A–5 Semicircle $\qquad w = \left(\dfrac{1+z}{1-z}\right)^2$

Fig. 8-23 z plane

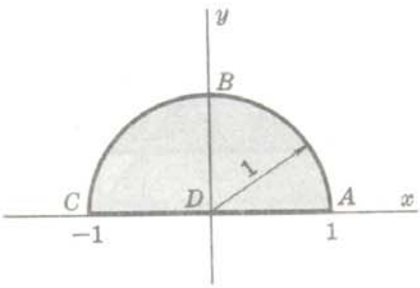
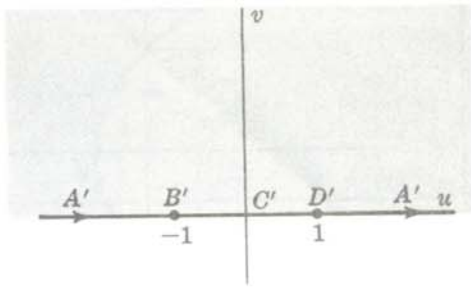


Fig. 8-24 w plane



A–6 Sector of a circle $\qquad w = \left(\dfrac{1+z^m}{1-z^m}\right)^2, \quad m \geqq \frac{1}{2}$

Fig. 8-25 z plane

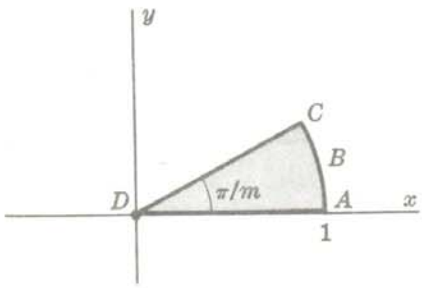
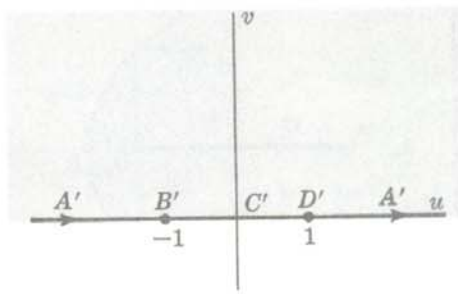


Fig. 8-26 w plane



A–7 Lens-shaped region of angle π/m [ABC and CDA are circular arcs.] $\qquad w = e^{2mi\cot^{-1}p}\left(\dfrac{z+1}{z-1}\right)^m, \quad m \geqq 2$

Fig. 8-27 z plane

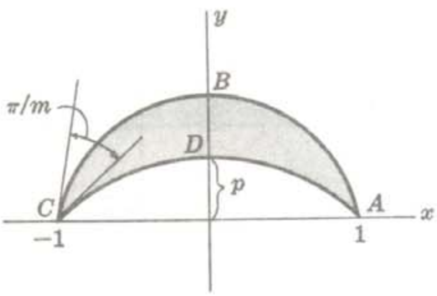
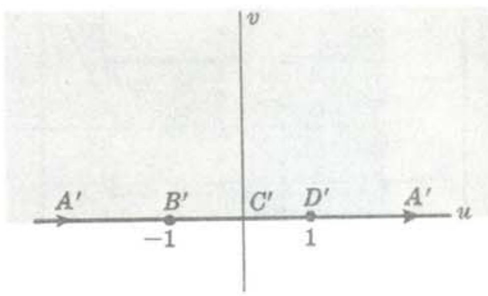


Fig. 8-28 w plane



A–8 Half plane with circle removed $\qquad w = \coth(\pi/z)$

Fig. 8-29 z plane

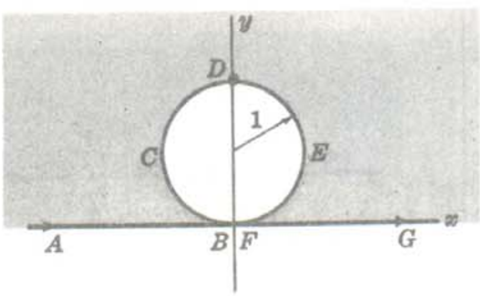
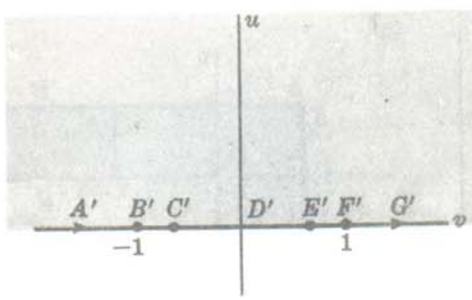


Fig. 8-30 w plane



A-9 **Exterior of parabola** $y^2 = 4p(p - x)$ $\qquad w = i(\sqrt{z} - \sqrt{p})$

Fig. 8-31 z plane

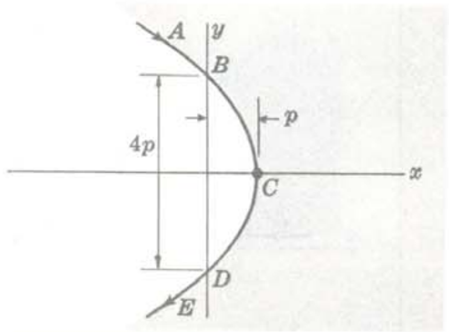
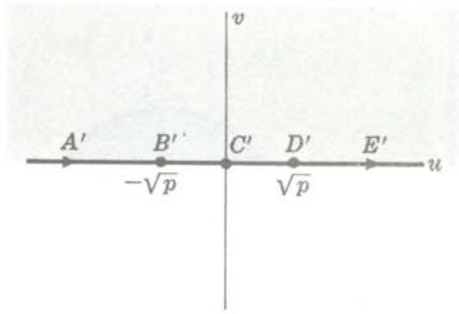


Fig. 8-32 w plane



A-10 **Interior of the parabola** $y^2 = 4p(p - x)$ $\qquad w = e^{\pi i \sqrt{z/p}}$

Fig. 8-33 z plane

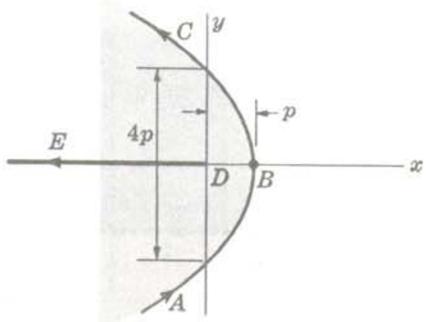
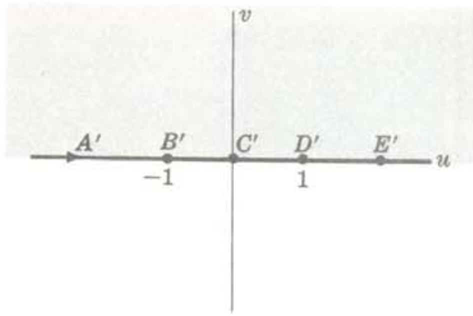


Fig. 8-34 w plane



A-11 **Plane with two semi-infinite parallel cuts** $\qquad w = -\pi i + 2 \ln z - z^2$

Fig. 8-35 w plane

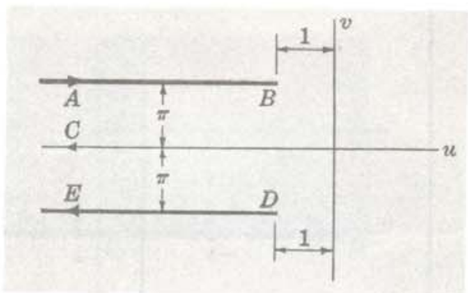
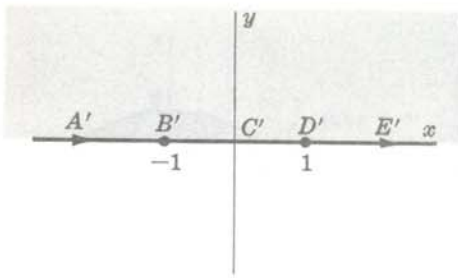


Fig. 8-36 z plane



A-12 **Channel with right angle bend** $\qquad w = \frac{2}{\pi}\{\tanh^{-1} p\sqrt{z} - p \tan^{-1} \sqrt{z}\}$

Fig. 8-37 w plane

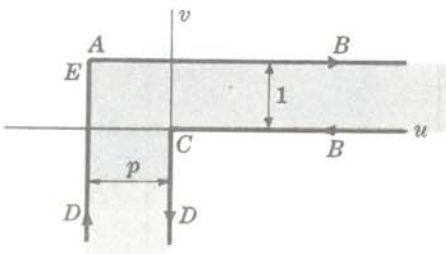
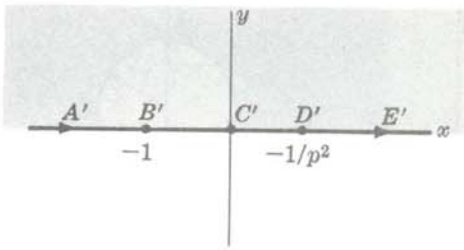


Fig. 8-38 z plane



A-13 Interior of triangle $\qquad w = \int_0^z t^{\alpha/\pi - 1}(1-t)^{\beta/\pi - 1}\,dt$

Fig. 8-39 w plane

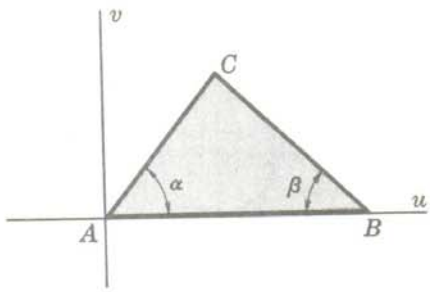
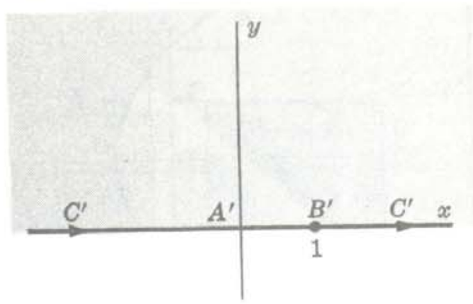


Fig. 8-40 z plane



A-14 Interior of rectangle $\qquad w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1$

Fig. 8-41 w plane

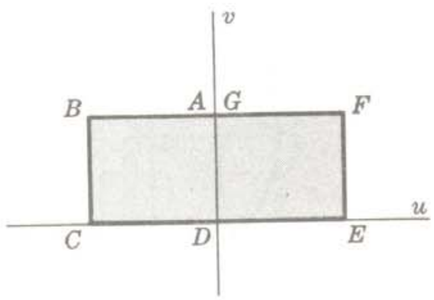
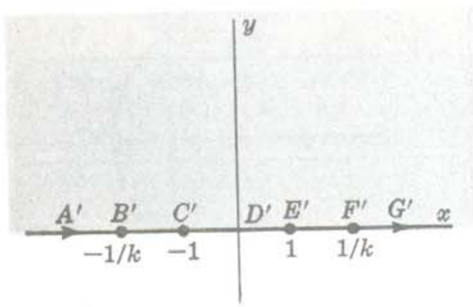


Fig. 8-42 z plane



B. Mappings on the Unit Circle

B-1 Exterior of unit circle $\qquad w = 1/z$

Fig. 8-43 w plane

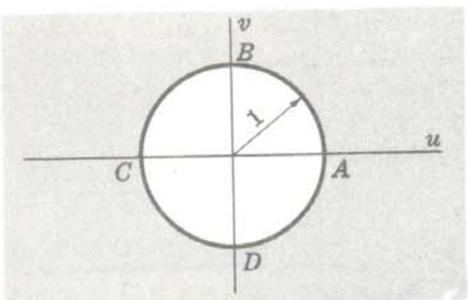
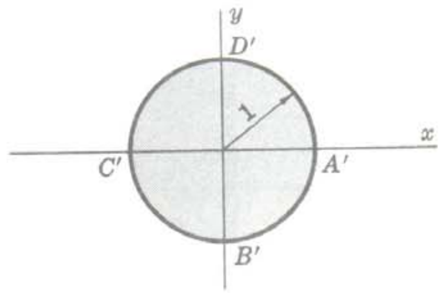


Fig. 8-44 z plane



B-2 Exterior of ellipse $\qquad w = \frac{1}{2}(ze^{-\alpha} + z^{-1}e^{\alpha})$

Fig. 8-45 w plane

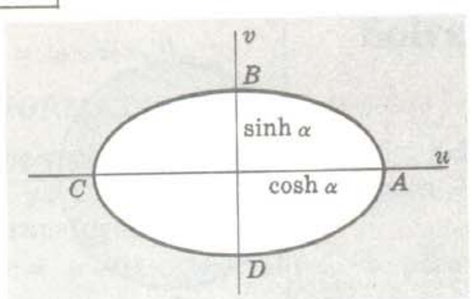
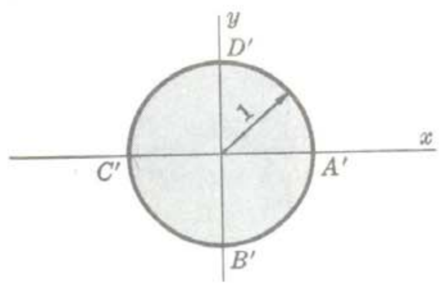


Fig. 8-46 z plane



B–3 Exterior of parabola $y^2 = 4p(p-x)$

$$w = 2\sqrt{\frac{p}{z}} - 1$$

Fig. 8-47

z plane

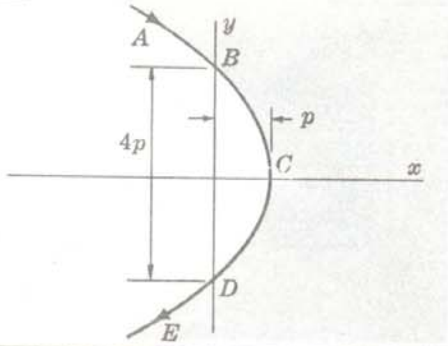
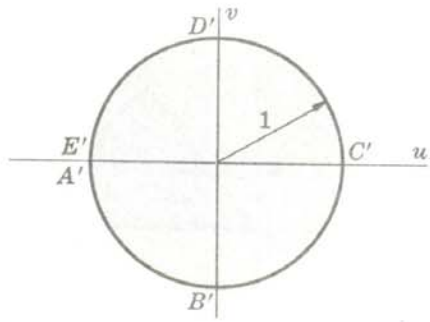


Fig. 8-48

w plane



B–4 Interior of parabola $y^2 = 4p(p-x)$

$$w = \tan^2\frac{\pi}{4}\sqrt{\frac{z}{p}}$$

Fig. 8-49

z plane

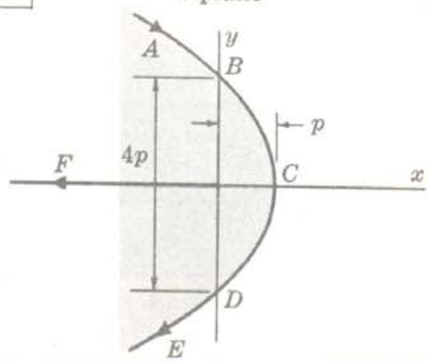
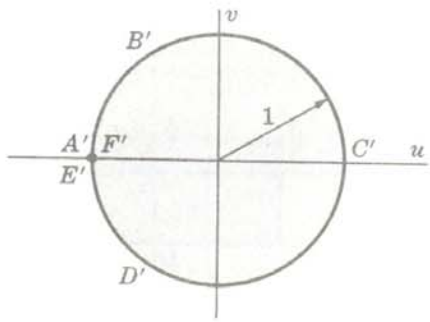


Fig. 8-50

w plane



C. Miscellaneous Mappings

C–1 Semi-infinite strip of width a on to quarter plane

$$w = \sin\frac{\pi z}{2a}$$

Fig. 8-51

z plane

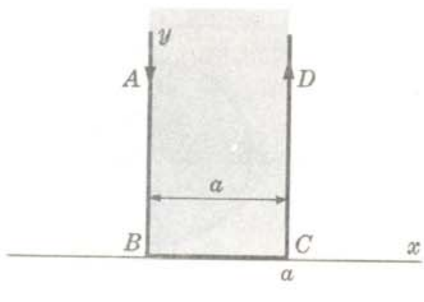
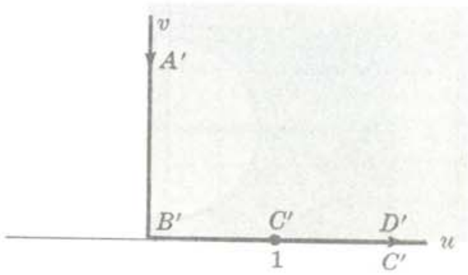


Fig. 8-52

w plane



C–2 Interior of cardioid on to circle

$$w = z^2$$

Fig. 8-53

w plane

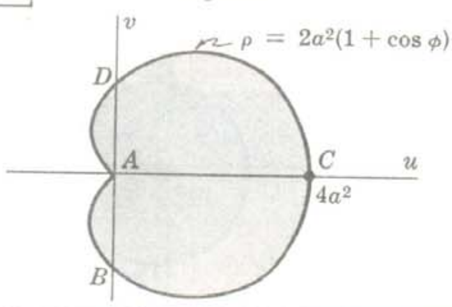
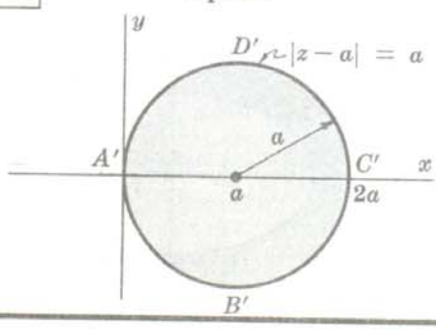


Fig. 8-54

z plane



C–3 Annulus on to rectangle $w = \ln z$

Fig. 8-55 z plane

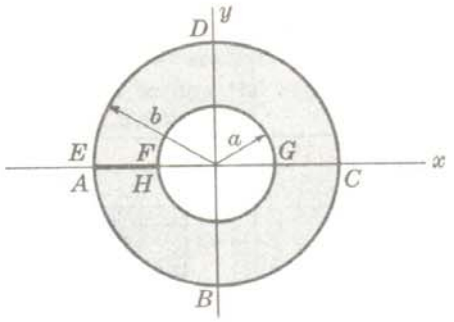
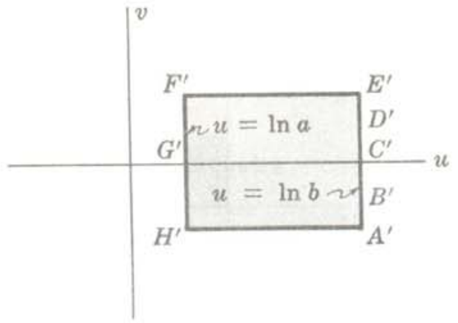


Fig. 8-56 w plane



C–4 Semi-infinite strip on to infinite strip $w = \ln \coth (z/2)$

Fig. 8-57 z plane

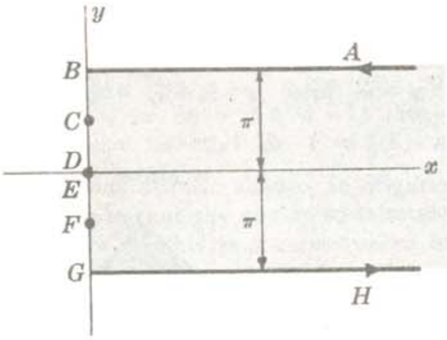
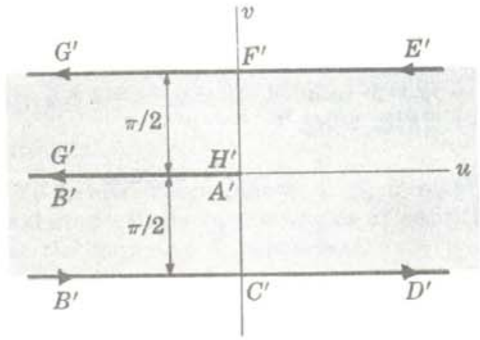


Fig. 8-58 w plane



C–5 Plane with two semi-infinite cuts on to infinite strip $w = z + e^z$

Fig. 8-59 w plane

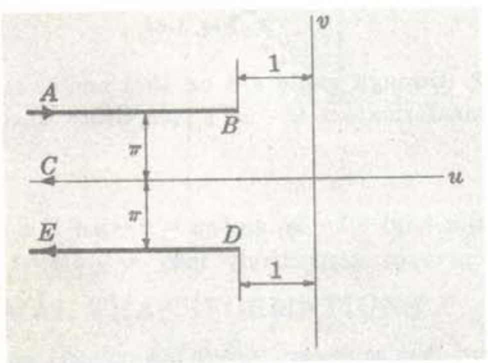
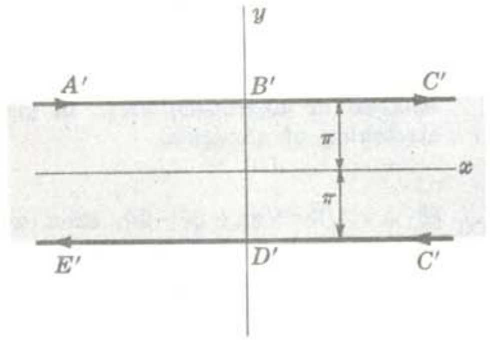


Fig. 8-60 z plane



Solved Problems

TRANSFORMATIONS

1. Let the rectangular region $\mathcal{R}$ [Fig. 8-61 below] in the z plane be bounded by $x = 0$, $y = 0$, $x = 2$, $y = 1$. Determine the region $\mathcal{R}'$ of the w plane into which $\mathcal{R}$ is mapped under the transformations:
(a) $w = z + (1 - 2i)$, (b) $w = \sqrt{2}\, e^{\pi i/4} z$, (c) $w = \sqrt{2}\, e^{\pi i/4} z + (1 - 2i)$.

(a) If $w = z + (1 - 2i)$, then $u + iv = x + iy + 1 - 2i = (x + 1) + i(y - 2)$ and $u = x + 1$, $v = y - 2$.