Given a set of data (x_i, f_i) $i = 0,1,\ldots n$. To construct a cubic spline which fits the data.

To find $y = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) \quad + d_i$ for each subinterval $[x_i, x_{i+1}]$.

We should find a_i, b_i, c_i, d_i for $i = 0, 1, \ldots, n - 1$.

Choosing the second derivatives, S_i, at the data points as auxiliary unknowns.

Writing a_i, b_i, c_i, d_i for $i = 0, 1, \ldots, n - 1$ in terms of S_i.

In

$y = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) \quad + d_i$.

Let $x = x_i$ then $f_i = d_i$.

Let $x = x_{i+1}$ then $f_{i+1} = a_i h_i^3 + b_i h_i^2 + c_i h_i + f_i$.

Denote $s = y'' = 6a_i(x - x_i) + 2b_i$.

Then at $x = x_i$ $S_i = 2b_i \Rightarrow b_i = \frac{S_i}{2}$ and at $x = x_{i+1} \Rightarrow$

$S_{i+1} = 6a_i h_i + S_i \Rightarrow a_i = (S_{i+1} - S_i)/6h_i$.

Upon substitution we obtain c_i in terms of S_i and S_{i+1}.

$c_i = \frac{f_{i+1} - f_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6}$

Now to calculate all S_i 's we use the continuity of the slope at x_i, that is

$y'(left) = y'(right)$ at x_i.

For the interval $[x_i, x_{i+1}]$: $y' = 3a_i(x - x_i)^2 + 2b_i(x - x_i) \quad + c_i$.

For the interval $[x_{i-1}, x_i]$: $y' =$
$3a_{i-1}(x - x_{i-1})^2 + 2b_{i-1}(x - x_{i-1}) \quad + c_{i-1}$

When $x = x_i$ in both left and right slopes yields:

$$3a_{i-1}h_{i-1}^2 + b_{i-1}h_{i-1} + c_{i-1} = c_i$$

Upon substitution of S_i 's and rearrangements, we'll have

$$h_{i-1}S_{i-1} + (2h_{i-1} + 2h_i)S_i + h_iS_{i+1} = 6\left(\frac{f_{i+1}-f_i}{h_i} - \frac{f_i-f_{i-1}}{h_{i-1}}\right)$$

Here $i = 1, \ldots, n-1$ and the unknowns are $S_0, S_1, \ldots, S_n$.

For the extra two unknowns we consider the end point conditions.

1. Take $S_0 = 0$ and $S_n = 0$. This condition is called *Natural Spline* or *Free Spline* .

2. If $f'(x_0) = A$ and $f'(x_n) = B$ are given then

$$2h_0S_0 + h_0S_1 = 6\left[\frac{f_1-f_0}{h_0} - A\right]$$ and

$$h_{n-1}S_{n-1} + 2h_{n-1}S_n = 6\left[B - \frac{f_n-f_{n-1}}{h_{n-1}}\right]$$

Are two extra equations. This condition is called *Clamped Spline*.

3. Take $S_0 = S_1$ and $S_n = S_{n-1}$.