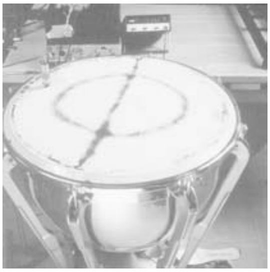
Answers to Odd-Numbered Exercises



Chapter 0

Section 0.1

1. $\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$.

3. The equation has constant coefficients $k = 0$, $p = 0$; $u(t) = c_1 + c_2 t$.

5. $w(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$.

7. Integrate, solve for dv/dx, and integrate again:

   $v(x) = c_1 + c_2 \ln|h + kx|$.

9. $u(x) = c_1 + c_2/x^2$.

11. $u(r) = c_1 + c_2 \ln(r)$.

13. Characteristic polynomial $m^4 - \lambda^4 = 0$; roots $m = \pm\lambda, \pm i\lambda$. General solution $u(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) + c_3 \cosh(\lambda x) + c_4 \sinh(\lambda x)$.

15. Characteristic polynomial $(m^2 + \lambda^2)^2 = 0$; roots $m = \pm i\lambda$ (double). General solution $u(x) = (c_1 + c_2 x)\cos(\lambda x) + (c_3 + c_4 x)\sin(\lambda x)$.

17. $v(t) = \ln(t)$ and $u_2(t) = t^b \ln(t)$.

19. $u'' + \lambda^2 u = 0$; $R(\rho) = (a\cos(\lambda\rho) + b\sin(\lambda\rho))/\rho$.

21. $t^2 d^2u/dt^2 = v'' - v'$; $t\,du/dt = v'$; $v'' + (k-1)v' + pv = 0$ (constant coefficients).

23. Roots of characteristic equation:

$$m = -\alpha \pm i\beta, \quad \beta = \sqrt{\sigma^2 - \alpha^2}.$$

Solution of differential equation:

$$y(t) = e^{-\alpha t}\big(c_1 \cos(\beta t) + c_2 \sin(\beta t)\big).$$

Initial conditions give: $c_1 = -0.001h$, $c_2 = (\alpha/\beta)c_1$.

25. $v = 2.62$ m/s.

Section 0.2

1. $u(t) = T + ce^{-at}$.
3. $u(t) = te^{-at} + ce^{-at}$.
5. $u(t) = \dfrac{1}{2} t \sin(t) + c_1 \cos(t) + c_2 \sin(t)$.
7. $u(t) = \dfrac{1}{12} e^t + \dfrac{1}{2} te^{-t} + c_1 e^{-t} + c_2 e^{-2t}$.
9. $u(\rho) = -\dfrac{1}{6}\rho^2 + \dfrac{c_1}{\rho} + c_2$.
11. $h(t) = -320t + c_1 + c_2 e^{-0.1t}$, $c_1 = h_0 + 3200$, $c_2 = -3200$.
13. $v(t) = t$, $u_p(t) = te^{-at}$.
15. $v_1(x) = \sin(x) - \ln|\sec(x) + \tan(x)|$, $v_2 = -\cos(x)$;

    $u_p(x) = -\cos(x) \ln|\sec(x) + \tan(x)|$.
17. $v_1(t) = t^2/2$, $v_2(t) = -t$; $u_p(t) = -t^2/2$.
19. $v_1(t) = -1/2t$, $v_2(t) = -t/2$, $u_p(t) = -1$.
23. $\beta = 1/\alpha$, $K = R\alpha/\rho c$.
25. $T = \beta(\exp(KI^2_{\max}(1 - e^{-2\lambda t})/2\lambda) - 1)$.

Section 0.3

1. a. $u(x) = c_2 \sin(x)$, c_2 arbitrary;

   b. $u(x) = 1 - \cos(x) - \dfrac{1 - \cos(1)}{\sin(1)} \sin(x)$ (unique);

   c. No solution exists.
3. a. and b. $\lambda = \pm(2n-1)\dfrac{\pi}{2a}$, $n = 1, 2, \ldots$;

c. $\lambda = \pm\dfrac{n\pi}{a}$, $n = 0, 1, 2, \ldots$.

5. $c = -a/2$, $c' = h - \dfrac{1}{\mu}\cosh\left(\dfrac{\mu a}{2}\right)$.

7. $u(x) = T + c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x)$, where

$\gamma = \sqrt{\dfrac{hC}{\kappa A}}$ and $c_1 = T_0 - T$, $c_2 = -\dfrac{\kappa\gamma \sinh(\gamma a) + h\cosh(\gamma a)}{\kappa\gamma \cosh(\gamma a) + h\sinh(\gamma a)} c_1$.

9. $u(x) = T + H\left(1 - \cosh(\gamma x) - \dfrac{1-\cosh(\gamma a)}{\sinh(\gamma a)}\sinh(\gamma x)\right)$,

where $H = \dfrac{I^2 R}{hC}$ and $\gamma = \sqrt{\dfrac{hC}{\kappa A}}$.

11. $u(y) = y(L-y)g/2\mu$.

13. $P = EI(n\pi/L)^2$, $n = 1, 2, \ldots$.

15. $u(x) = T + A\left(1 - \cosh(\gamma x) - \dfrac{1-\cosh(\gamma a)}{\sinh(\gamma a)}\sinh(\gamma x)\right)$,

$A = g/\kappa\gamma^2$, and $\gamma = \sqrt{\dfrac{hC}{\kappa A}}$.

17. $u(r) = c_1 \ln(r/a) + c_2$, $c_1 = h_0 h_1 (T_a - T_W)/D$, $c_2 = [h_0(\kappa/b + h_1 \ln(b/a))T_W + (\kappa/a)h_1 T_a]/D$, $D = h_1\kappa/a + h_0\kappa/b + h_0 h_1 \ln(b/a)$.

19. $u(x) = \dfrac{w_0}{EI}\left(\dfrac{x^4}{24} - \dfrac{ax^3}{6} + \dfrac{a^2x^2}{2}\right)$.

Section 0.4

1. a. $u'' + \dfrac{1}{r}u' - u = 0$, $r = 0$;

b. $u'' - \dfrac{2x}{1-x^2}u' = 0$, $x = \pm 1$;

c. $u'' + \cot(\phi)u' - u = 0$, $\phi = 0, \pm\pi, \pm 2\pi, \ldots$;

d. $u'' + \dfrac{2}{\rho}u' + \lambda^2 u = 0$, $\rho = 0$.

3. $u(0)$ bounded; $u(\rho) = \dfrac{H}{6\kappa}\left(c^2 - \rho^2\right) + \dfrac{Hc}{3h} + T$.

5. $u(\rho) = \dfrac{1}{\rho}\left(c_1 \cos(\mu\rho) + c_2 \sin(\mu\rho)\right)$,

$u(\rho) \equiv 0$ unless $\mu a = \pi, 2\pi, \ldots$. The critical radius is $a = \dfrac{\pi}{\mu}$.

7. $u(r) = 325 + 10^4(0.25 - r^2)/4$; $u(0) = 950$.

9. $u(x) = T_0 + AL^2(1 - e^{-x/L})$.

Section 0.5

1. $G(x,z) = \begin{cases} z(a-x)/(-a), & 0 < z \le x, \\ x(a-z)/(-a), & x \le z < a. \end{cases}$

3. $G(x,z) = \begin{cases} \cosh(\gamma z)\sinh(\gamma(a-x))/(-\gamma\cosh(\gamma a)), & 0 < z \le x, \\ \cosh(\gamma x)\sinh(\gamma(a-z))/(-\gamma\cosh(\gamma a)), & x \le z < a. \end{cases}$

5. $G(\rho,z) = \begin{cases} \dfrac{(c-\rho)/\rho}{-c/z^2}, & 0 \le z < \rho, \\ \dfrac{(c-z)/z}{-c/z^2}, & \rho \le z < c. \end{cases}$

7. $G(x,z) = \begin{cases} \dfrac{\sinh(\gamma z)e^{-\gamma x}}{-\gamma}, & 0 < z \le x, \\ \dfrac{\sinh(\gamma x)e^{-\gamma z}}{-\gamma}, & x \le z. \end{cases}$

9. $u(\rho) = (\rho^2 - c^2)/6$.

11. $u(x) = \displaystyle\int_0^a G(x,z)f(z)dz = \int_0^x \frac{z(a-x)}{-a}f(z)dz + \int_x^a \frac{x(a-z)}{-a}f(z)dz.$

There are two cases:

(i) $x \le a/2$, so $u(x) = \displaystyle\int_{a/2}^a \frac{x(a-z)}{-a}dz;$

and

(ii) $x > a/2$, so $u(x) = \displaystyle\int_x^a \frac{x(a-z)}{-a}dz.$

Results: $u(x) = \begin{cases} -ax/8, & 0 < x < a/2, \\ -x(a-x)^2/2a, & a/2 < x < a. \end{cases}$

13. (i) At the left boundary, $x = l < z$, so the second line of Eq. (17) holds. The boundary condition (2) is satisfied by v because it is satisfied by u_1. At the right boundary, use the first line of Eq. (17).

(ii) At $x = z$, both lines of Eq. (17) give the same value.

(iii) $v'(z+h) - v'(z-h) = \dfrac{u_1(z)u_2'(z+h) - u_1'(z-h)u_2(z)}{W(z)}$.

As h approaches 0, the numerator approaches $W(z)$.

(iv) This is true because $u_1(x)$ and $u_2(x)$ are solutions of the homogeneous equation.

Chapter 0 Miscellaneous Exercises

1. $u(x) = T_0 \cosh(\gamma x) + \big(T_1 - T_0 \cosh(\gamma a)\big)\dfrac{\sinh(\gamma x)}{\sinh(\gamma a)}$.

3. $u(x) = T_0$.

5. $u(r) = p(a^2 - r^2)/4$.

7. $u(\rho) = H(a^2 - \rho^2)/6 + T_0$.

9. $u(x) = T + (T_1 - T)\cosh(\gamma x)/\cosh(\gamma a)$.

11. $u(x) = T_0 + (T - T_0)e^{-\gamma x}$.

13. $h(x) = \sqrt{ex(a - x) + h_0^2 + (h_1^2 - h_0^2)(x/a)}$.

15. $u(x) = w(1 - e^{-\gamma x}\cos(\gamma x))EI/k$, where $\gamma = (k/4EI)^{1/4}$.

17. $u(x) = \begin{cases} T_0 + Ax, & 0 < x < \alpha a, \\ T_1 - B(a - x), & \alpha a < x < a, \end{cases}$

$$A = \frac{\kappa_2}{\kappa_1(1-\alpha) + \kappa_2\alpha}\frac{T_1 - T_0}{a}, \qquad B = \frac{\kappa_1}{\kappa_2}A.$$

19. $u(x) = \dfrac{1}{2}\big(1 - e^{-2x}\big) - \dfrac{1}{2}\big(1 - e^{-2a}\big)\dfrac{1 - e^{-x}}{1 - e^{-a}}$.

21. a. $u(x) = \sinh(px)/\sinh(pa)$;

    b. $u(x) = \cosh(px) - \dfrac{\cosh(pa)}{\sinh(pa)}\sinh(px) = \sinh\big(p(a - x)\big)/\sinh(pa)$;

    c. $u(x) = \cosh(px)/\cosh(pa)$;

    d. $u(x) = \cosh(p(a - x))/\cosh(pa)$;

    e. $u(x) = -\cosh(p(a - x))/p\sinh(pa)$;

    f. $u(x) = \cosh(px)/p\sinh(pa)$.

23. $u(x) = \dfrac{x}{2}\ln\left|\dfrac{1 + x}{1 - x}\right| - 1$.

25. Multiply by u' and integrate: $\frac{1}{2}(u')^2 = \frac{1}{5}\gamma^2 u^5 + c_1$. Since $u(x) \to 0$ as $x \to \infty$, also $u'(x) \to 0$; thus $c_1 = 0$. Now $u' = -\sqrt{2\gamma^2/5}u^{5/2}$ or $u^{-5/2}u' = -\sqrt{2\gamma^2/5}$ (the negative root makes u decrease) can be integrated to result in $(-2/3)u^{-3/2} = -\sqrt{2\gamma^2/5x} + c_2$. The condition at $x = 0$ gives $c_2 = (-3/2)U^{-3/2}$.

Finally $u(x) = (U^{-3/2} + (3/2)\sqrt{2\gamma^2/5x})^{-2/3}$.

27. 459.77 rad/s.

29. $u(x) = C_0 e^{-ax}$.

31. $w(x) = \dfrac{P}{2\gamma^2}\left[\dfrac{1}{4} - x^2 + \dfrac{\cosh(\gamma x) - \cosh(\gamma/2)}{\gamma \sinh(\gamma/2)}\right]$.

33. The solution breaks down (buckling occurs) if $\tan(\lambda/2) = \gamma/2$.

Chapter 1

Section 1.1

1. a. $2\left(\sin(x) - \dfrac{1}{2}\sin(2x) + \dfrac{1}{3}\sin(3x) - \cdots\right)$;

   b. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos(x) + \dfrac{1}{9}\cos(3x) + \dfrac{1}{25}\cos(5x) + \cdots\right)$;

   c. $\dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin(x) + \dfrac{1}{3}\sin(3x) + \dfrac{1}{5}\sin(5x) + \cdots\right)$;

   d. $\dfrac{2}{\pi} - \dfrac{4}{\pi}\left(\dfrac{1}{3}\cos(2x) + \dfrac{1}{15}\cos(4x) + \dfrac{1}{35}\cos(6x) + \cdots\right)$.

3. $f(x+p) = 1 = f(x)$ for any p and all x.

5. If c is a multiple of p, the graph of $f(x)$ between c and $c+p$ is the same as that between 0 and p. Otherwise, let k be the integer such that kp lies between c and $c+p$:

$$\int_c^{c+p} f(x)dx = \int_c^{kp} f(x)dx + \int_{kp}^{c+p} f(x)dx = \int_{c^*}^{p} f(x)dx + \int_0^{c^*} f(x)dx,$$

   where $c^* = c - (k-1)p$.

7. a. $\cos^2(x) = \dfrac{1}{2} + \dfrac{1}{2}\cos(2x)$;

   b. $\sin\left(x - \dfrac{\pi}{6}\right) = \cos\left(\dfrac{\pi}{6}\right)\sin(x) - \sin\left(\dfrac{\pi}{6}\right)\cos(x)$;

   c. $\sin(x)\cos(2x) = -\dfrac{1}{2}\sin(x) + \dfrac{1}{2}\sin(3x)$.

Section 1.2

1. a. $\dfrac{1}{2} - \dfrac{4}{\pi^2}\left[\cos(\pi x) + \dfrac{1}{9}\cos(3\pi x) + \dfrac{1}{25}\cos(5\pi x) + \cdots\right]$;

b. $\frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)+\cdots\right]$;

c. $\frac{1}{12}-\frac{1}{\pi^2}\left[\cos(2\pi x)-\frac{1}{4}\cos(4\pi x)+\frac{1}{9}\cos(6\pi x)-\cdots\right]$.

3. $\bar{f}(x)=f(x-2na)$, $2na<x<2(n+1)a$,

$$f(x)\sim a_0+\sum_1^\infty a_n\cos(n\pi x/a)+b_n\sin(n\pi x/a),$$

$$a_0=\frac{1}{2a}\int_0^{2a}f(x)dx,\ a_n=\frac{1}{a}\int_0^{2a}f(x)\cos(n\pi x/a)dx,$$

$$b_n=\frac{1}{a}\int_0^{2a}f(x)\sin(n\pi x/a)dx.$$

5. Odd: (a), (d), (e); even: (b), (c); neither: (f).

7. a. $\frac{2}{\pi}\left(\sin(\pi x)-\frac{1}{2}\sin(2\pi x)+\cdots\right)$;

b. This function is its own Fourier series;

c. $\frac{4}{\pi^2}\left(\sin(\pi x)-\frac{1}{9}\sin(3\pi x)+\frac{1}{25}\sin(5\pi x)-\cdots\right)$.

9. If $f(-x)=-f(x)$ and $f(x)=f(a-x)$ for $0<x<a$, sine coefficients with even indices are zero. Example: square wave.

11. a. $f(x)=1=\frac{2}{\pi}\sum_1^\infty\frac{1-\cos(n\pi)}{n}\sin\left(\frac{n\pi x}{a}\right)$;

b. $$f(x)=\frac{a}{2}-\frac{2a}{\pi^2}\sum_1^\infty\frac{1-\cos(n\pi)}{n^2}\cos\left(\frac{n\pi x}{a}\right)$$
$$=\frac{2a}{\pi}\sum_1^\infty\frac{-\cos(n\pi)}{n}\sin\left(\frac{n\pi x}{a}\right);$$

c. $$f(x)=\sum_1^\infty(-1)^{n+1}\sin(1)\frac{2n\pi}{(n\pi)^2-1}\sin(n\pi x),\quad 0<x<1$$
$$=\sum_1^\infty\left((-1)^n\cos(1)-1\right)\frac{2}{(n\pi)^2-1}\cos(n\pi x),\quad 0<x<1;$$

d. $f(x)=\frac{2}{\pi}\left[1-\sum_1^\infty\frac{1+\cos(n\pi)}{n^2-1}\cos(nx)\right]=\sin(x)$.

13. Even, yes. Odd, yes only if $f(0)=f(a)=0$.

Section 1.3

1. a. sectionally smooth; b, c, d, e are not;

   b: vertical tangent at 0; c: vertical asymptote at $\pm\pi/2$; d, e: vertical asymptote at $\pi/2$.

3. To $f(x)$ everywhere.

5. b. Graph consists of straight-line segments. c. $x = 1$, sum $= 1/2$; $x = 2$, sum $= 0$; $x = 9.6$, sum $= -0.6$; $x = -3.8$, sum $= 0.2$. Use periodicity.

7. $B = 0$, $A = -\pi^2/12$, $C = 1/4$.

9. a. $\sqrt{1-x^2}$; b. $a_0 = \pi/4$; c. No; d. nothing.

Section 1.4

1. (c), (d), (f), (g) have uniformly convergent Fourier series.

3. All of the cosine series converge uniformly. The sine series converges uniformly only in case (b).

5. (a), (b), (d) converge uniformly; (c) does not.

Section 1.5

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

3. $f'(x) = 1$, $0 < x < \pi$. The sine series cannot be differentiated, because the odd periodic extension of f is not continuous. But the cosine series can be differentiated.

5. For the sine series: $f(0+) = 0$ and $f(a-) = 0$. For the cosine series no additional condition is necessary.

7. No. The function $\ln|2\cos(\frac{x}{2})|$ is not even sectionally continuous.

9. Since f is odd, periodic, and sectionally smooth, (c) follows, and also $b_n \to 0$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} |n^k b_n e^{-n^2 t}|$ converges for all integers k ($t > 0$) by the comparison test and ratio test:

   $$\left|n^k b_n e^{-n^2 t}\right| \le M n^k e^{-n^2 t} \quad \text{for some } M$$

   and

   $$\frac{M(n+1)^k e^{-(n+1)^2 t}}{M n^k e^{-n^2 t}} = \left(\frac{n+1}{n}\right)^k e^{-(2n+1)t} \to 0$$

   as $n \to \infty$. Then by Theorem 7, (a) is valid. Property (b) follows by direction substitution.

Section 1.6

1. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left(\ln\left|2\cos\left(\frac{x}{2}\right)\right|\right)^2 dx = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$

3. a. Coefficients tend to zero.

   b. Coefficients tend to zero, although $\displaystyle\int_{-1}^{1}|x|^{-1}\,dx$ is infinite.

5. The integral must be infinite, because $\displaystyle\sum_{n=1}^{\infty} a_n^2 + b_n^2 = \infty.$

Section 1.7

1. The equality to be proved is

$$2\sin\left(\frac{1}{2}y\right)\left(\frac{1}{2}+\sum_{n=1}^{N}\cos(ny)\right) = \sin\left(\left(N+\frac{1}{2}\right)y\right).$$

   The left-hand side is transformed as follows:

$$\begin{aligned}
&2\sin\left(\frac{1}{2}y\right)\left(\frac{1}{2}+\sum_{n=1}^{N}\cos(ny)\right)\\
&= \sin\left(\frac{1}{2}y\right)+\sum_{n=1}^{N}2\sin\left(\frac{1}{2}y\right)\cos(ny)\\
&= \sin\left(\frac{1}{2}y\right)+\sum_{n=1}^{N}\left(\sin\left(\left(n+\frac{1}{2}\right)y\right)-\sin\left(\left(n-\frac{1}{2}\right)y\right)\right)\\
&= \sin\left(\frac{1}{2}y\right)+\sum_{n=1}^{N}\sin\left(\left(n+\frac{1}{2}\right)y\right)-\sum_{n=0}^{N-1}\sin\left(\left(n+\frac{1}{2}\right)y\right)\\
&= \sin\left(\left(N+\frac{1}{2}\right)y\right)
\end{aligned}$$

   because all other terms cancel.

3. $\phi(0+) = 1$, $\phi(0-) = -1$. See Fig. 1.

5. a. $f'(x) = \frac{3}{4}x^{-1/4}$ for $0 < x < \pi$ (and f' is an odd function). Thus, f has a vertical tangent at $x = 0$, although it is continuous there.

   b. $\phi(y) = \dfrac{|y|^{3/4}}{2\sin(\frac{1}{2}y)}\cos\left(\dfrac{1}{2}y\right), \quad -\pi < y < \pi$

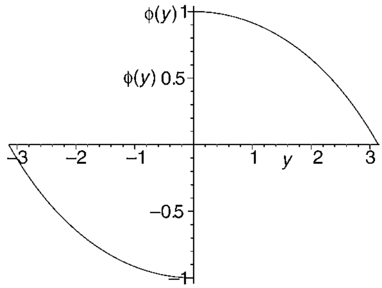


Figure 1 Graph for Exercise 3, Section 1.7.

is a product of continuous functions and is therefore continuous, except perhaps where the denominator is 0. At $y=0$, $\cos(\frac{1}{2}y)\cong 1$, $2\sin(\frac{1}{2}y)\cong y$, so $\phi(y)\cong |y|^{3/4}/y=\pm|y|^{-1/4}$ near $y=0$.

c. Now, $\int_{-\pi}^{\pi}\phi^2(y)dy$ is finite, so the Fourier coefficients of ϕ approach zero.

Section 1.8

1. $\hat{a}_6=-0.00701$, $a_6=-0.00569$.

3. $\hat{a}_0=1.367$,
$\hat{a}_1=-0.844$, $\hat{b}_1=-0.043$,
$\hat{a}_2=0.208$, $\hat{b}_2=-0.115$,
$\hat{a}_3=0.050$, $\hat{b}_3=-0.050$,
$\hat{a}_4=0.042$, $\hat{b}_4=0.00$,
$\hat{a}_5=-0.0064$, $\hat{b}_5=0.043$,
$\hat{a}_6=0.0167$.

Section 1.9

1. Each function has the representations (for $x>0$)

$$f(x)=\int_0^\infty A(\lambda)\cos(\lambda x)d\lambda=\int_0^\infty B(\lambda)\sin(\lambda x)d\lambda.$$

a. $A(\lambda)=2/\pi(1+\lambda^2)$, $B(\lambda)=2\lambda/\pi(1+\lambda^2)$;

b. $A(\lambda)=2\sin(\lambda)/\pi\lambda$, $B(\lambda)=2(1-\cos(\lambda))/\pi\lambda$;

c. $A(\lambda)=2(1-\cos(\lambda\pi))/\lambda^2\pi$, $B(\lambda)=2(\pi\lambda-\sin(\lambda\pi))/\pi\lambda^2$.

3. a. $\dfrac{1}{1+x^2}=\displaystyle\int_0^\infty e^{-\lambda}\cos(\lambda x)d\lambda$;

b. $\dfrac{\sin(x)}{x} = \displaystyle\int_0^\infty A(\lambda)\cos(\lambda x)d\lambda$, where $A(\lambda) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x. \end{cases}$

5. a. $A(\lambda) \equiv 0$, $B(\lambda) = \dfrac{2\sin(\lambda\pi)}{\pi(1-\lambda^2)}$;

b. $A(\lambda) = \dfrac{1+\cos(\lambda\pi)}{\pi(1-\lambda^2)}$, $B(\lambda) = \dfrac{\sin(\lambda\pi)}{\pi(1-\lambda^2)}$;

c. $A(\lambda) = \dfrac{2(1+\cos(\lambda\pi))}{\pi(1-\lambda^2)}$, $B(\lambda) \equiv 0$.

7. Change variable from x to λ with $x = \lambda z$.

Section 1.10

1. $e^{\alpha x} = 2\dfrac{\sinh(\alpha\pi)}{\pi}\left(\dfrac{1}{2\alpha} + \displaystyle\sum_{n=1}^\infty \dfrac{(-1)^n}{\alpha^2+n^2}\big(\alpha\cos(nx) - n\sin(nx)\big)\right)$.

3. $f(x) = \displaystyle\int_{-\infty}^\infty C(\lambda)e^{i\lambda x}d\lambda$.

a. $C(\lambda) = \dfrac{1}{2\pi(1+i\lambda)}$; b. $C(\lambda) = \dfrac{1+e^{-i\lambda\pi}}{2\pi(1-\lambda^2)}$.

5. a. $1 + \displaystyle\sum_{n=1}^\infty r^n\cos(nx) = \operatorname{Re}\sum_0^\infty \big(re^{ix}\big)^n = \operatorname{Re}\dfrac{1}{1-re^{ix}}$;

b. $\displaystyle\sum_{n=1}^\infty \dfrac{\sin(nx)}{n!} = \operatorname{Im}\sum_{n=1}^\infty \dfrac{e^{inx}}{n!} = \operatorname{Im}\exp\big(e^{ix}\big)$.

7. a. $f(x) = \dfrac{2\sin(x)}{x}$; b. $f(x) = \dfrac{2}{1+x^2}$.

Section 1.11

1. $u(t) = A_0 + \displaystyle\sum_{n=1}^\infty A_n\cos(nt/2) + B_n\sin(nt/2)$,

$A_0 = \dfrac{1}{2.08}$, $A_n = \dfrac{0.4/\pi}{(1.04-n^2)^2 + (0.4n)^2}$,

$B_n = -\dfrac{1}{n\pi}\dfrac{1.04-n^2}{(1.04-n^2)^2+(0.4n)^2}$.

3. $u(x) = \displaystyle\sum_{n=1}^\infty B_n\sin(n\pi x/L)$, $B_n = \dfrac{8K\sin(n\pi/2)}{((n\pi/L)^2+\gamma^2)n^2\pi^2}$,

$K = w/EI$, $\gamma^2 = T/EI$.

Chapter 1 Miscellaneous Exercises

1. $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$

$$b_n = \begin{cases} 0, & n \text{ even}, \\ \dfrac{4\sin(n\alpha)}{\pi\alpha n^2}, & n \text{ odd}. \end{cases}$$

3. Yes. As $\alpha \to 0$, $\sin(n\alpha)/n\alpha \to 1$.

5. $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/a),$

$$b_n = \frac{2h}{\pi^2}\frac{\sin(n\pi\alpha)}{n^2}\left(\frac{1}{\alpha} + \frac{1}{1-\alpha}\right).$$

7. a. $b_n = 0$, $a_n = 0$, $a_0 = 1$;

b. $\sum_{n=1}^{\infty} b_n \sin(n\pi x/a)$, $b_n = \dfrac{2(1-\cos(n\pi))}{n\pi}$;

c. and d. same as a;

e. same as b;

f. $a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/a) + b_n \sin(n\pi x/a),$

$$a_0 = \frac{1}{2},\ a_n = 0,\ b_n = \frac{1-\cos(n\pi)}{n\pi}.$$

9. $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/a) + b_n \sin(n\pi x/a),$

$$a_0 = \frac{1}{2}a,\ a_n = -\frac{2a(1-\cos(n\pi))}{n^2\pi^2},\ b_n = -\frac{2a\cos(n\pi)}{n\pi},$$

$$\begin{array}{llllll} x = -a, & -a/2, & 0, & a, & 2a, \\ \text{sum} = a, & 0, & 0, & a, & 0. \end{array}$$

11. $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$

$$a_0 = \frac{3}{4},\ a_n = \frac{\sin(n\pi/2)}{n\pi},$$

$$\begin{array}{llllll} x = 0, & \pi/2, & \pi, & 3\pi/2, & 2\pi, \\ \text{sum} = 1, & \frac{3}{4}, & \frac{1}{2}, & \frac{3}{4}, & 1. \end{array}$$

13. $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, $b_n = 2\big(1 + \cos(n\pi)\big)/n\pi$.

15. $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$,

$b_2 = \dfrac{1}{2}$, other $b_n = \dfrac{4\sin(n\pi/2)}{\pi(4-n^2)}$.

17. $\sum_{1}^{N} \cos(nx) = \operatorname{Re} \sum_{1}^{N} e^{inx} = \operatorname{Re} \dfrac{e^{ix} - e^{iNx}}{1 - e^{ix}} = \operatorname{Re} \dfrac{e^{ix/2} - e^{i(2N-1)x/2}}{e^{-ix/2} - e^{ix/2}}$.

The denominator is now $-2i\sin(x/2)$.

19. $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, $b_n = \dfrac{2a\sin(na+\pi)}{n^2a^2 - \pi^2}$.

21. $f(x) = \displaystyle\int_0^{\infty} \left(\frac{\sin(\lambda a)}{\lambda\pi} \cos(\lambda x) + \frac{1-\cos(\lambda a)}{\lambda\pi} \sin(\lambda x) \right) d\lambda$.

23. $f(x) = \displaystyle\int_0^{\infty} \frac{2\sin(\lambda\pi)}{\pi(1-\lambda^2)} \sin(\lambda x)\, d\lambda \quad (x > 0)$.

29. Use $\displaystyle\int_0^{\infty} \frac{\sin(\lambda t)}{\lambda} d\lambda = \frac{\pi}{2}$.

31. These answers are not unique.

a. $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$, $b_n = 2/n\pi$;

b. $a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$, $a_0 = \dfrac{1}{2}$, $a_n = 2\big(1 - \cos(n\pi)\big)/n^2\pi^2$;

c. $\displaystyle\int_0^{\infty} B(\lambda) \sin(\lambda x)\, d\lambda$, $B(\lambda) = 2\big(\lambda - \sin(\lambda)\big)/\big(\pi\lambda^2\big)$;

d. $\displaystyle\int_0^{\infty} A(\lambda) \cos(\lambda x)\, d\lambda$, $A(\lambda) = 2\big(1 - \cos(\lambda)\big)/\big(\pi\lambda^2\big)$.

The integrals of parts c. and d. converge to 0 for $x > 1$.

33. Use $s = 6$ in Eq. (7) of Section 8.

$\hat{a}_0 = 0.78424$, $\hat{a}_4 = -0.00924$,
$\hat{a}_1 = 0.22846$, $\hat{a}_5 = 0.00744$,
$\hat{a}_2 = -0.02153$, $\hat{a}_6 = -0.00347$,
$\hat{a}_3 = 0.01410$.

35. $a_0 = \dfrac{a}{6}$, $a_n = \dfrac{2a}{n^2\pi^2}\left(\cos\left(\dfrac{2n\pi}{3}\right) - \cos\left(\dfrac{n\pi}{3}\right)\right)$.

37. $a_0 = \dfrac{5}{8}$, $a_n = \dfrac{2}{n^2\pi^2}\left(3\cos\left(\dfrac{n\pi}{2}\right) - 2 - \cos(n\pi)\right)$.

39. $a_0 = \dfrac{1}{2}$, $a_n = \dfrac{2}{n^2\pi^2}\big(1 - \cos(n\pi)\big)$.

41. $a_0 = \dfrac{a^2}{6}$, $a_n = \dfrac{-2a^2}{n^2\pi^2}\big(1 + \cos(n\pi)\big)$.

43. $a_0 = \dfrac{1}{2}$, $a_n = \dfrac{-1}{n\pi} 2\sin\left(\dfrac{n\pi}{2}\right)$.

45. $b_n = \dfrac{1 + \cos(n\pi/2) - 2\cos(n\pi)}{n\pi}$.

47. $b_n = a\left(\dfrac{2\sin(n\pi/2)}{n^2\pi^2} - \dfrac{\cos(n\pi)}{n\pi}\right)$.

49. $b_n = \dfrac{2}{n\pi}\left(\cos\left(\dfrac{n\pi}{4}\right) - \cos\left(\dfrac{3n\pi}{4}\right)\right)$.

51. $b_n = 2n\pi\,\dfrac{(1 - e^{ka}\cos(n\pi))}{(a^2k^2 + n^2\pi^2)}$.

53. $A(\lambda) = \dfrac{2}{\pi(1+\lambda^2)}$.

55. $A(\lambda) = \dfrac{2\sin(\lambda b)}{\pi\lambda}$.

57. $A(\lambda) = \dfrac{2(1 - \cos(\lambda))}{\pi\lambda^2}$.

59. $B(\lambda) = \dfrac{2\lambda}{\pi(1+\lambda^2)}$.

61. $B(\lambda) = \dfrac{2(1 - \cos(\lambda b))}{\lambda\pi}$.

63. $B(\lambda) = \dfrac{2(\lambda - \sin(\lambda))}{\lambda^2\pi}$.

65. The term $a_n\cos(nx) + b_n\sin(nx)$ appears in $S_n, S_{n+1}, \ldots, S_N$, and thus $N + 1 - n$ times in σ_N.

67. Use Eq. (13) of Section 7 and the identity in Exercise 66.

69. a. Use $x = 0$; b. $x = 1/2$; c. $x = 0$.

Chapter 2

Section 2.1

1. One possibility: $u(x, t)$ is the temperature in a rod of length a whose lateral surface is insulated. The temperature at the left end is held constant at T_0. The right end is exposed to a medium at temperature T_1. Initially the temperature is $f(x)$.

3. $A\,\Delta x g = hC\Delta x(U - u(x, t))$, where h is a constant of proportionality and C is the circumference. Eq. (4) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{hC}{\kappa A}(U - u) = \frac{1}{k}\frac{\partial u}{\partial t}.$$

5. If $\frac{\partial x}{\partial u}(0, t)$ is positive, then heat is flowing to the left, so $u(0, t)$ is greater than $T(t)$.

7. The second factor is approximately constant if T is much larger than u or if T and u are approximately equal.

Section 2.2

1. $v'' - \gamma^2(\nu - U) = 0,\ 0 < x < a,$

   $v(0) = T_0,\ v(a) = T_1,$

   $v(x) = U + A\cosh(\gamma x) + B\sinh(\gamma x),$

   $A = T_0 - U,\ B = \dfrac{(T_1 - U) - (T_0 - U)\cosh(\gamma a)}{\sinh(\gamma a)}.$

   One interpretation: u is the temperature in a rod, with convective heat transfer from the cylindrical surface to a medium at temperature U.

3. $v(x) = T$. Heat is being generated at a rate proportional to $u - T$. If $\gamma = \pi/a$, the steady-state problem does not have a unique solution.

5. $v(x) = A\ln(\kappa_0 + \beta x) + B,\ A = (T_1 - T_0)/\ln(1 + a\beta/\kappa_0),$

   $B = T_0 - A\ln(\kappa_0).$

7. $v(x) = T_0 + r(2a - x)x/2.$

9. $Du'' - Su' = 0,\ 0 < x < a;\ u(0) = U,\ u(a) = 0,$

   $u(x) = U(e^{Sx/D} - e^{Sa/D})/(1 - e^{Sa/D}).$

Section 2.3

1. $$w(x,t) = -\frac{2}{\pi}(T_0 + T_1)\sin\left(\frac{\pi x}{a}\right)\exp\left(-\frac{\pi^2 kt}{a^2}\right) - \frac{2}{\pi}\left(\frac{T_0 - T_1}{2}\right)\sin\left(\frac{2\pi x}{a}\right)\exp\left(-\frac{4\pi^2 kt}{a^2}\right) - \cdots$$

3. The partial differential equation is

$$\frac{\partial^2 U}{\partial \xi^2} = \frac{\partial U}{\partial \tau}, \quad 0 < \xi < 1, \quad 0 < \tau.$$

5. $w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-n^2\pi^2 kt/a^2\right)$, $b_n = T_0 \dfrac{2(1 - \cos(n\pi))}{\pi n}$.

7. $w(x,t)$ as in the answer to Exercise 5, with $b_n = \dfrac{2\beta a}{\pi} \cdot \dfrac{1}{n}$.

9. a. $v(x) = C_1$;

   b. $\dfrac{\partial w}{\partial t} = D\dfrac{\partial^2 w}{\partial x^2}, \quad 0 < x < a, \quad 0 < t,$

   $w(0,t) = 0, \quad w(a,t) = 0, \quad 0 < t,$

   $w(x,0) = C_0 - C_1$;

   c. $C(x,t) = C_1 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-n^2\pi^2 kt/a^2\right),$

   $b_n = (C_0 - C_1)\dfrac{2(1 - \cos(n\pi))}{\pi n}$;

   d. $t = \dfrac{-a^2}{D\pi^2}\ln\left(\dfrac{\pi}{40}\right)$;

   e. $t = 6444 \quad \mathrm{s} = 107.4$ min.

Section 2.4

1. $a_0 = T_1/2$, $a_n = 2T_1(\cos(n\pi) - 1)/(n\pi)^2$.

3. $u(x,t)$ as given in Eq. (9), with $\lambda_n = n\pi/a$, $a_0 = T_0/2$, and $a_n = 4T_0(2\cos(n\pi/2) - 1 - \cos(n\pi))/n^2\pi^2$.

5. a. The general solution of the steady-state equation is $v(x) = c_1 + c_2 x$. The boundary conditions are $c_2 = S_0$, $c_2 = S_1$; thus there is a solution if $S_0 = S_1$. If heat flux is different at the ends, the temperature cannot approach a steady state. If $S_0 = S_1$, then $v(x) = c_1 + S_0 x$, c_1 undefined.

c. $A = (S_1 - S_0)/a$, $B = S_0$. If $S_0 \neq S_1$, then $\dfrac{\partial u}{\partial t} = kA$ for all t.

7. $\phi'' + \lambda^2\phi = 0$, $0 < x < a$,

$\phi(0) = 0$, $\phi(a) = 0$.

Solution: $\phi_n = \sin(\lambda_n x)$, $\lambda_n = n\pi/a (n = 1, 2, \ldots)$.

9. The series $\displaystyle\sum_{n=1}^{\infty} |A_n(t_1)|$ converges.

11. No. $u(0, t)$ is constant if $u_t(0, t) = 0$.

Section 2.5

1. $v(x, t) = T_0$.

3. $u(x, t) = T_0 + \displaystyle\sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\bigl(-\lambda_n^2 kt\bigr)$, $\lambda_n = (2n - 1)\pi/2a$,

$$b_n = \frac{8T(-1)^{n+1}}{\pi^2(2n-1)^2} - \frac{4T_0}{\pi(2n-1)}.$$

5. The steady-state solution is $v(x) = T_0 - Tx(x - 2a)/2a^2$. The transient satisfies Eqs. (5)–(8) with

$$g(x) = T_0 - v(x) = \frac{Tx(x-2a)}{2a^2}.$$

7. $u(x, t) = T_0 + \displaystyle\sum_{n=1}^{\infty} c_n \cos(\lambda_n x) \exp\bigl(-\lambda_n^2 kt\bigr)$,

$$\lambda_n = (2n-1)\pi/2a, \quad c_n = \frac{4(T_1 - T_0)(-1)^{n+1}}{\pi(2n-1)}.$$

9. $u(x, t) = T_1 \cos(\pi x/2a) \exp\left(-\left(\dfrac{\pi}{2a}\right)^2 kt\right)$.

11. The graph of G in the interval $0 < x < 2a$ is made by reflecting the graph of g in the line $x = a$ (like an even extension).

13. a. $u(x, t) = T_0 + \displaystyle\sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\bigl(-\lambda_n^2 kt\bigr)$, $\lambda_n = (2n - 1)\pi/2a$,

$$b_n = \frac{1}{a}\int_0^{2a} g(x) \sin\left(\frac{n\pi x}{2a}\right) dx.$$

In the integral for b_n, break the interval of integration at a; in the second integral, make the change of variable $y = 2a - x$. The two integrals cancel if n is even, and the coefficient is the same as Eq. (18) if n is odd.

b. In the solution of Eqs. (1)–(4), the eigenfunction $\phi(x) = \sin((2n-1)\pi x/2a)$ has the property $\phi(2a-x) = \phi(x)$, so the sum of the series has the same property. This implies 0 derivative at $x = a$.

15. $W(t) = C_0 LA\left[1 - \sum_{n=1}^{\infty} \frac{2e^{-(\lambda_n^2 Dt)}}{(n-1/2)^2\pi^2}\right]$.

Section 2.6

1. The graph of $v(x)$ is a straight line from T_0 at $x = 0$ to T^* at $x = a$, where

$$T^* = T_0 + \frac{ha}{k + ha}(T_1 - T_0).$$

In all cases, T^* is between T_0 and T_1.

3. Negative solutions provide no new eigenfunctions.

7. $b_m = \dfrac{2(1 - \cos(\lambda_m a))}{\lambda_m[a + (\kappa/h)\cos^2(\lambda_m a)]}$.

9. $b_m = \dfrac{-2(\kappa + ah)\cos(\lambda_m a)}{\lambda_m(ah + \kappa\cos^2(\lambda_m a))}$.

Section 2.7

1. $\lambda_n = n\pi/\ln 2$, $\phi_n = \sin(\lambda_n \ln(x))$.

3. a. $\sin(\lambda_n x)$, $\lambda_n = (2n-1)\pi/2a$;

   b. $\cos(\lambda_n x)$, $\lambda_n = (2n-1)\pi/2a$;

   c. $\sin(\lambda_n x)$, λ_n a solution of $\tan(\lambda a) = -\lambda$;

   d. $\lambda_n\cos(\lambda_n x) + \sin(\lambda_n x)$, λ_n a solution of $\cot(\lambda a) = \lambda$;

   e. $\lambda_n\cos(\lambda_n x) + \sin(\lambda_n x)$, λ_n a solution of $\tan(\lambda a) = 2\lambda/(\lambda^2 - 1)$.

5. The weight functions in the orthogonality relations and limits of integration are:

   a. $1 + x$, 0 to a; b. e^x, 0 to a; c. $\frac{1}{x^2}$, 1 to 2; d. e^x, 0 to a.

7. Because λ appears in a boundary condition.

9. The negative value of μ does not contradict Theorem 2 because the coefficient α_2 is not positive.

Section 2.8

1. $x = \sum_{n=1}^{\infty} c_n \phi_n$, $1 < x < b$; $c_n = 2n\pi \dfrac{1 - b\cos(n\pi)}{n^2\phi^2 + \ln^2(b)}$.

3. $1 = \sum_{n=1}^{\infty} c_n \phi_n$, $0 < x < a$; $c_n = 2n\pi \dfrac{1 - e^{a/2}\cos(n\pi)}{n^2\pi^2 + a^2/4}$.

   (Hint: Find the sine series of $e^{x/2}$.)

5. $b_n = \int_l^r f(x)\psi_n(x)p(x)dx$.

7. 1 and $\sqrt{2}\cos(n\pi x)$, $n = 1, 2, \ldots$.

Section 2.9

1. a. $v(x) = $ constant; b. $v(x) = AI(x) + B$.

3. If $\partial u/\partial x = 0$ at both ends, then the steady-state problem is indeterminate. But Eqs. (1)–(3) are homogeneous, so separation of variables applies directly. Note that $\lambda_0 = 0$ and $\phi_0 = 1$. The constant term in the series for $u(x, t)$ is

$$a_0 = \frac{\int_l^r p(x)f(x)dx}{\int_l^r p(x)dx}.$$

Section 2.10

1. The solution is as in Eq. (9), with $B(\lambda) = 2T(\cos(\lambda a) - \cos(\lambda b))/\lambda\pi$.

3. $u(x, t)$ is given by Eq. (6) with $B(\lambda) = \dfrac{2T_0\lambda}{\pi(\alpha^2 + \lambda^2)}$.

5. $u(x, t) = \int_0^{\infty} A(\lambda)\cos(\lambda x)\exp\big(-\lambda^2 kt\big)d\lambda$;

   $A(\lambda) = \dfrac{2T}{\pi\lambda}\big(\sin(\lambda b) - \sin(\lambda a)\big)$.

7. $u(x, t) = T_0 + \int_0^{\infty} B(\lambda)\sin(\lambda x)\exp\big(-\lambda^2 kt\big)d\lambda$;

   $B(\lambda) = \dfrac{2}{\pi}\int_0^{\infty}\big(f(x) - T_0\big)\sin(\lambda x)dx$.

9. a. $v(x) = C_0 e^{-ax}$;

   b. $\dfrac{\partial w}{\partial t} = D\left(\dfrac{\partial^2 w}{\partial x^2} - a^2 w\right), \quad 0 < x, \quad 0 < t,$

$w(0, t) = 0, \quad 0 < t,$

$w(x, 0) = -C_0 e^{-ax}, \quad 0 < x;$

c. $w(x, t) = e^{-a^2 Dt} \int_0^\infty B(\lambda) \sin(\lambda x) e^{-\lambda^2 Dt}\, d\lambda,$

$B(\lambda) = -2C_0\lambda / \left(\pi\left(\lambda^2 + a^2\right)\right).$

Section 2.11

1. Break the interval of integration at $x' = 0$.

3. $B(\lambda) = 0, A(\lambda) = \dfrac{2T_0 a}{\pi(1 + \lambda^2 a^2)}$.

5. The function $u(x, t)$, as a function of x, is the famous "bell-shaped" curve. The smaller t is, the more sharply peaked the curve.

7. In Eq. (3) replace both $f(x')$ and $u(x, t)$ by 1.

9. Using the integral given, obtain

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{1}{\lambda} \sin(\lambda x) e^{-\lambda^2 kt}\, d\lambda.$$

Note, however, that $B(\lambda) = 2/\lambda\pi$ is *not* found using the usual formulas for Fourier coefficient functions.

Section 2.12

5. As $t \to 0+, x/\sqrt{4\pi kt} \to \begin{cases} +\infty & \text{if } x > 0, \\ -\infty & \text{if } x < 0, \end{cases}$

so $\operatorname{erf}(x/\sqrt{4\pi kt}\,) \to \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$

7. Make the substitution $x = y^2$. Then $I(x) = \sqrt{\pi}\operatorname{erf}(\sqrt{x}) + c$.

9. Let z be defined by $\operatorname{erf}(z) = -U_b/(U_i - U_b)$. Then $x(t) = z\sqrt{4kt}$.

Chapter 2 Miscellaneous Exercises

1. SS: $v(x) = T_0, 0 < x < a$.

EVP: $\phi'' + \lambda^2\phi = 0$, $\phi(0) = 0$, $\phi(a) = 0$, $\lambda_n = n\pi/a$, $\phi_n = \sin(\lambda_n x)$, $n = 1, 2, \ldots.$

$$u(x,t) = T_0 + \sum_1^{\infty} b_n \sin(\lambda_n x) e^{-\lambda_n^2 kt},$$

$$b_n = \frac{2}{a} \int_0^a (T_1 - T_0) \sin\left(\frac{n\pi x}{a}\right) dx.$$

3. SS: $v(x) = T_0 + \frac{r}{2} x(x-a)$, $0 < x < a$.

   EVP: $\phi'' + \lambda^2 \phi = 0$, $\phi(0) = 0$, $\phi(a) = 0$, $\lambda_n = n\pi/a$, $\phi_n = \sin(\lambda_n x)$, $n = 1, 2, \ldots$.

$$u(x,t) = T_0 - \frac{r}{2} x(x-a) + \sum_1^{\infty} b_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right),$$

$$b_n = \frac{2}{a} \int_0^a \left[T_1 - T_0 + \frac{r}{2} x(x-a)\right] \sin\left(\frac{n\pi x}{a}\right) dx.$$

5. SS: not needed.

   (Hint: Put $-\gamma^2 u$ on the other side of the equation. Separation of variables gives $\phi''/\phi = \gamma^2 + T'/kT = -\lambda^2$.)

   EVP: $\phi'' + \lambda^2 \phi = 0$, $\phi'(0) = 0$, $\phi'(a) = 0$, $\lambda_0 = 0$, $\phi_0 = 1$; $\lambda_n = n\pi/a$, $\phi_n = \cos(\lambda_n x)$, $n = 1, 2, \ldots$.

$$u(x,t) = e^{-\gamma^2 kt}\left(a_0 + \sum a_n \cos(\lambda_n x) \exp\left(-\lambda_n^2 kt\right)\right).$$

$$a_0 = T_1/2,\ a_n = -2T_1\left(1 - \cos(n\pi)\right)/n^2\pi^2.$$

7. $u(x,t) = T_0$.

9. $u(x,t) = T_0 + \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right)$,

$$\lambda_n = \frac{(2n-1)\pi}{2a},\ c_n = \frac{(T_1 - T_0) \cdot 4}{(2n-1)\pi}.$$

11. $u(x,t) = T_0 + \int_0^{\infty} B(\lambda) \sin(\lambda x) \exp\left(-\lambda^2 kt\right) d\lambda$, $B(\lambda) = \dfrac{-2\lambda T_0}{\pi(\alpha^2 + \lambda^2)}$.

13. $u(x,t) = \int_0^{\infty} A(\lambda) \cos(\lambda x) \exp\left(-\lambda^2 kt\right) d\lambda$, $A(\lambda) = \dfrac{2T_0 \sin(\lambda a)}{\pi\lambda}$.

15. $u(x,t) = \int_0^{\infty} \left(A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\right) \exp\left(-\lambda^2 kt\right) d\lambda$,

$$A(\lambda) = \frac{T_0 \sin(\lambda a)}{\pi\lambda},\ B(\lambda) = \frac{T_0(1 - \cos(\lambda a))}{\pi\lambda}$$

or

$$u(x,t) = \frac{T_0}{\sqrt{4\pi kt}} \int_0^a \exp\left(-\frac{(x'-x)^2}{4kt}\right) dx'$$
$$= \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{a-x}{\sqrt{4kt}}\right) + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right].$$

17. Interpretation: u is the temperature in a rod with insulation on the cylindrical surface and on the left end. At the right end, heat is being forced into the rod at a constant rate (because $q(a,t) = -\kappa \frac{\partial u}{\partial x}(a,t) = -\kappa S$, so heat is flowing to the left, into the rod). The accumulation of heat energy accounts for the steady increase of temperature.

19. $(1/6ka)u_3 - (a/6k)u_1$ satisfies the boundary conditions.

21. $w(x,t) = -\frac{2}{u}\frac{\partial u}{\partial x}$, where $u(x,t) = a_0 + \sum a_n \cos(n\pi x)\exp(-n^2\pi^2 t)$,
where $a_0 = 2(1 - e^{-1/2})$ and $a_n = \dfrac{1 - e^{-1/2}\cos(n\pi)}{\frac{1}{4} + (n\pi)^2}$.

23. $u_2 = T_0 \dfrac{\beta_2 V}{\beta_1 + \beta_2}$, $u_1 = T_0\left(1 - \dfrac{\beta_1 V}{\beta_1 + \beta_2}\right)$,
where $V = 1 - \exp(-(\beta_1 + \beta_2)t)$ and $\beta_i = h/c_i$.

25. $u(\rho,t) = \dfrac{1}{\rho}\displaystyle\sum_{n=1}^{\infty} b_n \sin(\lambda_n \rho)\exp(-\lambda_n^2 kt)$,
$\lambda_n = n\pi/a$, $b_n = \dfrac{2}{a}\displaystyle\int_0^a \rho T_0 \sin(\lambda_n \rho) d\rho$.

27. $v(x) = T_0 + Sx - S\dfrac{\sinh(\lambda x)}{\gamma\cosh(\gamma a)}$.

29. If $\lambda = 0$, the differential equation is $\phi'' = 0$ with general solution $\phi(x) = c_1 + c_2 x$. The boundary conditions require $c_2 = 0$ but allow $c_1 \neq 0$. Thus, this value of λ permits the existence of a nonzero solution, and therefore $\lambda = 0$ is an eigenvalue.

31. Choose $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)dt$. If f has a Fourier integral representation, then this choice of B will make $u(0,t) = f(t)$, $0 < t$.

33. a. $v(x) = -Ix/aK + c_1 + c_2(1 - e^{-aKx/T})$,
$c_1 = h_1$, $c_2 = (h_2 - h_1 + IL/aK)/(1 - e^{-aKL/T})$.

b. $\dfrac{\partial^2 w}{\partial x^2} + \mu\dfrac{\partial w}{\partial x} = \dfrac{1}{k}\dfrac{\partial w}{\partial t}, \quad 0 < x < L, \quad 0 < t,$
$w(0,t) = 0, \quad w(L,t) = 0, \quad 0 < t,$
$w(x,0) = h_0(x) - v(x), \quad 0 < x < L,$
where $\mu = aK/T$, $k = T/S$.

c. $w(x,t) = \sum c_n \phi_n(x) e^{-\lambda_n^2 kT}, \quad \phi_n(x) = e^{-\mu x/2} \sin(n\pi x/L),$

$\lambda_n^2 = \left(\frac{n\pi}{L}\right)^2 + \frac{\mu^2}{4};$

d. $\lambda_n^2 = (7.30n^2 + 0.0133) \times 10^{-4}\ \mathrm{m}^{-1}.$

35. a. $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad 0 < t,$

$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = S_0, \quad 0 < t;$

$u(0,t) = 0, \quad 0 < x < L;$

b. $u(x,t) = S_0 + \sum_{n=1}^{\infty} c_n \cos(\lambda_n x) \exp\left(-\lambda_n^2 Dt\right),$

$c_n = 4S_0(-1)^n/(2n-1).$

37. $T(y,t) = 300 - 150y/c + \sum b_n \sin\lambda_n(y+c)\exp(-\lambda_n^2 kt)$, $\lambda_n = n\pi/2c$, $b_n = (400\cos(n\pi) + 1000)/n\pi$. c. Just before time $t = 0$, the three terms add to 0. Just after time $t = 0$, the integrated terms do not change sensibly, but in the first term, near $y = c$, $T(y,t)$ changes suddenly.

Chapter 3

Section 3.1

1. $[u] = L$, $[c] = L/t$.

3. $v(x) = \frac{(x^2 - ax)g}{2c^2}.$

Section 3.2

3. $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right),$

$b_n = \frac{2a(1-\cos(n\pi))}{n^2\pi^2 c}.$

5. $u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$, $a_n = 2U_0\frac{1-\cos(n\pi/2)}{n\pi}.$

7. a. $\sin\left(\frac{n\pi x}{a}\right)$; b. $\sin\left(\frac{2n-1}{2}\frac{\pi x}{a}\right).$

9. Product solutions are $\phi_n(x)T_n(t)$, where

$$\phi_n(x) = \sin(\lambda_n x), \quad T_n(t) = \exp\big(-kc^2t/2\big) \times \begin{cases} \sin(\mu_n t) \\ \cos(\mu_n t), \end{cases}$$

$$\lambda_n = \frac{n\pi}{a}, \quad \mu_n = \sqrt{\lambda_n^2 c^2 - \frac{1}{4}k^2 c^4}.$$

11. Product solutions are $\phi_n(x)T_n(t)$, where

$$\phi_n(x) = \sin\left(\frac{n\pi x}{a}\right),$$

$$T_n(t) = \sin \text{ or } \cos\left(\frac{n^2\pi^2 ct}{a^2}\right).$$

Frequencies $n^2\pi^2 c/a^2$.

13. The general solution of the differential equation is $\phi(x) = A\cos(\lambda x) + B\sin(\lambda x) + C\cosh(\lambda x) + D\sinh(\lambda x)$. Boundary conditions at $x = 0$ require $A = -C$, $B = -D$; those at $x = a$ lead to $C/D = -(\cosh(\lambda a) + \cos(\lambda a))/(\sinh(\lambda a) - \sin(\lambda a))$ and $1 + \cos(\lambda a)\cosh(\lambda a) = 0$. The first eigenvalues are $\lambda_1 = 1.875/a$, $\lambda_2 = 4.693/a$, and the eigenfunctions are similar to the functions shown in the figure.

15. $u(x,t) = \sum_{n=1}^{\infty} \big(a_n \cos(\mu_n t) + b_n(\sin \mu_n t)\big) \sin(\lambda_n x)$: $\lambda_n = n\pi/a$,

$\mu_n = \sqrt{\lambda_n^2 + \gamma^2}c$, $a_n = 2h\big(1 - \cos(n\pi)\big)/n\pi$, $b_n = 0$, $n = 1, 2, \ldots$.

17. Convergence is uniform because $\sum |b_n|$ converges.

Section 3.3

1. Table shows $u(x,t)/h$.

	t				
x	0	$0.2a/c$	$0.4a/c$	$0.8a/c$	$1.4a/c$
$0.25a$	0.5	0.5	0.2	−0.5	−0.2
$0.5a$	1.0	0.6	0.2	−0.6	−0.2

3. $u(0, 0.5a/c) = 0$; $u(0.2a, 0.6a/c) = 0.2\alpha a$; $u(0.5a, 1.2a/c) = -0.2\alpha a$. (Hint: $G(x) = \alpha x$, $0 < x < a$.)

5. $G(x) = \begin{cases} 0, & 0 < x < 0.4a, \\ 5(x - 0.4a), & 0.4a < x < 0.6a, \\ a, & 0.6a < x < a. \end{cases}$

Notice that G is a continuous function whose graph is composed of line segments.

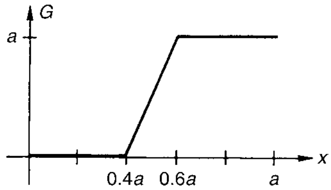


Figure 2 Solution for Exercise 7, Section 3.3.

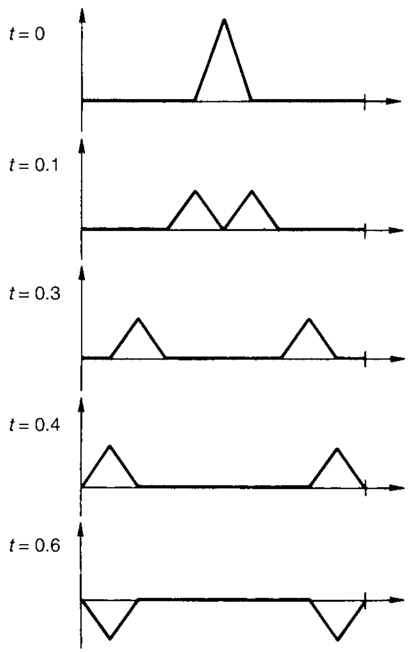


Figure 3 Solution for Exercise 9, Section 3.3.

7. See Fig. 2.
9. See Fig. 3.
11. By the chain rule we calculate

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial v}{\partial w} + \frac{\partial v}{\partial z},$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial w}\left(\frac{\partial v}{\partial w} + \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial w} + \frac{\partial v}{\partial z}\right)\frac{\partial z}{\partial x} \\ &= \frac{\partial^2 v}{\partial w^2} + 2\frac{\partial^2 v}{\partial w \partial z} + \frac{\partial^2 v}{\partial z^2}\end{aligned}$$

and similarly

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial w^2} - 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}\right).$$

(We have assumed that the two mixed partials $\partial^2 v/\partial z\partial w$ and $\partial^2 v/\partial w\partial z$ are equal.) If $u(x, t)$ satisfies the wave equation, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In terms of the function v and the new independent variables this equation becomes

$$\frac{\partial^2 v}{\partial w^2} + 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial w^2} - 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}$$

or, simply,

$$\frac{\partial^2 v}{\partial z \partial w} = 0.$$

13. $u(x, t) = -c^2 \cos(t) + \phi(x - ct) + \psi(x + ct)$.

Section 3.4

1. If f and g are sectionally smooth and f is continuous.
3. The frequency is $c\lambda_n$ rads/sec, and the period is $2\pi/c\lambda_n$ sec.
5. Separation of variables leads to the following in place of Eqs. (11) and (12):

$$T'' + \gamma T' + \lambda^2 c^2 T = 0, \tag{11'}$$

$$\big(s(x)\phi'\big)' - q(x)\phi + \lambda^2 p(x)\phi = 0. \tag{12'}$$

The solutions of Eq. (11′) all approach 0 as $t \to \infty$, if $\gamma > 0$.

7. The period of $T_n(t) = a_n \cos(\lambda_n ct) + b_n \sin(\lambda_n ct)$ is $2\pi/\lambda_n c$. All T_n's have a common period p if and only if for each n there is an integer m such that $m(2\pi/\lambda_n c) = p$, or $m = (pc/2\pi)\lambda_n$ is an integer. For λ_n as shown and $\beta = q/r$, where q and r are integers, this means

$$m = \left(\frac{pc}{2\pi}\right)\alpha\left(n + \frac{q}{r}\right)$$

or

$$m = \left(\frac{pc}{2\pi}\right)\frac{\alpha}{r}(rn + q).$$

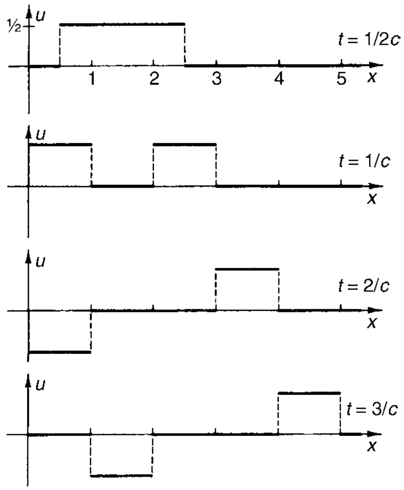


Figure 4 Solution for Exercise 3, Section 3.6.

Given α, p can be adjusted so that m is an integer whenever n is an integer.

Section 3.5

1. If $q \geq 0$, the numerator in Eq. (3) must also be greater than or equal to 0, since $\phi_1(x)$ cannot be identically 0.

3. $2\pi^2/3$ is one estimate from $y = \sin(\pi x)$.

5. $\displaystyle\int_1^2 (y')^2 dx = \frac{1}{3}, \int_1^2 \frac{y^2}{x^4} dx = \frac{25}{6} - 6\ln 2;$

   $N(y)/D(y) = 42.83$; $\lambda_1 \leq 6.54$.

Section 3.6

1. $u(x,t) = \frac{1}{2}[f_e(x+ct) + G_o(x+ct)] + \frac{1}{2}[f_e(x-ct) - G_o(x-ct)]$, where f_e is the even extension of f and G_o is the odd extension of G.

3. See Fig. 4.

5. See Fig. 5.

7. $\displaystyle u(x,t) = \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy.$

Chapter 3 Miscellaneous Exercises

1. $\displaystyle u(x,t) = \sum_1^\infty b_n \sin(\lambda_n x)\cos(\lambda_n ct),\ b_n = 2\big(1-\cos(n\pi)\big)/n\pi,\ \lambda_n = n\pi/a.$

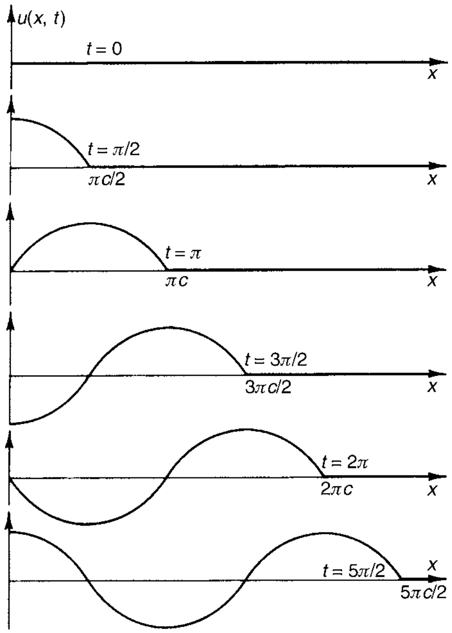


Figure 5 Solution for Exercise 5, Section 3.6.

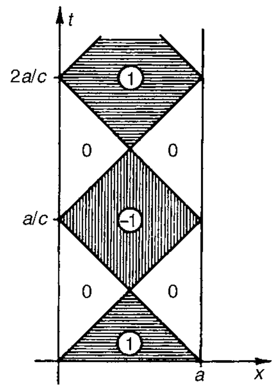


Figure 6 Solution of Miscellaneous Exercise 3, Chapter 3.

3. See Fig. 6.
5. See Fig. 7.
7. See Fig. 8.
9. See Fig. 9.
11. See Fig. 10.

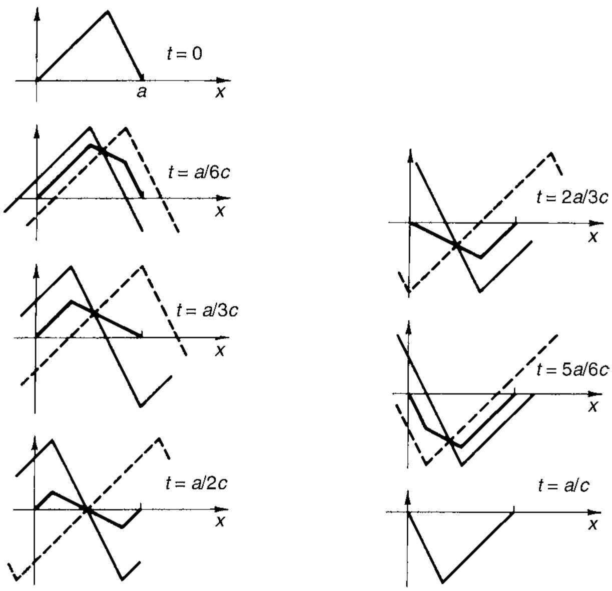


Figure 7 Solution of Miscellaneous Exercise 5, Chapter 3.

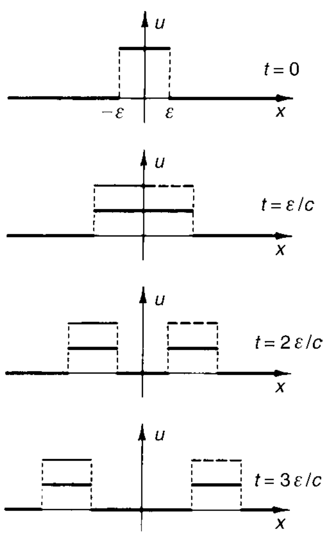


Figure 8 Solution of Miscellaneous Exercise 7, Chapter 3.

13. See Fig. 11.
15. Using $y(x) = x(1-x)$, find $\lambda_1^2 \leq 10.5$.
17. $f(q) = 12a^2\text{sech}^2(aq)$, $c = 4a^2$.

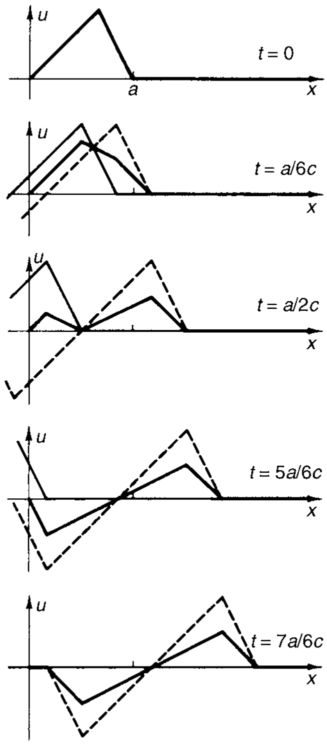


Figure 9 Solution of Miscellaneous Exercise 7, Chapter 3.

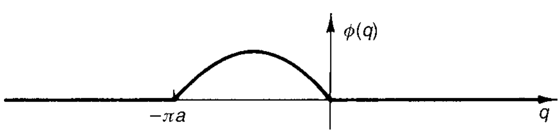


Figure 10 Solution for Miscellaneous Exercise 11, Chapter 3.

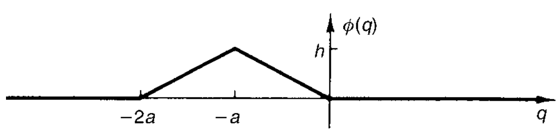


Figure 11 Solution of Miscellaneous Exercise 13, Chapter 3.

21. $v(x, t) = \sum_{n=1}^{\infty} \big(a_n \cos(\lambda_n ct) + b_n \sin(\lambda_n ct)\big) \sin(\lambda_n x),$

$\lambda_n = (2n - 1)\pi/2a,$

$$a_n = \frac{8aU_0(-1)^{n+1}}{\pi^2(2n-1)^2},\ b_n = 0.$$

23. $\dfrac{Y''}{Y} = \dfrac{2V}{k}\dfrac{\psi'}{\psi}$. The function $\phi(x - Vt)$ cancels from both sides.

25. $\phi_n(-Vt) = T_0 \exp(\lambda_n^2 kt/2) b_n,\ t > 0,$

$\phi_n(x) = T_1 \exp(\lambda_n^2 kx/2V) b_n,\ x > 0,$

where $\displaystyle\sum_{n=1}^{\infty} b_n \sin(\lambda_n y) = 1,\ 0 < y < b.$

27. $\phi(x - ct) = e^{-c(x-ct)/k} = e^{(c^2t-cx)/k}$. The given c satisfies $c^2 = i\omega k$, so $\phi(x - ct) = e^{i\omega t-(1+i)px} = e^{-px}e^{i(\omega t-px)}$. Now form $\frac{1}{2}(\phi(x - ct) + \phi(x - ct)) = e^{-px}\cos(\omega t - px)$ and so forth.

29. Differentiate and substitute.

31. $\phi^{(2)} - \epsilon\phi^{(4)} + \lambda^2\phi = 0,$

$\phi(0) = 0, \quad \phi(a) = 0,$

$\phi''(0) = 0, \quad \phi''(a) = 0.$

33. $\lambda_n = \dfrac{n\pi}{a}\sqrt{1 + \epsilon\left(\dfrac{n\pi}{a}\right)^2} \cong \dfrac{n\pi}{a}.$

Chapter 4

Section 4.1

1. $f + d = 0.$

3. $Y(y) = A\sinh(\pi y),\ A = 1/\sinh(\pi).$

5. $v(r) = a\ln(r) + b.$

7. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial r}\cos(\theta) - \dfrac{\partial v}{\partial \theta}\dfrac{\sin(\theta)}{r},$

$\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial r}\sin(\theta) + \dfrac{\partial v}{\partial \theta}\dfrac{\cos(\theta)}{r}.$

9. a. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$

$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b,$

$u(x, 0) = f(x), \quad u(x, b) = f(x), \quad 0 < x < a.$

Membrane is attached to a frame that is flat on the left and right but has the shape of the graph of $f(x)$ at top and bottom.

b. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$

$\dfrac{\partial u}{\partial x}(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b,$

$u(x, 0) = 0, \quad u(x, b) = 100, \quad 0 < x < a.$

The bar is insulated on the left; the temperature is fixed at 100 on the top, at 0 on the other two sides.

c. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$

$u(0, y) = 0, \quad u(a, y) = 100, \quad 0 < y < b,$

$\dfrac{\partial u}{\partial y}(x, 0) = 0, \quad \dfrac{\partial u}{\partial y}(x, b) = 0, \quad 0 < x < a.$

The sheet is electrically insulated at top and bottom. The voltage is fixed at 0 on the left and 100 on the right.

d. $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$

$\dfrac{\partial \phi}{\partial x}(0, y) = 0, \quad \dfrac{\partial \phi}{\partial x}(a, y) = -a, \quad 0 < y < b,$

$\dfrac{\partial \phi}{\partial y}(x, 0) = 0, \quad \dfrac{\partial \phi}{\partial y}(x, b) = b, \quad 0 < x < a.$

The velocities, given by $\mathbf{V} = -\nabla\phi$, are $V_x = a$, $V_y = 0$ on the right, $V_x = 0$, $V_y = -b$ on the top; and walls on the other two sides make velocities 0 there.

Section 4.2

1. Show by differentiating and substituting that both are solutions of the differential equation. The Wronskian of the two functions is

$$\begin{vmatrix} \sinh(\lambda y) & \sinh(\lambda(b-y)) \\ \lambda\cosh(\lambda y) & -\lambda\cosh(\lambda(b-y)) \end{vmatrix} = -\lambda\sinh(\lambda b) \neq 0.$$

3. In the case $b = a$, use two terms of the series: $u(a/2, a/2) = 0.32$.

5. $u(x, y) = \displaystyle\sum_1^\infty b_n \sin\left(\frac{n\pi x}{a}\right)\frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}$, $b_n = \dfrac{8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)$.

7. a. See Eq. (11). $a_n = 0$, $c_n = 200(1 - \cos(n\pi))/n\pi$;

b. $u(x,y) = u_1(x,y) + u_2(x,y)$, $u_1(x,y)$ is the solution to Part a,

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \frac{\sinh(\mu_n x)}{\sinh(\mu_n a)} \sin(\mu_n y),$$

$\mu_n = n\pi/b$, $c_n = 200(1 - \cos(n\pi))/n\pi$.

c. $u(x,y) = u_1(x,y) + u_2(x,y)$, where

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} \sin(\lambda_n x),$$

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \frac{\sinh(\mu_n x)}{\sinh(\mu_n a)} \sin(\mu_n y).$$

In both series, $c_n = 2ab(-1)^{n+1}/n\pi$. Also note $u(x,y) = xy$.

Section 4.3

1. a. $u(x,y) = 1$, but the form found by applying the methods of this section is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(\lambda_n y) + \sinh(\lambda_n (b-y))}{\sinh(\lambda_n b)} \cos(\lambda_n x) + \sum_{n=1}^{\infty} b_n \frac{\cosh(\mu_n x)}{\cosh(\mu_n a)} \sin(\mu_n y),$$

where

$$\lambda_n = \frac{(2n-1)\pi}{2a}, \quad a_n = \frac{4\sin\left(\frac{(2n-1)\pi}{2}\right)}{\pi(2n-1)},$$

$$\mu_n = \frac{n\pi}{b}, \quad b_n = \frac{2(1-\cos(n\pi))}{n\pi}.$$

b. $u(x,y) = y/b$, and this is found by the methods of this section. In this case, 0 is an eigenvalue.

c. $\displaystyle \frac{4}{\pi}\sum_{1}^{\infty} \frac{(-1)^{n+1}\cos(\lambda_n y)}{(2n-1)} \frac{\sinh(\lambda_n(a-x))}{\sinh(\lambda_n a)}, \ \lambda_n = \left(\frac{2n-1}{2}\frac{\pi}{b}\right).$

3. $\displaystyle b_0 b = \frac{V_0}{2}, \ b_n \sinh(\lambda_n b) = \frac{2V_0(\cos(n\pi) - 1)}{n^2\pi^2}.$

5. Check zero boundary conditions by substituting. At $x = a$, find

$$A_n \cosh(\mu_n a) = \frac{2}{b} \int_0^b Sy \cos(\mu_n y) dy.$$

7. $w(x, y) = \sum_{n=1}^{\infty} a_n \cosh(\lambda_n y) \cos(\lambda_n x)$. From the condition at $y = b$,

$$a_n \cosh(\lambda_n b) = \frac{2}{a} \int_0^a \frac{Sb}{a}(x - a) \cos(\lambda_n x) dx.$$

9. $w(x, y) = \sum_{n=1}^{\infty} \frac{c_n \sinh(\lambda_n y) + a_n \sinh(\lambda_n (b - y))}{\sinh(\lambda_n b)} \sin(\lambda_n x),$

$$a_n = c_n = -\frac{2}{a} \int_0^a Hx(a - x) \sin(\lambda_n x) dx = -2Ha^2 \frac{1 - \cos(n\pi)}{n^3 \pi^3}.$$

11. $12A + 2C = -K$, $12E + 2C = -K$. There are many solutions.

Section 4.4

1. $a_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$

3. $A(\mu) = \frac{2}{\pi} \int_0^\infty g_2(y) \sin(\mu y) dy.$

5. a. $u(x, y) = \sum c_n \cos(\lambda_n x) \exp(-\lambda_n y)$, $\lambda_n = (2n - 1)\pi/2a$,

   $c_n = 4(-1)^{n+1}/\pi(2n - 1)$;

   b. $u(x, y) = \int_0^\infty B(\lambda) \cosh(\lambda x) \sin(\lambda y) d\lambda$, $B(\lambda) = \frac{2\lambda}{\pi(\lambda^2 + 1) \cosh(\lambda a)}$;

   c. $u(x, y) = \int_0^\infty A(\lambda) \cos(\lambda y) \sinh(\lambda x) d\lambda$, $A(\lambda) = \frac{2 \sin(\lambda b)}{\pi \lambda \sinh(\lambda a)}$.

7. $$u(x, y) = \sum_1^\infty b_n \sin(\lambda_n x) \exp(-\lambda_n y) + \int_0^\infty \left(A(\mu) \frac{\sinh(\mu x)}{\sinh(\mu a)} + B(\mu) \frac{\sinh(\mu(a - x))}{\sinh(\mu a)} \right) \sin(\mu y) d\mu,$$

   $\lambda_n = n\pi/a$, $b_n = 2(1 - \cos(n\pi))/n\pi$, $A(\mu) = B(\mu) = 2\mu/\pi(\mu^2 + 1)$.

   Also see Exercise 8.

9. a. $u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda a)}{\lambda} \sin(\lambda x) \frac{\sinh(\lambda y)}{\sinh(\lambda b)} d\lambda$;

b. $u(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\lambda}{1+\lambda^2}\sin(\lambda x)\frac{\sinh(\lambda(b-y))}{\sinh(\lambda b)}d\lambda.$

11. $u(x,y) = \displaystyle\int_0^\infty \frac{2}{\pi(1+\lambda^2)}\frac{\sinh(\lambda x)}{\sinh(\lambda a)}\cos(\lambda y)d\lambda.$

13. $e^{-\lambda y}\sin(\lambda x)$, $\lambda > 0$.

15. $e^{-\lambda y}\sin(\lambda x)$, $e^{-\lambda y}\cos(\lambda x)$, $\lambda > 0$.

17. $u(x,y) = \dfrac{1}{\pi}\left[\dfrac{\pi}{2} + \tan^{-1}(x/y)\right].$

19. This solution is unbounded as x tends to infinity and cannot be found by the method of this section.

Section 4.5

1. $v(r,\theta)$ is given by Eq. (10) with $b_n = 0$, $a_0 = \pi/2$,

   $a_n = -2\,(1-\cos(n\pi))/\pi n^2 c^n$.

3. The solution is as in Eq. (10) with $b_n = 0$, $a_0 = 1/\pi$, $a_1 = 1/2$, and

   $$a_n = \frac{2\sin((n-1)\pi/2)}{\pi(n^2-1)} \text{ for } n \neq 1.$$

5. Convergence is uniform in θ.

7. $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)d\theta,\ a_n = \frac{c^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta,$

   $b_n = \dfrac{c^n}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta.$

9. $\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-\cos(n\pi)}{nc^{2n}}r^{2n}\sin(2n\theta) = v(r,\theta).$

11. $v_n(r,\theta) = r^{n/\alpha}\sin(n\theta/\alpha)$ has $\partial v/\partial r$ unbounded as $r \to 0+$, if $n = 1$.

Section 4.6

1. Hyperbolic (a) and (e); elliptic (b) and (c); parabolic (d).

3. Only (e).

5. a. $u(x,y) = \displaystyle\sum_1^\infty a_n \sin(n\pi x)e^{-n\pi y};$

   b. $u(x,y) = \displaystyle\sum_1^\infty a_n \sin(n\pi x)\cos(n\pi y);$

c. $u(x, y) = \sum_{1}^{\infty} a_n \sin(n\pi x) \exp(-n^2\pi^2 y)$,

$a_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$.

7. $X''/X = -\lambda^2$, $T''/T = -\lambda^2/(1 + \epsilon\lambda^2)$.

Chapter 4 Miscellaneous Exercises

1. $u(x, y) = \sum_{1}^{\infty} b_n \frac{\sinh(\lambda_n(a - x))}{\sinh(\lambda_n a)} \sin(\lambda_n y)$,

   $\lambda_n = n\pi/b$, $b_n = 2(1 - \cos(n\pi))/n\pi$.

3. $u(x, y) = 1$. Note that 0 is an eigenvalue.

5. $u(x, y) = \sum_{n=1}^{\infty} \frac{a_n \sinh(\lambda_n x) + b_n \sinh(\lambda_n(a - x))}{\sinh(\lambda_n a)} \cos(\lambda_n y)$,

   $\lambda_n = (2n - 1)\pi/2b$, $a_n = b_n = 4(-1)^{n+1}/\pi(2n - 1)$.

7. $u(x, y) = w(x, y) + w(y, x)$, where

   $w(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sinh(\lambda_n(a - y))}{\sinh(\lambda_n a)} \sin(\lambda_n x)$,

   $\lambda_n = n\pi/a$, $b_n = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$.

9. $u(x, y) = \int_0^{\infty} A(\lambda) \frac{\sinh(\lambda(b - y))}{\sinh(\lambda b)} \cos(\lambda x) d\lambda$, $A(\lambda) = 2\sin(\lambda a)/\lambda\pi$.

11. $u(x, y) = \int_0^{\infty} A(\lambda) \cos(\lambda x) e^{-\lambda y} d\lambda$, $A(\lambda) = 2\alpha/\pi\left(\alpha^2 + \lambda^2\right)$.

13. $u(x, y) = \frac{-1}{\pi} \tan^{-1}\left(\frac{x - x'}{y}\right)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$.

15. $u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{c}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$,

    $a_0 = \frac{1}{2}$, $a_n = 0$, $b_n = \frac{1 - \cos(n\pi)}{n\pi}$.

17. Same form as Exercise 15, but $a_0 = 2/\pi$,

    $a_n = 2(1 + \cos(n\pi))/(1 - n^2)$, $b_n = 0$ (and $a_1 = 0$).

19. $u(r, \theta) = (\ln(r) - \ln(b))/(\ln(a) - \ln(b))$.

21. $u(r,\theta) = \sum_{1}^{\infty} b_n \left(\frac{r}{c}\right)^{n/2} \sin(n\theta/2)$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta/2)d\theta$.

23. $u(x,y) = \sum c_n \sinh(\lambda_n y)\sin(\lambda_n x)$, $\lambda_n = (2n-1)\pi/2a$,

    $c_n = 2\sin(\lambda_n a)/(a\lambda_n^2 \sinh(\lambda_n b))$.

25. w satisfies the potential equation in the rectangle with boundary conditions

    $w(0,y) = 0, \quad w_x(a,y) = ay/b, \quad 0 < y < b,$

    $w(x,0) = 0, \quad w(x,b) = 0, \quad 0 < x < a.$

    $$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n y)\cosh(\lambda_n x),$$

    $\lambda_n = n\pi/b$, $b_n = 2a(-1)^{n+1}/n^2\pi^2\cosh(\lambda_n a)$.

27. The equations become

    $$\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\phi}{\partial x\partial y}, \qquad \left(1-M^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

29. $$\phi(x,y) = \int_0^{\infty} \big(A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\big)e^{-\beta y}d\alpha + c,$$

    where $\beta = \alpha\sqrt{1-M^2}$, c is an arbitrary constant, and

    $$\left.\begin{matrix} A(\alpha) \\ B(\alpha) \end{matrix}\right\} = -\frac{U_0}{\beta\pi}\int_{-\infty}^{\infty} f'(x)\left\{\begin{matrix} \cos(\alpha x) \\ \sin(\alpha x) \end{matrix}\right\} dx.$$

31. If $(x(s), y(s))$ is the parametric representation for the boundary curve $\mathcal{C}$, then the vector $y'\mathbf{i} - x'\mathbf{j}$ is normal to $\mathcal{C}$, and

    $$\int_C \frac{\partial u}{\partial n}ds = \int_C \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx.$$

    By Green's theorem,

    $$\int_C \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx = \iint_{\mathcal{R}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)dA,$$

    which is 0, since u satisfies the potential equation in $\mathcal{R}$.

33. Substitute directly.

35. $-\nabla u = -(x\mathbf{i} + y\mathbf{j})/(x^2+y^2)$.