Advanced Statistics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- ❑ **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- ❑ **Probability Demystified**, Allan G. Bluman

- ❑ **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- ❑ **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- ❑ **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

- ❑ Elementary Statistics, 14th Edition, Mario F. Triola

These notes contain material from the above resources.

The Least Squares Criterion

- **Objective:** To provide a fitted line that ensures "closeness" between the line and the plotted data points.
- **Method:** Minimizes the **sum of squared residuals** to achieve this closeness.

SSE = $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

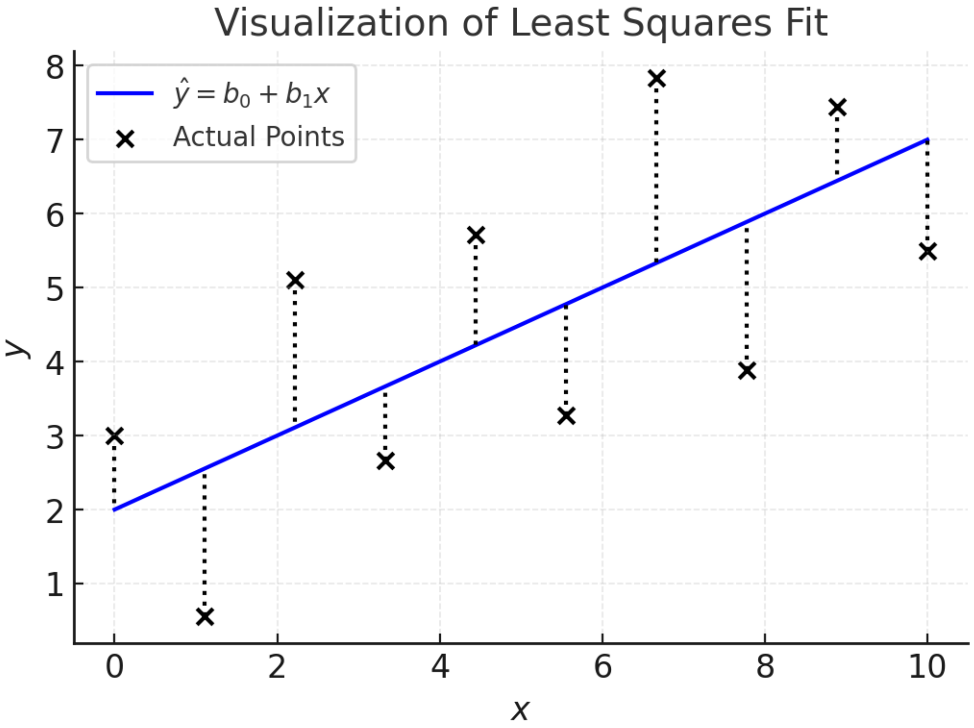
Residuals and Fitted Line in Least Squares

Predicted Values: Points on the fitted line represent predicted values based on the model.

Residuals: Vertical deviations from the observed data points to the line.

Key Insight:

The least squares procedure generates a line that **minimizes the sum of squares of these vertical deviations**.


Visualization of Least Squares Fit
$\hat{y} = b_0 + b_1 x$
Actual Points
8
7
6
5
4
3
2
1
y
0
2
4
6
8
10
x

Closeness and Alternative Measures

There are various ways to measure closeness between the line and data points:

- **Minimizing the sum of absolute residuals:**

$$\sum_{i=1}^{n} \left| y_i - \widehat{y}_i \right|$$

- **Minimizing the sum of residuals raised to a power:**

$$\sum_{i=1}^{n} \left| y_i - \widehat{y}_i \right|^{1.5}$$

These approaches, like least squares, aim to make residuals **"small".**

Benefits of Least Squares

- Provides a systematic and consistent method to fit a line.

- Ensures **residuals** are minimized in a **squared sense**, reducing the **impact of larger errors**.

- Widely used due to simplicity and statistical properties, such as **unbiased estimators**.

Definitions

Given a collection of paired sample data, the **regression equation**

$$\hat{y} = \boldsymbol{b_0} + \boldsymbol{b_1}\mathbf{x}$$

algebraically describes the relationship **between the two variables**. The graph of the **regression equation** is called the **regression line** (or ***line of best fit***, or ***least-squares line***).

Notation for Regression Equation

	Population Parameter	Sample Statistic
y-intercept of regression equation	β_0	b_0
Slope of regression equation	β_1	b_1
Equation of the regression line	$Y = \beta_0 + \beta_1 x$	$\hat{y} = b_0 + b_1 x$

Finding the slope b_1 and y-intercept b_0 in the regression equation $\hat{y} = b_0 + b_1 x$

Slope	$b_1 = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$
***y*-intercept:**	$b_0 = \bar{y} - b_1\bar{x}$ **or** $b_0 = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n(\sum x^2) - (\sum x)^2}$

Finding the slope b_1 and *y*-intercept b_0 in the regression equation $\hat{y}$ = $\mathbf{b_0 + b_1 x}$

Slope: b_1 = $r\frac{s_y}{s_x}$

where ***r*** is the linear correlation coefficient, s_y **is the standard deviation** of the *y* values, and s_x **is the standard deviation** of the *x* values.

***y*-intercept: $\mathbf{b_0}$ = $\bar{y}$ - $\mathbf{b_1}\bar{x}$**

Example: Table 1 is reproduced here. (Jackpot amounts are in millions of dollars and numbers of tickets sold are in millions.) Find the equation of the **regression line in which the explanatory variable (or *x* variable)** is the amount of the lottery jackpot and the response variable (or *y* variable) is the corresponding number of lottery tickets sold.

Table 1 Powerball Tickets Sold and Jackpot Amounts

Jackpot	334	127	300	227	202	180	164	145	255
Tickets	54	16	41	27	23	18	18	16	26

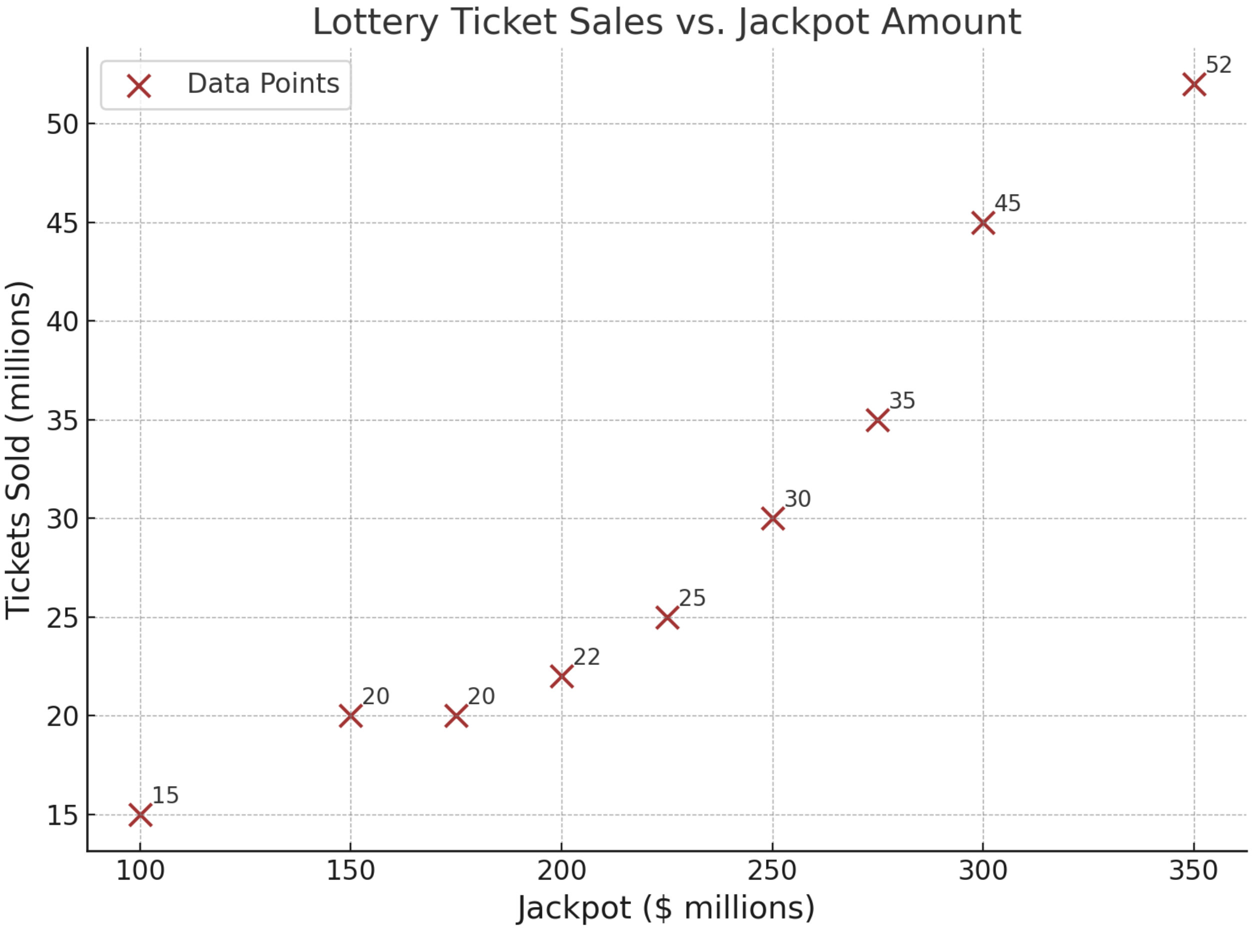


FIGURE 1 Scatterplot from Table 1

1. The data are a simple random sample.

2. The scatterplot in Figure 1 on previous slide shows that the pattern of points is reasonably close to a **straight-line pattern**.

3. The scatterplot also shows that there are **no outliers**.

The requirements are satisfied.

x(Jackpot)	y(Tickets)	x^2	y^2	xy
334	54	111,556	2916	18,036
127	16	16,129	256	2032
300	41	90,000	1681	12,300
227	27	51,529	729	6129
202	23	40,804	529	4646
180	18	32,400	324	3240
164	18	26,896	324	2952
145	16	21,025	256	2320
255	26	65,025	676	6630
$\sum x$ = 1934	$\sum y$ = 239	$\sum x^2$= 455,364	$\sum y^2$= 7691	$\sum xy$ = 58,285

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2}\sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$r = \frac{9(158{,}2852) - (1934)(239)}{\sqrt{9(455{,}364) - (1943)^2}\sqrt{9(7651) - (239)^2}}$$

$$r = \frac{62{,}339}{\sqrt{357{,}920}\sqrt{12{,}098}} = 0.947$$

$\boldsymbol{b_1} = \boldsymbol{r\frac{s_y}{s_x}}$

$s_y = \sqrt{\frac{1}{n(n-1)}\{n \sum_{i=1}^{n} \mathrm{y}^2{}_i \; - (\sum_{i=1}^{n} y_i \;)^2\}}$

$s_y = \sqrt{\frac{1}{9(9-1)}\{9(7691) - (239)^2\}}$

$s_y = 70.50611$

$s_x = \sqrt{\frac{1}{n(n-1)}\{n \sum_{i=1}^{n} x^2{}_i \; - (\sum_{i=1}^{n} x_i \;)^2\}}$

$s_x = \sqrt{\frac{1}{9(9-1)}\{9(455{,}364\;) - (1934\;)^2\}}$

$= 12.96255$

$$\boldsymbol{b_1} = \boldsymbol{r}\frac{\boldsymbol{s_y}}{\boldsymbol{s_x}}$$

$$= 0.947 \times \frac{12.9625}{70.5061}$$

$$= 0.1742$$

$$\bar{\boldsymbol{x}} = \frac{1934}{9} = 214.8889$$

$$\bar{\boldsymbol{y}} = \frac{239}{9} = 26.5556$$

$$\mathbf{b_0 = \bar{y} - b_1\bar{x}}$$

$$\mathrm{b}_0 = 26.5556 - (0.1742)(214.8889)$$

$$\mathbf{b_0} = -10.8716$$

$$\hat{y} = b_0 + b_1 x$$

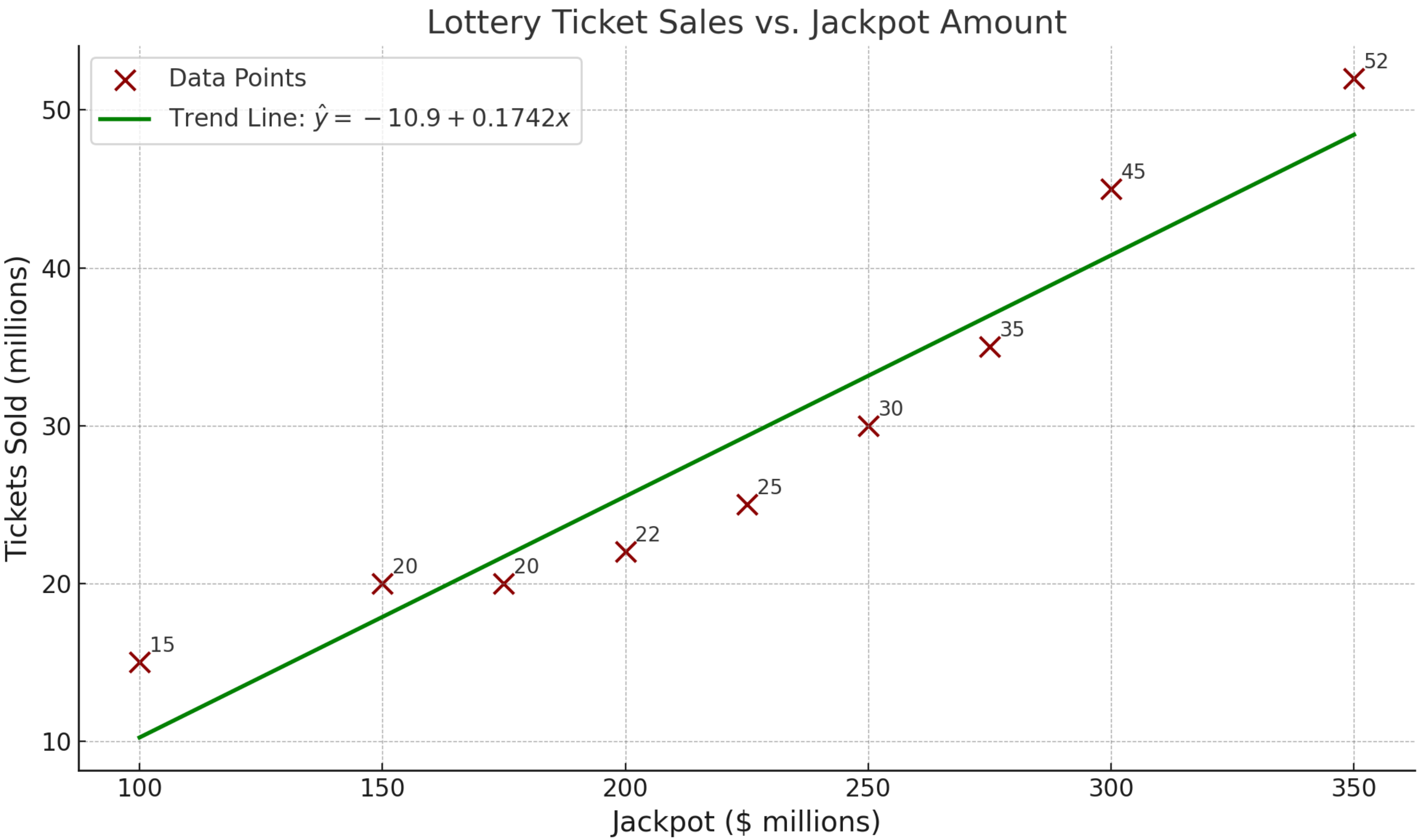
$$\hat{y} = -10.8716 + (0.1742)x$$

Or

$$\hat{y} = -10.9 + (0.1742)x$$

where $\hat{y}$ is the predicted number of tickets sold and ***x*** is the amount of the jackpot.

Graph the regression equation $\widehat{y}$ =-10.9 $+(0.1742)x$ on the scatterplot of the jackpot/tickets data from Table 1 and examine the graph to subjectively determine how well the regression line fits the data.



Problem: A study was conducted at Virginia Tech to determine if certain **static arm-strength measures** have an influence on the **"dynamic lift"** characteristics of an individual. **Twenty-five** individuals were subjected to strength tests and then were asked to perform a **weightlifting test** in which weight was dynamically lifted overhead. The data are given in next two slides.

(a) Estimate β_0 and β_1 for the linear regression curve $\mu_{Y|x} = \beta_0 + \beta_1 x$.

(b) Find a point estimate of $\mu_{Y|30}$.

(c) Plot the residuals versus the *x*'s (arm strength).

Comment.

Individual	Arm Strength (x)	Dynamic Lift (y)
1	17.3	71.7
2	19.3	48.3
3	19.5	88.3
4	19.7	75.0
5	22.9	91.7
6	23.1	100.0
7	26.4	73.3
8	26.8	65.0
9	27.6	75.0
10	28.1	88.3
11	28.2	68.3
12	28.7	96.7

Individual	Arm Strength (x)	Dynamic Lift (y)
13	29.0	76.7
14	29.6	78.3
15	29.9	60.0
16	29.9	71.7
17	30.3	85.0
18	31.3	85.0
19	36.0	88.3
20	39.5	100.0
21	40.4	100.0
22	44.3	100.0
23	44.6	91.7
24	50.4	100.0
25	55.9	71.7

Individual	Arm Strength (x)	Dynamic Lift (y)	xy	x^2
1	17.3	71.7	1240.41	299.29
2	19.3	48.3	932.19	372.49
3	19.5	88.3	1721.85	380.25
4	19.7	75.0	1477.5	388.09
5	22.9	91.7	2099.93	524.41
6	23.1	100.0	2310.0	533.61
7	26.4	73.3	1935.12	696.96
8	26.8	65.0	1742.0	718.24
9	27.6	75.0	2070.0	761.76
10	28.1	88.3	2481.23	789.61
11	28.2	68.3	1926.06	795.24
12	28.7	96.7	2775.29	823.69

Individual	Arm Strength (x)	Dynamic Lift (y)	xy	x^2
13	29.0	76.7	2224.3	841.0
14	29.6	78.3	2317.68	876.16
15	29.9	60.0	1794.0	894.01
16	29.9	71.7	2143.83	894.01
17	30.3	85.0	2575.5	918.09
18	31.3	85.0	2660.5	979.69
19	36.0	88.3	3178.8	1296.0
20	39.5	100.0	3950.0	1560.25
21	40.4	100.0	4040.0	1632.16
22	44.3	100.0	4430.0	1962.49
23	44.6	91.7	4089.82	1989.16
24	50.4	100.0	5040.0	2540.16
25	55.9	71.7	4008.03	3124.81
	$\sum_{i=1}^{n} x_i$**=778.7**	$\sum_{i=1}^{n} y_i$ **= 2050.0**	$\sum_{i=1}^{n} x_i y_i$**=** **65164.04**	$\sum_{i=1}^{n} x_i^2$**=** **26591.63**

$$b_1 = \frac{n\sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$= \frac{(25)(65164.04)-(778.7)(2020)}{(25)(26591.63)-(778.7)^2}$$

$$\mathbf{b_1 = 0.5609}$$

$$b_0 = \bar{y} - b_1\bar{x}$$

or

$$b_0 = \frac{\sum_{i=1}^{n} y_i - b_1\sum_{i=1}^{n} x_i}{n}$$

$$= \frac{2020-(0.5609)(778.7)}{25}$$

$$b_0 = 64.53$$

$\mu_{Y|x}$ = β_0+ β_1x

Estimate of linear regression line is

$\boldsymbol{\mu_{Y|x}}$ = 64.53 +0.5609 x

Or

$\hat{\boldsymbol{y}}$ = 64.53 +0.5609 x

b) At x = 30

$\mu_{Y|30}$ = 64.53 +(0.5609) (30)

= 81.40

c) No Clear Pattern:

The residuals appear randomly scattered around the horizontal line at zero.

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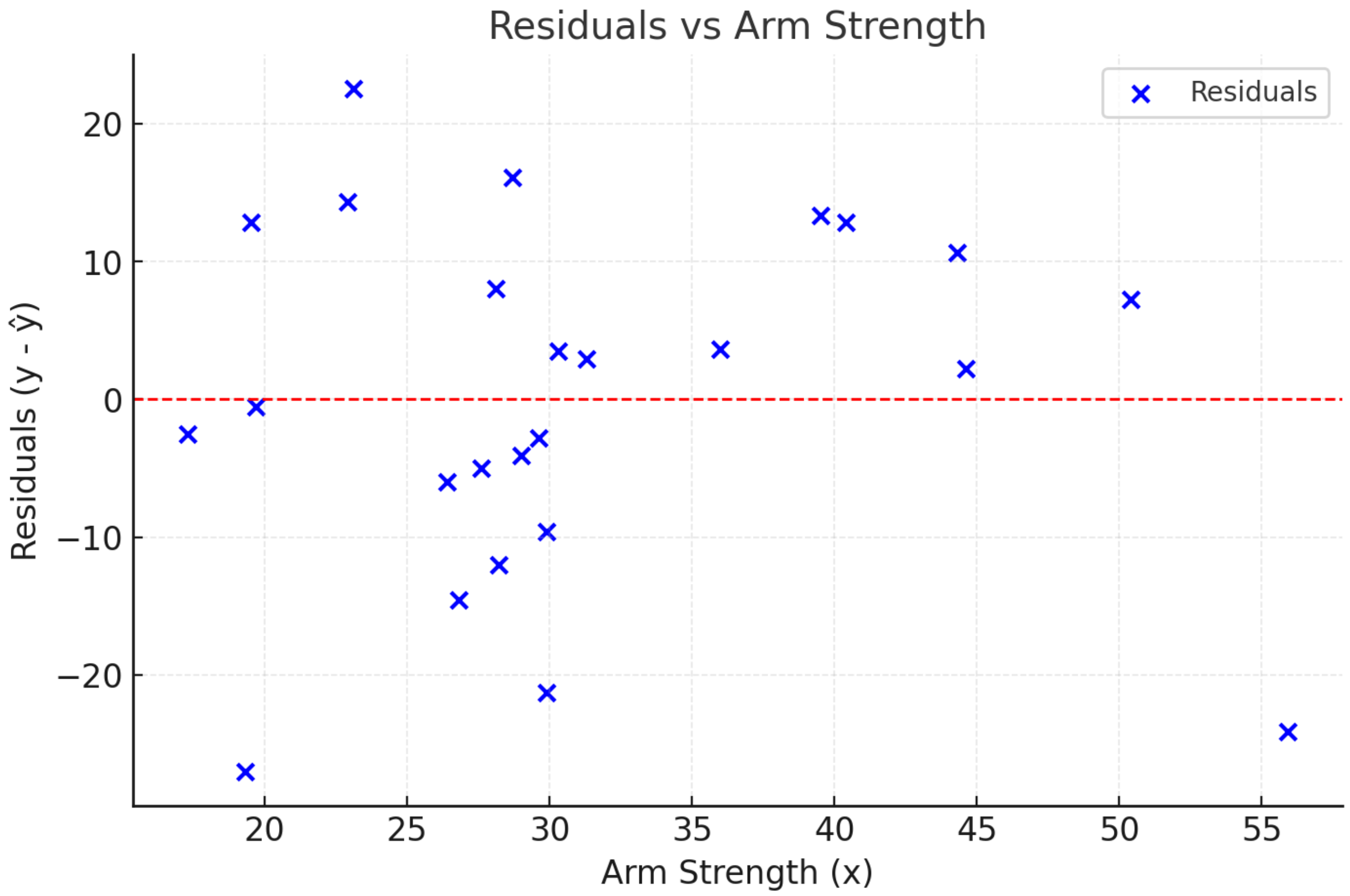


1. No Clear Pattern:

The residuals appear randomly scattered around the horizontal line at zero.

This suggests that the regression model adequately captures the linear relationship between arm strength and dynamic lift.

2. Heteroscedasticity:

Some variability in the spread of residuals is noticeable, particularly for smaller and larger values of arm strength.

This might indicate a **non-constant variance in the data**, suggesting that a **transformation or weighted** regression might improve the model.

3. Outliers:

A few points deviate significantly from zero.

These outliers could be investigated for potential measurement errors or unique conditions.

What is Heteroscedasticity?

Heteroscedasticity occurs when the variance of residuals (errors) is **not constant across all levels** of the independent variable(s).

- **Homoscedasticity:** Constant residual variance.
- **Heteroscedasticity:** Residual variance changes with predictor values.

Impact of Heteroscedasticity

Effects of heteroscedasticity on regression analysis:

1. Does not bias the coefficients (e.g., β_0, β_1).
2. Increases standard errors, making hypothesis tests unreliable.
3. Can lead to incorrect conclusions about variable significance.

Causes of Heteroscedasticity

- Presence of outliers.
- Skewed distribution of the dependent variable.
- Non-linear relationships not captured by the model.
- Measurement errors in the data.

Detecting Heteroscedasticity

1. Residual Plot:

Plot residuals vs. fitted values or predictors.

Look for a funnel-shaped pattern (increasing or decreasing spread).

2. Statistical Tests:

Breusch-Pagan Test: Regress squared residuals on predictors.

White's Test: Tests for heteroscedasticity in a model.

3. Visual Inspection:

Look for systematic patterns in residuals.

Solutions for Heteroscedasticity

1. Transformations:

Apply log, square root, or inverse transformations to stabilize variance.

2. Weighted Least Squares (WLS):

Assign weights inversely proportional to residual variance.

3. Robust Standard Errors:

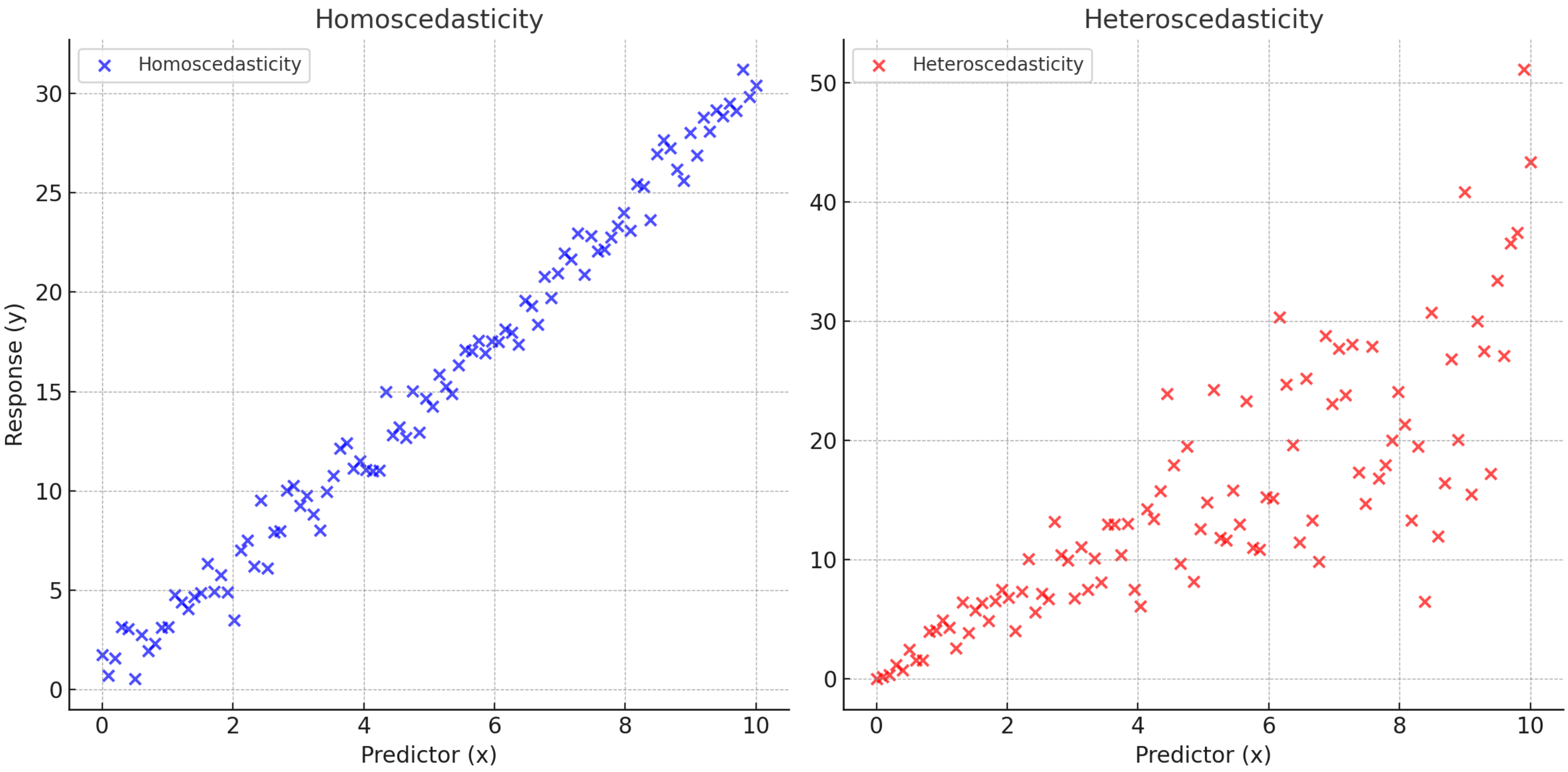
Use heteroscedasticity-robust standard errors to adjust hypothesis tests.

Why Does It Matter?

- Heteroscedasticity undermines **reliability of statistical inference** (e.g., confidence intervals, p-values).

- Addressing it ensures accurate and meaningful interpretations of regression results.

Homoscedasticity vs Heteroscedasticity



- **Homoscedasticity (Left Graph):**
  - Residual variance remains **constant** across all values of the predictor.
  - Points are evenly spread around the trend line.
- **Heteroscedasticity (Right Graph):**
  - Residual variance **increases** with the value of the predictor.
  - Points show a fan-like spread, indicating changing variance.