April 18, 2023

THE LORENZ EQUATIONS & THE BUTTERFLY EFFECT
MATH 481A - FINAL PROJECT

Abstract. In this project, we will study the Lorenz Equations; the famous system of ordinary differential equations first studied by mathematician and meteorologist Edward Lorenz. It is a good example of a chaotic system in the sense that the solution is sensitive to the choice of parameter values and initial conditions. Keep in mind, this is research-type of project, so you should use all your sources and not only reply on this draft. Type your answers for all nine problems in Section 2, include the figures and label them as suggested. The report should not exceed more than 2 pages. Print your report (preferably colorful) two sided, attache the code you use (you do not need to attach all the codes used, only one is enough), and turn it in before the final exam on Thursday, 5/18/23 at 3:00 pm.

1. Introduction

Chaotic Systems. Chaos arises in several disciplines, including meteorology, physics, environmental science, engineering, economics, biology and ecology. Chaotic systems are difficult to define. Essentially they are dynamic systems that are highly sensitive to small differences in their initial conditions and also to rounding errors in numerical computation. These differences produce widely diverging outcomes for such systems. In 1972 Edward Lorenz presented a talk to the American Association for the Advancement of Science entitled "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set a Tornado in Texas?". The theme of this presentation was that small changes in the state of the atmosphere can result in large differences in its later states. The term butterfly effect has entered popular culture as a symbol of chaos.

Chaos occurs even though such systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved. Given exactly the same initial conditions, the same result is obtained. Thus, despite its "random" appearance, chaos is a deterministic outcome. An intuitive example of a chaotic system quoted by Lorenz (1993) is a pin ball machine. At the start of the game, the ball is given an initial velocity. The ball then rolls through an array of pins. Striking a pin changes the ball's direction and the ball subsequently strikes other pins in the array. The slightest change in the initial velocity of the ball will cause it to strike different pins and take a totally different route through the array of pins.

Lorenz Equations. The Lorenz equations are a set of three nonlinear differential equations first introduced by mathematician and meteorologist Edward Lorenz in 1963. These equations were derived as a simplified mathematical model for atmospheric convection, but they are also used to model other physical systems that exhibit chaotic behavior. The Lorenz equations are:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(\rho-z)-y\\ \frac{dz}{dt} &= xy-\beta z\end{aligned} \tag{Lorenz equations}$$

which describe the evolution of a three-dimensional system with variables $x = x(t), y = y(t)$ and $z = z(t)$ over time. The dynamics of the system depend on the values of the parameters σ, ρ and β. When certain values of these parameters are used, the system exhibits chaotic behavior, meaning that small changes in the initial conditions can lead to large differences in the behavior of the system over time. These equations were initially introduced as a severe truncation of the Navier–Stokes equations describing Rayleigh–Benard convection in a fluid (like Earth's atmosphere), which is heated from below in a gravitational field. Now let's briefly review some of its interesting properties:

- Nonlinear: The Lorenz equations are nonlinear, meaning that the behavior of the system is not proportional to the input. This leads to a wide range of possible behaviors, including chaotic behavior.
- Chaotic: When certain parameter values are used, the Lorenz equations exhibit chaotic behavior. This means that the system is highly sensitive to initial conditions, and small changes in the initial conditions can lead to very different behavior over time.
- Attractor: If $\sigma, \rho, \beta > 0$ then all solution of the Lorenz equation will enter an ellipsoid centered at $(0, 0, 2\rho)$ in finite time. In addition the solution will remain inside the ellipsoid once it has entered. It follows by definition that the ellipsoid is an attracting set. The Lorenz equations have a strange attractor, which is a geometric shape in three-dimensional space that describes the long-term behavior of the system. The strange attractor is a complex, self-similar structure that exhibits many interesting properties.
- Bifurcations: As the parameter values are changed, the Lorenz equations can undergo bifurcations, which are sudden changes in the behavior of the system. These bifurcations can lead to the creation of new attractors or the destruction of existing ones.
- Butterfly effect: The Lorenz equations are often used as an example of the butterfly effect, which refers to the idea that small changes in one part of a system can have large and unpredictable effects on the system as a whole. This is because the Lorenz equations are highly sensitive to initial conditions, and even small changes can lead to very different behavior over time.

Runge-Kutta RK4 Method for solving ODEs. The Runge-Kutta RK4 method is a widely used numerical method for solving ordinary differential equations (ODEs) numerically. It is a fourth-order method, meaning that the error of the method is proportional to the fourth power of the step size used in the method. The RK4 method works by using a series of approximations to estimate the value of the solution at each time step. At each step, the method uses four approximations to the solution to estimate the value of the solution at the next time step. These four approximations are weighted and combined to obtain the final estimate of the solution at the next time step.

To explain the method, consider an ordinary differential equation of the form

$$\frac{dy}{dt} = f(t, y)$$

with initial condition $y(t_0) = y_0$. We can then define the formulas for Runge-Kutta RK4 methods as follows. Given the initial condition y_0, we pick a step size $h > 0$ and for $t_{n+1} = t_n + h$ define:

(Runge-Kutta RK4)
$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where for $n = 1, 2, 3, ...$

$$\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}) \\
k_3 &= f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}) \\
k_4 &= f(t_n + h, y_n + hk_3).
\end{aligned}$$

Here y_{n+1} is the RK4 approximation of of the unknown exact solution $y(t_{n+1})$. The RK4 method has a local truncation error of order h^5. This means that as the step size is decreased, the error of the method decreases faster than other lower-order methods. In Matlab, we use Matlab function ode45 for RK4.

2. Project

Instruction: Type your answers for the following nine problems, include the Figures 1:5 and label them as suggested. For all the figures, you could either plot the three dimensional $(x(t), y(t), z(t))$ or two dimensional $(x(t), z(t))$ trajectories. The report should not exceed more than 2 pages. Print your report (preferably colorful) two sided, attache the code you use (you do not need to attach all the codes used, only one is enough), and turn it in before the final exam on Thursday, 5/18/23 at 3:00 pm.

(1) Show that Lorenz equations are invariant under $(x, y) \to (-x, -y)$. Hence if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$.

(2) Show that the $z-$axis is invariant; meaning that a solution that starts on the $z-$axis (i.e. $x(0) = y(0) = 0$ but $z(0) = z_0 \neq 0$) will remain on the $z-$axis. In addition, show that the solution will tend toward the origin exponentially fast if the initial condition are on the $z-$axis.

(3) Recall that in solving ordinary differential equations, an equilibrium point (also known as a critical point or a stationary point) is a point where the derivative of the solution with respect to time is zero. In other words, it is a point where the solution is constant and does not change over time. Equilibrium points are important in the study of ODEs because they can give insight into the behavior of the system near that point. Depending on the stability of the equilibrium point, the system may converge to that point over time, or it may diverge away from it. Find the equilibrium points for Lorenz equations.

(4) Write a code and use the Runge-Kutta RK4 scheme to solve the Lorenz equations with $\sigma = 10, \beta = 8/3, \rho = 28$ with initial condition $(x_0 = -8, y_0 = 8, z_0 = 27)$ and step size $h = 0.01$ and for the time interval $[0, 8]$. Plot the numerical solutions/trajectories $(x(t), y(t), z(t))$ in one graph, label it as Figure 1.

(5) Keep all as same as question 4 but cut the step size by half, $h = 0.005$, and repeat the simulation. Re-plot it on Figure 1 with a different color then you can see the difference, label it as Figure 2.

(6) Keep all as same as question 4 but change the parameters to $\sigma = 10, \beta = 8/3, \rho = 14$ and repeat the simulation, plot it and label it as Figure 3.

(7) Keep all as same as question 4 but change the initial condition to $(x_0 = 0, y_0 = 0, z_0 = 10)$, and repeat the simulation, plot it and label it as Figure 4.

(8) Now we want to see if the solution is sensitive to initial condition. Repeat problem 4 with two initial conditions $(x_0 = 5, y_0 = 5, z_0 = 5)$ and $(x_0 = 5.0091, y_0 = 4.9997, z_0 = 5.0060)$ for the time interval $[0, 15]$. Plot both simulation on one graph but use different colors then you can see each clearly, label it as Figure 5.

(9) Write a short paragraph to describe your observation and understanding of the above simulations and project.

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