COMP30026 Models of Computation Assignment 1

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Challenge 1

a.

P	Q	R	$((\neg P \land Q) \Rightarrow \neg R) \Rightarrow R$	R
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

$\therefore ((\neg P \land Q) \Rightarrow \neg R) \Rightarrow R \equiv R$

b.

P	Q	R	$(P \Rightarrow (Q \lor R)) \land (P \Leftrightarrow Q)$	$P \Leftrightarrow Q$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

$\therefore (P \Rightarrow (Q \lor R)) \land (P \Leftrightarrow Q) \equiv P \Leftrightarrow Q$

c.

P	Q	$P \Rightarrow (P \Leftrightarrow Q)$	$P \Rightarrow Q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

$\therefore P \Rightarrow (P \Leftrightarrow Q) \equiv P \Rightarrow Q$

d.

P	Q	R	$P \vee (R \Rightarrow Q)$	$(Q \Rightarrow P) \Rightarrow (R \Rightarrow P)$
0	0	0	1	1
0	0	1	0	0
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

$\therefore P \vee (R \Rightarrow Q) \equiv (Q \Rightarrow P) \Rightarrow (R \Rightarrow P)$

Challenge 2

Firstly, let us define both inhabitant's statements in propositional logic. Inhabitant P's statements are equivalent to $\neg Q$ and Q', respectively. Inhabitant Q's statements are equivalent to P and $\neg P'$, respectively.

We are given information that the statements given by the inhabitants P and Q respectively must be either true or false, but *cannot* be a mixture of both. In other words, if an inhabitant lies, then every statement they say is guaranteed to be a false statement.

This means that if inhabitant P is telling the truth, then $\neg Q \wedge Q'$ is true, otherwise $Q \wedge \neg Q'$ must be true. Similarly, if inhabitant Q is telling the truth, then $P \wedge \neg P'$ must be true, otherwise $\neg P \wedge P'$ must be true.

Let A mean P *is telling the truth*, and B mean Q is *telling the truth*. We can surmise when P and Q are lying or telling the truth via truth tables.

P	P'	A
0	0	1
0	1	0
1	0	0
1	1	1

Q	Q'	B
0	0	1
0	1	0
1	0	0
1	1	1

It can be seen that P is truthful when $P \Leftrightarrow P'$ is true, and lies when $P \oplus P'$ is true. The same notions hold for inhabitant Q, with respect to literals Q and Q'.

Using the formulas generated above, we can now extract the maximal amount of information provided by the two inhabitants, in the form of propositional formulas (1) and (2), both of which must be true.

$$((P \Leftrightarrow P') \wedge (\neg Q \wedge Q')) \oplus ((P \oplus P') \wedge (Q \wedge \neg Q')) \tag{1}$$

$$((Q \Leftrightarrow Q') \wedge (P \wedge \neg P')) \oplus ((Q \oplus Q') \wedge (\neg P \wedge P')) \tag{2}$$

Using a truth table, we can determine for each of P and Q whether they are a knight or a knave, and whether they are healthy or sick.

P	P'	Q	Q'	$((P \Leftrightarrow P') \wedge (\neg Q \wedge Q')) \oplus ((P \oplus P') \wedge (Q \wedge \neg Q'))$	$((Q \Leftrightarrow Q') \wedge (P \wedge \neg P')) \oplus ((Q \oplus Q') \wedge (\neg P \wedge P'))$
0	0	0	0	0	0
0	0	0	1	1	0
0	0	1	0	0	0
0	0	1	1	0	0
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	1	1
0	1	1	1	0	0
1	0	0	0	0	1
1	0	0	1	0	0
1	0	1	0	1	0
1	0	1	1	0	1
1	1	0	0	0	0
1	1	0	1	1	0
1	1	1	0	0	0
1	1	1	1	0	0

We know that both formulas must be true. Therefore, there is only *one* possible scenario. That is, P is a *healthy knave* and Q is a *sick knight*.

Challenge 3

a. Consider the interpretation I with domain $D = \{0,1\}$ and $P(0)$. Additionally, let Q always be *false*.

   For this interpretation, it can be seen that $I \vDash \forall x(P(x)) \Rightarrow Q$, but $I \nvDash \forall x(P(x) \Rightarrow Q)$. Hence, $\forall x(P(x)) \Rightarrow Q \nvDash \forall x(P(x) \Rightarrow Q)$.

   Therefore, $\forall x(P(x)) \Rightarrow Q$ is *not* equivalent to $\forall x(P(x) \Rightarrow Q)$.

b.

Challenge 4

a. $\forall x \forall y(M(x, y) \Rightarrow F(x))$

b. $\forall x \forall y((U(x) \wedge M(y, x)) \Rightarrow U(y))$

c. $\forall x((F(x) \wedge U(x) \wedge B(x)) \Rightarrow D(x))$

d. $\forall x \forall y((U(x) \wedge M(x, y) \wedge D(x)) \Rightarrow D(y))$

e. $\forall x((S(x) \wedge U(x)) \Rightarrow B(x))$

The *Horn clauses* generated are as follows:

a. $\{\neg M(x, y), F(x)\}$

b. $\{\neg U(x), \neg M(y, x), U(y)\}$

c. $\{\neg F(x), \neg U(x), \neg B(x), D(x)\}$

d. $\{\neg U(x), \neg M(x,y), \neg D(x), D(y)\}$

e. $\{\neg S(x), \neg U(x), B(x)\}$

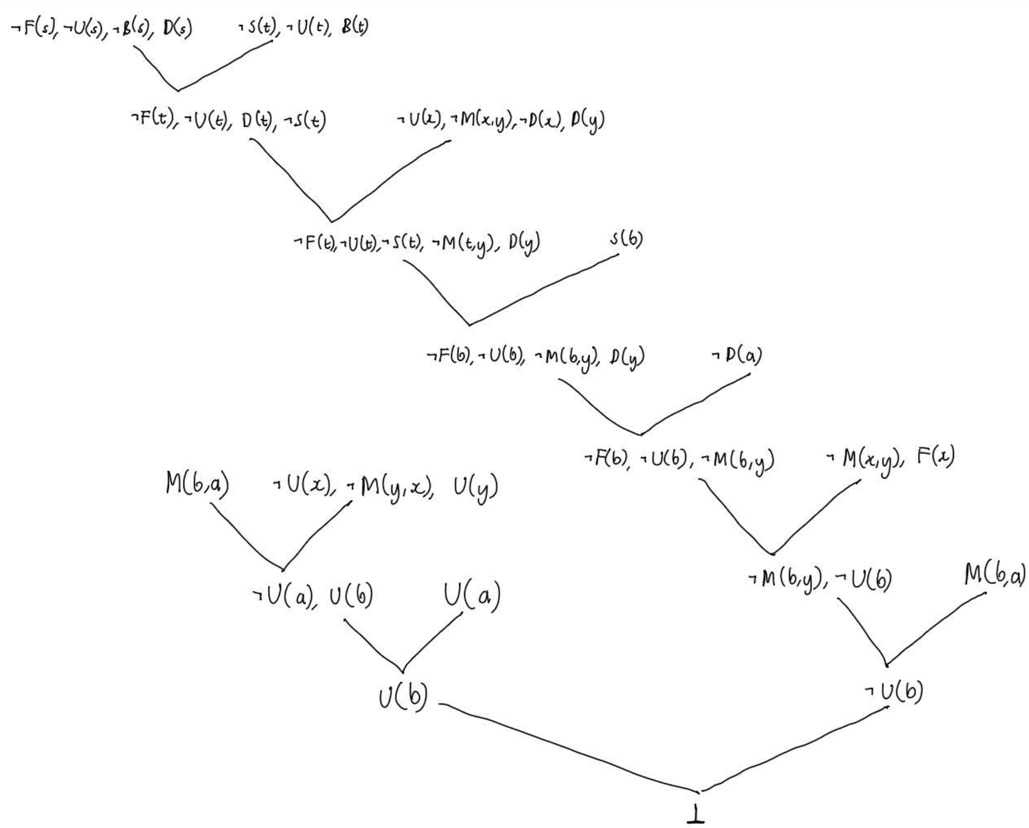
Now, we must express statement (f) in first-order predicate logic.

f. $\forall x \forall y((U(x) \wedge M(y,x) \wedge S(y)) \Rightarrow D(x))$

The negation of (f) in clausal form is:
$\neg\forall x \forall y((U(x) \wedge M(y,x) \wedge S(y)) \Rightarrow D(x))$
$\rightsquigarrow \exists x \neg\forall y((U(x) \wedge M(y,x) \wedge S(y)) \Rightarrow D(x))$
$\rightsquigarrow \exists x \exists y(U(x) \wedge M(y,x) \wedge S(y) \wedge \neg D(x))$
$\rightsquigarrow \exists y(U(a) \wedge M(y,a) \wedge S(y) \wedge \neg D(a))$
$\rightsquigarrow (U(a) \wedge M(b,a) \wedge S(b) \wedge \neg D(a))$

Which produces *four* clauses:
$\{U(a)\}, \{M(b,a)\}, \{S(b)\}, \{\neg D(a)\}$

Now, we can find a refutation of the set of nine clauses.



By the refutation proof shown above, we have proven that the conjunction of premise statements (a)-(e) and the negation of conclusion statement (f) is unsatisfiable.

$\therefore$ (f) is a logical consequence of statements (a)-(e)

Challenge 5

a. The statements expressed in first-order predicate logic are as follows:

(a1) $\forall x \forall y (N(x,y) \Rightarrow N(y,x))$

(a2) $\forall x \forall y (\exists u (M(u,x) \wedge \neg M(u,y)) \Rightarrow N(x,y))$

(a3) $\forall x (\neg \exists u (M(u,x)) \Rightarrow E(x))$

(a4) $\forall x \forall y \forall u ((D(x,y) \wedge M(u,x)) \Rightarrow \neg M(u,y))$

b. The set of clauses produced by statements (a1)-(a4) are as follows:

(a1) $\{N(x,y), N(y,x)\}$

(a2) $\{\neg M(u,x), M(u,y), N(x,y)\}$

(a3) $\{M(f(x),x), E(x)\}$

(a4) $\{\neg D(x,y), \neg M(u,x), \neg M(u,y)\}$

c. Firstly, we must transform the statement into first-order predicate logic. The statement is equivalent to the formula $\forall x \forall y (D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y))))$.

Since we would like to use this formula in a refutation proof as the logical consequence of other statements, we must negate it and then convert its negation to clausal form.

The negation of the statement in clausal form is:
$\neg \forall x \forall y (D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y))))$
$\rightsquigarrow \exists x \neg \forall y (D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y))))$
$\rightsquigarrow \exists x \exists y (\neg (D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y)))))$
$\rightsquigarrow \exists x \exists y (\neg (D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y)))))$
$\rightsquigarrow \exists x \exists y (D(x,y) \wedge \neg (N(x,y) \vee (E(x) \wedge E(y)))))$
$\rightsquigarrow \exists x \exists y (D(x,y) \wedge ((\neg N(x,y) \wedge \neg E(x)) \vee (\neg N(x,y) \wedge \neg E(y))))$
$\rightsquigarrow \exists x \exists y ((D(x,y) \wedge \neg N(x,y) \wedge \neg E(x)) \vee (D(x,y) \wedge \neg N(x,y) \wedge \neg E(y)))$
$\rightsquigarrow \exists x \exists y (D(x,y) \wedge \neg N(x,y) \wedge (\neg E(x) \vee \neg E(y)))$
$\rightsquigarrow (D(a,b) \wedge \neg N(a,b) \wedge (\neg E(a) \vee \neg E(b)))$

Hence, the fifth statement produces *three* clauses:
$\{D(a,b)\}, \{\neg N(a,b)\}, \{\neg E(a), \neg E(b)\}$

d. Refutation Proof:

Challenge 6

See Grok.