Dig-In:

The Squeeze Theorem

The Squeeze theorem allows us to compute the limit of a difficult function by "squeezing" it between two easy functions.

In mathematics, sometimes we can study complex functions by relating them for simpler functions. The *Squeeze Theorem* tells us one situation where this is possible.

Theorem 1 (Squeeze Theorem). *Suppose that*

$$g(x) \leq f(x) \leq h(x)$$

for all x close to a but not necessarily equal to a. If

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} f(x) = L$.

Question 1 *I'm thinking of a function f. I know that for all x*

$$0 \leq f(x) \leq x^2.$$

What is $\lim_{x \to 0} f(x)$*?*

Multiple Choice:

(a) $f(x)$

(b) $f(0)$

(c) 0 ✓

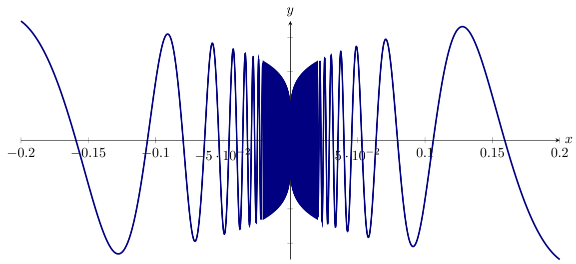
(d) *impossible to say*

Example 1. *Consider the function*

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

Learning outcomes: Understand the Squeeze Theorem and how it can be used to find limit values. Calculate limits using the Squeeze Theorem.

Author(s):



Is this function continuous at $x = 0$?

Explanation. *We must show that* $\lim_{x\to 0} f(x) = \underbrace{\boxed{0}}_{\text{given}}$. *First, let's assume that* $x \geq 0$ *and small. Since*

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

by multiplying these inequalities by $\sqrt[5]{x} \geq 0$ *, we obtain*

$$-\sqrt[5]{x} \leq \sqrt[5]{x}\sin\left(\frac{1}{x}\right) \leq \sqrt[5]{x}$$

which can be written as

$$-\sqrt[5]{x} \leq f(x) \leq \sqrt[5]{x}.$$

Now, let's assume that $x \leq 0$ *and small. Since*

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

by multiplying these inequalities by $\sqrt[5]{x} \leq 0$ *, we obtain*

$$\sqrt[5]{x} \leq \sqrt[5]{x}\sin\left(\frac{1}{x}\right) \leq -\sqrt[5]{x}$$

which can be written as

$$\sqrt[5]{x} \leq f(x) \leq -\sqrt[5]{x}.$$

Therefore for all small values of x

$$-\left|\sqrt[5]{x}\right| \leq f(x) \leq \left|\sqrt[5]{x}\right|.$$

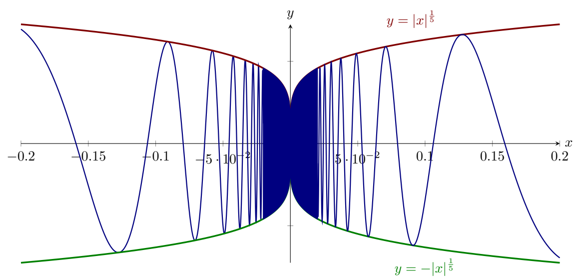
Since

$$\lim_{x\to 0}\left(-\left|\sqrt[5]{x}\right|\right) = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0}\left|\sqrt[5]{x}\right|$$

we apply the Squeeze Theorem and obtain that

$\lim_{x\to 0} f(x) = \underbrace{\boxed{0}}_{\text{given}}$. *Hence $f(x)$ is continuous.*

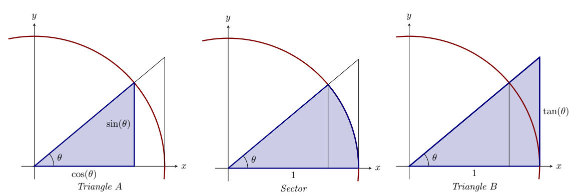
Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.



Example 2. *Compute:*

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

Explanation. *To compute this limit, use the Squeeze Theorem. First note that we only need to examine $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and for the present time, we'll assume that θ is positive. Consider the diagrams below:*



From our diagrams above we see that

$$\textit{Area of Triangle A} \leq \textit{Area of Sector} \leq \textit{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(\theta)\sin(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}.$$

Multiplying through by 2, and recalling that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ we obtain

$$\cos(\theta)\sin(\theta) \leq \theta \leq \frac{\sin(\theta)}{\cos(\theta)}.$$

Dividing through by $\sin(\theta)$ and taking the reciprocals (reversing the inequalities), we find

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

Note, $\cos(-\theta) = \cos(\theta)$ *and* $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin(\theta)}{\theta}$*, so these inequalities hold for all* $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{\theta\to 0}\cos(\theta) = \underbrace{\boxed{1}}_{\text{given}} = \lim_{\theta\to 0}\frac{1}{\cos(\theta)},$$

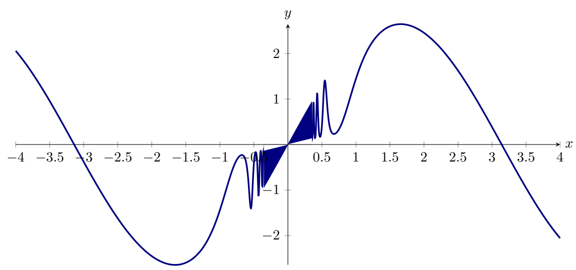
and so we conclude by the Squeeze Theorem, $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \underbrace{\boxed{1}}_{\text{given}}$.

When solving a problem with the Squeeze Theorem, one must write a sort of mathematical poem. You have to tell your friendly reader exactly which functions you are using to "squeeze-out" your limit.

Example 3. *Compute:*

$$\lim_{x\to 0}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right)$$

Explanation. *Let's graph this function to see what's going on:*



The function $\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}$ *has two factors:*

$$\overbrace{\sin(x)}^{\textit{goes to zero as } x\to 0}\cdot\underbrace{e^{\cos\left(\frac{1}{x^3}\right)}}_{\textit{bounded between } e^{-1} \textit{ and } e}$$

Hence we have that when $0 < x < \pi$

$$0 \le \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \le \sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and we see

$$\lim_{x\to 0^+} 0 = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0^+}\sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and so by the Squeeze theorem,

$$\lim_{x\to 0^+}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underset{\text{given}}{\boxed{0}}.$$

In a similar fashion, when $-\pi < x < 0$,

$$\sin(x)\underset{\text{given}}{\boxed{e}} \leq \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \leq 0$$

and so

$$\lim_{x\to 0^-}\sin(x)\underset{\text{given}}{\boxed{e}} = \underset{\text{given}}{\boxed{0}} = \lim_{x\to 0^-} 0,$$

and again by the Squeeze Theorem $\lim\limits_{x\to 0^-}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0$. *Hence we see that*

$$\lim_{x\to 0}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underset{\text{given}}{\boxed{0}}.$$