Dig-In:

Limits of the form zero over zero

We want to evaluate limits for which the Limit Laws do not apply.

In the last section we computed limits using continuity and the limit laws. What about limits that cannot be directly computed using these methods? Consider the following limit,

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}.$$

Here

$$\lim_{x\to 2}\left(x^2-3x+2\right)=0 \qquad \text{and} \qquad \lim_{x\to 2}(x-2)=0$$

in light of this, you may think that the limit is one or zero. **Not so fast**. This limit is of an *indeterminate form*. What does this mean? Read on, young mathematician.

Definition 1. *A limit*

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is said to be of the form $\frac{\mathbf{0}}{\mathbf{0}}$ *if*

$$\lim_{x\to a} f(x)=0 \quad \textit{and} \quad \lim_{x\to a} g(x)=0.$$

Question 1 *Which of the following limits are of the form* $\frac{\mathbf{0}}{\mathbf{0}}$*?*

Select All Correct Answers:

(a) $\lim_{x\to 0}\frac{\sin(x)}{x}$ ✓

(b) $\lim_{x\to 0}\frac{\cos(x)}{x}$

(c) $\lim_{x\to 0}\frac{x^2-3x+2}{x-2}$

(d) $\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$ ✓

Learning outcomes: Understand what is meant by the form of a limit. Calculate limits of the form zero over zero. Identify determinate and indeterminate forms. Distinguish between determinate and indeterminate forms. Discuss why infinity is not a number.

Author(s):

(e) $\displaystyle\lim_{x\to 3} \frac{x^2-3x+2}{x-3}$

Warning 1. *The symbol* $\mathbf{\frac{0}{0}}$ *is* ***not*** *the number 0 divided by 0. It is simply short-hand and means that a limit* $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ *has the property that*

$$\lim_{x\to a} f(x) = 0 \quad \textit{and} \quad \lim_{x\to a} g(x) = 0.$$

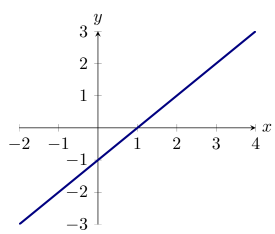
Let's finish the example with the function above.

Example 1. *Compute:*

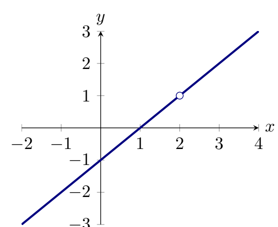
$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$$

Explanation. *This limit is of the form* $\mathbf{\frac{0}{0}}$. *However, note that if we assume* $x \neq 2$, *then we can write*

$$\frac{x^2-3x+2}{x-2} = \frac{(x-2)(\underset{\text{given}}{\boxed{x-1}})}{(x-2)} = \underset{\text{given}}{\boxed{x-1}}.$$



$y = x - 1$



$y = \dfrac{x^2-3x+2}{x-2}$

So, "our function" is equal to the polynomial $x-1$ *everywhere, except at* $x=2$. *Therefore, for all values of* x *near 2, "our function" is equal to the polynomial* $x-1$! *This means that*

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2}(x-1).$$

But now, we have to take the limit of a polynomial, $\displaystyle\lim_{x\to 2}(x-1) = \underset{\text{given}}{\boxed{1}}$.

Hence

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \underset{\text{given}}{\boxed{1}}.$$

Let's consider a few more examples of the form $\frac{\mathbf{0}}{\mathbf{0}}$.

Example 2. *Compute:*

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3}.$$

Explanation. *First note that*

$$\lim_{x\to 1}(x-1)=0 \quad \textit{and} \quad \lim_{x\to 1}\left(x^2+2x-3\right)=0.$$

Hence, this limit is of the form $\frac{\mathbf{0}}{\mathbf{0}}$. *Again, we cannot apply the Quotient Law or any other Limit Law. We cannot use continuity, either. Namely, "our function" is not continuous at* $x=1$, *since it is not defined at* $x=1$.

What can be done? We can hope to be able to cancel a factor going to 0 *out of the numerator and denominator. Since* $\underset{\text{given}}{\boxed{(x-1)}}$ *is a factor going to* 0 *in the numerator, let's see if we can factor a* $\underset{\text{given}}{\boxed{(x-1)}}$ *out of the denominator as well.*

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2+2x-3} &= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}\underset{\text{given}}{\boxed{(x+3)}}} \\ &= \lim_{x\to 1} \frac{1}{\underset{\text{given}}{\boxed{x+3}}} \\ &= \frac{1}{4}.\end{aligned}$$

Example 3. *Compute:*

$$\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{3}{x+5}}{x-1}.$$

Explanation. *We find the form of this limit by looking at the limits of the numerator and denominator separately. The limit of the numerator is:*

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x+1}-\frac{3}{x+5}\right) &= \lim_{x\to 1}\frac{x+5-3(x+1)}{(x+1)(x+5)} \\ &= \lim_{x\to 1}\frac{-2x+2}{(x+1)(x+5)} \\ &= \frac{0}{12} \\ &= 0\end{aligned}$$

The limit of the denominator is:

$$\lim_{x\to 1}(x-1)=0$$

Our limit is therefore of the form $\mathbf{\frac{0}{0}}$ *and we can probably factor a term going to* 0 *out of both the numerator and denominator.*

$$\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1}$$

When looking at the denominator, we hope that this term is $(x-1)$*. Unfortunately, it is not immediately obvious how to factor an* $(x-1)$ *out of the numerator. So, we should first simplify the complex fraction by multiplying it by*

$$1 = \frac{(x+1)(x+5)}{(x+1)(x+5)}$$

this will allow us to cancel immediately

$$\begin{aligned}
&\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1} \cdot \frac{(x+1)(x+5)}{(x+1)(x+5)} \\
&= \lim_{x\to 1} \frac{(x+5) - 3(x+1)}{(x+1)(x+5)(x-1)}.
\end{aligned}$$

Now we will multiply out the numerator. Note that we do not want to multiply out the denominator because we already have an $(x-1)$ *factored out of the denominator and that was the goal.*

$$\begin{aligned}
&\lim_{x\to 1} \frac{(x+5) - 3(x+1)}{(x+1)(x+5)(x-1)} \\
&= \lim_{x\to 1} \frac{x+5-3x-3}{(x+1)(x+5)(x-1)} \\
&= \lim_{x\to 1} \frac{-2x+2}{(x+1)(x+5)(x-1)} \\
&= \lim_{x\to 1} \frac{-2\cancel{(x-1)}}{(x+1)(x+5)\cancel{(x-1)}} \\
&= \lim_{x\to 1} \frac{-2}{(x+1)(x+5)}.
\end{aligned}$$

Now, we can see that the limit of "our function" is equal to the limit of a rational function $\frac{-2}{(x+1)(x+5)}$*. This rational function is continuous on its domain, and, therefore, at* $x = 1$*. Hence*

$$\begin{aligned}
\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1} &= \lim_{x\to 1} \frac{-2}{(x+1)(x+5)} \\
&= \frac{-2}{(1+1)(1+5)} \\
&= \underbrace{\boxed{\frac{-1}{6}}}_{\text{given}}.
\end{aligned}$$

We will look at one more example.

Example 4. *Compute:*

$$\lim_{x \to -1} \frac{\sqrt{x+5}-2}{x+1}.$$

Explanation. *Note that*

$$\lim_{x \to -1} \left(\sqrt{x+5}-2\right) = 0 \quad \textit{and} \quad \lim_{x \to -1} (x+1) = 0.$$

Our limit is, again, of the form $\frac{\mathbf{0}}{\mathbf{0}}$ *and we can probably factor a term going to* 0 *out of both the numerator and denominator. We suspect from looking at the denominator that this term is* $(x+1)$*. Unfortunately, it is not immediately obvious how to factor an* $(x+1)$ *out of the numerator.*

We will use an algebraic technique called ***multiplying by the conjugate****. This technique is useful when you are trying to simplify an expression that looks like*

$$\sqrt{\textit{something}} \pm \textit{something else}.$$

It takes advantage of the difference of squares rule

$$a^2 - b^2 = (a-b)(a+b).$$

In our case, we will use $a = \sqrt{x+5}$ *and* $b = 2$*. Write*

$$\lim_{x \to -1} \frac{\sqrt{x+5}-2}{x+1}$$

$$\begin{aligned}
&= \lim_{x \to -1} \frac{\left(\sqrt{x+5}-2\right)}{(x+1)} \cdot \frac{\left(\sqrt{x+5}+2\right)}{\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x \to -1} \frac{\underset{\text{given}}{\boxed{\left(\sqrt{x+5}\right)^2 - 2^2}}}{(x+1)\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x \to -1} \frac{x+5-4}{(x+1)\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x \to -1} \frac{\cancel{(x+1)}}{\cancel{(x+1)}\left(\sqrt{x+5}+2\right)} \\
&= \lim_{x \to -1} \frac{1}{\sqrt{x+5}+2} \\
&= \frac{1}{\sqrt{-1+5}+2} \\
&= \underset{\text{given}}{\boxed{\frac{1}{4}}}.
\end{aligned}$$

All of the examples in this section are limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$. When you come across a limit of the form $\frac{\mathbf{0}}{\mathbf{0}}$, you should try to use algebraic techniques to come up with a continuous function whose limit you can evaluate.

Notice that we solved multiple examples of limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$ and we got different answers each time. This tells us that just knowing that the form of the limit is $\frac{\mathbf{0}}{\mathbf{0}}$ is not enough to compute the limit. The moral of the story is

Limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$ can take any value.

Definition 2. *A form that gives us no information about whether the limit exists or not, and if the limit exists, no information about the value of the limit, is called an **indeterminate form**.*

*A form that gives information about whether the limit exists or not, and if it exists gives information about the value of the limit, is called a **determinate form**.*

Finally, you may find it distressing that we introduced a form, namely $\frac{\mathbf{0}}{\mathbf{0}}$, only to end up saying they give no information on the value of the limit. But this is precisely what makes indeterminate forms interesting... they're a mystery!