Dig-In:

Continuity

The limit of a continuous function at a point is equal to the value of the function at that point.

Limits are simple to compute when they can be found by plugging the value into the function. That is, when

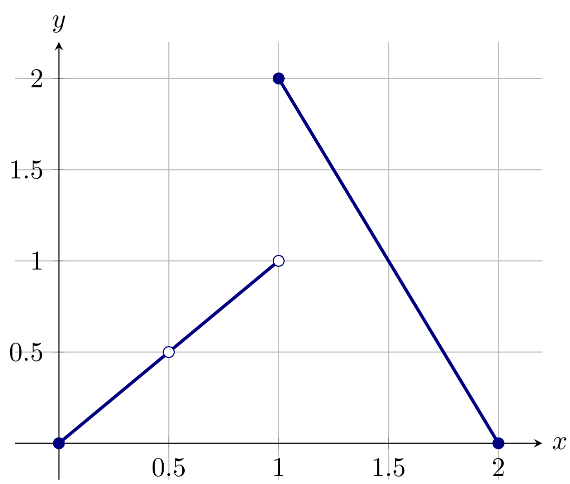
$$\lim_{x\to a} f(x) = f(a).$$

We call this property *continuity*.

Definition 1. *A function f is* ***continuous at a point*** *a if*

$$\lim_{x\to a} f(x) = f(a).$$

Question 1 *Consider the graph of the function f*



Which of the following are true?

Multiple Choice:

(a) *f is continuous at $x = 0.5$*

Learning outcomes: Define continuity in terms of limits. Calculate limits using the limit laws. Famous functions are continuous on their domains.

Author(s):

(b) f *is continuous at* $x = 1$

(c) f *is continuous at* $x = 1.5$ ✓

It is very important to note that saying

“a function f is continuous at a point a”

is really making **three** statements:

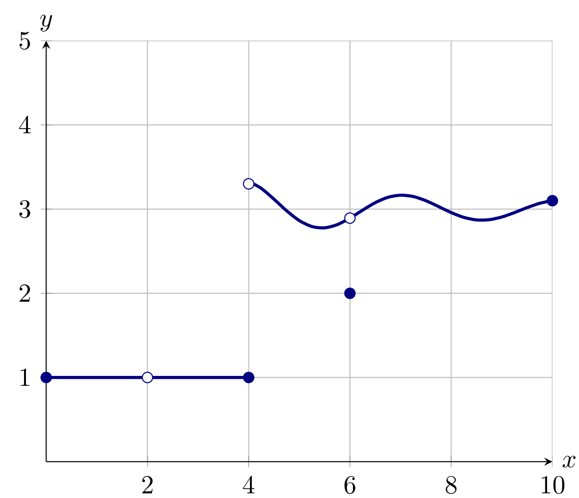
(a) $f(a)$ is defined. That is, a is in the domain of f.

(b) $\lim_{x \to a} f(x)$ exists.

(c) $\lim_{x \to a} f(x) = f(a)$.

The first two of these statements are implied by the third statement.

Example 1. *Find the discontinuities (the points x where a function is not continuous) for the function described below:*



Explanation. *To start, f is not even defined at $x = \underset{\text{given}}{\boxed{2}}$, hence f cannot be continuous at $x = \underset{\text{given}}{\boxed{2}}$.*

Next, from the plot above we see that $\lim_{x \to 4} f(x)$ does not exist because

$$\lim_{x \to 4^-} f(x) = \underset{\text{given}}{\boxed{1}} \qquad \textit{and} \qquad \lim_{x \to 4^+} f(x) \approx 3.3$$

Since $\lim_{x\to 4} f(x)$ *does not exist,* f *cannot be continuous at* $x = 4$.

We also see that $\lim_{x\to 6} f(x) \approx 2.9$ *while* $f(6) = \underbrace{\boxed{2}}_{\text{given}}$. *Hence* $\lim_{x\to 6} f(x) \neq f(6)$, *and so* f *is not continuous at* $x = 6$.

Building from the definition of *continuity at a point*, we can now define what it means for a function to be *continuous* on an open interval.

Definition 2. *A function* f *is* ***continuous on an open interval*** I *if* $\lim_{x\to a} f(x) = f(a)$ *for all* a *in* I.

Loosely speaking, a function is continuous on an interval I if you can draw the function on that interval without any breaks in the graph. This is often referred to as being able to draw the graph "without picking up your pencil."

Theorem 1 (Continuity of Famous Functions)**.** *The following functions are continuous on their natural domains, for* k *a real number and* b *a positive real number:*

Constant function $f(x) = k$

Identity function $f(x) = x$

Power function $f(x) = x^b$

Exponential function $f(x) = b^x$

Logarithmic function $f(x) = \log_b(x)$

Sine and cosine *Both* $\sin(x)$ *and* $\cos(x)$

In essence, we are saying that the functions listed above are continuous wherever they are defined.

Question 2 *Compute:* $\lim_{x\to\pi} x^3 = \boxed{\pi^3}$

Feedback(`attempt`): *The function* $f(x) = x^3$ *is of the form* x^b *for a positive real number* b. *Therefore,* $f(x) = x^3$ *is continuous for all positive real values of* x. *In particular,* $f(x)$ *is continuous at* $x = \pi$. *Since* x^3 *is continuous at* π, *we know that* $\lim_{x\to\pi} f(x) = f(\pi)$. *That is,* $\lim_{x\to\pi} x^3 = \pi^3$.

Question 3 *Compute:* $\lim_{x\to\pi} \sqrt{2} = \boxed{\sqrt{2}}$

Feedback(*attempt*): *The function $f(x) = \sqrt{2}$ is a constant. Therefore, $f(x) = \sqrt{2}$ is continuous for all real values of x. In particular, $f(x)$ is continuous at $x = \pi$. Since f is continuous at π, we know that $\lim_{x\to\pi} f(x) = f(\pi) = \sqrt{2}$. That is, $\lim_{x\to\pi} \sqrt{2} = \sqrt{2}$.*

Question 4 *Compute:* $\lim_{x\to\pi} \cos x = \boxed{-1}$

Feedback(*attempt*): *The function $f(x) = \cos x$ is contionuous for all real values of x. In particular, $f(x)$ is continuous at $x = \pi$. Since f is continuous at π, we know that $\lim_{x\to\pi} f(x) = f(\pi) = -1$. That is, $\lim_{x\to\pi} \cos x = -1$.*

Left and right continuity

At this point we have a small problem. For functions such as $\sqrt{x}$, the natural domain is $0 \le x < \infty$. This is not an open interval. What does it mean to say that $\sqrt{x}$ is continuous at 0 when $\sqrt{x}$ is not defined for $x < 0$? To get us out of this quagmire, we need a new definition:

Definition 3. *A function f is **left continuous** at a point a if $\lim_{x\to a^-} f(x) = f(a)$.*

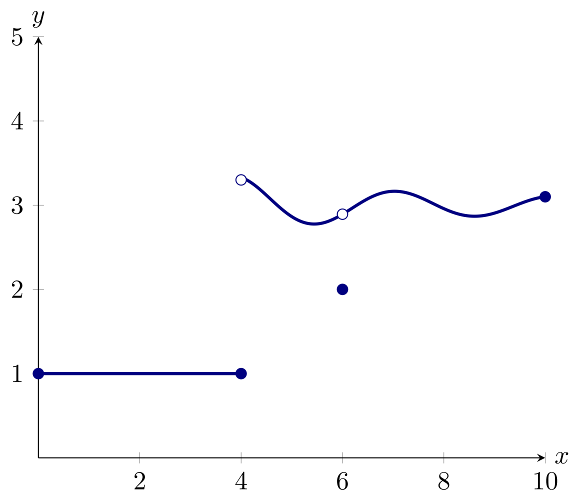
*A function f is **right continuous** at a point a if $\lim_{x\to a^+} f(x) = f(a)$.*

This allows us to talk about continuity on closed and half-closed intervals.

Definition 4. *A function f is*

- ***continuous on a closed interval** $[a, b]$ if f is continuous on (a, b), right continuous at a, and left continuous at b;*
- ***continuous on a half-closed interval** $[a, b)$ if f is continuous on (a, b) and right continuous at a;*
- ***continuous on a half-closed interval** $(a, b]$ if f is f is continuous on (a, b) and left continuous at b.*

Question 5 *Here we give the graph of a function defined on* $[0, 10]$.



Select all intervals for which the following statement is true.

The function f is continuous on the interval I.

Select All Correct Answers:

(a) $I = [0, 10]$

(b) $I = [0, 4]$ ✓

(c) $I = [4, 6]$

(d) $I = [4, 6)$

(e) $I = (4, 6]$

(f) $I = (4, 6)$ ✓

(g) $I = (6, 10]$ ✓

(h) $I = [6, 10)$

Feedback(`attempt`): *Notice that our function is left continuous at $x = 4$ so we can include 4 in the interval $[0, 4]$. Four is not included in the interval $(4, 6)$ because our function is not right continuous at $x = 4$. Similarly, our function is neither right or left continuous at $x = 6$, so 6 is not included in any intervals. Our function is left continuous at $x = 0$ and right continuous at $x = 10$ so we included these endpoints in our intervals.*