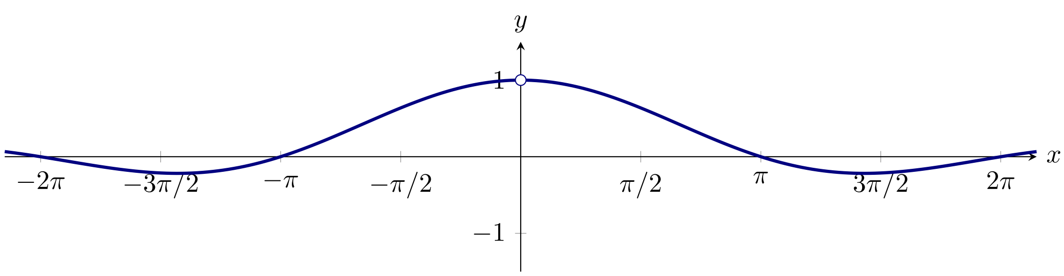
Dig-In:

What is a limit?

We introduce limits.

The basic idea

Consider the function
$$f(x) = \frac{\sin(x)}{x}.$$
While $f(x)$ is undefined at $x = 0$, we can still plot $f(x)$ at other values near $x = 0$.



Question 1 *Use the graph of $f(x) = \frac{\sin(x)}{x}$ above to answer the following question: What is $f(0)$?*

Multiple Choice:

(a) 0

(b) $f(0)$

(c) 1

(d) $f(0)$ *is undefined* ✓

(e) *it is impossible to say*

Learning outcomes: Consider values of a function at inputs approaching a given point. Understand the concept of a limit. Limits as understanding local behavior of functions. Calculate limits from a graph (or state that the limit does not exist). Define a one-sided limit.

Author(s):

Nevertheless, we can see that as x approaches zero, $f(x)$ approaches one. From this setting we come to our definition of a limit.

Definition 1. *Intuitively,*

*the **limit** of $f(x)$ as x approaches a is L,*

written

$$\lim_{x\to a} f(x) = L,$$

if the value of $f(x)$ is as close as one wishes to L for all x sufficiently close, but not equal to, a.

Question 2 *Use the graph of $f(x) = \dfrac{\sin(x)}{x}$ above to finish the following statement: "A good guess is that…"*

Multiple Choice:

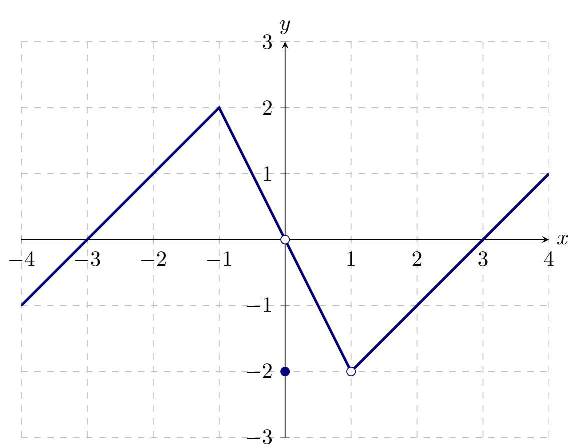
(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$ ✓

(b) $\displaystyle\lim_{x\to 1} \frac{\sin(x)}{x} = 0.$

(c) $\displaystyle\lim_{x\to 1} f(x) = \frac{\sin(1)}{1}.$

(d) $\displaystyle\lim_{x\to 0} f(x) = \frac{\sin(0)}{0} = \infty.$

Question 3 *Consider the following graph of $y = f(x)$*



Use the graph to evaluate the following. Write DNE if the value does not exist.

(a) $f(-2) = \boxed{1}$

(b) $\lim_{x \to -2} f(x) = \boxed{1}$

(c) $f(-1) = \boxed{2}$

(d) $\lim_{x \to -1} f(x) = \boxed{2}$

(e) $f(0) = \boxed{-2}$

(f) $\lim_{x \to 0} f(x) = \boxed{0}$

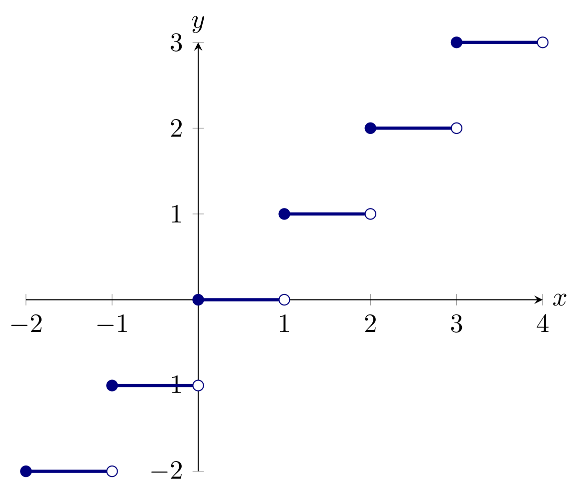
(g) $f(1) = \boxed{DNE}$

(h) $\lim_{x \to 1} f(x) = \boxed{-2}$

Limits might not exist

Limits might not exist. Let's see how this happens.

Example 1. *Consider the graph of* $f(x) = \lfloor x \rfloor$.



Explain why the limit

$$\lim_{x\to 2} f(x)$$

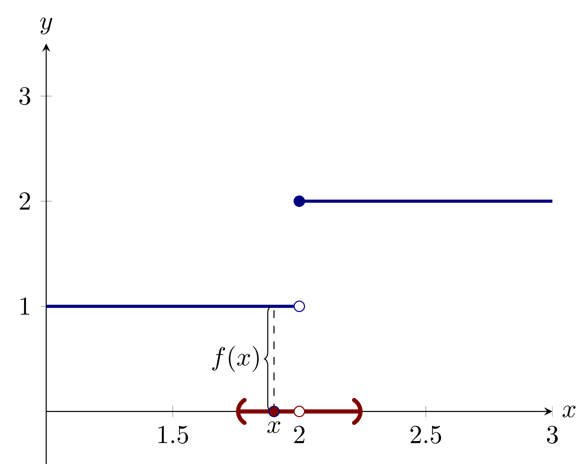
does not exist.

Explanation. *The function* $\lfloor x \rfloor$ *is the function that returns the greatest integer less than or equal to* x. *Recall that*

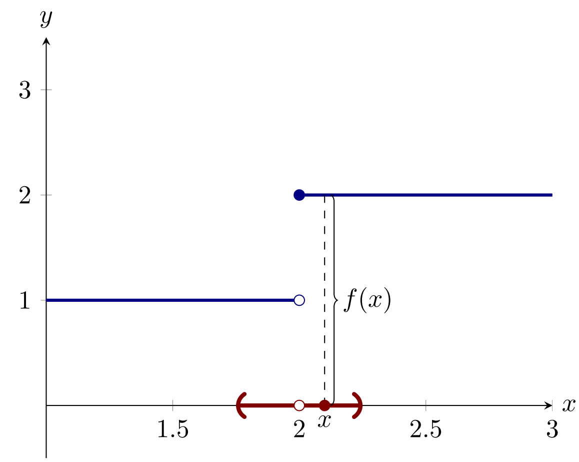
$$\lim_{x\to 2} \lfloor x \rfloor = L$$

if $\lfloor x \rfloor$ *can be made arbitrarily close to* L *by making* x *sufficiently close, but not equal to,* 2. *So let's examine* x *near, but not equal to,* 2. *Now the question is: What is* L*?*

If this limit exists, then we should be able to look sufficiently close, but not at, $x = 2$, *and see that* f *is approaching some number. Let's look at a graph:*



If we allow x *values on the left of 2 to get closer and closer to 2, we see that* $f(x) = 1$. *However, if we allow the values of* x *on the right of 2 to get closer and closer to 2 we see*



So just to the right of 2, $f(x) = 2$. We cannot find a single number that $f(x)$ approaches as x approaches 2, and so the limit does not exists.

Tables can be used to help guess limits, but one must be careful.

Question 4 *Consider $f(x) = \sin\left(\frac{\pi}{x}\right)$. Fill in the tables below rounding to three decimal places:*

x	$f(x)$
0.1	$\boxed{0}$
0.01	$\boxed{0}$
0.001	$\boxed{0}$
0.0001	$\boxed{0}$

We may rush and say that, based on the table above,

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0.$$

But, recall the definition of the limit: L is the limit if the value of $f(x)$ is as close as one wishes to L for ***all*** *x sufficiently close, but not equal to, a.*

From this table we can see that $f(x)(= 0)$ is as close as one wishes to $L(= 0)$ for ***some*** *values x that are sufficiently close to $a(= 0)$. But this does does not satisfy the definition of the limit, at least, not yet.*

But, wait! Fill in another table.

x	$f(x)$
0.3	$\boxed{-.866}$
0.03	$\boxed{-.866}$
0.003	$\boxed{-.866}$
0.0003	$\boxed{-.866}$

What do these two tables tell us about

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)?$$

Multiple Choice:

(a) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 0$

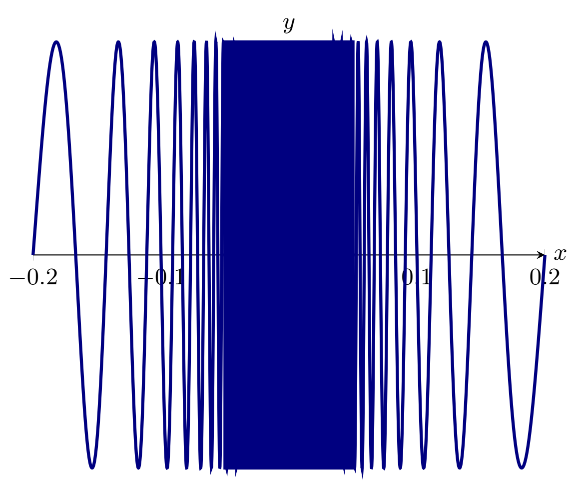
(b) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 1$

(c) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = -.866$

(d) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = -.433$

(e) *The limit does not exist.* ✓

Feedback(*`attempt`*): *The limit does not exist. The first table shows that we can always find a value of x as close as we want to 0 such that $f(x) = 0$. However, the limit is not equal to 0, since the second table shows that we can also find a value of x as close as we want to 0 such that $f(x) = -\frac{\sqrt{3}}{2}$. It turns out that for any number y, $-1 \le y \le 1$, we can find a value of x as close as we want to 0 such that $f(x) = y$. Check the graph of the function f.*



We see that $f(x)$ oscillates "wildly" as x approaches 0, and hence does not approach any one number.

One-sided limits

While we have seen that $\lim_{x\to 2}\lfloor x\rfloor$ does not exist, its graph looks much "nicer" near $a = 2$ than does the previous graph near $a = 0$. More can be said about the function $\lfloor x\rfloor$ and its behavior near $a = 2$.

Definition 2. *Intuitively,*

for the function f, L is the ***limit from the right*** *as x approaches a,*

written

$$\lim_{x\to a^+} f(x) = L,$$

if the value of $f(x)$ is as close as one wishes to L for all $x > a$ sufficiently close to a.

Similarly,

for the function f, L is the ***limit from the left*** *as x approaches a,*

written

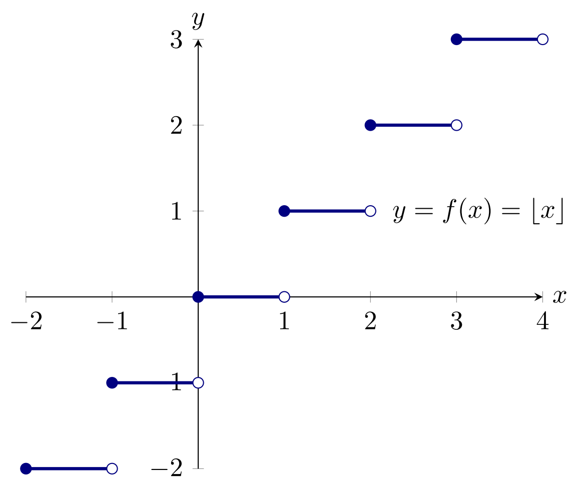
$$\lim_{x\to a^-} f(x) = L,$$

if the value of $f(x)$ is as close as one wishes to L for all $x < a$ sufficiently close to a.

Example 2. *Compute:*

$$\lim_{x\to 2^-} f(x) \qquad \textit{and} \qquad \lim_{x\to 2^+} f(x)$$

by using the graph below



Explanation. *From the graph we can see that as x approaches 2 from the left, $\lfloor x \rfloor$ remains at $y = 1$ up until the exact point that $x = 2$. Hence*

$$\lim_{x \to 2^-} f(x) = 1.$$

Also from the graph we can see that as x approaches 2 from the right, $\lfloor x \rfloor$ remains at $y = 2$ up to $x = 2$. Hence

$$\lim_{x \to 2^+} f(x) = 2.$$

When you put this all together

One-sided limits help us talk about limits.

Theorem 1. *A limit*

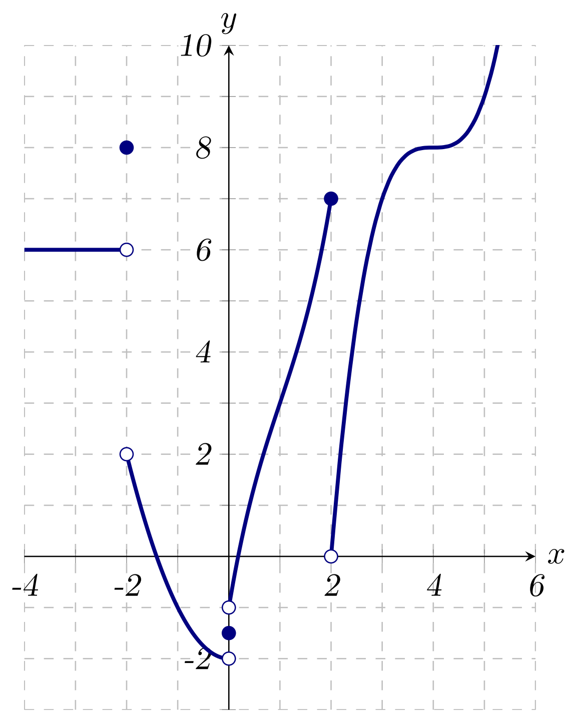
$$\lim_{x \to a} f(x)$$

exists if and only if

- $\lim_{x \to a^-} f(x)$ *exists*
- $\lim_{x \to a^+} f(x)$ *exists*
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

In this case, $\lim_{x \to a} f(x)$ is equal to the common value of the two one sided limits.

Question 5 *Evaluate the expressions by referencing the graph below. Write DNE if the limit does not exist.*



(a) $\lim\limits_{x\to 4} f(x) = \boxed{8}$

(b) $\lim\limits_{x\to -3} f(x) = \boxed{6}$

(c) $\lim\limits_{x\to 0} f(x) = \boxed{DNE}$

(d) $\lim\limits_{x\to 0^-} f(x) = \boxed{-2}$

(e) $\lim\limits_{x\to 0^+} f(x) = \boxed{-1}$

(f) $f(-2) = \boxed{8}$

(g) $\lim\limits_{x\to 2^-} f(x) = \boxed{7}$

(h) $\lim\limits_{x\to -2^-} f(x) = \boxed{6}$

(i) $\lim\limits_{x\to 1} f(x) = \boxed{3}$

(j) $f(0) = \boxed{-3/2}$

(k) $\lim_{x \to -3^-} f(x) = \boxed{6}$

(l) $\lim_{x \to -2^+} f(x) = \boxed{2}$