Dig-In:

The limit laws

We give basic laws for working with limits.

In the previous section were able to compute the limits

$$\lim_{x\to\pi} x^3 = \pi^3, \quad \lim_{x\to\pi} \sqrt{2} = \sqrt{2}, \quad \lim_{x\to\pi} \cos x = -1,$$

using continuity of the functions x^3, $\sqrt{2}$, and $\cos x$, at $x = \pi$. Does this imply that we can compute the limits

$$\begin{aligned}
&\lim_{x\to\pi} (x^3 - \cos x), \\
&\lim_{x\to\pi} \frac{\sqrt{2}}{\cos x}, \\
&\lim_{x\to\pi} (\sqrt{2} \cdot x^3 \cdot \cos x), \text{ or} \\
&\lim_{x\to\pi} \cos(x^3)?
\end{aligned}$$

Well, we cannot use continuity here, because we don't know if the functions $x^3 - \cos x$, $\dfrac{\sqrt{2}}{\cos x}$, $\sqrt{2} \cdot x^3 \cdot \cos x$, and $\cos(x^3)$ are continuous at $x = \pi$, and we have no other tools available, since the graphs and tables are not reliable. Obviously, we need more tools to help us with computation of limits.

In this section, we present a handful of rules, called the *Limit Laws*, that allow us to find limits of various combinations of functions.

Theorem 1 (Limit Laws)**.** *Suppose that* $\lim_{x\to a} f(x) = L$*,* $\lim_{x\to a} g(x) = M$*.*

Sum/Difference Law $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M.$

Product Law $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM.$

Quotient Law $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$*, if* $M \neq 0$*.*

In plain language, "limit of a sum equals the sum of the limits," "limit of a product equals the product of the limits," etc.

Let's examine how the Limit Laws can be used in computation of limits.

Learning outcomes: Calculate limits using the limit laws.
Author(s):

Example 1. *Compute the following limits using Limit Laws:*

(a) $\lim_{x\to\pi}(x^3 - \cos x)$

(b) $\lim_{x\to\pi} \frac{\sqrt{2}}{\cos x}$

(c) $\lim_{x\to\pi}(\sqrt{2}\cdot x^3 \cdot \cos x)$

(d) $\lim_{x\to\pi} \cos(x^3)$

Explanation. *For* $\lim_{x\to\pi}(x^3 - \cos x)$*, write:*

$$\begin{aligned}
\lim_{x\to\pi}(x^3 - \cos x) &= \lim_{x\to\pi} x^3 - \lim_{x\to\pi}\cos x && \textit{by Difference Law}\\
&= \pi^3 - (-1)\\
&= \pi^3 + 1
\end{aligned}$$

For $\lim_{x\to\pi} \frac{\sqrt{2}}{\cos x}$*, write:*

$$\begin{aligned}
\lim_{x\to\pi} \frac{\sqrt{2}}{\cos x} &= \frac{\lim_{x\to\pi}\sqrt{2}}{\lim_{x\to\pi}\cos x} && \textit{by Quotient Law}\\
&= \frac{\sqrt{2}}{-1}\\
&= -\sqrt{2}
\end{aligned}$$

For $\lim_{x\to\pi}(\sqrt{2}\cdot x^3 \cdot \cos x)$*, using the Product Law, we can write:*

$$\begin{aligned}
\lim_{x\to\pi}(\sqrt{2}\cdot x^3 \cdot \cos x) &= \lim_{x\to\pi}\sqrt{2}\cdot \lim_{x\to\pi} x^3 \cdot \lim_{x\to\pi}\cos x\\
&= \sqrt{2}\cdot\pi^3\cdot(-1)\\
&= -\sqrt{2}\pi^3
\end{aligned}$$

For $\lim_{x\to\pi}\cos(x^3)$*, write:*

$$\lim_{x\to\pi}\cos(x^3) = \cdots$$

The function $\cos(x^3)$ *is a combination of the functions* $\cos(x)$ *and* x^3*, but it is neither a sum/difference, nor a product, nor a quotient of these two functions, so we cannot apply any of the Limit Laws. This function is a composition of the two functions,* $\cos(x)$ *and* x^3*.*

Example 2. (a) *Compute the following limit using limit laws:*

$$\lim_{x\to 1}(5x^2 + 3x - 2)$$

Explanation. *Well, get out your pencil and write with me:*

$$\lim_{x\to 1}(5x^2 + 3x - 2) = \lim_{x\to 1} 5x^2 + \lim_{x\to 1} \underbrace{\boxed{3x}}_{\text{given}} - \lim_{x\to 1} 2$$

by the Sum/Difference Law. So now

$$= 5\lim_{x\to 1} x^2 + 3\lim_{x\to 1} x - \lim_{x\to 1} \underbrace{\boxed{2}}_{\text{given}}$$

by the Product Law. Finally by continuity of x^b and k,

$$= 5(1)^2 + 3(1) - 2 = \underbrace{\boxed{6}}_{\text{given}}.$$

We can check our answer by looking at the graph of $y = f(x)$:

Graph of $5x^2 + 3x - 2$

(b) *Is the polynomial function $f(x) = 5x^2 + 3x - 2$ continuous at $x = 1$?*

Explanation. *We have to check whether*

$$\lim_{x\to 1} f(x) = f(1).$$

Since we already know that $\lim_{x\to 1}(5x^2+3x-2) = 6$, we only have to compute $f(1)$.

$$\begin{aligned} f(1) &= 5(1)^2 + 3(1) - 2 \\ &= 6 \end{aligned}$$

Therefore, the polynomial function f is continuous at $x = 1$.

But what about continuity at any other value x? Is the function f continuous on its entire domain?

And what about any other polynomial function?

Are polynomials continuous on their domains?

We can generalize the example above to get the following theorems.

Theorem 2 (Continuity of Polynomial Functions)**.** ***All polynomial functions****, meaning functions of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a positive integer and each coefficient a_i, $i = 0, 1, ..., n$, is a real number, are ***continuous for all real numbers****.*

Explanation. *In order to show that any polynomial, f, is continuous at any real number, a, we have to show that*

$$\lim_{x\to a} f(x) = f(a).$$

Write with me:

$$\lim_{x\to a} f(x) = \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$$

Now by the Sum Law,

$$= \lim_{x\to a} a_n x^n + \lim_{x\to a} a_{n-1}x^{n-1} + \cdots + \lim_{x\to a} a_1 x + \lim_{x\to a} a_0$$

and by the Product Law,

$$\begin{aligned} = \lim_{x\to a} a_n \cdot \lim_{x\to a} x^n + \lim_{x\to a} a_{n-1} \cdot \lim_{x\to a} x^{n-1} + \dots \\ + \lim_{x\to a} a_1 \cdot \lim_{x\to a} x + \lim_{x\to a} a_0 \end{aligned}$$

and by Continuity

$$= a_n \cdot a^n + a_{n-1} \cdot a^{n-1} + \cdots + a_1 \cdot a + a_0$$

And this equals...

$$= f(a)$$

Since we have shown that $\lim_{x\to a} f(x) = f(a)$, we have shown that f is continuous at $x = a$.

Theorem 3 (Continuity of Rational Functions)**.** *A* ***rational function*** *h, meaning a function of the form*

$$h(x) = \frac{f(x)}{g(x)}$$

where f and g are polynomials, is ***continuous*** *for all real numbers except where $g(x) = 0$. That is, rational functions are continuous wherever they are defined.*

Explanation. *Let a be a real number such that $g(a) \neq 0$. Then, since $g(x)$ is continuous at a, $\lim_{x\to a} g(x) \neq 0$. Therefore, write with me,*

$$\lim_{x\to a} h(x) = \lim_{x\to a} \frac{f(x)}{g(x)}$$

and now by the Quotient Law,

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

and by the continuity of polynomials we may now set $x = a$

$$\frac{f(a)}{g(a)} = h(a).$$

Since we have shown that $\lim_{x \to a} h(x) = h(a)$*, we have shown that* h *is continuous at* $x = a$*.*

Question 1 *Where is* $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ *continuous?*

Multiple Choice:

(a) *for all real numbers*

(b) *at* $x = 2$

(c) *for all real numbers, except* $x = 2$ ✓

(d) *impossible to say*

Question 2 *True or false: If* f *and* g *are continuous functions on an interval* I*, then* $f \pm g$ *is continuous on* I*.*

Multiple Choice:

(a) *True* ✓

(b) *False*

Feedback(*`attempt`*): *Let's assume that* I *is an open interval and* a *is a number in* I*. Remember, since* f *and* g *are both continuous on* I*, they are both continuous at* a*.*

This means that $\lim_{x \to a} f(x) = f(a)$ *and* $\lim_{x \to a} g(x) = g(a)$*.*

Now, define a new function, h*, where* $h(x) = f(x) + g(x)$*, for all* x *in* I*. We have to show that* h *is continuous at* a*, or that*

$$\lim_{x \to a} h(x) = h(a).$$

Lat's start with

$$\lim_{x \to a} h(x) = \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \text{ by Sum Law},$$

and, therefore,

$$\lim_{x \to a} h(x) = f(a) + g(a) = h(a) \checkmark$$

We have proved that h is continuous at any number a in I. Therefore, h is continuous on I. Similarly, we can prove that $f+g$ is continuous on any interval I, by showing it is left-or right-continuous at the endpoints. We can adjust the proof for the function $f-g$.

Question 3 *True or false: If f and g are continuous functions on an interval I, then f/g is continuous on I.*

Multiple Choice:

(a) *True*

(b) *False* ✓

Feedback(`attempt`): *In this case, f/g will not be continuous for x where $g(x) = 0$.*

We still don't know how to compute a limit of a composition of two functions. Our next theorem provides basic rules for how limits interact with composition of functions.

Theorem 4 (Composition Limit Law)**.** *If $f(x)$ is continuous at $b = \lim_{x\to a} g(x)$, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Because the limit of a continuous function is the same as the function value, we can now pass limits inside continuous functions.

Corollary 1 (Continuity of Composite Functions)**.** *If g is continuous at $x = a$, and if f is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$.*

Using the Composition Limit Law, we can compute the last example from the beginning of this section.

Example 3. *Compute the following limit using limit laws:*

$$\lim_{x\to\pi} \cos(x^3)$$

Explanation. *We will use the Composition Limit Law. Let*

$$f(g(x)) = \cos(x^3),$$

where $f(x) = \cos x$*, and* $g(x) = x^3$*. Now, continuity of* x^3 *implies that* $\lim_{x\to\pi} g(x) = \pi^3$*. Continuity of* $\cos x$ *implies that* f *is continuous at* $\pi^3 = \lim_{x\to\pi} g(x)$*. Now, the Composition Limit Law implies that*

$$\lim_{x\to\pi} \cos(x^3) = \cos\left(\lim_{x\to\pi} x^3\right) = \cos(\pi^3) \approx 0.91726.$$

We can confirm our results by checking out the graph of $y = f(g(x))$*:*

Graph of $\cos(x^3)$

Many of the Limit Laws and theorems about continuity in this section might seem like they should be obvious. You may be wondering why we spent an entire section on these theorems. The answer is that these theorems will tell you exactly when it is easy to find the value of a limit, and exactly what to do in those cases.

The most important thing to learn from this section is whether the limit laws can be applied for a certain problem, and when we need to do something more interesting. We will begin discussing those more interesting cases in the next section.

A list of questions

Let's try this out.

Question 4 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2} \frac{x^2+3x+2}{x+2}$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 4.1 *Compute:*

$$\lim_{x\to 2} \frac{x^2+3x+2}{x+2} = \boxed{3}$$

Feedback(*`attempt`*): *Since* $f(x) = \dfrac{x^2+3x+2}{x+2}$ *is a rational function, and the denominator does not equal* 0*, we see that* $f(x)$ *is continuous at* $x = 2$*. Thus, to find this limit, it suffices to plug* 2 *into* $f(x)$*.*

Question 5 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(*`attempt`*): $f(x) = \dfrac{x^2-3x+2}{x-2}$ *is a rational function, but the denominator* $x-2$ *equals* 0 *when* $x = 2$*. None of our current theorems address the situation when the denominator of a fraction approaches* 0*.*

Question 6 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} x\sin(1/x)$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(*`attempt`*): *If we are trying to use limit laws to compute this limit, we would first have to use the Product Law to say that*

$$\lim_{x\to 0} x\sin(1/x) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin(1/x).$$

We are only allowed to use this law if both limits exist, so we must check this first. We know from continuity that

$$\lim_{x\to 0} x = 0.$$

However, we also know that $\sin(1/x)$ *oscillates "wildly" as* x *approaches* 0*, and so the limit*

$$\lim_{x\to 0} \sin(1/x)$$

does not exist. Therefore, we cannot use the Product Law.

Question 7 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \cot(x^3)$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(`attempt`): *Notice that*

$$\cot(x^3) = \frac{\cos(x^3)}{\sin(x^3)}.$$

If we are trying to use limit laws to compute this limit, we would like to use the Quotient Law to say that

$$\lim_{x\to 0} \frac{\cos(x^3)}{\sin(x^3)} = \frac{\lim_{x\to 0} \cos(x^3)}{\lim_{x\to 0} \sin(x^3)}.$$

We are only allowed to use this law if both limits exist and the denominator is not 0. *We suspect that the limit in the denominator might equal* 0, *so we check this limit.*

$$\begin{aligned}\lim_{x\to 0} \sin(x^3) &= \sin(\lim_{x\to 0} x^3)\\ &= \sin(0)\\ &= 0.\end{aligned}$$

This means that the denominator is zero and hence we cannot use the Quotient Law.

Question 8 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \sec^2(e^x - 1)$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 8.1 *Compute:*

$$\lim_{x\to 0} \sec^2(e^x - 1) = \boxed{1}$$

Feedback(`attempt`): *Notice that*

$$\lim_{x\to 0} \sec^2(e^x - 1) = \lim_{x\to 0} \frac{1}{\cos^2(e^x - 1)}.$$

If we are trying to use Limit Laws to compute this limit, we would now have to use the Quotient Law to say that

$$\lim_{x\to 0} \frac{1}{\cos^2(e^x - 1)} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \cos^2(e^x - 1)}.$$

We are only allowed to use this law if both limits exist and the denominator is not 0*. Let's check the denominator and numerator separately. First we'll compute the limit of the denominator:*

$$\begin{aligned}
\lim_{x\to 0} \cos^2(e^x - 1) &= (\lim_{x\to 0} \cos(e^x - 1))^2 \\
&= \left(\cos\left(\lim_{x\to 0}(e^x - 1)\right)\right)^2 \\
&= \cos^2\left(\lim_{x\to 0}(e^x - 1)\right) \\
&= \cos^2\left(\lim_{x\to 0}(e^x) - \lim_{x\to 0}(1)\right) \\
&= \cos^2(1 - 1) \\
&= \cos^2(0) \\
&= 1
\end{aligned}$$

Therefore, the limit in the denominator exists and does not equal 0*. We can use the Quotient Law, so we will compute the limit of the numerator:*

$$\lim_{x\to 0} 1 = 1$$

Hence

$$\frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \cos^2(e^x - 1)} = \frac{1}{1} = 1$$

Question 9 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 1} (x - 1) \cdot \csc(\ln(x))$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(*`attempt`*): *If we are trying to use limit laws to compute this limit, we would have to use the Product Law to say that*

$$\lim_{x\to 1}(x-1)\cdot\csc(\ln(x)) = \lim_{x\to 1}(x-1)\cdot\lim_{x\to 1}\csc(\ln(x)).$$

We are only allowed to use this law if both limits exist. Let's check each limit separately.

$$\begin{aligned}\lim_{x\to 1}(x-1) &= \lim_{x\to 1}(x) - \lim_{x\to 1}(1)\\ &= 1-1\\ &= 0.\end{aligned}$$

So this limit exists. Now we check the other factor. Notice that

$$\lim_{x\to 1}\csc(\ln(x)) = \frac{1}{\sin(\ln(x))}.$$

If we are trying to use limit laws to compute this limit, we would now have to use the Quotient Law to say that

$$\frac{1}{\sin(\ln(x))} = \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}\sin(\ln(x))}.$$

We are only allowed to use this law if both limits exist and the denominator does not equal 0*. The limit in the numerator definitely exists, so let's check the limit in the denominator.*

$$\begin{aligned}\lim_{x\to 1}\sin(\ln(x)) &= \sin(\lim_{x\to 1}\ln(x))\\ &= \sin(\ln(1))\\ &= \sin(0)\\ &= 0\end{aligned}$$

Since the denominator is 0*, we cannot apply the Quotient Law.*

Question 10 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} x\ln x$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(*`attempt`*): *If we are trying to use limit laws to compute this limit, we would have to use the Product Law to say that*

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} x \cdot \lim_{x\to 0} \ln x.$$

We are only allowed to use this law if both limits exist. We know $\lim_{x\to 0} x = 0$, *but what about* $\lim_{x\to 0} \ln x$*? We do not know how to find* $\lim_{x\to 0} \ln x$ *using limit laws because* 0 *is not in the domain of* $\ln x$.

Question 11 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}}$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 11.1 *Compute:*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}} = \boxed{0}$$

Feedback(*`attempt`*): *If we are trying to use limit laws to compute this limit, we would have to use the Quotient Law to say that*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}} = \frac{\lim_{x\to 0}(2^x - 1)}{\lim_{x\to 0}(3^{x-1})}.$$

We are only allowed to use this law if both limits exist and the denominator does not equal 0*. Let's check each limit separately, starting with the denominator*

$$\begin{aligned}\lim_{x\to 0}(3^{x-1}) &= 3^{\lim_{x\to 0}(x-1)}\\ &= 3^{-1}\\ &= \frac{1}{3}\end{aligned}$$

On the other hand the limit in the numerator is

$$\begin{aligned}\lim_{x\to 0}(2^x - 1) &= \lim_{x\to 0}(2^x) - \lim_{x\to 0}(1)\\ &= 1 - 1\\ &= 0\end{aligned}$$

The limits in both the numerator and denominator exist and the limit in the denominator does not equal 0*, so we can use the Quotient Law. We find:*

$$\frac{\lim_{x\to 0}(2^x - 1)}{\lim_{x\to 0}(3^{x-1})} = \frac{0}{\frac{1}{3}} = 0.$$

Question 12 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0}(1+x)^{1/x}$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback(*`attempt`*): *We do not have any limit laws for functions of the form* $f(x)^{g(x)}$*, so we cannot compute this limit.*