Dig-In:

Limits of the form nonzero over zero

What can be said about limits that have the form nonzero over zero?

Let's cut to the chase:

Definition 1. *A limit*

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is said to be of the form $\frac{\#}{\mathbf{0}}$ *if*

$$\lim_{x\to a} f(x) = k \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

where k is some nonzero constant.

Question 1 *Which of the following limits are of the form* $\frac{\#}{\mathbf{0}}$*?*

Select All Correct Answers:

(a) $\displaystyle\lim_{x\to -1} \frac{1}{(x+1)^2}$ ✓

(b) $\displaystyle\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

(d) $\displaystyle\lim_{x\to 2} \frac{x^2-3x-2}{x-2}$ ✓

(e) $\displaystyle\lim_{x\to 1} \frac{e^x}{\ln(x)}$ ✓

In our next example, let's see what is going on with limits of the form $\frac{\#}{\mathbf{0}}$.

Example 1. *Consider the function*

$$f(x) = \frac{1}{(x+1)^2}.$$

Use a table of values to investigate $\displaystyle\lim_{x\to -1} \frac{1}{(x+1)^2}$.

Learning outcomes: Calculate limits of the form number over zero. Identify determinate and indeterminate forms. Distinguish between determinate and indeterminate forms.

Author(s):

Explanation. *Fill in the table below:*

x	$(x+1)^2$	$f(x) = \dfrac{1}{(x+1)^2}$
-1.1	0.01	100
-1.01	0.0001	10000
-1.001	0.000001	$\boxed{1000000}$
-1.0001	0.00000001	$\boxed{100000000}$
-0.9	0.01	$\boxed{100}$
-0.99	0.0001	$\boxed{10000}$
-0.999	0.000001	$\boxed{1000000}$
-0.9999	0.00000001	$\boxed{100000000}$

What does the table tell us about

$$\lim_{x \to -1} \frac{1}{(x+1)^2}?$$

It appears that the limit does not exist, since the expression

$$\frac{1}{(x+1)^2}$$

becomes larger and larger as x approaches -1 . So,

$$\lim_{x \to -1} \frac{1}{(x+1)^2} \qquad \text{is of the form } \frac{\#}{\mathbf{0}}$$

as

$$\lim_{x \to -1} 1 = 1 \qquad \text{and} \qquad \lim_{x \to -1} (x+1)^2 = 0.$$

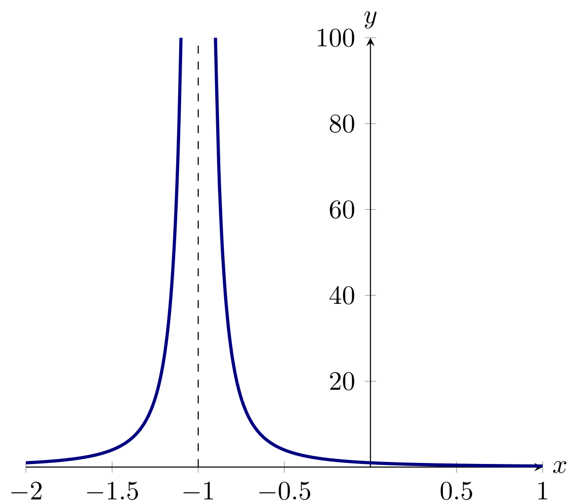
Moreover, as x approaches -1:

- *The numerator is positive.*
- *The denominator approaches zero and is positive.*

Hence, the expression

$$\frac{1}{(x+1)^2}$$

will become arbitrarily large as x approaches -1. We can see this in the graph of f.



We are now ready for our next definition.

Definition 2. *If $f(x)$ grows arbitrarily large as x approaches a, we write*

$$\lim_{x \to a} f(x) = \infty$$

and say that the limit of $f(x)$ is ***infinity*** *as x goes to a.*

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say that the limit of $f(x)$ is ***negative infinity*** *as x goes to a.*

Note: Saying "the limit is equal to infinity" describes more precisely the behavior of the function f near a, than just saying "the limit does not exist".

Let's consider a few more examples.

Example 2. *Compute:*

$$\lim_{x \to -2} \frac{e^x}{(x+2)^4}$$

Explanation. *First let's look at the form of this limit. We do this by taking the limits of both the numerator and denominator:*

$$\lim_{x \to -2} e^x = \underbrace{\boxed{\frac{1}{e^2}}}_{\text{given}} \qquad \text{and} \lim_{x \to -2} \left((x+2)^4\right) = 0.$$

So, this limit is of the form $\frac{\#}{\mathbf{0}}$. This form is ***determinate****, since it implies that the limit does not exist.*
But, we can do better than that! As x approaches -2:

- *The numerator is a (positive ✓/ negative) number.*
- *The denominator is (positive ✓/ negative) and is approaching zero.*

This means that

$$\lim_{x\to -2} \frac{e^x}{(x+2)^4} = \infty.$$

Example 3. *Compute:*

$$\lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6}$$

Explanation. *First let's look at the form of this limit, which we do by taking the limits of both the numerator and denominator.*

$$\lim_{x\to 3^+} \left(x^2-9x+14\right) = \underbrace{\boxed{-4}}_{\text{given}} \qquad \textit{and} \lim_{x\to 3^+} \left(x^2-5x+6\right) = 0$$

This limit is of the form $\frac{\#}{\mathbf{0}}$*. Next, we should factor the numerator and denominator to see if we can simplify the problem at all.*

$$\begin{aligned} \lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6} &= \lim_{x\to 3^+} \frac{\cancel{(x-2)}(x-7)}{\cancel{(x-2)}(x-3)} \\ &= \lim_{x\to 3^+} \frac{x-7}{x-3} \end{aligned}$$

Canceling a factor of $x-2$ *in the numerator and denominator means we can more easily check the behavior of this limit. As* x *approaches* 3 *from the right:*

- *The numerator is a (positive/ negative ✓) number.*
- *The denominator is (positive ✓/ negative) and approaching zero.*

This means that

$$\lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6} = -\infty.$$

Here is our final example.

Example 4. *Compute:*

$$\lim_{x\to 3} \frac{x^2-9x+14}{x^2-5x+6}$$

Explanation. *We've already considered part of this example, but now we consider the two-sided limit. We already know that*

$$\lim_{x\to 3} \frac{x^2-9x+14}{x^2-5x+6} = \lim_{x\to 3} \frac{x-7}{x-3},$$

and that this limit is of the form $\frac{\#}{\mathbf{0}}$*. We also know that as* x *approaches* 3 *from the right,*

- *The numerator is a negative number.*
- *The denominator is positive and approaching zero.*

Hence our function is approaching $-\infty$ from the right.

As x approaches 3 from the left,

- *The numerator is negative.*
- *The denominator is negative and approaching zero.*

Hence our function is approaching ∞ from the left. This means

$$\lim_{x\to 3} \frac{x^2-9x+14}{x^2-5x+6} = \underbrace{\boxed{DNE}}_{\text{given}}.$$

We can confirm our results of the previous two examples by looking at the graph of $y = \dfrac{x^2-9x+14}{x^2-5x+6}$:

Graph of $\dfrac{x^2-9x+14}{x^2-5x+6}$

Some people worry that the mathematicians are passing into mysticism when we talk about infinity and negative infinity. However, when we write

$$\lim_{x\to a} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = -\infty$$

all we mean is that as x approaches a, $f(x)$ becomes arbitrarily large and $|g(x)|$ becomes arbitrarily large, with $g(x)$ taking negative values.