5.3 The Parabolic Elevator

Theorem 5.8. *The domain* Exterior($\mathcal{J}$) *is quasiconvex with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let $d \gg 1$ be a sufficiently large integer and let $z_1, z_2 \in \mathcal{J}$ be a pair of points. Repeatedly apply f on z_1, z_2 until one of the following two stopping conditions happens:

(i) $\left|f^{\circ N}(z_1) - f^{\circ N}(z_2)\right| > \epsilon$, where $\epsilon = \epsilon(d) > 0$ is a constant; or

(ii) $f^{\circ N}(z_1) \in I_n$ and $f^{\circ N}(z_2) \in I_m$ with $|m - n| \geq d$.

Denote $w_i = f^{\circ N}(z_i)$. We already proved that the itinerary η_{w_1,w_2} satisfies

$$\text{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|$$

for some $A > 0$. We now deduce that the original points z_1, z_2 enjoy a similar estimate,

$$\text{Length}(\eta_{z_1,z_2}) \leq C|z_1 - z_2|,$$

where C depends only on A. This follows from Koebe's distortion theorem applied N times on the univalent function f^{-1}, but some care is due in constructing the inverse branches in a neighborhood of the itineraries.

Fix a small ball $B = B(\frac{1}{2}, 0.1)$ around the main cusp. Its preimages are topological balls centered at cusps q of $\mathcal{J}$, and we index them by B_ζ where $\zeta = \frac{q}{|q|}$.

Each ball B_ζ has a *level*, which is the first iterate k for which $f^{\circ k}(B_\zeta) = B$. Repeatedly applying f^{-1} on the pair (w_1, w_2) until we reach (z_1, z_2), we get two sequences of balls $B = B_1^1, B_2^1, \ldots$ and $B = B_1^2, B_2^2, \ldots$. We now consider several cases.

Case (i). Suppose stopping criterion (i) occured. We choose a neighborhood N_1 of w_1, w_2 disjoint from the post-critical set $\wp$ and from B on which f is univalent. Next choose a neighborhood $N_2 \subset N_1$ such that for every $w_1, w_2 \in N_2$ we have $\eta_{w_1,w_2} \subset N_1$. Finally choose a neighborhood $N_3 \supset N_1$ disjoint from $\wp$ on which f is univalent such that there is a definite modulus to the annulus $N_3 \setminus N_1$.

Choosing a branch of $f^{\circ -N}$ in N_3 that sends η_{w_1,w_2} to η_{z_1,z_2}, Koebe's distortion theorem gives

$$\frac{\text{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|}. \tag{5.9}$$

By compactness of $\mathcal{J} \setminus B$, these neighborhoods can be chosen from a finite collection, hence the constant is uniform in the points z_1, z_2 and we obtain quasiconvexity of η_{z_1,z_2} in case (i).

Case (ii). Suppose that stopping criterion (ii) occured. We first claim that the two sequences of balls $(B_k^1), (B_k^2)$ coincide. Indeed, suppose otherwise and let j be the first index for which $B_j^1 \neq B_j^2$. The two points $f^{-j}(z_1), f^{-j}(z_2)$ belong to two distinct preimages of the same ball $B_{j-1}^1 = B_{j-1}^2$, hence they are a positive distance apart and stopping criterion i applies. This is a contradiction, hence the two sequences coincide.

There is thus a branch of $f^{\circ -N}$ sending (w_1, w_2) to (z_1, z_2), and the necessary bound on η_{z_1,z_2} follows from applying Koebe's distortion theorem on this branch as in case (i).

$\square$

References

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