1 Introduction

Let $f_c(z) = z^2 + c$ be a quadratic polynomial. Its *filled Julia set* consists of points in the complex plane with bounded orbit under iteration by f_c:

$$\mathcal{K}(f_c) = \{z \in \mathbb{C} : \sup_{n \geq 0} f_c^{\circ n}(z) < \infty\}.$$

Its complement $A_\infty(f_c) = \mathbb{C} \setminus \mathcal{K}(f_c)$ forms the *attracting basin of infinity*. The *Julia set* $\mathcal{J}(f_c) = \partial\mathcal{K}(f_c)$ is defined as the boundary of the filled Julia set. It is well known in complex dynamics that $\mathcal{J}(f_c), \mathcal{K}(f_c)$ are closed sets while $A_\infty(f_c)$ is an open set. Furthermore, each of the three sets $\mathcal{J}(f_c), \mathcal{K}(f_c)$ and $A_\infty(f_c)$ are both forward and backward invariant under the dynamics of f.

The *Mandelbrot set* $\mathcal{M}$ consists of parameters $c \in \mathbb{C}$ for which $\mathcal{J}_c$ is connected. The *main cardioid*

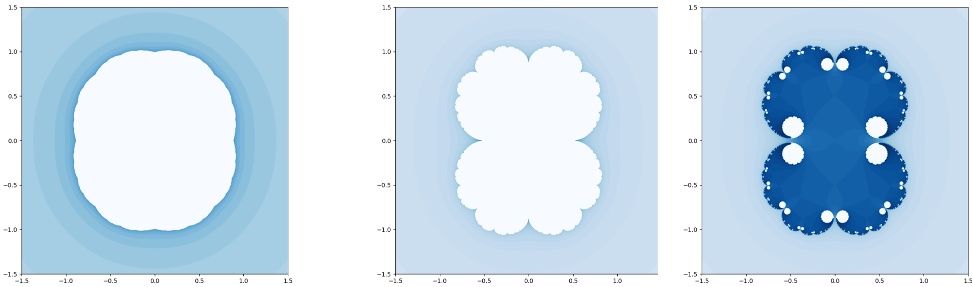
$$\heartsuit = \left\{c \in \mathbb{C} : c = \lambda/2 - \lambda^2/4,\ \lambda \in \mathbb{D}\right\}$$

is a connected component of the interior of $\mathcal{M}$, which consists of parameters $c \in \mathbb{C}$ for which f_c has an attracting fixed point. When $c \in \heartsuit$, the Julia set $\mathcal{J}(f_c)$ is a Jordan curve. In fact, more is true. Parameters $c \in \heartsuit$ are *hyperbolic*. Loosely speaking, this means that $\mathcal{J}(f_c)$ satisfies the principle of the conformal elevator: up to bounded distortion, a small piece of $\mathcal{J}(f_c)$ looks like a large piece $\mathcal{J}(f_c)$. Using the principle of the conformal elevator, it can be show that is $\mathcal{J}(f_c)$ a quasicircle: a condition which roughly means that $\mathcal{J}(f_c)$ has no "cusps".

In this work, we study the *Cauliflower*, which is the filled Julia set of $f_{1/4}(z) = z^2 + \frac{1}{4}$. In this case the Julia set $\mathcal{J}(f_{1/4})$ is no longer a quasicircle, as it has an inward-pointing cusp at the point $p = 1/2$. However, according to a theorem of Carleson, Jones and Yoccoz [1, Theorem 6.1], the Cauliflower is a so-called John domain, which roughly says that it has no outward-pointing cusps.

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ are connected by a rectifiable path $\gamma : [0,1] \to \Omega$ which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2|\,.$$



(a) $c = 0.1$. (b) $c = 1/4$, the Cauliflower. (c) $c = 0.26$.

Figure 1: The Julia set $\mathcal{J}_c$ of f_c for different values of c. When $c > 1/4$, the Julia set is no longer connected.

We call such a path γ a *quasiconvexity certificate* for z_1 and z_2.

If Ω is the interior of a Jordan curve, then by [3, Corollary F], it is enough to find certificates for points z_1, z_2 that lie on the boundary curve $\partial\Omega$.

If Ω is a quasiconvex Jordan domain, then its complement has a John interior; see [3, Corollary 3.4] for a proof. In this work, we strengthen the result of [1, Theorem 6.1] by showing:

Theorem 1.1. *The exterior of the Cauliflower, $A_\infty(f_{1/4})$, is quasiconvex.*

Our result also has a function-theoretic interpretation. A domain Ω is called a *$W^{1,1}$-extension domain* if every $u \in W^{1,1}(\Omega)$ extends to a function in $W^{1,1}(\mathbb{C})$.

In [2, Equation (1.1) and Theorem 1.4], it is shown that a bounded, simply connected domain is a $W^{1,1}$ extension domain if and only if its complement is quasiconvex. Thus our result can be rephrased as follows:

Theorem 1.2. *The Cauliflower is a $W^{1,1}$ extension domain.*

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of certificates connecting any given pair of points on the Julia set. We first build the certificates in the exterior unit disk $\mathbb{D}^*$, then we transport them to the exterior of the Cauliflower by the Böttcher coordinate $\psi : \mathbb{D}^* \to A_\infty(f_{1/4})$, which is a conjugacy between the map f_0 and $f_{1/4}$.

To retain control of the certificates after applying ψ, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This makes the image of a certificate η in $\mathbb{D}^*$ under the conjugacy ψ invariant under $f_{1/4}$. We use this invariance to show that $\psi(\eta)$ is indeed a certificate, by employing a parabolic variant of the principle of the conformal elevator.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps $f_c(z) = z^2 + c$ where $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$. In this case, the usual conformal elevator applies. We subsequently treat the parabolic case of $c = \frac{1}{4}$.

2 The exterior disk

We connect boundary points in $\partial\mathbb{D}^*$ by moving along the boundaries of Carleson boxes:

Definition 2.1. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{1/2^{n+1}}, 2^{1/2^n}\right], \qquad \arg(z) \in \left(\frac{k}{2^n}\,2\pi, \frac{k+1}{2^n}\,2\pi\right] \right\}$$

a *Carleson box*. Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{1/2^{n+1}} < |z| \leq 2^{1/2^n} \right\}$$

into 2^n equally-spaced sectors.

The *Carleson box decomposition* is the partition of $\mathbb{D}^*$ into Carleson boxes:

$$\mathbb{D}^* = \{\zeta : |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under f_0, stemming from the relation

$$f_0\left(B_{n+1,k}\right) = B_{n,k\,(\mathrm{mod}\ 2^n)}.$$

We describe paths that go along boundaries of Carleson boxes using the metaphor of a passenger who travels by train:

Definition 2.2. A *terminal* is a point $\zeta \in \partial\mathbb{D}^*$ on the unit circle. *Stations* are the iterated preimages of the central station under the map $f_0 : \zeta \mapsto \zeta^2$. We index them as

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\},$$

and refer to n as the *generation* of the station $s_{n,k}$. The 2^n stations of generation n are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$. The *central station* is the point $s_{0,0} = 2$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition 2.3. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are the two stations $s_{n,(k\pm1)(\text{mod } 2^n)}$ adjacent to $s_{n,k}$ on C_n.
2. The *peripheral track* $\gamma_{s,s'}$ from s to a peripheral neighbor s' is the shorter arc of the circle C_n connecting s to s'.
3. The *radial successor* of s is $\text{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.
4. The *express track* $\gamma_{s,s'}$ from s to its radial successor s' is the radial segment $[s, s']$.

Notice that the tracks preserve the dynamics: applying f_0 to a peripheral track between stations s, s' gives a peripheral track between the "parents" of s, s', and likewise for an express track. This holds unless one of the stations is of generation 0; this will bear no consequence for the proof.

When a passenger travels between two stations s_1 and s_2, they must follow a particular itinerary from s_1 to s_2. If s_1 is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination s_2 in the angular distance. This also determines how to travel from the central station to a terminal $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 2 and the next definition.

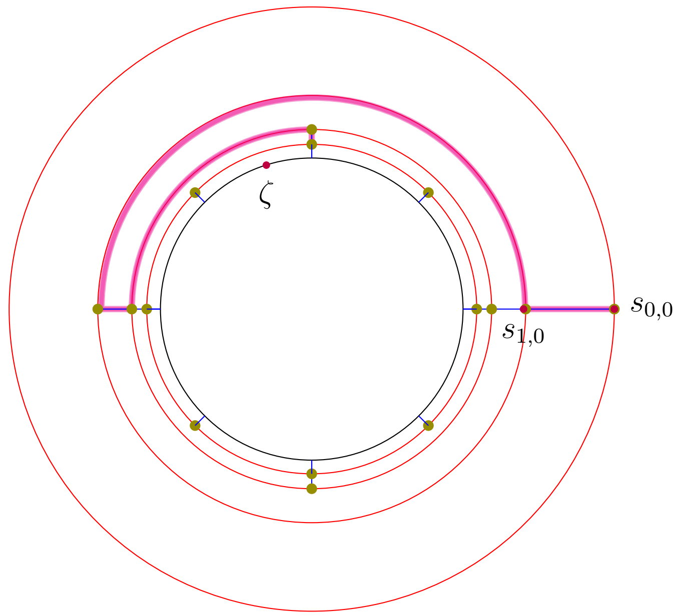


Figure 2: The central journey to a point ζ.

Definition 2.4. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \dots$ from the central station to ζ, made of tracks between stations $\sigma_0, \sigma_1, \dots$. It is defined inductively as follows:

Start at the central station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \dots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma)| < |\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

We identify the central itinerary η_ζ with its sequence of stations $(\sigma_0, \dots)$. We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in η_ζ, and in particular
$$\mathrm{Generation}(\sigma_k) \geq \frac{k}{2}; \tag{2.1}$$
- Central itineraries are essentially invariant under f_0, in the sense that
$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

for every $\zeta \in \partial\mathbb{D}^*$.

Definition 2.5. Given two distinct terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^{\infty}$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^{\infty}$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\ldots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \ldots\right).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2, see Figure 3. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{2.2}$$

for every pair of terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ with $|\zeta_1 - \zeta_2| < \sqrt{2}$.

3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$. For these values of c, the Julia set of $f_c : z \mapsto z^2 + c$ is a Jordan curve, and f_c has a Böttcher coordinate ψ at infinity; namely, ψ is the unique conformal map $\mathbb{D}^* \to \text{Exterior}(\mathcal{J}_c)$ which fixes ∞ and satisfies the conjugacy relation

$$f \circ \psi = \psi \circ f_0.$$

The map ψ is constructed explicitly by the limit

$$\psi(z) = \lim_{n\to\infty} (f_0)^{\circ(-n)} \circ f^{\circ n} = \lim_{n\to\infty} \sqrt[n]{f^{\circ n}},$$

and it extends to a homeomorphism between the circle $\partial\mathbb{D}$ and $\mathcal{J}_c$ by Carathéodory's theorem.

We apply ψ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in Exterior$(\mathcal{J}_c)$:

Definition 3.1.

1. The *stations* of f_c are the points $\psi(s_{n,k})$.

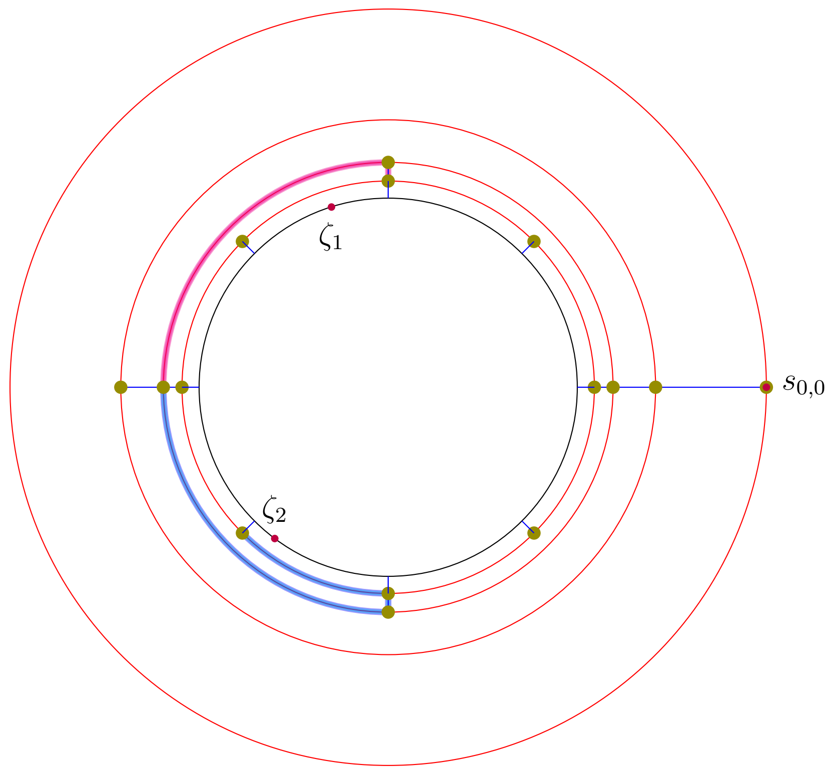


Figure 3: A quasiconvexity certificate between two points ζ_1, ζ_2 in $\mathbb{D}^*$. Only the first two steps are shown.

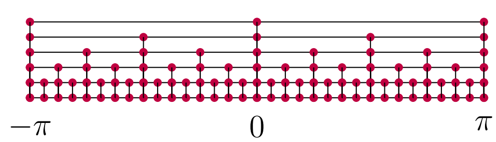


Figure 4: A convenient representation of the dyadic grid in the Böttcher coordinates. The horizontal axis is the external ray angle, and the vertical axis is the equipotential $|\psi^{-1}(z)|$, plotted on a log scale. The rightmost edge is glued to the leftmost edge. Stations are marked in red, and the segments connecting adjacent stations are tracks. An express track is a vertical segment, and a peripheral track is horizontal.

2. The *tracks* of f_c are the curves of the form $\psi(\gamma_{s,s'})$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on the external rays of the filled Julia set $\mathcal{K}_c$, while peripheral tracks lie on the equipotentials of $\mathcal{K}_c$.

3. The *itinerary* between a pair of points (z_1, z_2) on $\mathcal{J}_c$ is $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$, where $\zeta_i = \psi^{-1}(z_i)$ are the corresponding points on $\partial\mathbb{D}^*$.

We omit c and ψ from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior($\mathcal{J}$).

Note that $\psi((1,\infty)) \subseteq \mathbb{R}$ since $\mathcal{J}$ is symmetric with respect to the real line, and in particular the central station $\psi(s_{0,0})$ lies on the real axis.

4 Hyperbolic Maps

A quadratic map f_c is (dynamically) *hyperbolic* if its postcritical set $\mathcal{P}$ is disjoint from its Julia set $\mathcal{J}$. This is equivalent to f being expanding on its Julia set:

Theorem 4.1. *Let f_c be a hyperbolic quadratic map, let*

$$\mathcal{P} = \overline{\{f^{\circ n}(0) : n \geq 1\}}$$

be the (closed) post-critical set of f, and let $\|\cdot\|_{\text{hyp}}$ be the hyperbolic (Poincaré) metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$. Suppose f is not conjugate to $f_0 : z \mapsto z^2$. Then we have

$$\|f'(z)\|_{\text{hyp}} > 1 \tag{4.1}$$

for every $z \in f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$, and in particular for every $z \in \mathcal{J}$.

For two proofs, see [**?**, Theorem 19.1], which also proves the converse.

Corollary 4.2. *Let f_c be a hyperbolic quadratic map. There exists $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}$ has a forward iterate $f^{\circ n}$ for which*

$$|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon.$$

Definition 4.3. A point $z \in \mathcal{J}$ is *rectifiably accessible* from $\operatorname{Exterior}(\mathcal{J})$ if there is a rectifiable curve $\gamma : [0,1) \to \operatorname{Exterior}(\mathcal{J})$ such that $\gamma(t) \to z$ as $t \to 1$.

We are now ready to show quasiconvexity in the hyperbolic case:

Theorem 4.4. *Let $f : z \mapsto z^2 + c$ be a hyperbolic map.*

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots .$$

We have the estimate

$$\operatorname{Length}(\gamma_k) \lesssim \theta^k,$$

uniformly in z, for some constant $\theta = \theta(c) < 1$. In particular,any point on $\mathcal{J}$ is rectifiably accessible.

(ii) *The domain* $\operatorname{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (i) For $c = 0$ this is a direct computation. Suppose $c \neq 0$, and let $\mathcal{P}$ as before be the post-critical set of f.

Any inverse branch $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus \mathcal{P}$, potentially defined only in a subdomain, is a strict hyperbolic contraction by Theorem 4.1.

Let $B(0,R) \subset \mathbb{C}$ be a ball large enough that it contains every central itinerary. By hyperbolicity, $\hat{\mathbb{C}} \setminus \mathcal{P}$ contains $\overline{\operatorname{Exterior}(\mathcal{J})}$. Thus $\operatorname{Exterior}(\mathcal{J}) \cap B(0,R)$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, and there is a constant $\theta < 1$ such that $\|(f^{-1})'\|_{\mathrm{hyp}} < \theta$ on $\operatorname{Exterior}(\mathcal{J}) \cap B(0,R)$. Therefore,

$$\begin{aligned}
\operatorname{HypLength}(\gamma_k) &\leq \theta \cdot \operatorname{HypLength}(f(\gamma_k)) \\
&\leq \dots \\
&\leq \theta^k \cdot \operatorname{HypLength}(f^{\circ k}(\gamma_k)), \\
&\lesssim \theta^k,
\end{aligned}$$

where the last inequality holds since $f^{\circ k}(\gamma_k)$ lies on the real axis in case γ_k is an express track, or on the equipotetial $\psi(\{|z| = \sqrt{2}\})$ otherwise.

As the hyperbolic metric is equivalent to the Euclidean metric on compact subsets, we conclude that $\mathrm{Length}(\gamma_k) \lesssim \theta^k$ as well.

Thus any point on $\mathcal{J}$ can be reached from the central station $s_{0,0}$ by a curve of bounded length.

(ii) By Corollary 4.2, there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate $f^{\circ n}$. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of η_{z_1,z_2} is bounded from above uniformly by part (i). Thus we may assume that $|z_1 - z_2| < \epsilon$.

We have

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon, \tag{4.2}$$

and by Koebe's distortion theorem

$$\frac{\mathrm{Length}\,(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\mathrm{Length}\,(\eta_{w_1,w_2})}{|w_1 - w_2|}. \tag{4.3}$$

We already know that the itineraries η_{w_1,w_2} are certificates, so it follows that the original itineraries η_{z_1,z_2} are also certificates. $\square$

5 The Cauliflower

In this section, $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior$(\mathcal{J})$, Theorem 5.11. This is more complicated than the hyperbolic case, because the postcritical set $\mathcal{P}$ of f accumulates at the parabolic fixed point $p = \frac{1}{2}$. One no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce the quasiconvexity of the itinerary η_{z_1,z_2} from the quasiconvexity of η_{w_1,w_2} using Koebe's distortion theorem. As a substitute, we present an analogue of the principle of the conformal elevator in this parabolic setting.

5.1 Itineraries have finite length

We first show that each itinerary η_{z_1,z_2} has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary that lies on the real axis, $\eta_{1/2}$, have only a quadratic rate of length decay. To fix this, we introduce:

Definition 5.1. The *reduced decomposition* of an itinerary η is the unique decomposition $\eta = \gamma_1 + \delta_1 + \dots$ where each γ_i is a concatenation of express tracks and is followed by a single peripheral track δ_i.

Proposition 5.2. *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of its itinerary. Then* $\mathrm{Length}(\gamma_k) \lesssim \theta^k$ *and* $\mathrm{Length}(\delta_k) \lesssim \theta^k$ *for some $\theta < 1$. In particular,* $\mathrm{Length}(\eta_z) < \infty$ *and all points $z \in \mathcal{J}$ are rectifiably accessible.*

For the proof, call $s_{-1} := s_{1,1}$ the *pre-central station* and let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-central station and the leftmost one. This domain is constructed so that it contains all itineraries that start at the pre-central station.

Lemma 5.3. *Let $\gamma = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of an itinerary γ. Then for every $k > 1$, there exist $k-1$ iterates $n_1 < \dots < n_{k-1}$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

Proof. Every station $s \notin (0, \infty)$ has a first iterate $f^{\circ n_s}(s)$ lying on the negative real axis $(-\infty, 0)$. For any $i \in \{2, \dots, k-1\}$, let s_i be the first station of γ_i and take $n_i := n_{s_i}$. By the definition of $\mathcal{U}_{-1}$, the itinerary $f^{\circ n_i}(\gamma)$ is contained in $\mathcal{U}_{-1}$ from the station $f^{\circ n_i}(s_i)$ onwards, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. $\square$

Proof (Proposition 5.2). As in the previous hyperbolic case, since $\mathcal{U}_{-1}$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, there is a uniform bound $\|(f^{-1})'\|_{\mathrm{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ with respect to the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$, for both branches $f^{-1} : \mathcal{U}_{-1} \to \mathcal{U}_{\pm i}$. Then, in the notation of Lemma 5.3, we have

$$\begin{aligned}
\mathrm{HypLength}(\gamma_k) &\leq \mathrm{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\
&\leq \theta \cdot \mathrm{HypLength}(f^{\circ n_1}(\gamma_k)) \\
&\leq \dots \\
&\leq \theta^k \cdot \mathrm{HypLength}(f^{\circ n_k}(\gamma_k)) \\
&\lesssim \theta^k.
\end{aligned} \tag{5.1}$$

As in the hyperbolic case, we infer that $\mathrm{Length}(\gamma_k) \leq C\theta^k$ by the local equivalence of the Euclidean metric and the hyperbolic metric on $B(0, R) \setminus \mathcal{P}$. $\square$

5.2 Some Estimates and Notation

To estimate the length of express tracks, we introduce the notations $s_n := s_{n,0}$ and

$$\ell_n := \mathrm{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{5.2}$$

Lemma 5.4. *The lengths ℓ_n satisfy:*

(i)

$$\frac{|p - s_n|}{\ell_n} \to \infty, \tag{5.3}$$

(ii)

$$\frac{\ell_n}{\ell_{n+1}} \to 1. \tag{5.4}$$

In particular, for any $C > 0$, there is a sufficiently large integer d such that

$$\ell_m + \ldots + \ell_n \geq C(\ell_m + \ell_n)$$

whenever $|m - n| \geq d$.

Proof. Using the affine conjugacy of the map f to the map $g : z \mapsto z^2 + z$, which sends the parabolic fixed point $\frac{1}{2}$ of f to 0, one can show that

$$\ell_n \asymp \frac{1}{n^2} \qquad \text{and} \qquad |p - s_n| \asymp \frac{1}{n}.$$

After a little arithmetic, we get (5.3) and (5.4). $\square$

Definition 5.5. The *relative distance* of a curve γ to the post-critical set $\mathcal{P}$ is

$$\Delta(\gamma, \mathcal{P}) = \frac{\mathrm{dist}(\gamma, \mathcal{P})}{\min(\mathrm{diam}(\gamma), \mathrm{diam}(\mathcal{P}))}.$$

We say that the curve γ is *η-relatively separated* from the post-critical set if $\Delta(\gamma, \mathcal{P}) \geq \eta$.

If an itinerary γ is relatively separated from $\mathcal{P}$, then the preimages of γ under f have bounded distortion. In particular, if γ is a quasiconvexity certificate, then Koebe's distortion theorem implies that $f^{-1}(\gamma)$ is also a certificate with a comparable constant.

Lemma 5.6. *There exists a constant $k > 0$ such that for any pair of points $z_1, z_2 \in \mathcal{J}$, we have $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$.*

Proof. By Lagrange's theorem, we may take $k = \max_{z \in B} |f'(z)|$, where B is any ball containing $\mathcal{J}$. □

5.3 Dynamics near the parabolic fixed point

The purpose of the following definition is to organize points on the Julia set $\mathcal{J}$ according to their distance from the main cusp in an f-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure*: the first time that the central itinerary makes a turn.

Definition 5.7. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 6.

This decomposition is invariant under f_0 in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of η_ζ. Applying the Böttcher map ψ, we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under f.

We now use this decomposition to analyze the case where the points w_1, w_2 lie in "well-separated cusps". Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \tag{5.5}$$

where d is a sufficiently large integer, to be chosen later. This gives some control from below on $|w_1 - w_2|$. We represent the itinerary $\eta = \eta_{w_1,w_2}$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, γ_m and γ_n. See Figure 7 for the picture in the exterior unit disk. Thus we have

$$\text{Length}(\eta) = \text{Length}(\gamma_m) + \text{Length}(\gamma_{m,n}) + \text{Length}(\gamma_n). \tag{5.6}$$

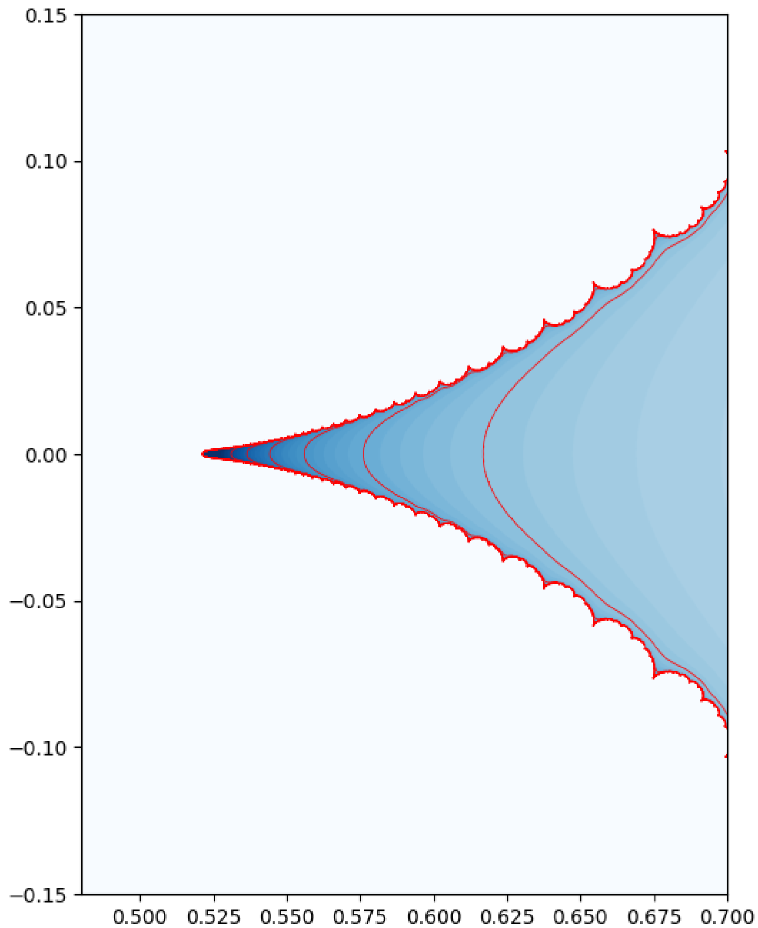


Figure 5: The Cauliflower near the parabolic point $p = 1/2$.

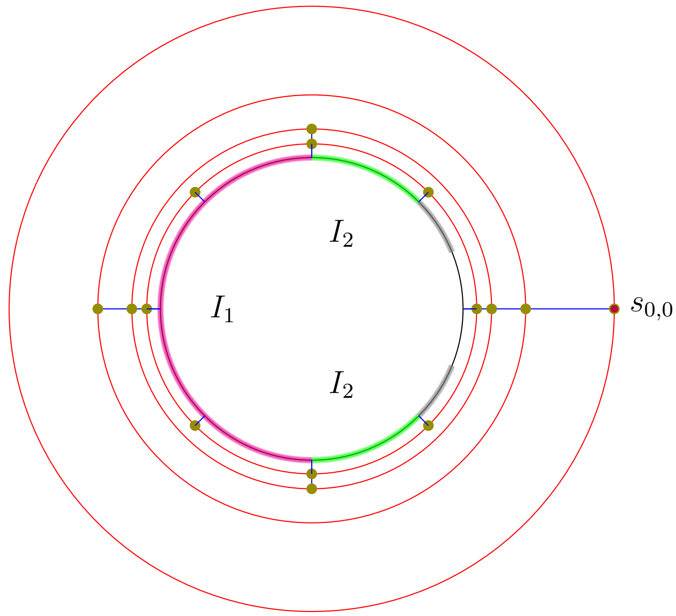


Figure 6: First few parts of the departure decomposition I_m of the circle.

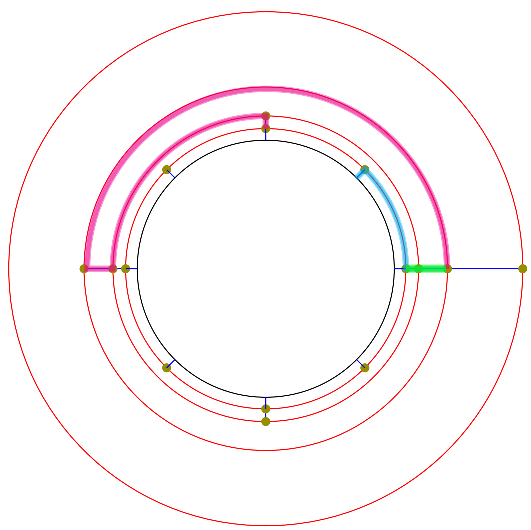


Figure 7: The three parts of an itinerary η. The green path is $\gamma_{m,n}$, the cyan and magenta are γ_m and γ_n.

The condition $m - n \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to γ_m and γ_n:

Proposition 5.8. *There exists a sufficiently large integer d so that*

$$\mathrm{Length}(\gamma_{m,n}) \asymp |w_1 - w_2|, \tag{5.7}$$

whenever $m - n \geq d$.

We henceforth fix a value of d as in the proposition.

Proof. We first make two elementary observations. Koebe's distortion theorem applied to the iterates of f^{-1} shows that

$$\mathrm{Length}(\gamma_m) \leq C\ell_m, \tag{5.8}$$

for some constant $C \geq 0$. Notice that (5.8) holds for $m = 1$ by Proposition 5.2, which gives a uniform bound on the length of an itinerary.

Meanwhile, by Lemma 5.4, there exists an integer d such that

$$C(\ell_m + \ell_n) \leq \frac{\mathrm{Length}(\gamma_{m,n})}{2} \tag{5.9}$$

whenever $m - n \geq d$.

By the triangle inequality, we have

$$\begin{aligned} \big|\mathrm{Length}(\gamma_{m,n}) - |w_1 - w_2|\big| &\leq \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_n) \\ &\leq \frac{\mathrm{Length}(\gamma_{m,n})}{2}, \end{aligned}$$

which clearly implies (5.7). $\square$

5.4 Quasiconvexity: three special cases

We now show that the itineraries η_{w_1,w_2} are certificates in three special cases. To state them, we introduce some notation.

5.4.1 Notation

For each n, we denote by α_n the union of the two outermost tracks emanating from the station $s_{n,0}$. Notice that the curves α_n are pairwise disjoint since this is true for their pullbacks to the exterior unit disk.

We define the constants C_1, C_2, ϵ as follows. We first choose $C_1 \geq 2$, then we let $C_2 = C_1 + d + 2$ and choose $\epsilon > 0$ small enough so that we have

$$\operatorname{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k\epsilon. \tag{5.10}$$

The constant C_2 was chosen so that for any pair (m, n) of integers, we have at least one of the following three cases: either m, n are both greater than C_1, or both are smaller than C_2, or $|m - n| > d$.

5.4.2 Three Special Cases

In this section we treat the following special cases:

1. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n < C_2, \quad m, n \geq 2;$
2. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n > C_1;$
3. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

Notice that Case 2 overlaps with Case 1. We denote the domain enclosed by α_m, α_n and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and denote the domain enclosed by $\mathcal{J}$ and α_n by $\mathcal{K}_n$.

Lemma 5.9. *Let $w_1 \in I_m$ and $w_2 \in I_n$, for $n \geq m \geq 2$. Then the itinerary η_{w_1,w_2} is contained in the domain $\mathcal{K}_{m,n+1}$.*

Lemma 5.10. *Let $w_1, w_2 \in \mathcal{J}$. In each of the three special cases, the itinerary γ_{w_1,w_2} is a quasiconvexity certificate. In Cases 1 and 2, γ_{w_1,w_2} is relatively separated.*

Proof. Case 1. In this case, the itinerary is contained in the domain $\mathcal{K}_{2,C_2+1}$. Since $\operatorname{dist}(\mathcal{K}_{2,C_2+1}, \mathcal{P}) > 0$, γ_{w_1,w_2} is η-relatively separated for some $\eta > 0$.

Case 2. Assuming without loss of generality that $n \geq m$, the itinerary is contained in $\mathcal{K}_{m,n+1}$. By Koebe's distortion theorem, γ_{w_1,w_2} is also relatively separated.

Case 3 is the content of Proposition 5.8. □

5.5 Quasiconvexity: general case

We apply a stopping time argument to promote the quasiconvexity of η_{w_1,w_2} to the quasiconvexity of η_{z_1,z_2}, thereby proving the following theorem:

Theorem 5.11. *The domain* $\mathrm{Exterior}(\mathcal{J})$ *is quasiconvex, with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let (z_1, z_2) be a pair of points in $\mathcal{J}$. Repeatedly apply f to (z_1, z_2) until either of the three special cases occurs. Denote by $w_i = f^{\circ N}(z_i)$ the resulting points. We have already proved that the itinerary η_{w_1,w_2} satisfies

$$\mathrm{Length}(\eta_{w_1,w_2}) \le A|w_1 - w_2|,$$

for some $A > 0$. We deduce that the original pair of points (z_1, z_2) enjoys a similar estimate,

$$\mathrm{Length}(\eta_{z_1,z_2}) \le C|z_1 - z_2|,$$

where C depends only on A.

In Cases 1 and 2, we are done by Lemma 5.10. In Case 3, the itinerary η_{w_1,w_2} is contained in $\mathcal{K}_2$. Let $\mathcal{K}_{-2}$ be the preimage of $\mathcal{K}_2$ under f that contains the negative preimage $f^{-1}(p) = -\frac{1}{2}$ of the cusp p. As the domain $\mathcal{K}_{-2}$ is relatively separated from $\mathcal{P}$ and contains the curve $f^{\circ(N-1)}(\eta_{z_1,z_2}) = \eta_{f^{-1}(w_1),f^{-1}(w_2)}$, we may use Koebe's distortion theorem to conclude that

$$\frac{\mathrm{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\mathrm{Length}(\eta_{f^{-1}(w_1),f^{-1}(w_2)})}{|f^{-1}(w_1) - f^{-1}(w_2)|} \asymp \frac{\mathrm{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|} \tag{5.11}$$

as desired. □

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