1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\mathrm{Length}(\gamma) \leq A \cdot |z_1 - z_2| \,.$$

We call such a path γ a *quasiconvexity witness* for z_1, z_2.

If Ω is the interior of a Jordan curve, then by [1, Corollary F] it is enough to find certificates for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

Our interest in quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then the interior of its complement is John. See [1, Corollary 3.4] for details.

We want to show that the exterior of the developed deltoid is quasiconvex.

We show that the exterior of the cauliflower, $\mathcal{J}^{\text{exterior}}(z^2 + 1/4)$, is quasiconvex. We then adapt our argument to the exterior of the developed deltoid.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of a certificate connecting two given points on the Julia set.

We first build such certificates for the exterior unit disk $\mathbb{D}^*$, and then transport them by the Böttcher coordinate ψ of $f_{1/4}$ to the exterior of the cauliflower.

In order to retain control on the certificates after applying ψ, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This is done by only traveling along the boundaries of a suitable Carleson box decomposition of $\mathbb{D}^*$.

The image of a certificate c of $\mathbb{D}^*$ under the conjugacy ψ is invariant under $f_{1/4}$. We use this invariance to show that $\psi(c)$ is indeed a certificate, by a parabolic variant of the principle of the conformal elevator: We repeatedly apply $f_{1/4}$ on $\psi(c)$ until either the distance between the endpoints grows to a definite size or one endpoint is attracted sufficiently fast to the parabolic fixed point $1/2$. The latter case requires a more delicate treatment.

To facilitate the reading, we separate this latter difficulty by first demonstrating the proof in the hyperbolic case of maps f_c where $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$. In this case the usual conformal elevator applies. We then embark on the $c = \frac{1}{4}$ case.

2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

Definition. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{2^{-n}}, 2^{2^{-n-1}} \right], \qquad \arg(z) \in \left(\frac{k}{2^n} 2\pi, \frac{(k+1)}{2^n} 2\pi \right] \right\}$$

an ***f_0-Carleson box***.

Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{2^{-n-1}} < |z| \leq 2^{2^{-n}} \right\}$$

into 2^n equally-spaced sectors.

The ***Carleson f_0-box decomposition*** is the partition of $\mathbb{D}^*$ obtained by f_0-Carleson boxes:

$$\mathbb{D}^* = \{z : |z| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this partition is its invariance under f_0, stemming from the relation

$$f_0 \left(B_{n+1,k} \right) = B_{n, \lfloor \frac{k}{2} \rfloor}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels using trains. When a passenger purchases a ticket between two stations s_1 and s_2, they purchase a particular way to get from s_1 to s_2. We proceed to define "stations" and "tracks".

Definition. A *terminal* is a point $\zeta \in \partial\mathbb{D}$ on the unit circle. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : z \mapsto z^2$. We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|z| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are $s_{n,(k\pm1) \mod 2^n}$, the two stations adjacent to $s_{n,k}$ on C_n.
2. Given a peripheral neighbor s' of s, the *peripheral track* $\gamma^{\text{peripheral}}_{s,s'}$ between these stations is the short arc of the circle C_n connecting s to s'.
3. The *radial successor* of s is $\text{RadicalSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.
4. The *Express track* $\gamma^{\text{express}}_s$ from s is the radial segment $[s, \text{RadicalSuccessor}(s)]$.

Notice that the tracks preserve the dynamics: applying $z \mapsto z^2$ on a peripheral track between s, s' gives a peripheral track between the parents of s, s' in the tree, and likewise for an express track.

Definition. Let $z \in \partial\mathbb{D} = z_1 = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The itinerary η_z from z to the central station $\sigma_0 = s_{0,0} = 2$ is a sequence of stations $(\sigma_0, \sigma_1, \ldots)$ from the central station $\sigma_0 = s_{0,0} = 2$ to $\lim \sigma_k = z$, defined as follows.

We choose the stations σ_i inductively in pairs, in a greedy manner. In each step we drive peripherally to the station closest to z_1 and then drive to its radial successor. See Figure 1.

For the first station σ_1 we have no choice and we drive to the station $\sigma_1 = s_{1,0} = \sqrt{2}$.

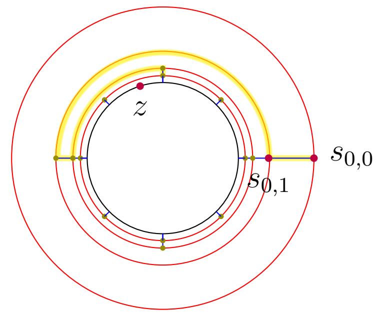

Figure 1: The central journey to a point z.

Suppose that we already chose the stations $(\sigma_0, \ldots, \sigma_{2k-1})$. Then from σ_{2k-1} we drive to the station σ_{2k} on the same circle, $|\sigma_{2k-1}| = |\sigma_{2k}|$, that minimizes the angular distance $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})|$.

We call η_z the *central itinerary* of z. Note that there are no two consecutive peripheral tracks in η_z, and that the itineraries are invariant under f_0, in the sense that

$$f_0(\eta_z) = \eta_{f_0(z)} \cup [2, 4]$$

for every $z \in \partial\mathbb{D}$.

Lemma 2.1. *Let $z \in \partial\mathbb{D}$. Decompose the central itinerary η_z into its constituent tracks,*

$$\eta_z = \gamma_1 + \gamma_2 \ldots .$$

Then we have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim 2^{-k}$$

uniformly in z.

Proof. The station σ_{2k} is adjacent peripherally to σ_{2k-1}, since the angular distance between stations on C_n is $\frac{2\pi}{2^n}$ and we maintain the invariant $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})| \leq \frac{2\pi}{2^k}$ throughout the itinerary. Thus the length of the peripheral track $\gamma^p_{\sigma_{2k-1},\sigma_{2k}}$ is either $r_n \cdot \frac{2\pi}{2^k} = 2^{1/2^k} \frac{2\pi}{2^k}$ or 0 (in case $\sigma_{2k-1} = \sigma_{2k}$), and in any case the length is at most $\lesssim \frac{1}{2^k}$ for a global hidden constant. The length of the k-th express track

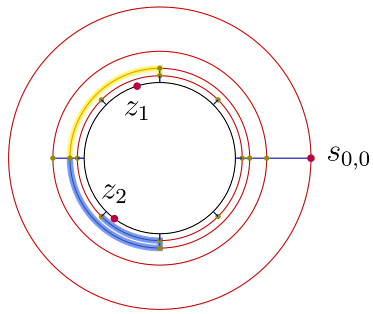

Figure 2: A quasiconvexity certificate between two points z_1, z_2.

decays exponentially due to the invariance under f_0. Explicitly it is $2^{1/2^k} - 2^{1/2^{k+1}} \leq 2^{2^{-k}} - 1 \lesssim 2^{-k}$ since $\lim_{x \to 0} \frac{2^x - 1}{x} = \log 2$.

Thus the total length of the itinerary is bounded uniformly in z. $\square$

Theorem 2.2. *The domain $\mathbb{D}^*$ is quasiconvex with quasiconvexity certificates that are an itineraries.*

Proof. Fix two points ("terminal stations") $z_1, z_2 \in \partial\mathbb{D}$. Let $\eta_{z_1} = (\sigma_n^1)_{n=0}^{\infty}, \eta_{z_2} = (\sigma_n^2)_{n=0}^{\infty}$ be their central an itineraries, connecting each terminal to the central station.

Let $(\sigma_0, \ldots, \sigma_N)$ be the maximal common prefix of η_{z_1} and η_{z_2}. Let $\eta_{z_i}^{\text{truncated}} = \left(\sigma_N, \sigma_{N+1}^i, \ldots\right)$ be the truncated paths. By the maximality of N, we have that $\eta_{z_1}^{\text{truncated}}$ and $\eta_{z_2}^{\text{truncated}}$ are two an itineraries with a common starting point, so we can concatenate them to obtain a bi-infinite itinerary

$$\eta_{z_1,z_2} = \left(\ldots \sigma_{N+2}^2, \sigma_{N+1}^2, \sigma_N, \sigma_{N+1}^1, \sigma_{N+2}^1, \ldots\right)$$

connecting z_1 and z_2.

We conclude the proof by showing that $\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim |z_1 - z_2|$.

As $|z_1 - z_2| \asymp |\theta_1 - \theta_2|$ and $\text{Arg}\left(z_i\right) \propto \theta_i$, it is equivalent to show

$$\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim \left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right|.$$

By the choice of N,

$$\left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that Length $(\eta_{z_1,z_2}) \lesssim 2^{-N}$. But

$$\text{Length}\,(\eta_{z_1,z_2}) = \text{Length}\left(\eta_{z_1}^{\text{truncated}}\right) + \text{Length}\left(\eta_{z_2}^{\text{truncated}}\right),$$

so it is enough to observe that

$$\text{Length}\left(\eta_{z_i}^{\text{truncated}}\right) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma.

□

3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$, and denote by ψ the Böttcher coordinate of $f : z \mapsto z^2 + c$ at infinity. This means that ψ is the unique conformal map $\mathbb{D}^* \to \text{Exterior}(\mathcal{J})$ which fixes ∞ and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the julia set $\mathcal{J}$ is a Jordan domain, the map ψ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f)$ by Carathéodory's theorem.

We apply ψ on the rails that we constructed in $\mathbb{D}^*$ to obtain corresponding rails in Exterior$(\mathcal{J})$. In particular, we obtain the following:

Definition. 1. The c-stations are the points $s_{n,k,c} = \psi(s_{n,k})$.

2. The c-express tracks are the curves of the form $\psi\gamma_s^{\text{express}}$. These tracks lie on external rays of the filled Julia set $\mathcal{K}$.

3. The c-peripheral tracks are the curves of the form $\gamma_{s,s',c}^{\text{peripheral}} = \psi(\gamma_{s,s'}^{\text{peripheral}})$. These tracks lie on the level sets of ψ, or equivalently on the equipotentials of $\mathcal{K}$.

4. The c-central an itineraries are $\eta_{z,c} = \psi(\eta_z)$. Every c-central itinerary is a c-itinerary $(\sigma_0, \sigma_1, \ldots)$ from the central station $\sigma_0 = f(s_{0,0})$ to $\lim_{k\to\infty} \sigma_k = f(z)$.

We observe that the c-central station is still on the real line:

Lemma 3.1. $\psi\left((1,\infty)\right) \subseteq \mathbb{R}$. *In particular,* $\psi(s_{0,0}) \in \mathbb{R}$.

Proof. This is true by the symmetry of $\mathbb{D}^*$ and Exterior($\mathcal{J}$) with respect to $\mathbb{R}$. Formally, $\overline{\psi}(\overline{z})$ is another conformal conjugacy between f and f_0 which fixes infinity, so by uniqueness of the Böttcher coordinate we obtain $\psi(z) = \overline{\psi}(\overline{z})$, hence $\psi(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. □

4 Quasiconvexity for a hyperbolic Map

A rational map is said to be *hyperbolic* if every critical point converges to an attracting cycle, and no critical point is on the Julia set.

Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size.

Proposition 4.1 (The Principle of the Conformal Elevator)**.** *Let f be a rational hyperbolic map, let $z \in \mathcal{J}$ be a point on the Julia set of f and let $r > 0$. Consider the ball $B = B(z, r)$. Then there exists some forward iterate $f^{\circ n}$ of f which is injective on $B(z, 2r)$ and under whom* $\operatorname{diam} f^{\circ n}(B(z, r))$ *is bounded below, uniformly in z and r.*

See [2] for a stronger version, details and a proof.

We need a corollary of this principle:

Corollary 4.2. *Let f be a rational hyperbolic map. There exists a constant ϵ such that for every two points $z, w \in \mathcal{J}(f)$, there is a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

Proof. Apply proposition 4.1 to a ball centered on the Julia set which contains z, w on its boundary at roughly antipodal points. After blowing up we get points $f^{\circ n} z, f^{\circ w}$ which are a definite distance apart by Koebe's distortion theorem. /TODO: make this correct/ □

We are now ready to show the analogue of theorem 2.2 for a hyperbolic map. /TODO: We use more than hyperbolicity, also that the Julia set is a Jordan domain/

Theorem 4.3. *a. Let $\zeta \in \partial D$, and decompose its central itinerary into tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 + \dots .$$

Then we have the estimate

$$\mathrm{Length}(\psi(\gamma_k)) \lesssim \theta^{-k}$$

uniformly in ζ*, for some constant* $\theta = \theta(c) > 1$*.*

b. The domain Exterior$(\mathcal{J})$ *is quasiconvex.*

Proof. The map f has some iterate $f^{\circ N}$ that is pointwise expanding on the Julia set $\mathcal{J}(f)$, where N is independent of z. By compactness of $\mathcal{J}$, the map $f^{\circ N}$ is also uniformly expanding there, i.e. there are a constant $\theta > 1$ and a neighborhood $\mathcal{U}$ of $\mathcal{J}(f)$ on which $|(f^{\circ n})'| = |\prod f'(f^{\circ} k)| > \theta$. /TODO: Make this correct/

Thus the length of peripheral tracks decays exponentially at rate θ, and likewise for express tracks.

b. /TODO: delete?/ Let z_1, z_2 be two points on $\mathcal{J}(f)$. We construct a quasiconvexity certificate curve connecting z_1 and z_2. Let $\zeta_i = \psi^{-1}(z_i)$ be the corresponding points on the unit circle, then we have a quasiconvexity certificate η_{ζ_1,ζ_2} between them in $\mathbb{D}^*$. We need to show that the curves $\eta_{z_1,z_2,c} = \psi(\eta_{\zeta_1,\zeta_2})$ are certificates. By the invariance of the construction, this is an itinerary on the f-Carleson decomposition which can similarly be described directly in terms of a common ancestor in the tree structure, since ψ is a bijective correspondence between the two decompositions. Since we already know that the lengths of tracks in the itinerary decay exponentially, with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case. □

We give a second proof, relying on the previous claim on the separation of points under iteration. This proof will better prepare us for the parabolic $c = 1/4$ case, in which we don't have uniform expansion of f on its Julia set.

Proof. By the claim, there exists some ϵ such that any two points are ϵ-apart under some iteration of f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$ then there is nothing to prove, since we may just concatenate η_{z_1} and η_{z_2} and absorb this bounded length into the quasiconvexity constant A. Explicitly, if Length$(\eta_z) \leq L$ for all $z \in \mathcal{J}$ then we take $A \geq \frac{2L}{\epsilon}$ and then automatically Length$(\eta_{z_1} + \eta_{z_2}) \leq A\,|z_1 - z_2|$.

If, on the other hand, $|z_1 - z_2| < \epsilon$, then we may use the claim to find an iterate $f^{\circ n}$ such that $|f^{\circ n}(z_0) - f^{\circ n}(z_1)| \geq \epsilon$. Then there is a certificate itinerary

$\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}$ between them, and we take the certificate η_{z_0,z_1} between the original points to be the component of $f^{\circ -n}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)$ that connects the points z_0, z_1.

A distortion estimate:

$$\text{Length}\left(\eta_{z_0,z_1,c}\right) \asymp \frac{\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)}{\left|\left(f^{\circ n}\right)'(\zeta)\right|}$$

for some point ζ on $\mathcal{J}$. The denominator grows with n exponentially at rate θ, while the numerator has a bound of the form

$$\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right) \lesssim \left|f^{\circ n}(z_0) - f^{\circ n}(z_1)\right| \lesssim \theta^n \left|z_1 - z_2\right|$$

so altogether

$$\text{Length}\left(\eta_{z_0,z_1,c}\right) \lesssim \frac{\theta^n \left|z_1 - z_2\right|}{\theta^n} = \left|z_1 - z_2\right|$$

so $\eta_{z_0,z_1,c}$ is a quasiconformality certificate.

$\square$

References

[1] Hakobyan, Hrant, and Herron, David A.. ”Euclidean quasiconvexity..” Annales Academiae Scientiarum Fennicae. Mathematica 33.1 (2008): 205-230.

[2] Mario Bonk, Mikhail Lyubich, Sergei Merenkov, Quasisymmetries of Sierpiński carpet Julia sets, Advances in Mathematics, Volume 301, 2016, Pages 383-422, ISSN 0001-8708, https://doi.org/10.1016/j.aim.2016.06.007.