1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2| \, .$$

We call such a path γ a *quasiconvexity witness* for z_1, z_2.

If Ω is the interior of a Jordan curve, then by [1, Corollary F] it is enough to find witnesses for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

Our interest in quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then the interior of its complement is John. See [1, Corollary 3.4] for details.

We want to show that the exterior of the developed deltoid is quasiconvex.

Before that, we show that the exterior of the cauliflower, $\mathcal{J}^{\text{exterior}}(z^2 + 1/4)$ is quasiconvex. The proof for the deltoid is similar, but requires some extra work.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of paths connecting given points on the boundary.

We use the conjugation of $f_c : z \mapsto z^2 + c$ on the exterior of the Julia set to the map z^2 on the exterior of the unit disk $\mathbb{D}^*$. We decompose $\mathbb{D}^*$ into Carleson boxes invariant under z^2 and connect points on the unit circle by traveling on the boundary of these boxes. Both the boxes and the path on their boundary respect the dynamics of z^2 there, which allow us to transport these paths from $\mathbb{D}^*$ to the exterior of $\mathcal{J}(f_c)$

We construct two collections of curves, the so-called "express" and "peripheral" tracks. We use the metaphor of a train traveling between the endpoints and switching between tracks. We stitch the quasiconvex paths from these tracks.

The construction of the train tracks is done first in the case of the exterior unit disk ($c = 0$), and then transported to the nontrivial $c = 1/4$ case using the Böttcher linearization. Thus we first explain the method in the case of $c = 0$.

For clarity of exposition, we then show how the method works in the case of $c = 0.1$, in which the conformal elevator makes the argument simpler, and only then prove the parabolic case $c = 1/4$. We do this even though $J_{0.1}^{\text{exterior}}$ is known in advance to be quasiconvex, in virtue of being a quasidisk.

2 Power map

The exterior $\mathbb{D}^* = \{|z| > 1\}$ of the unit disk is trivially quasiconvex by connecting points along the perimeter of the circle. However, these paths follow the boundary too closely and their length would blow up if we transport them to the exterior of $\mathcal{J}(f_c)$, $c \neq 0$, via the Riemann map. Instead, we connect points by traveling along the boundaries of invariant boxes which we now define.

Definition. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \exp\left(\left(2^{-n-1}\log 2,\ 2^{-n}\log 2\right] \times \left(\frac{k}{2^n}2\pi,\ \frac{(k+1)}{2^n}2\pi\right]\right)$$

an ***f_0-Carleson box***.

Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{2^{2^{-n-1}} < |z| \leq 2^{2^{-n}}\right\}$$

into 2^n equally-spaced sectors.

The ***Carleson f_0-box decomposition*** is the partition of $\mathbb{D}^*$ obtained by f_0-Carleson boxes:

$$\mathbb{D}^* = \{z : |z| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this partition is its invariance under f_0, stemming from the relation

$$f_0\left(B_{n+1,k}\right) = B_{n,\lfloor \frac{k}{2} \rfloor}.$$

Definition. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : z \mapsto z^2$. We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi\right), \qquad n \in \mathbb{N}_0,\ k \in \{0, \dots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|z| = 2^{1/2^n}\right\}$.

The *radial successor* of a station $s_{n,k}$ of generation n is $s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.

The *peripheral neighbors* of a station $s_{n,k}$ are $s_{n,(k\pm 1) \mod 2^n}$, the two stations adjacent to $s_{n,k}$ on C_n.

We next lay two types of "train tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition. Let $s_{n,k}$ be a station.

1. Given a peripheral neighbor s' of s, the *peripheral track* $\gamma^{\text{peripheral}}_{s,s'}$ between these stations is the short arc of the circle C_n connecting s to s'.
2. The *Express track* $\gamma^{\text{express}}_{s,s'}$ between these two stations is the radial segment $[s, s']$.

We refer to a concatenation of tracks as a *train journey.* A journey is identified with its sequence of stations (σ_i).

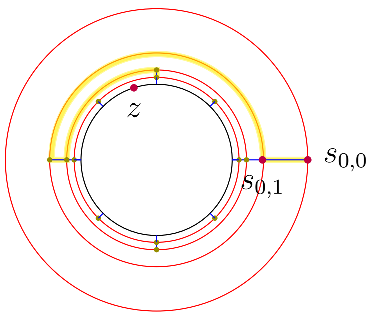
Notice that the tracks preserve the dynamics: applying $z \mapsto z^2$ on a peripheral track between s, s' gives a peripheral track the parents of s, s' in the tree, and likewise for an express track.

Lemma 2.1. *There is a family $\{\eta_z : z \in \partial\mathbb{D}\}$ of journeys with the following properties:*

1. *Every η_z is a journey $(\sigma_0, \sigma_1, \dots)$ from the central station $\sigma_0 = s_{0,0} = 2$ to $\lim \sigma_k = z$.*
2. *There are no two consecutive peripheral tracks in η_z.*
3. $\text{Length}(\sigma_k) \lesssim 2^{-k}$ *uniformly in z.*

4. The journeys are invariant under f_0, in the sense that $f_0(\eta_z) = \eta_{f_0(z)} \cup [2, 4]$ for every $z \in \partial\mathbb{D}$.

Proof. Let $z = z_1 = \exp(2\pi i\theta) \in \partial\mathbb{D}$. We choose the stations σ_i inductively in pairs, in a greedy manner. In each step we drive peripherally to the station closest to z_1 and then drive to its radial successor. See Figure. $\square$



Proof. For the first station σ_1 we have no choice and we drive to the station $\sigma_1 = s_{1,0} = \sqrt{2}$.

Suppose that we already chose the stations $(\sigma_0, \dots, \sigma_{2k-1})$. Then from σ_{2k-1} we drive to the station σ_{2k} on the same circle, $|\sigma_{2k-1}| = |\sigma_{2k}|$, that minimizes the angular distance $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})|$.

The minimizer σ_{2k} is adjacent peripherally to σ_{2k-1}, since the angular distance between stations on C_n is $\frac{2\pi}{2^n}$ and we maintain the invariant $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})| \leq \frac{2\pi}{2^k}$ throughout the journey. Thus the length of the peripheral track $\gamma^p_{\sigma_{2k-1},\sigma_{2k}}$ is either $r_n \cdot \frac{2\pi}{2^k} = 2^{1/2^k}\frac{2\pi}{2^k}$ or 0 (in case $\sigma_{2k-1} = \sigma_{2k}$), and in any case the length is at most $\lesssim \frac{1}{2^k}$ for a global hidden constant. The length of the k-th express track decays exponentially due to the invariance under f_0. Explicitly it is $2^{1/2^k} - 2^{1/2^{k+1}} \leq 2^{2^{-k}} - 1 \lesssim 2^{-k}$ since $\lim_{x\to 0} \frac{2^x-1}{x} = \log 2$.

Thus the total length of the journey is bounded uniformly in z. $\square$

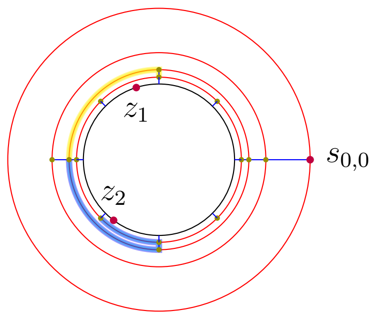
We call η_z the *central journey* of z.

Theorem 2.2. *The domain $\mathbb{D}^*$ is quasiconvex with quasiconvexity witnesses that are journeys.*

Proof. Fix two points ("terminal stations") $z_1, z_2 \in \partial\mathbb{D}$. Let $\eta_{z_1} = (\sigma_n^1)_{n=0}^{\infty}, \eta_{z_2} = (\sigma_n^2)_{n=0}^{\infty}$ be their central journeys, connecting each terminal to the central station.

Let $(\sigma_0, \ldots, \sigma_N)$ be the maximal common prefix of η_{z_1} and η_{z_2}. Let $\eta_{z_i}^{\text{truncated}} = \left(\sigma_N, \sigma_{N+1}^i, \ldots\right)$ be the truncated paths. By the maximality of N, we have that $\eta_{z_1}^{\text{truncated}}$ and $\eta_{z_2}^{\text{truncated}}$ are two journeys with a common starting point, so we can concatenate them to obtain a bi-infinite journey

$$\eta_{z_1,z_2} = \left(\ldots \sigma_{N+2}^2, \sigma_{N+1}^2, \sigma_N, \sigma_{N+1}^1, \sigma_{N+2}^1, \ldots\right)$$



connecting z_1 and z_2.

We conclude the proof by showing that Length $(\eta_{z_1,z_2}) \lesssim |z_1 - z_2|$.

As $|z_1 - z_2| \asymp |\theta_1 - \theta_2|$ and Arg $(z_i) \propto \theta_i$, it is equivalent to show

$$\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim \left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right|.$$

By the choice of N,

$$\left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that Length $(\eta_{z_1,z_2}) \lesssim 2^{-N}$. But

$$\text{Length}\left(\eta_{z_1,z_2}\right) = \text{Length}\left(\eta_{z_1}^{\text{truncated}}\right) + \text{Length}\left(\eta_{z_2}^{\text{truncated}}\right),$$

so it is enough to observe that

$$\text{Length}\left(\eta_{z_i}^{\text{truncated}}\right) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma.

□

3 Hyperbolic Map

Throughout this section we take $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$. Since the critical point 0 is in the filled Julia set of f_c, there is a conformal map $\psi_c : \mathbb{D}^* \to \mathcal{J}^{\text{exterior}}(f_c)$ conjugating f_c to z^2, i.e. $f_c \circ \psi_c(z) = \psi_c \circ f_0(z)$ for every $z \in \mathbb{D}^*$.

The map ψ_c extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f_c)$ by Carathéodory's theorem, since $\mathcal{J}$ is a Jordan curve.

All f_0-invariant constructions carry over from $\mathbb{D}^*$ to $\mathcal{J}^{\text{exterior}}(f_c)$, and now they are f_c-invariant: We have f_c-Carelson boxes $B_{n,k,c} = \psi_c(B_{n,k})$ and an f_c-Carelson decomposition

$$\mathcal{J}^{\text{exterior}}(f_c) = \psi_c\left(\{|z| > 2\}\right) \sqcup \bigsqcup_{n,k,} B_{n,k,c},$$

we have stations $s_{n,k,c} = \psi_c(s_{n,k})$ and likewise tracks. The express tracks lie on the external rays of ψ_c, and the peripheral tracks are on the level sets of ψ_c, or equivalently on the equipotentials of Green's function.

As an example of how invariance carries other, applying ψ_c to both sides of the equation

$$f_0\left(B_{n+1,k}\right) = B_{n,\lfloor \frac{k}{2} \rfloor}$$

gives the corresponding relation

$$f_c\left(B_{n+1,k,c}\right) = B_{n,\lfloor \frac{k}{2} \rfloor,c}.$$

We observe that the central station is still on the real line:

Lemma 3.1. $\psi_c(\mathbb{D}^* \cap \mathbb{R}) \subseteq \mathbb{R}$. *In particular,* $\psi_c(s_{0,0}) \in \mathbb{R}$.

Proof. This is true by symmetry of $\mathbb{D}^*$ and $\mathcal{J}^{\text{exterior}}(f_c)$ with respect to $\mathbb{R}$. Formally, $\overline{\psi_c}(\overline{z})$ is another conformal map with the same conjugation relation, so by uniqueness of the Böttcher coordinate (with a given derivative at ∞) we obtain $\psi_c(z) = \overline{\psi_c}(\overline{z})$, hence $\psi_c(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. $\square$

Note that this lemma remains true for $c = 1/4$.

As before, we have:

Lemma 3.2. *There is a family* $\{\eta_{z,c} : z \in \mathcal{J}(f_c)\}$ *of journeys with the following properties:*

1. Every $\eta_{z,c}$ *is a journey* $(\sigma_0, \sigma_1, \dots)$ *from the central station* $\sigma_0 = s_{0,0,c}$ *to* $\lim \sigma_k = z$*.*

2. There are no two consecutive peripheral tracks in η_z*.*

3. $\mathrm{Length}(\sigma_k) \lesssim \theta^{-k}$ *uniformly in* z*, for a constant* $\theta = \theta(c) > 1$*.*

4. The journeys are invariant under f_c*, in the sense that* $f_c(\eta_{z,c}) = \eta_{f_c(z)} \cup f_c([2,4])$ *for every* $z \in \partial\mathbb{D}$*.*

Proof. Let $z \in \mathcal{J}(f_c)$. Let $\zeta = \psi_c^{-1}(z)$ be the corresponding point on $\partial\mathbb{D}$,

which has a central journey η_ζ in $\mathbb{D}^*$. We choose the central journey of z to be $\eta_{z,c} = f_c(\eta_\zeta)$.

We take the images of the corresponding journeys from the case of $\mathbb{D}^*$. Parts $1, 2, 4$ are then automatic. We now check part (3).

The map f_c is pointwise expanding on the Julia set $\mathcal{J}(f_c)$, so by compactness f_c is uniformly expanding there, i.e. there are a constant $\theta > 1$ and a neighborhood $\mathcal{U}$ of $\mathcal{J}(f_c)$ on which $|f'| > \theta$. Since every journey is eventually contained in $\mathcal{U}$, we have for $k \gg 1$ that $\mathrm{Length}(f(\gamma)) \geq \theta \cdot \mathrm{Length}(\gamma)$. Thus the length of peripheral tracks decays exponentially at rate θ, and likewise for express tracks. □

To take advantage of the preceding lemma for showing quasiconvexity, we use the following claim.

Claim 3.3. *Let* f *be a rational map that is expanding on its Julia set* $\mathcal{J}(f)$*. Then there exists a constant* ϵ *such that for every two points* $z, w \in \mathcal{J}(f)$*, there exists* $n \in \mathbb{N}$ *for which* $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$*.*

Proof. (sketch.) This claim follows from the condition that $|f'| > 1$ on $\mathcal{J}$ since as long as two points $z, w \in \mathcal{J}$ are close enough we can approximate f linearly to see that the images $f(z), f(w)$ must be further apart. □

Remark. This claim has some similarity to the *principle of the conformal elevator*, which we now recall.

A rational map is said to be *hyperbolic* if every critical point converges to an attracting cycle, and no critical point is on the Julia set.

The principle of the conformal elevator: Let f be a hyperbolic map, let $\zeta \in \mathcal{J}(f)$ and let $r > 0$ be sufficiently small. Then there exists an iterate $f^{\circ n}(B(\zeta, r))$ of the ball $B(\zeta, r)$ which is a set of diameter bounded below uniformly in ζ, r and which is "almost round". Since we do not need this control on the distortion of the balls, we do not state the precise form of this latter constraint and refer the reader to [2] for details.

Theorem 3.4. *The domain $\mathcal{J}^{exterior}(f_c)$ is quasiconvex.*

Proof. Let z_1, z_2 be two points on $\mathcal{J}(f_c)$. We construct a quasiconformality witness curve connecting z_1 and z_2. We use the obvious candidate: let $\zeta_i = \psi^{-1}(z_i) \in \partial\mathbb{D}$, then we have a quasiconformality witness for them η_{ζ_1,ζ_2} from the $c = 0$ case. We choose $\eta_{z_1,z_2,c} = \psi_c(\eta_{\zeta_1,\zeta_2})$ to be the witnesses. By the invariance of the construction, this is a journey on the f_c-Carleson decomposition which can similarly be described directly in terms of a common ancestor in the tree structure, since ψ_c is a bijective correspondence between the two decompositions. Since we already know that the lengths of tracks in the journey decay exponentially, with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case. □

We give a second proof, relying on the previous claim on separation of points under iteration. This proof will better prepare us to the parabolic $c = 1/4$ case, in which we don't have uniform expansion of f_c on its Julia set.

Proof. By the claim, there exists some ϵ such that any two points are ϵ-apart under some iteration of f_c. Let $z_1, z_2 \in \mathcal{J}(f_c)$. If $|z_0 - z_1| \geq \epsilon$ then there is nothing to prove, since we may just concatenate η_{z_1} and η_{z_2} and absorb this bounded length into the quasiconformality constant A. Explicitly, if $\text{Length}(\eta_z) \leq L$ for all $z \in \mathcal{J}$ then we take $A \geq \frac{2L}{\epsilon}$ and then automatically $\text{Length}(\eta_{z_1} + \eta_{z_2}) \leq A|z_0 - z_1|$.

If, on the other hand, $|z_0 - z_1| < \epsilon$, then we may use the claim to find an iterate $f^{\circ n}$ such that $|f^{\circ n}(z_0) - f^{\circ n}(z_1)| \geq \epsilon$. Then there is a witness journey $\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}$ between them, and we take the witness η_{z_0,z_1} between the original points to be the component of $f^{\circ -n}\left(\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}\right)$ that connects the points z_0, z_1.

A distortion estimate:

$$\text{Length}(\eta_{z_0,z_1,c}) \asymp \frac{\text{Length}\left(\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}\right)}{\left|\left(f_c^{\circ n}\right)'(\zeta)\right|}$$

for some point ζ on $\mathcal{J}$. The denominator grows with n exponentially at rate θ, while the numerator has a bound of the form

$$\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right) \lesssim |f^{\circ n}(z_0) - f^{\circ n}(z_1)| \lesssim \theta^n |z_0 - z_1|$$

so altogether

$$\text{Length}\left(\eta_{z_0,z_1,c}\right) \lesssim \frac{\theta^n |z_0 - z_1|}{\theta^n} = |z_0 - z_1|$$

so $\eta_{z_0,z_1,c}$ is a quasiconformality witness.

$\square$

References

[1] Hakobyan, Hrant, and Herron, David A.. "Euclidean quasiconvexity.." Annales Academiae Scientiarum Fennicae. Mathematica 33.1 (2008): 205-230.

[2] Mario Bonk, Mikhail Lyubich, Sergei Merenkov, Quasisymmetries of Sierpiński carpet Julia sets, Advances in Mathematics, Volume 301, 2016, Pages 383-422, ISSN 0001-8708, https://doi.org/10.1016/j.aim.2016.06.007.