1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\mathrm{Length}(\gamma) \leq A \cdot |z_1 - z_2| .$$

We call such a path γ a *quasiconvexity witness* for z_1, z_2.

We call η_z the *central journey* of z.

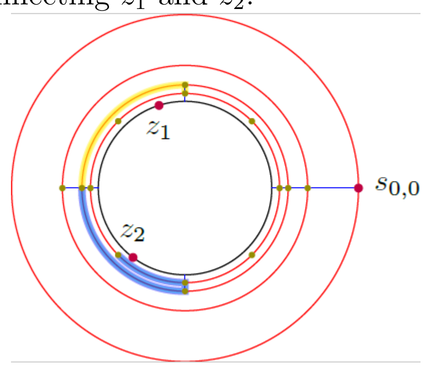
Theorem 1.1. *The domain $\mathbb{D}^*$ is quasiconvex with quasiconvexity witnesses that are central journeys.*

Proof. Fix two points ("terminal stations") $z_1, z_2 \in \partial\mathbb{D}$. Let $\eta_{z_1} = (\sigma_n^1)_{n=0}^{\infty}, \eta_{z_2} = (\sigma_n^2)_{n=0}^{\infty}$ be their central journeys, connecting each terminal to the central station.

Let $(\sigma_0, \ldots, \sigma_N)$ be the maximal common prefix of η_{z_1} and η_{z_2}. Let $\eta_{z_i}^{\text{truncated}} = \left(\sigma_N, \sigma_{N+1}^i, \ldots\right)$ be the truncated paths. By the maximality of N, we have that $\eta_{z_1}^{\text{truncated}}$ and $\eta_{z_2}^{\text{truncated}}$ are two journeys with a common starting point, so we can concatenate them to obtain a bi-infinite journey

$$\eta_{z_1,z_2} = \left(\ldots \sigma_{N+2}^2, \sigma_{N+1}^2, \sigma_N, \sigma_{N+1}^1, \sigma_{N+2}^1, \ldots\right)$$

connecting z_1 and z_2.



We conclude the proof by showing that $\mathrm{Length}\left(\eta_{z_1,z_2}\right) \lesssim |z_1 - z_2|$.

As $|z_1 - z_2| \asymp |\theta_1 - \theta_2|$ and $\mathrm{Arg}\left(z_i\right) \propto \theta_i$, it is equivalent to show

$$\mathrm{Length}\left(\eta_{z_1,z_2}\right) \lesssim \left|\mathrm{Arg}\left(z_1\right) - \mathrm{Arg}\left(z_2\right)\right| .$$

By the choice of N,

$$|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(z_2)| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that $\mathrm{Length}\,(\eta_{z_1,z_2}) \lesssim 2^{-N}$. But

$$\mathrm{Length}\,(\eta_{z_1,z_2}) = \mathrm{Length}\left(\eta_{z_1}^{\mathrm{truncated}}\right) + \mathrm{Length}\left(\eta_{z_2}^{\mathrm{truncated}}\right),$$

so it is enough to observe that

$$\mathrm{Length}\left(\eta_{z_i}^{\mathrm{truncated}}\right) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma.

□

References

[1] Hakobyan, Hrant, and Herron, David A.. " Euclidean quasiconvexity.." Annales Academiae Scientiarum Fennicae. Mathematica 33.1 (2008): 205-230.