1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\mathrm{Length}(\gamma) \leq A \cdot |z_1 - z_2| \,.$$

We call such a path γ a *quasiconvexity certificate* for z_1, z_2.

If Ω is the interior of a Jordan curve, then by [3, Corollary F] it is enough to find certificates for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

The *cauliflower* is the filled Julia set of the map $z^2 + \frac{1}{4}$. We show that its complement, Exterior$(\mathcal{J}(z^2 + 1/4))$, is quasiconvex. We then adapt our argument to establish that the exterior of the developed deltoid is quasiconvex.

One motivation to study quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then its complement has a John interior. See [3, Corollary 3.4] for a proof.

Thus this result is a strengthening of [2, Theorem 6.1], which shows that the cauliflower is a John domain directly.

This result also has a function-theoretic interpretation: By [4, Theorem 1.1], it shows that the cauliflower is a BV-extension domain.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of a certificate connecting two given points on the Julia set.

We first build certificates for the exterior unit disk $\mathbb{D}^*$ and then transport them by the Böttcher coordinate ψ of $f_{1/4}$ to the exterior of the cauliflower.

In order to retain control on the certificates after applying ψ, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This is done by only traveling along the boundaries of Carleson boxes in $\mathbb{D}^*$.

The image of a certificate c in $\mathbb{D}^*$ under the conjugacy ψ is invariant under $f_{1/4}$. We use this invariance to show that $\psi(c)$ is indeed a certificate, by employing a parabolic variant of the principle of the conformal elevator: We repeatedly apply $f_{1/4}$ on $\psi(c)$ until either the distance between the endpoints grows to a definite size or one endpoint is attracted sufficiently quickly to the parabolic fixed point $1/2$. The latter case requires a more delicate treatment.

To facilitate the reading, we separate this latter difficulty by first demonstrating the proof in the hyperbolic case of maps f_c where $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$. In this case the usual conformal elevator applies. We then treat the $c = \frac{1}{4}$ case.

2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

Definition 2.1. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{2^{-n}}, 2^{2^{-n-1}}\right], \qquad \arg(z) \in \left(\frac{k}{2^n} 2\pi, \frac{(k+1)}{2^n} 2\pi\right] \right\}$$

a *Carleson box*. Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{2^{-n-1}} < |z| \leq 2^{2^{-n}} \right\}$$

into 2^n equally-spaced sectors.

The *Carleson box decomposition* is the partition of $\mathbb{D}^*$ which consists of f_0-Carleson boxes:

$$\mathbb{D}^* = \{\zeta : |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under f_0, stemming from the relation

$$f_0\left(B_{n+1,k}\right) = B_{n,\lfloor \frac{k}{2} \rfloor}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels by trains. We now define “stations” and “tracks”.

Definition 2.2. A *terminal* is a point $\zeta \in \partial\mathbb{D}^*$ on the unit circle. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : \zeta \mapsto \zeta^2$. We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition 2.3. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are $s_{n,(k\pm1) \mod 2^n}$, the two stations adjacent to $s_{n,k}$ on C_n.
2. Given a peripheral neighbor s' of s, the *peripheral track* $\gamma_{s,s'}$ from s to s' is the short arc of the circle C_n connecting s to s'.
3. The *radial successor* of s is $\mathrm{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n + 1$ on the radial segment $[0, s]$.
4. Denote $s' = \mathrm{RadialSuccessor}(s)$, then the *Express track* $\gamma_{s,s'}$ from s to s' is the radial segment $[s, s']$.

Notice that the tracks preserve the dynamics: applying $\zeta \mapsto \zeta^2$ to a peripheral track between s, s' gives a peripheral track between the parents of s, s' in the tree, and likewise for an express track.

When a passenger purchases a ticket between two stations s_1 and s_2, they must follow a particular itinerary to get from s_1 to s_2. If s_1 is the central station, then this compulsory itinerary is determined by the rule that the passenger always stays as close as possible to its destination in the peripheral distance. This also determines how to travel from the central station to a terminal $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 1 and the next definition.

Definition 2.4. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *itinerary* η_ζ from the central station to ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \ldots$ made of tracks between stations $\sigma_0, \sigma_1, \ldots$, defined inductively as follows:

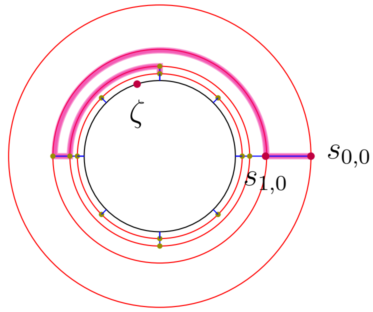

Figure 1: The central journey to a point ζ.

Start at the main station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \ldots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma)| < |\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

We call η_ζ the *central itinerary* of ζ and identify η_ζ with its sequence of stations $(\sigma_0, \ldots)$.

Note that there are no two consecutive peripheral tracks in η_ζ and thus

$$\mathrm{Generation}(\sigma_k) \geq \frac{k}{2}. \tag{2.1}$$

Also note that the itineraries are invariant under f_0, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

for every $\zeta \in \partial\mathbb{D}$.

Lemma 2.5. *Let $\zeta \in \partial\mathbb{D}^*$. Decompose the central itinerary η_ζ into its constituent tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 \ldots.$$

Then we have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim 2^{-k}$$

uniformly in ζ. In particular, the total length of η_ζ is bounded uniformly in ζ.

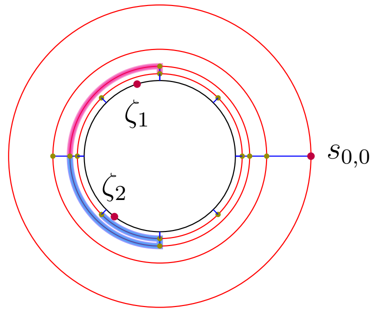

Figure 2: A quasiconvexity certificate between two points ζ_1, ζ_2.

We now define the would-be certificates. We do this by traveling as efficiently as possible, constrained to staying on the central itineraries:

Definition 2.6. Let $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ be two distinct terminals, let $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^\infty$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^\infty$ be the corresponding central itineraries and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. Note that σ is well-defined. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\dots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2. See Figure 2.

Theorem 2.7. *The domain $\mathbb{D}^*$ is quasiconvex with the itineraries η_{ζ_1,ζ_2} as certificates.*

Proof. We keep the notations of Definition 2.6. We decompose the itinerary into two paths, so that

$$\mathrm{Length}(\eta_{\zeta_1,\zeta_2}) = \mathrm{Length}\left(\sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right) + \mathrm{Length}\left(\sigma, \sigma_{i+1}^1, \sigma_{i+2}^1, \dots\right), \tag{2.2}$$

and bound each summand using Lemma 2.5: if Generation$(\sigma) = N$, then

$$\mathrm{Length}(\eta_{\zeta_1,\zeta_2}) \lesssim 2 \sum_{k=N}^{\infty} \frac{1}{2^k} \tag{2.3}$$

$$\lesssim 2^{-N}. \tag{2.4}$$

On the other hand

$$|\zeta_1 - \zeta_2| \asymp |\mathrm{Arg}\,(\zeta_1) - \mathrm{Arg}\,(\zeta_2)|\,, \tag{2.5}$$

and

$$|\mathrm{Arg}\,(\zeta_1) - \mathrm{Arg}\,(\zeta_2)| \geq \frac{2\pi}{2^{N+2}}. \tag{2.6}$$

Combining the previous estimates gives

$$\mathrm{Length}\,(\eta_{\zeta_1,\zeta_2}) \lesssim |\zeta_1 - \zeta_2|$$

as desired. □

3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$ and denote by ψ the Böttcher coordinate of $f : z \mapsto z^2 + c$ at infinity. This means that ψ is the unique conformal map $\mathbb{D}^* \to \mathrm{Exterior}(\mathcal{J})$ which fixes ∞ and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the Julia set $\mathcal{J}$ is a Jordan curve, the map ψ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f)$ by Carathéodory's theorem.

We apply ψ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in Exterior($\mathcal{J}$):

Definition 3.1. 1. The *stations* of f_c are the points $s_{n,k,c} = \psi(s_{n,k})$.

2. The *c-tracks* are the curves of the form $\psi\,(\gamma_{s,s'})$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$.

Express tracks lie on *external* rays of the filled Julia set $\mathcal{K}$. Peripheral tracks lie on the level sets of ψ, or equivalently on the *equipotentials* of $\mathcal{K}$.

3. Let $z_1, z_2 \in \mathcal{J}$ and let $\zeta_i = \psi^{-1}(z_i)$ be the corresponding points on $\partial\mathbb{D}^*$. The *c-itineraries* are $\eta_{z,z'} = \psi(\eta_{\zeta,\zeta'})$.

We omit c from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior($\mathcal{J}$).

Note that $\psi\,((1,\infty)) \subseteq \mathbb{R}$ since ($\mathcal{J}$) is symmetric with respect to the real line. In particular $\psi(s_{0,0}) \in \mathbb{R}$, i.e. the c-central station is real.

4 Hyperbolic Maps

A rational map is *hyperbolic* if every critical point converges to an attracting cycle. Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size. More precisely, we have the following:

Proposition 4.1 (The Principle of the Conformal Elevator)**.** *Let f be a rational hyperbolic map, let $z \in \mathcal{J}$ be a point on the Julia set of f and let $r > 0$. Then there exists some forward iterate $f^{\circ n}$ of f which is injective on the ball $B(z, 2r)$ and such that* $\operatorname{diam} f^{\circ n}(B(z,r))$ *is bounded below uniformly in z and r.*

Corollary 4.2. *Let f be a rational hyperbolic map. There exists ϵ such that for every two points $z, w \in \mathcal{J}(f)$, there is a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

We are now ready to show the analogue of Theorem 2.7:

Theorem 4.3. (i) *Let $z \in \mathcal{J}$ and decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots .$$

Then we have the estimate

$$\operatorname{Length}(\gamma_k) \lesssim \theta^{-k}$$

uniformly in z, for some constant $\theta = \theta(c) > 1$.

(ii) *The domain* $\operatorname{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries η_{z_1,z_2} as certificates.*

Proof. (i) The map f has some iterate $f^{\circ N}$ such that $|f^{\circ N}| > 1$ on the Julia set $\mathcal{J}$, where N is independent of z. By the compactness of $\mathcal{J}$, there is some $\theta > 1$ uniform in z such that $\left|(f^{\circ N})'\right| > \theta$ on some neighborhood $\mathcal{U}$ of $\mathcal{J}$. Every itinerary is eventually contained in $\mathcal{U}$, so for almost all itineraries γ we have

$$\operatorname{Length}(f^{\circ N}(\gamma)) \geq \theta \cdot \operatorname{Length}(\gamma).$$

For every value k modulo N, the peripheral tracks on circles C_n of index $n \equiv k$ modulo N have a total length bounded by a geometric series of rate θ, hence finite. The lengths of the express tracks can be bounded in the same way.

(ii) Since we already know that the lengths of tracks in the itinerary decay exponentially with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case.

We give a second proof, relying on Corollary 4.2. This proof will better prepare us for the parabolic $c = 1/4$ case, in which we don't have expansion of f on its Julia set.

By Corollary 4.2, there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$ then there is nothing to prove since we may just concatenate η_{z_1} and η_{z_2} and absorb this bounded length into the quasiconvexity constant A.

On the other hand, if $|z_1 - z_2| < \epsilon$, then we use Corollary 4.2 to find an iterate $f^{\circ n}$ such that

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon. \tag{4.1}$$

There is a certificate η_{w_1,w_2} between them and we take the certificate η_{z_1,z_2} between the original points to be the component of $f^{\circ -n}\left(\eta_{w_1,w_2}\right)$ that connects the points z_1, z_2. Now, by a distortion estimate

$$\mathrm{Length}\left(\eta_{z_0,z_1}\right) \asymp \frac{\mathrm{Length}\left(\eta_{w_0,w_1}\right)}{\left|\left(f^{\circ n}\right)'(\zeta)\right|}$$

for some point $\zeta \in \mathcal{J}$. The denominator grows with n exponentially at rate θ, while the numerator has a bound of the form

$$\mathrm{Length}\left(\eta_{w_1,w_2}\right) \lesssim |w_1 - w_2| \lesssim \theta^n \left|z_1 - z_2\right|.$$

Altogether

$$\mathrm{Length}\left(\eta_{z_1,z_2}\right) \lesssim \frac{\theta^n \left|z_1 - z_2\right|}{\theta^n} = |z_1 - z_2|$$

so η_{z_1,z_2} is a quasiconvexity certificate.

□

5 The Cauliflower

In this section $c = \frac{1}{4}$, so $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$ etc.

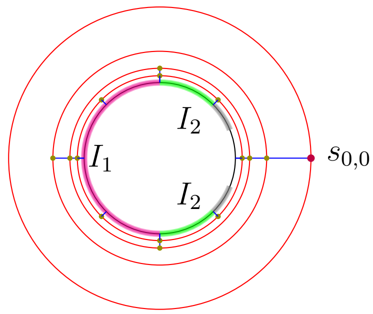

Figure 3: First few parts of the departure decomposition I_m of the circle.

The cauliflower has cusps at the landing points of the dyadic external rays, i.e. the points whose central itinerary has only finitely-many peripheral tracks.

We prove the following:

Theorem 5.1. (i) *Let* $\zeta \in \partial\mathbb{D}^*$ *and* $z = \psi(\zeta)$. *Then* $\text{Length}(\eta_z) < \infty$.

(ii) *The domain* $\text{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries* η_{z_1,z_2} *as certificates.*

To prove Theorem 5.1 we develop some machinery. We decompose the points of the circle according to the first *departure*: the first time that the central itinerary made a turn.

Definition 5.2. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_n \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 3.

This departure decomposition is invariant under f_0 in the sense that $f_0(I_{n+1}) = I_n$, because of the invariance of η_ζ. Thus by applying the Böttcher map ψ we obtain a corresponding departure decomposition of $\mathcal{J}$ that is invariant under f.

5.1 Building the Elevator

The map f has a parabolic fixed point at $\frac{1}{2}$, so the usual conformal elevator does not apply. We now develop a substitute.

Theorem 5.3 (Parabolic Conformal Elevator on $\mathcal{J}$)**.** *Let $d \geq 2$ be an integer. Every pair of points $z_1, z_2 \in \mathcal{J}$ has a forward iterate $f^{\circ N}$ satisfying at least one of the following two conditions:*

(i) $\left|f^{\circ N}(z_1) - f^{\circ N}(z_2)\right| > \epsilon$, *where $\epsilon = \epsilon(d) > 0$ is a constant;*

or

(ii) $f^{\circ N}(z_1) \in \psi(I_n)$ *and* $f^{\circ N}(z_2) \in \psi(I_m)$ *with* $|m - n| \geq d$.

5.2 Climbing the Elevator

The following lemma shows that to check quasiconvexity it is enough to connect points after replacing them with a forward-image of our choice.

Lemma 5.4. *Let $z_1, z_2 \in \mathcal{J}$, and suppose that for some forward iterates*

$$w_1 = f^{\circ N}(z_1), \quad w_2 = f^{\circ N}(z_2)$$

the itinerary η_{w_1,w_2} satisfies

$$\mathrm{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|.$$

Then the original points enjoy a similar estimate,

$$\mathrm{Length}(\eta_{z_1,z_2}) \leq B|z_1 - z_2|$$

where B depends only on A.

Proof. By Koebe's distortion theorem,

$$|w_1 - w_2| \asymp |z_1 - z_2| \cdot |(f^{\circ n})'|$$

and likewise

$$\mathrm{Length}(f^{\circ n}(\eta_{z_1,z_2})) \asymp \mathrm{Length}(\eta_{z_1,z_2}) \cdot |(f^{\circ n})'|$$

but $f^{\circ n}(\eta_{z_1,z_2})$ is longer than η_{w_1,w_2}. $\square$

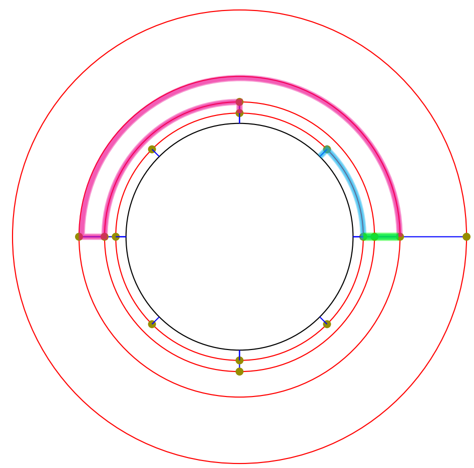

Figure 4: The three parts of an itinerary η. The green path is $\gamma_{m,n}$, the cyan and magenta are γ_m and γ_n.

By Lemma 5.4 and Theorem 5.3, it is enough to show that the itinerary η_{z_1,z_2} is a certificate for points that satisfy

$$z_1 \in \psi(I_n), \quad z_2 \in \psi(I_m), \quad m - n \geq d. \tag{5.1}$$

where d is a large enough integer.

Condition (5.1) gives us some control from below on $|z_1 - z_2|$. Our next step is to control the length of the itinerary $\psi(\eta) = \psi(\eta_{z_1,z_2})$ from above.

We represent η as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, γ_m and γ_n. See Figure 4. Thus we have

$$\text{Length}(\psi(\eta)) = \text{Length}(\psi(\gamma_m)) + \text{Length}(\psi(\gamma_{m,n})) + \text{Length}(\psi(\gamma_n)). \tag{5.2}$$

Notice that $\psi(\gamma_{m,n})$ is a line segment. To estimate its length, denote $\ell_k = \psi(s_{k,0}) - \psi(s_{k+1,0})$. Then

$$\text{Length}(\psi(\gamma_{m,n})) = \ell_m + \ldots + \ell_n.$$

Lemma 5.5. *The lengths ℓ_n satisfy*

$$\frac{\ell_n}{\ell_{n+1}} \asymp 1. \tag{5.3}$$

In particular, for every constant C there is some large enough integer d such that

$$\ell_{m+1} + \ldots + \ell_{n-1} \geq C(\ell_m + \ell_n)$$

whenever $|m - n| \geq d$.

Proof. These segments are images of one another emanating from the cusp, so $\ell_n \asymp \frac{1}{n}$. □

Lemma 5.6.

$$\text{InnerDiameter}(\psi(I_n)) \lesssim \ell_n$$

where the hidden constant is independent of n.

The condition $|m - n| \geq d$ prevents the segment $\gamma_{m,n}$ from being too small in comparison to the other two paths:

Proposition 5.7. *For $d \gg 1$,*

$$\text{Length}(\psi(\gamma_m)) \lesssim \text{Length}(\psi(\gamma_{m,n})). \tag{5.4}$$

Proposition 5.8. *For $d \gg 1$,*

$$\text{Length}(\psi(\gamma_{m,n})) \lesssim |z_1 - z_2|. \tag{5.5}$$

References

[1] Bonk, M., Lyubich, M., and Merenkov, S. Quasisymmetries of Sierpiński carpet Julia sets. *Advances in Mathematics 301* (Oct. 2016), 383–422.

[2] Carleson, L., Jones, P. W., and Yoccoz, J.-C. Julia and John. *Bol. Soc. Bras. Mat 25*, 1 (Mar. 1994), 1–30.

[3] Hakobyan, H., and Herron, D. Euclidean quasiconvexity. *Annales Academiae Scientiarum Fennicae Mathematica 33* (Jan. 2008).

[4] Koskela, P., Miranda, M., and Shanmugalingam, N. Geometric Properties of Planar BV -Extension Domains. In *Around the Research of Vladimir Maz'ya I: Function Spaces*, A. Laptev, Ed., International Mathematical Series. Springer, New York, NY, 2010, pp. 255–272.