1 Introduction

Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial. Its *filled Julia set* consists of the points in the complex plane with bounded orbit under iteration by f_c:

$$\mathcal{K}_c = \{z \in \mathbb{C} : \sup_{n \geq 0} f_c^{\circ n}(z) < \infty\}.$$

Its boundary $\mathcal{J}_c = \partial\mathcal{K}(f_c)$ is known as the *Julia set*, and its complement Exterior$(\mathcal{J}_c) = \mathbb{C} \setminus \mathcal{K}(f_c)$ forms the *attracting basin of infinity*.

The set $\mathcal{K}_c$ is compact, and each of the three sets $\mathcal{J}_c, \mathcal{K}_c$ and Exterior$(\mathcal{J}_c)$ are both forward and backward invariant under the dynamics of f.

The *main cardioid*

$$\heartsuit = \left\{c \in \mathbb{C} : c = \lambda/2 - \lambda^2/4,\ \lambda \in \mathbb{D}\right\}$$

is the set of parameters $c \in \mathbb{C}$ for which f_c has an attracting fixed point. When $c \in \heartsuit$, the Julia set $\mathcal{J}_c$ is a *quasidisk*, the image of a round disk under a quasiconformal map. This intuitively means that $\mathcal{K}_c$ has no "cusps".

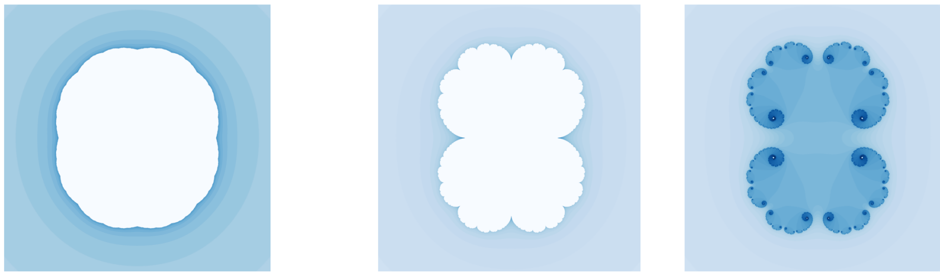
In this work we take $c = 1/4$, which lies on the boundary of $\heartsuit$. The filled Julia set $\mathcal{K}_{1/4}$, also called the *Cauliflower*, is a Jordan domain with an inward-pointing cusp at the point $p = 1/2$. However, according to a theorem of Carleson, Jones and Yoccoz [1, Theorem 6.1], the Cauliflower is a *John domain*, a condition which rules out "outward-pointing cusps". Formally, a domain Ω is John if there exists a "center" point $z_0 \in \Omega$ that can be connected to any other point $z_1 \in \Omega$ by a curve γ which stays away from the boundary:

$$\operatorname{dist}(z, \partial\Omega) \gtrsim |z_1 - z| \tag{1.1}$$

for all $z \in \gamma$. See Figure 1.

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ are connected by a rectifiable path $\gamma_{z_1,z_2} : [0,1] \to \Omega$ which satisfies

$$\operatorname{Length}(\gamma_{z_1,z_2}) \leq A \cdot |z_1 - z_2|\,. \tag{1.2}$$



(a) $c = 0.1$. (b) $c = 1/4$, the Cauliflower. (c) $c = 0.3$.

Figure 1: The Julia set $\mathcal{J}_c$ of f_c for different values of c. When $c > 1/4$, the Julia set is no longer connected.

We refer to such a family of paths γ_{z_1,z_2} as *quasiconvexity certificates* for Ω.

If Ω is a quasiconvex Jordan domain, then its complement has a John interior; see [3, Corollary 3.4] for a proof. In this work, we strengthen the result of [1, Theorem 6.1] by showing:

Theorem 1.1. *The exterior of the Cauliflower is quasiconvex.*

Our result also has a function-theoretic interpretation. For a planar domain $\Omega \subset \mathbb{R}^2$, the *Sobolev space* $W^{1,1}(\Omega)$ is the set of functions $u \in L^1(\Omega)$ for which both weak derivatives $\partial_1 u, \partial_2 u$ exist and are in $L^1(\Omega)$.

We call Ω a *$W^{1,1}$ extension domain* if every $u \in W^{1,1}(\Omega)$ extends to a function in $W^{1,1}(\mathbb{C})$.

In [2, Equation (1.1) and Theorem 1.4], it is shown that a bounded, simply connected domain is a $W^{1,1}$ extension domain if and only if its complement is quasiconvex. Thus our result can be rephrased as follows:

Theorem 1.2. *The Cauliflower is a $W^{1,1}$ extension domain.*

1.1 Sketch of the argument

To show that a Jordan domain Ω is quasiconvex, it is enough to find certificates for points z_1, z_2 that lie on the boundary curve $\partial\Omega$. For a proof, see [3, Corollary F].

We show quasiconvexity by an explicit construction of certificates connecting any given pair of points on the Julia set. We first build the certificates in the exterior unit disk $\mathbb{D}^*$, then we transport them to the exterior of the Cauliflower by the Riemann map $\psi : \mathbb{D}^* \to \mathrm{Exterior}(\mathcal{J}_{1/4})$, which conjugates f_0 with $f_{1/4}$.

To retain control of the certificates after applying ψ, we build the certificates of $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This makes the image of a certificate η in $\mathbb{D}^*$ under the conjugacy ψ invariant under $f_{1/4}$. We use this invariance to show that $\psi(\eta)$ are indeed certificates for $\mathrm{Exterior}(\mathcal{J}_{/4})$, by employing a parabolic variant of the so-called principle of the conformal elevator.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps $f_c(z) = z^2 + c$ where $c \in \heartsuit$), in which the usual conformal elevator applies, and we subsequently treat the parabolic case of $c = \frac{1}{4}$.

2 Complex-analytic preliminaries

2.1 The Distortion Principle

We record here for convenience a form of Koebe's distortion principle that will be used repeatedly.

Definition 2.1. Every topological annulus $A \subset \hat{\mathbb{C}}$ is biholomorphic to a unique round annulus of the form $\{1 < |z| < R\}$. The (conformal) *modulus* of A is the value $\mathrm{Mod}(A) = \frac{1}{2\pi} \log R$.

Theorem 2.2 (Koebe's Distortion Principle, [4, Theorem 2.9])**.** *Let $D \subset U$ be topological disks with $\mathrm{Mod}(U \setminus D) \geq m > 0$ and let f be a map univalent in U, then we have the bound*

$$\frac{|f(y) - f(z)|}{|y - z|} \asymp_m |f'(x)| \tag{2.1}$$

for all $x, y, z \in D$.

2.2 The hyperbolic metric

Even though quasiconvexity is defined using the Euclidean metric, the arguments will involve the hyperbolic metric, which is better-behaved in our setting.

Theorem 2.3. *The hyperbolic metric*

$$\mathrm{ds} = \frac{|\mathrm{dz}|}{1 - |z|^2}$$

is the unique Riemannian metric on the unit disk $\mathbb{D}$, up to multiplication by a positive constant, which is invariant under conformal automorphisms.

Theorem 2.4. *Let U be a planar domain whose complement $\mathbb{C} \setminus U$ has at least two points. The universal covering $\tilde{U}$ of U is biholomorphic to $\mathbb{D}$; this defines the hyperbolic metric on U as the unique Riemannian metric for which the projection $\tilde{U} \to U$ is a local isometry.*

3 The exterior disk

We connect any two boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ by a path in $\mathbb{D}^*$ in a manner that respects the map $f_0 : \zeta \mapsto \zeta^2$. We describe these paths using the metaphor of a passenger who travels by train:

Definition 3.1. *Stations* are the points in $\mathbb{D}^*$ of the form

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

These are the iterated preimages of the *central station* $s_{0,0} = 2$ under the map f_0. We refer to n as the *generation* of the station $s_{n,k}$. The 2^n stations of generation n are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks", which we use to travel between stations.

Definition 3.2. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are the two stations $s_{n,(k\pm1)(\mathrm{mod}\ 2^n)}$ adjacent to $s_{n,k}$ on C_n.

2. The *peripheral track* $\gamma_{s,s'}$ from s to a peripheral neighbor s' is the shorter arc of the circle C_n connecting s to s'.

3. The *radial successor* of s is $\text{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.

4. The *express track* $\gamma_{s,s'}$ from s to its radial successor s' is the radial segment $[s, s']$.

Notice that the tracks respect the dynamics: applying f_0 to a track gives a track of the previous generation.

When a passenger travels between two stations s_1 and s_2, they must follow a particular itinerary from s_1 to s_2. If s_1 is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination s_2 in the angular distance. This also determines how to travel from the central station to a boundary point $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 2 and the next definition.

Definition 3.3. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \ldots$ from the central station to ζ, made of tracks between the stations $\sigma_0, \sigma_1, \ldots$. It is defined inductively as follows:

Start at the central station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \ldots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\text{Arg}\,(\zeta) - \text{Arg}\,(\sigma)| < |\text{Arg}\,(\zeta) - \text{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \text{RadialSuccessor}(\sigma)$.

We identify the central itinerary η_ζ with its sequence of stations $(\sigma_0, \ldots)$. We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in η_ζ, and in particular
$$\text{Generation}(\sigma_k) \geq \frac{k}{2}; \tag{3.1}$$
- Central itineraries are essentially equivariant under f_0, in the sense that
$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$
for every $\zeta \in \partial\mathbb{D}^*$.

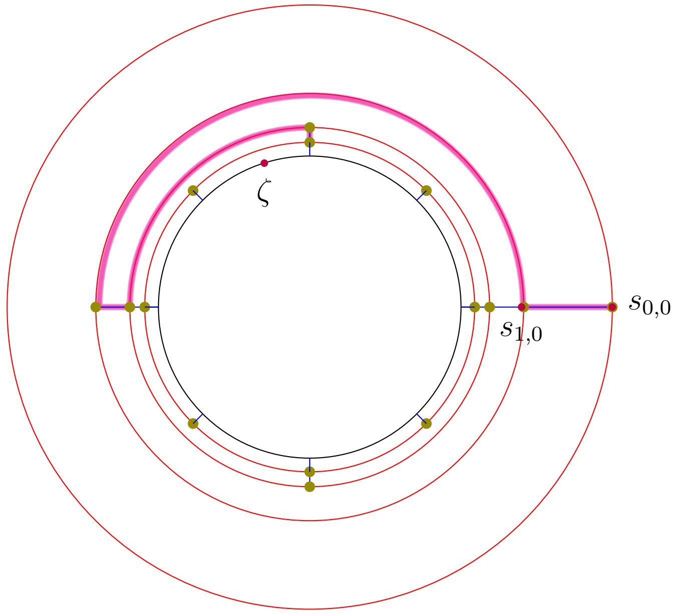


Figure 2: The central itinerary to a point ζ.

Definition 3.4. Given two distinct boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^{\infty}$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^{\infty}$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\dots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2, see Figure 3. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{3.2}$$

for every pair of boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ with $|\zeta_1 - \zeta_2| < \sqrt{2}$.

4 Transporting the Rails

Let $c \in \heartsuit$. For these values of c, the Julia set of $f_c : z \mapsto z^2 + c$ is a Jordan curve, and f_c has a Böttcher coordinate ψ at infinity; namely, ψ is the unique conformal

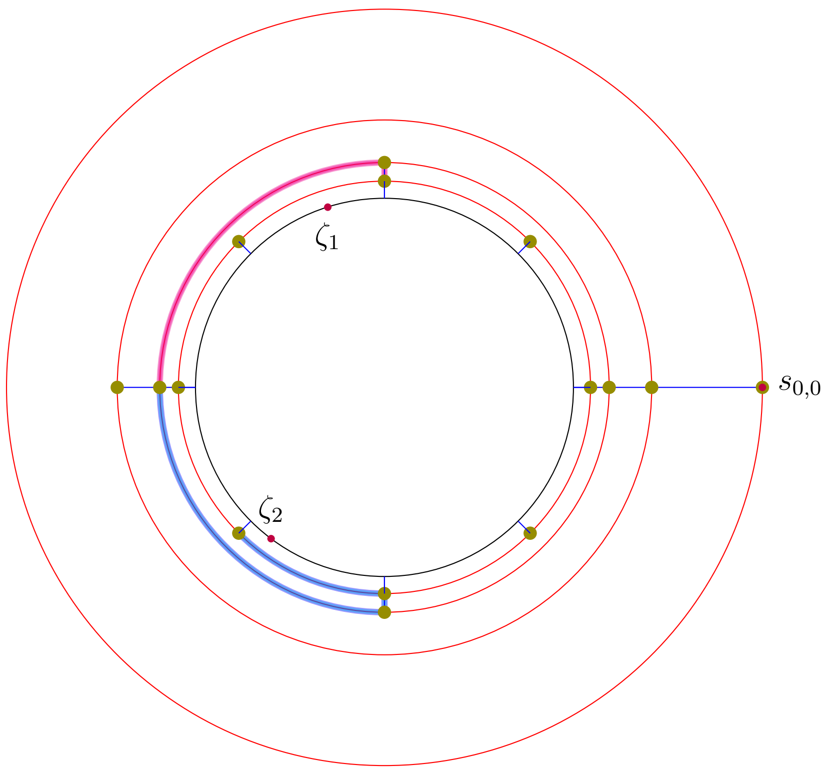


Figure 3: A quasiconvexity certificate between two points ζ_1, ζ_2 in $\mathbb{D}^*$. Only the first two steps are shown.

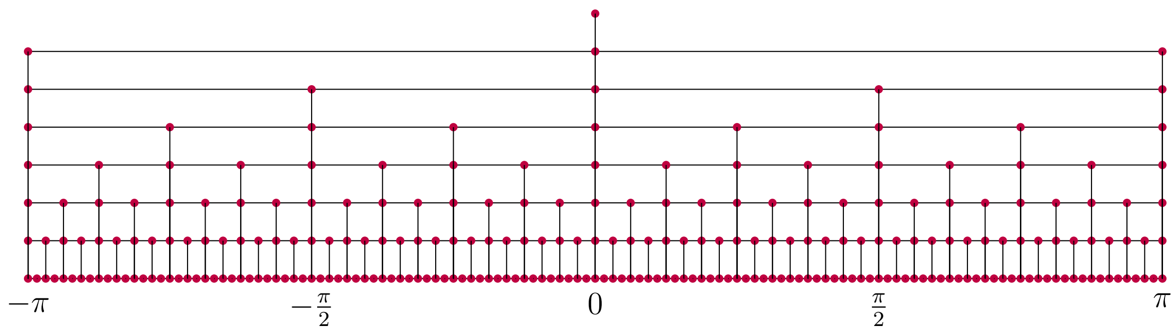


Figure 4: A convenient representation of the dyadic grid in the Böttcher coordinates. The horizontal axis is the external angle $\mathrm{Arg}(\psi^{-1}(z))$, and the vertical axis is the equipotential $|\psi^{-1}(z)|$, plotted on a log scale. The rightmost edge is glued to the leftmost edge. Stations are marked in red, and the segments connecting adjacent stations are tracks. An express track is a vertical segment, while a peripheral track is a horizontal segment.

map $\mathbb{D}^* \to \mathrm{Exterior}(\mathcal{J}_c)$ which fixes ∞ and satisfies the conjugacy relation

$$f \circ \psi = \psi \circ f_0.$$

The Böttcher coordinate ψ extends to a homeomorphism between the unit circle $\partial\mathbb{D}$ and $\mathcal{J}_c$ by Carathéodory's theorem.See [5, Theorem 9.5] for a proof of existence, relying on the explicit construction

$$\psi(z) = \lim_{n\to\infty} (f_0)^{\circ(-n)} \circ f^{\circ n} = \lim_{n\to\infty} \left(f^{\circ n}\right)^{1/2^n}. \tag{4.1}$$

We apply ψ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in $\mathrm{Exterior}(\mathcal{J}_c)$:

Definition 4.1.

1. The *stations* of f_c are the points $\psi(s_{n,k})$.

2. The *tracks* of f_c are the curves of the form $\psi\left(\gamma_{s,s'}\right)$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on the external rays of the filled Julia set $\mathcal{K}_c$, while peripheral tracks lie on the equipotentials of $\mathcal{K}_c$.

3. The *itinerary* between a pair of points (z_1, z_2) on $\mathcal{J}_c$ is $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$, where $\zeta_i = \psi^{-1}(z_i)$ are the corresponding points on $\partial\mathbb{D}^*$.

We omit c and ψ from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior$(\mathcal{J})$.

Note that $\psi((1,\infty)) \subseteq \mathbb{R}$ since $\mathcal{J}$ is symmetric with respect to the real line, and in particular the central station $\psi(s_{0,0})$ lies on the real axis.

5 Hyperbolic Maps

In this section we prove quasiconvexity for parameters c in the main cardioid $\heartsuit$.

Definition 5.1. The *post-critical set* of f_c is the closure of the forward orbits of the critical points:

$$\mathcal{P} = \overline{\{f^{\circ n}(0) : n \geq 1\} \cup \{\infty\}}.$$

For every $c \neq 0$, the post-critical set $\mathcal{P}$ of f_c contains at least 3 points and consequently its complement $\hat{\mathbb{C}} \setminus \mathcal{P}$ is a hyperbolic domain by Theorem 2.4.

We call f_c *hyperbolic* if its post-critical set $\mathcal{P}$ is disjoint from its Julia set $\mathcal{J}$. This is equivalent to f_c being expanding on $\mathcal{J}$:

Theorem 5.2. *Let $f_c : z \mapsto z^2 + c$ be a map with $c \neq 0$. Let $\|\cdot\|_{\text{hyp}}$ be the norm induced by the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$, then we have*

$$\|Df_z(v)\|_{\text{hyp}} > \|v\|_{\text{hyp}} \tag{5.1}$$

for every $z \in f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$.

For two proofs of this theorem, see [5, Theorem 19.1], which also proves the converse.

Corollary 5.3. *If $c \in \heartsuit$ and $c \neq 0$, then we have*

$$\|Df_z(v)\|_{\text{hyp}} \geq \kappa \|v\|_{\text{hyp}} \tag{5.2}$$

for all $z \in \mathcal{J}$, for some constant $\kappa > 1$.

Proof. By hyperbolicity $\mathcal{J} \subseteq f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$, and the claim follows by compactness. $\square$

Corollary 5.4. *Let f_c be a hyperbolic quadratic map. There exists $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}$ has a forward iterate $f^{\circ n}$ for which*

$$|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon.$$

Proof. There is an iterate $g = f^{\circ m}$ of f for which there is a uniform bound $|g'| > \kappa$ on $\mathcal{J}$, for some constant $\kappa > 1$. By compactness, there exists $\epsilon > 0$ such that whenever $|z - w| < \epsilon$ on $\mathcal{J}$, we have $|g(z) - g(w)| \geq \kappa |z - w|$. The claim follows by iterating g. $\square$

Definition 5.5. A point $z \in \mathcal{J}$ is *rectifiably accessible* from $\mathrm{Exterior}(\mathcal{J})$ if there is a rectifiable curve $\gamma : [0, 1) \to \mathrm{Exterior}(\mathcal{J})$ such that $\gamma(t) \to z$ as $t \to 1$.

We are now ready to show quasiconvexity in the hyperbolic case:

Theorem 5.6. *Let $f : z \mapsto z^2 + c$ be a quadratic map with $c \in \heartsuit$.*

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots.$$

We have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim \theta^k,$$

uniformly in z, for some constant $\theta = \theta(c) < 1$. In particular, any point on $\mathcal{J}$ is rectifiably accessible.

(ii) *The domain* $\mathrm{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries η_{z_1,z_2} as certificates.*

Proof. (i) For $c = 0$, this is a direct computation. Suppose $c \neq 0$, and let $\mathcal{P}$ be the post-critical set of f.

Any branch of $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus \mathcal{P}$ is a strict hyperbolic contraction by Theorem 5.2.

Let $B(0, R) \subset \mathbb{C}$ be a ball large enough that it contains every central itinerary. By hyperbolicity, $\hat{\mathbb{C}} \setminus \mathcal{P}$ contains $\overline{\mathrm{Exterior}(\mathcal{J})}$. Thus $\mathrm{Exterior}(\mathcal{J}) \cap B(0, R)$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, and there is a constant $\theta < 1$ such that $\|(f^{-1})'\|_{\text{hyp}} < \theta$ on Exterior$(\mathcal{J}) \cap B(0, R)$. Therefore,

$$\begin{aligned}
\text{HypLength}(\gamma_k) &\leq \theta \cdot \text{HypLength}(f(\gamma_k)) \\
&\leq \dots \\
&\leq \theta^k \cdot \text{HypLength}(f^{\circ k}(\gamma_k)), \\
&\lesssim \theta^k,
\end{aligned}$$

where the last inequality holds since $f^{\circ k}(\gamma_k)$ lies on the real axis in case γ_k is an express track, or on the equipotetial $\psi(\{|z| = \sqrt{2}\})$ otherwise.

As the hyperbolic metric is equivalent to the Euclidean metric on compact subsets of $\hat{\mathbb{C}} \setminus \mathcal{P}$, we conclude that Length$(\gamma_k) \lesssim \theta^k$ as well.

Thus any point on $\mathcal{J}$ can be reached from the central station $s_{0,0}$ by a curve of bounded length.

(ii) By Corollary 5.4, there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate of f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of η_{z_1,z_2} is bounded from above uniformly by part (i). On the other hand, if $|z_1 - z_2| < \epsilon$, then there is an iterate $f^{\circ n}$ for which

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon \tag{5.3}$$

and we have a uniform bound on

$$\frac{\text{Length}\left(\eta_{w_1,w_2}\right)}{|w_1 - w_2|}$$

as before. Thus we are left with showing that

$$\frac{\text{Length}\left(\eta_{z_1,z_2}\right)}{|z_1 - z_2|} \asymp \frac{\text{Length}\left(\eta_{w_1,w_2}\right)}{|w_1 - w_2|}, \tag{5.4}$$

which we rewrite as

$$\frac{|z_1 - z_2|}{|w_1 - w_2|} \asymp \frac{\text{Length}\left(\eta_{z_1,z_2}\right)}{\text{Length}\left(\eta_{w_1,w_2}\right)}. \tag{5.5}$$

We deduce Equation (5.5) from a distortion argument.

Indeed, denote $g = f^{\circ n}$. We may find a topological ball B containing the points w_1, w_2 and the itinerary η_{w_1,w_2} but with a definite modulus inside $\hat{\mathbb{C}} \setminus \mathcal{P}(g)$.

We apply Theorem 2.2 on a branch of g^{-1} in B sending (w_1, w_2) to (z_1, z_2). It follows that

$$\frac{|z_1 - z_2|}{|w_1 - w_2|} \asymp |(g^{-1})'(x)| \asymp \frac{\text{Length}\,(\eta_{z_1,z_2})}{\text{Length}\,(g(\eta_{z_1,z_2}))}. \tag{5.6}$$

for any point $x \in B$, as needed. $\square$

6 The Cauliflower

In this section, $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior$(\mathcal{J})$, Theorem 6.11. This is more complicated than the hyperbolic case, because the post-critical set $\mathcal{P}$ of f accumulates at the parabolic fixed point $p = \frac{1}{2}$. One no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce the quasiconvexity of the itinerary η_{z_1,z_2} from the quasiconvexity of η_{w_1,w_2} using Koebe's distortion theorem. As a substitute, we present an analogue of the principle of the conformal elevator in this parabolic setting.

6.1 Itineraries have finite length

We first show that each itinerary η_{z_1,z_2} has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary that lies on the real axis, $\eta_{1/2}$, have only a quadratic rate of length decay. To fix this, we introduce:

Definition 6.1. The *reduced decomposition* of an itinerary η is the unique decomposition $\eta = \gamma_1 + \delta_1 + \ldots$ where each γ_i is a concatenation of express tracks and is followed by a single peripheral track δ_i.

Proposition 6.2. *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \ldots$ be the reduced decomposition of its itinerary. Then* Length$(\gamma_k) \lesssim \theta^k$ *and* Length$(\delta_k) \lesssim \theta^k$ *for some $\theta < 1$. In particular,* Length$(\eta_z) < \infty$ *and all points $z \in \mathcal{J}$ are rectifiably accessible.*

For the proof, let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-central station and the leftmost one. See

Figure 4. This domain is constructed so that it contains all itineraries that start at the station $s_{1,1} = \psi(-1/2)$, the preimage of the central station under f. Its crucial property is:

Lemma 6.3. *Let $\gamma = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of an itinerary γ. Then for every $k > 1$, there exist $k-1$ iterates $n_1 < \dots < n_{k-1}$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

Proof. Every station $s \notin (0,\infty)$ has a first iterate $f^{\circ n_s}(s)$ lying on the negative real axis $(-\infty, 0)$. For any $i \in \{2,\dots,k-1\}$, let s_i be the first station of γ_i and take $n_i := n_{s_i}$. By the definition of $\mathcal{U}_{-1}$, the itinerary $f^{\circ n_i}(\gamma)$ is contained in $\mathcal{U}_{-1}$ from the station $f^{\circ n_i}(s_i)$ onwards, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. $\square$

Proof (Proposition 6.2). There is a uniform bound $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ with respect to the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$, for both branches $f^{-1} : \mathcal{U}_{-1} \to \mathcal{U}_{\pm i}$. This follows from Theorem 5.2, in the slightly more general formulation of [4, Theorem 3.5], since $\mathcal{U}_{-1}$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$.

In the notation of Lemma 6.3, we then have

$$\begin{aligned} \text{HypLength}(\gamma_k) &\le \text{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\ &\le \theta \cdot \text{HypLength}(f^{\circ n_1}(\gamma_k)) \\ &\le \dots \\ &\le \theta^k \cdot \text{HypLength}(f^{\circ n_k}(\gamma_k)) \\ &\lesssim \theta^k. \end{aligned} \tag{6.1}$$

As in the hyperbolic case, we infer that $\text{Length}(\gamma_k) \lesssim \theta^k$ by the equivalence on $B(0,R) \setminus \mathcal{P}$ of the Euclidean metric and the hyperbolic metric. $\square$

6.2 Some Estimates and Notation

To estimate the length of express tracks, we introduce the notations $s_n := s_{n,0}$ and

$$\ell_n := \text{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{6.2}$$

Lemma 6.4. *The lengths ℓ_n satisfy:*

(i)
$$\frac{|p - s_n|}{\ell_n} \to \infty, \tag{6.3}$$

(ii)
$$\frac{\ell_n}{\ell_{n+1}} \to 1. \tag{6.4}$$

In particular, for any $C > 0$, there is a sufficiently large integer d such that

$$\ell_m + \ldots + \ell_n \geq C(\ell_m + \ell_n)$$

whenever $|m - n| \geq d$.

Proof. Using the affine conjugacy of the map f to the map $g : z \mapsto z^2 + z$, which sends the parabolic fixed point $\frac{1}{2}$ of f to 0, one can show that

$$\ell_n \asymp \frac{1}{n^2} \qquad \text{and} \qquad |p - s_n| \asymp \frac{1}{n}.$$

After a little arithmetic, we get (6.3) and (6.4). □

Definition 6.5. The *relative distance* of a curve γ to the post-critical set $\mathcal{P}$ is

$$\Delta(\gamma, \mathcal{P}) = \frac{\operatorname{dist}(\gamma, \mathcal{P})}{\min(\operatorname{diam}(\gamma), \operatorname{diam}(\mathcal{P}))}.$$

We say that the curve γ is *η-relatively separated* from the post-critical set if $\Delta(\gamma, \mathcal{P}) \geq \eta$.

If an itinerary γ is relatively separated from $\mathcal{P}$, then the preimages of γ under f have bounded distortion. In particular, if γ is a quasiconvexity certificate, then Koebe's distortion theorem implies that $f^{-1}(\gamma)$ is also a certificate with a comparable constant.

Lemma 6.6. *There exists a constant $k > 0$ such that for any pair of points $z_1, z_2 \in \mathcal{J}$, we have $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$.*

Proof. We have
$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} f \right| \leq k|z_1 - z_2| \tag{6.5}$$
for $k = \max_{z \in B} |f'(z)|$, where B is any ball containing $\mathcal{J}$. □

6.3 Dynamics near the parabolic fixed point

The purpose of the following definition is to organize points on the Julia set $\mathcal{J}$ according to their distance from the main cusp $z = 1/2$ in an f-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure*: the first time that the central itinerary makes a turn.

Definition 6.7. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 6.

This decomposition is invariant under f_0 in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of η_ζ. Applying the Böttcher map ψ, we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under f.

We now use this decomposition to analyze the case where the points w_1, w_2 lie in "well-separated cusps". Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \tag{6.6}$$

where d is a sufficiently large integer, to be chosen later. This gives some control from below on $|w_1 - w_2|$. We represent the itinerary $\eta = \eta_{w_1,w_2}$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, γ_m and γ_n. See Figure 7 for the picture in the exterior unit disk. Thus we have

$$\mathrm{Length}(\eta) = \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_{m,n}) + \mathrm{Length}(\gamma_n). \tag{6.7}$$

The condition $m - n \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to γ_m and γ_n:

Proposition 6.8. *There exists a sufficiently large integer d so that*

$$\mathrm{Length}(\gamma_{m,n}) \asymp |w_1 - w_2|, \tag{6.8}$$

whenever $m - n \geq d$.

We henceforth fix a value of d as in the proposition.

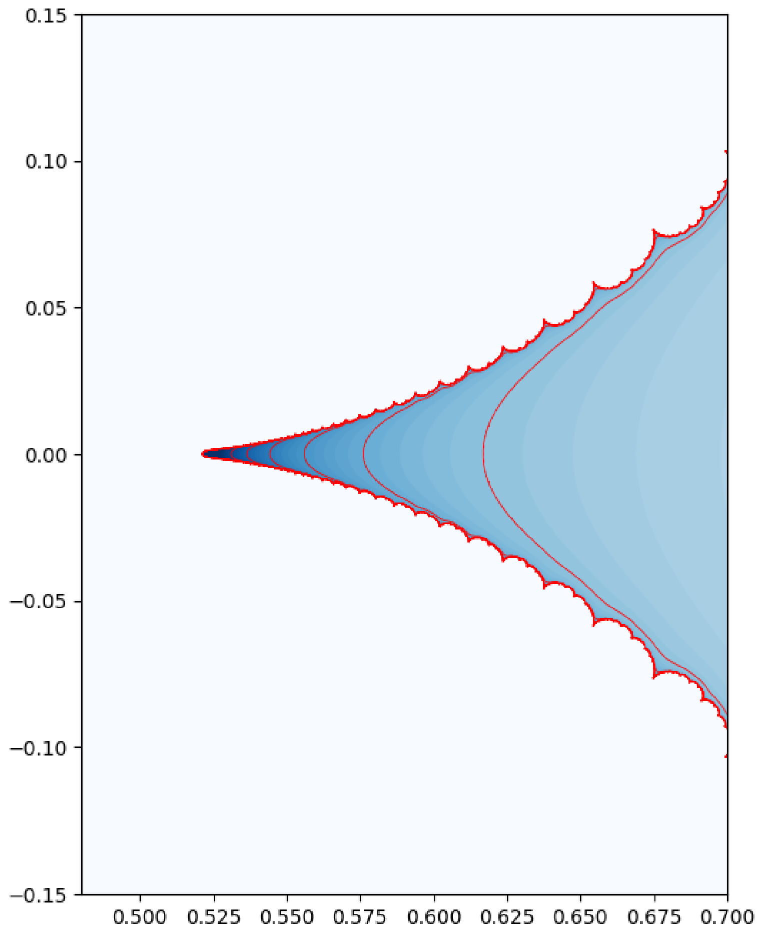


Figure 5: The Cauliflower near the parabolic point $p = 1/2$.

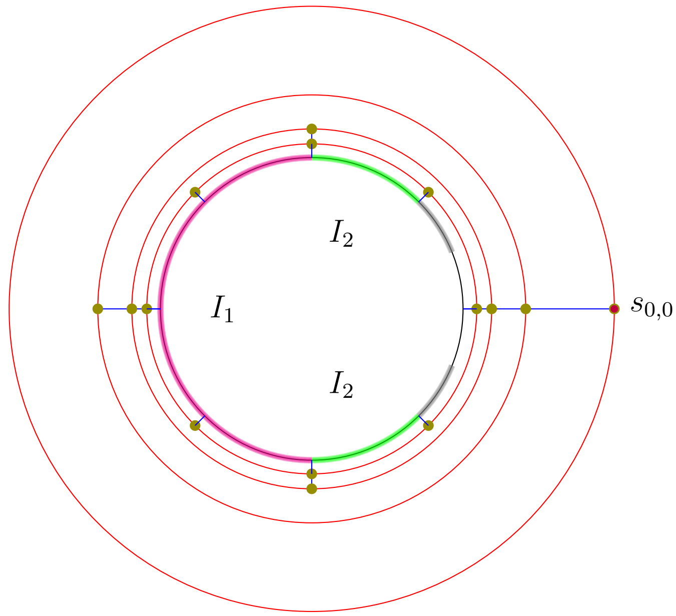


Figure 6: First few parts of the departure decomposition I_m of the circle.

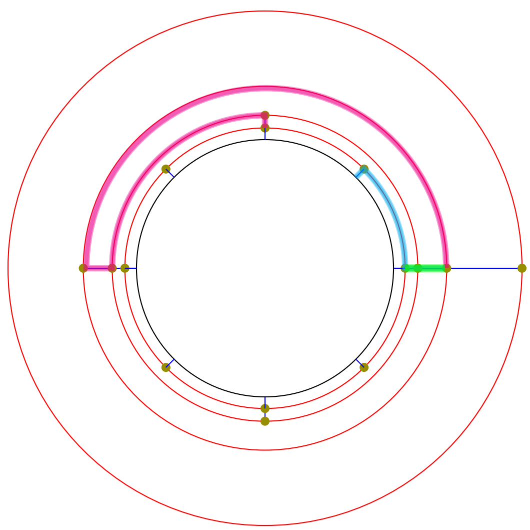


Figure 7: The three parts of an itinerary η. The green path is $\gamma_{m,n}$, the cyan and magenta are γ_m and γ_n.

Proof. We first make two elementary observations. Koebe's distortion theorem applied to the iterates of f^{-1} shows that

$$\mathrm{Length}(\gamma_m) \leq C\ell_m, \tag{6.9}$$

for some constant $C \geq 0$. Notice that (6.9) holds for $m = 1$ by Proposition 6.2, which gives a uniform bound on the length of an itinerary.

Meanwhile, by Lemma 6.4, there exists an integer d such that

$$C(\ell_m + \ell_n) \leq \frac{\mathrm{Length}(\gamma_{m,n})}{2} \tag{6.10}$$

whenever $m - n \geq d$.

By the triangle inequality, we have

$$\begin{aligned} \big|\mathrm{Length}(\gamma_{m,n}) - |w_1 - w_2|\big| &\leq \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_n) \\ &\leq \frac{\mathrm{Length}(\gamma_{m,n})}{2}, \end{aligned}$$

which clearly implies (6.8). □

6.4 Quasiconvexity: three special cases

We now show that the itineraries η_{w_1,w_2} are certificates in three special cases. To state them, we introduce some notation.

6.4.1 Notation

For each n, we denote by α_n the union of the two outermost tracks emanating from the station $s_{n,0}$. Notice that the curves α_n are pairwise disjoint since this is true for their pullbacks to the exterior unit disk.

We define the constants C_1, C_2, ϵ as follows. We first choose $C_1 \geq 2$, then we let $C_2 = C_1 + d + 2$ and choose $\epsilon > 0$ small enough so that we have

$$\mathrm{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k\epsilon. \tag{6.11}$$

The constant C_2 was chosen so that for any pair (m, n) of integers, we have at least one of the following three cases: either m, n are both greater than C_1, or both are smaller than C_2, or $|m - n| > d$.

6.4.2 Three Special Cases

In this section we treat the following special cases:

1. $|w_1 - w_2| \geq \epsilon, \quad |m-n| < d, \quad m, n < C_2, \quad m, n \geq 2;$

2. $|w_1 - w_2| \geq \epsilon, \quad |m-n| < d, \quad m, n > C_1;$

3. $|w_1 - w_2| \leq k\epsilon, \quad |m-n| \geq d.$

Notice that Case 2 overlaps with Case 1. We denote the domain enclosed by α_m, α_n and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and denote the domain enclosed by $\mathcal{J}$ and α_n by $\mathcal{K}_n$.

Lemma 6.9. *Let $w_1 \in I_m$ and $w_2 \in I_n$, for $n \geq m \geq 2$. Then the itinerary η_{w_1,w_2} is contained in the domain $\mathcal{K}_{m,n+1}$.*

Lemma 6.10. *Let $w_1, w_2 \in \mathcal{J}$. In each of the three special cases, the itinerary γ_{w_1,w_2} is a quasiconvexity certificate. In Cases 1 and 2, γ_{w_1,w_2} is relatively separated.*

Proof. Case 1. In this case, the itinerary is contained in the domain $\mathcal{K}_{2,C_2+1}$. Since $\mathrm{dist}(\mathcal{K}_{2,C_2+1}, \mathcal{P}) > 0$, γ_{w_1,w_2} is η-relatively separated for some $\eta > 0$.

Case 2. Assuming without loss of generality that $n \geq m$, the itinerary is contained in $\mathcal{K}_{m,n+1}$. By Koebe's distortion theorem, γ_{w_1,w_2} is also relatively separated.

Case 3 is the content of Proposition 6.8. □

6.5 Quasiconvexity: general case

We apply a stopping time argument to promote the quasiconvexity of η_{w_1,w_2} to the quasiconvexity of η_{z_1,z_2}, thereby proving the following theorem:

Theorem 6.11. *The domain* Exterior($\mathcal{J}$) *is quasiconvex, with the itineraries η_{z_1,z_2} as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let (z_1, z_2) be a pair of points in $\mathcal{J}$. Repeatedly apply f to (z_1, z_2) until either of the three special cases occurs. Denote by $w_i = f^{\circ N}(z_i)$ the resulting points. We have already proved that the itinerary η_{w_1,w_2} satisfies

$$\mathrm{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|,$$

for some $A > 0$. We deduce that the original pair of points (z_1, z_2) enjoys a similar estimate,

$$\text{Length}(\eta_{z_1,z_2}) \leq C|z_1 - z_2|,$$

where C depends only on A.

In Cases 1 and 2, we are done by Lemma 6.10. In Case 3, the itinerary η_{w_1,w_2} is contained in $\mathcal{K}_2$. Let $\mathcal{K}_{-2}$ be the preimage of $\mathcal{K}_2$ under f that contains the negative preimage $f^{-1}(p) = -\frac{1}{2}$ of the cusp p. As the domain $\mathcal{K}_{-2}$ is relatively separated from $\mathcal{P}$ and contains the curve $f^{\circ(N-1)}(\eta_{z_1,z_2}) = \eta_{f^{-1}(w_1),f^{-1}(w_2)}$, we may use Koebe's distortion theorem to conclude that

$$\frac{\text{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{f^{-1}(w_1),f^{-1}(w_2)})}{|f^{-1}(w_1) - f^{-1}(w_2)|} \asymp \frac{\text{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|} \tag{6.12}$$

as desired. □

References

[1] Carleson, L., Jones, P. W., and Yoccoz, J.-C. Julia and John. *Bol. Soc. Bras. Mat 25*, 1 (Mar. 1994), 1–30.

[2] García-Bravo, M., and Rajala, T. Strong BV-extension and $W^{1,1}$-extension domains.

[3] Hakobyan, H., and Herron, D. Euclidean quasiconvexity. *Annales Academiae Scientiarum Fennicae Mathematica 33* (Jan. 2008).

[4] McMullen, C. T. *Complex Dynamics and Renormalization (AM-135)*. Princeton University Press, 1994.

[5] Milnor, J. *Dynamics in One Complex Variable. (AM-160): Third Edition. (AM-160)*. Princeton University Press, 2006.

Nomenclature

α_n the union of the two outermost tracks emanating from the station $s_{n,0}$.

$\Delta(\gamma, \mathcal{P})$ The relative distance to the post-critical set.

ℓ_n Length$([s_n, s_{n+1}]) = s_{n,0} - s_{n+1,0}$.

η_{z_1,z_2} The itinerary connecting two points. When z_1 and z_2 are stations, this is the same as γ_{z_1,z_2}.

γ_{z_1,z_2} The track connecting z_1 and z_2. It can be either angular ("peripheral") or radial ("express").

$\mathcal{J}_c$ The Julia set of f_c.

Exterior$(\mathcal{J})$ An Alternative notation for $A_\infty(f_c)$.

ψ The Bottcher coordinate $\mathbb{D}^* \to A_\infty(f_{1/4})$ conjugating f_0 and $f_{1/4}$.

$A_\infty(f_c)$ The exterior of the Julia set of f_c. The complement of K_c.

f_c The map $z \mapsto z^2 + c$.

I_n The n-th departure set.

K_c The filled Julia set of f_c.

$s_{n,k}$ A station in $\mathbb{D}^*$ or its image under ψ.