

# 1 Introduction

A domain  $\Omega \subseteq \mathbb{C}$  is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant  $A \geq 1$  such that every two points  $z_1, z_2 \in \Omega$  have a rectifiable path  $\gamma : [0, 1] \rightarrow \Omega$  connecting them which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2|.$$

We call such a path  $\gamma$  a *quasiconvexity certificate* for  $z_1, z_2$ .

If  $\Omega$  is the interior of a Jordan curve, then by [HH08, Corollary F] it is enough to find certificates for points  $z_1, z_2$  that are on the boundary curve  $\partial\Omega$ .

The *cauliflower* is the filled Julia set of the map  $z^2 + \frac{1}{4}$ . We show that its complement,  $\text{Exterior}(\mathcal{J}(z^2 + 1/4))$ , is quasiconvex. We then adapt our argument to establish that the exterior of the developed deltoid is quasiconvex.

One motivation to study quasiconvexity stems from its connection with the John property: If  $\Omega$  is a quasiconvex Jordan domain, then its complement has a John interior. See [HH08, Corollary 3.4] for a proof.

Thus this result is a strengthening of [CJY94, Theorem 6.1], which shows that the cauliflower is a John domain directly.

This result also has a function-theoretic interpretation: By [KMS09, Theorem 1.1], it shows that the cauliflower is a BV-extension domain.

## 1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of a certificate connecting two given points on the Julia set.

We first build such certificates for the exterior unit disk  $\mathbb{D}^*$ , and then transport them by the Böttcher coordinate  $\psi$  of  $f_{1/4}$  to the exterior of the cauliflower.

In order to retain control on the certificates after applying  $\psi$ , we build the certificates on  $\mathbb{D}^*$  in a manner invariant under the map  $f_0 : z \mapsto z^2$ . This is done by only traveling along the boundaries of a suitable Carleson box decomposition of  $\mathbb{D}^*$ .

The image of a certificate  $c$  of  $\mathbb{D}^*$  under the conjugacy  $\psi$  is invariant under  $f_{1/4}$ . We use this invariance to show that  $\psi(c)$  is indeed a certificate, by a parabolic variant

of the principle of the conformal elevator: We repeatedly apply  $f_{1/4}$  on  $\psi(c)$  until either the distance between the endpoints grows to a definite size or one endpoint is attracted sufficiently fast to the parabolic fixed point  $1/2$ . The latter case requires a more delicate treatment.

To facilitate the reading, we separate this latter difficulty by first demonstrating the proof in the hyperbolic case of maps  $f_c$  where  $c \in [-\frac{3}{4}, \frac{1}{4}]$ . In this case the usual conformal elevator applies. We then embark on the  $c = \frac{1}{4}$  case.

## 2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

**Definition.** Let  $n \in \mathbb{N}_0$  and  $k \in \{0, \dots, 2^n - 1\}$ . We call the set

$$B_{n,k} = \left\{ z : |z| \in \left( 2^{2^{-n}}, 2^{2^{-n-1}} \right], \quad \arg(z) \in \left( \frac{k}{2^n} 2\pi, \frac{(k+1)}{2^n} 2\pi \right] \right\}$$

a *Carleson box*.

Observe that for a fixed  $n$ , the union  $\bigsqcup_{k=0}^{2^n-1} B_{n,k}$  is a partition of the annulus

$$\left\{ 2^{2^{-n-1}} < |z| \leq 2^{2^{-n}} \right\}$$

into  $2^n$  equally-spaced sectors.

The *Carleson box decomposition* is the partition of  $\mathbb{D}^*$  obtained by  $f_0$ -Carleson boxes:

$$\mathbb{D}^* = \{\zeta : |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under  $f_0$ , stemming from the relation

$$f_0(B_{n+1,k}) = B_{n, \lfloor \frac{k}{2} \rfloor}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels by trains. We now define "stations" and "tracks".

**Definition.** A *terminal* is a point  $\zeta \in \partial\mathbb{D}^*$  on the unit circle. The *central station* is the point  $s_{0,0} = 2$ . *Stations* are the iterated preimages of the central station under the map  $f_0 : \zeta \mapsto \zeta^2$ . We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi i\right), \quad n \in \mathbb{N}_0, \quad k \in \{0, \dots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The  $2^n$  stations of generation  $n$  in the tree are equally spaced on the circle  $C_n = \{|\zeta| = 2^{1/2^n}\}$ .

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

**Definition.** Let  $s = s_{n,k}$  be a station.

1. The *peripheral neighbors* of  $s$  are  $s_{n,(k\pm 1) \bmod 2^n}$ , the two stations adjacent to  $s_{n,k}$  on  $C_n$ .
2. Given a peripheral neighbor  $s'$  of  $s$ , the *peripheral track*  $\gamma_{s,s'}$  from  $s$  to  $s'$  is the short arc of the circle  $C_n$  connecting  $s$  to  $s'$ .
3. The *radial successor* of  $s$  is  $\text{RadicalSuccessor}(s) = s_{n+1,2k}$ , the unique station of generation  $n+1$  on the radial segment  $[0, s]$ .
4. Denote  $s' = \text{RadicalSuccessor}(s)$ , then the *Express track*  $\gamma_{s,s'}$  from  $s$  to  $s'$  is the radial segment  $[s, s']$ .

Notice that the tracks preserve the dynamics: applying  $\zeta \mapsto \zeta^2$  on a peripheral track between  $s, s'$  gives a peripheral track between the parents of  $s, s'$  in the tree, and likewise for an express track.

When a passenger purchases a ticket between two stations  $s_1$  and  $s_2$ , they must follow a particular itinerary to get from  $s_1$  to  $s_2$ . If  $s_1$  is the central station, then this compulsory itinerary is determined by the rule that the passenger always stays as close as possible to its destination in the peripheral distance. This also determines how to travel from the central station to a terminal  $\zeta \in \partial\mathbb{D}^*$ , by continuity. See figure 1 and the next definition.

**Definition.** Let  $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$ . The *itinerary*  $\eta_\zeta$  from the central station to  $\zeta$  is a sequence of stations  $(\sigma_0, \sigma_1, \dots)$  where  $\sigma_0 = s_{0,0} = 2$  and  $\lim_{k \rightarrow \infty} \sigma_k = \zeta$ , defined as follows.

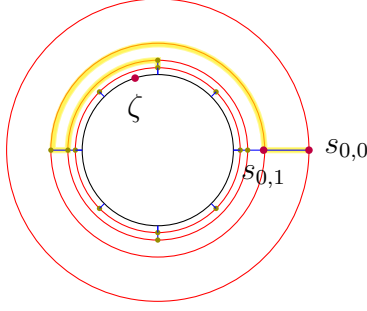


Figure 1: The central journey to a point  $\zeta$ .

We choose the stations  $\sigma_i$  greedily in pairs. In each step we travel peripherally to the station closest to  $\zeta$  and then travel to its radial successor. See Figure 1.

For the first station  $\sigma_1$  we have no choice and we travel to the station  $\sigma_1 = s_{1,0} = \sqrt{2}$ .

Suppose that we already chose the stations  $(\sigma_0, \dots, \sigma_{2k-1})$ . Then from  $\sigma_{2k-1}$  we travel to the station  $\sigma_{2k}$  on the same circle,  $|\sigma_{2k-1}| = |\sigma_{2k}|$ , that minimizes the angular distance  $|\text{Arg}(z_1) - \text{Arg}(\sigma_{2k})|$ .

We call  $\eta_\zeta$  the *central itinerary* of  $\zeta$ . Note that there are no two consecutive peripheral tracks in  $\eta_\zeta$ , and that the itineraries are invariant under  $f_0$ , in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

for every  $\zeta \in \partial\mathbb{D}$ .

**Lemma 2.1.** *Let  $\zeta \in \partial\mathbb{D}^*$ . Decompose the central itinerary  $\eta_\zeta$  into its constituent tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 \dots$$

*Then we have the estimate*

$$\text{Length}(\gamma_k) \lesssim 2^{-k}$$

*uniformly in  $\zeta$ .*

*Proof.* The station  $\sigma_{2k}$  is adjacent peripherally to  $\sigma_{2k-1}$ , since the angular distance between stations on  $C_n$  is  $\frac{2\pi}{2^n}$  and we maintain the invariant  $|\text{Arg}(\zeta) - \text{Arg}(\sigma_{2k})| \leq \frac{2\pi}{2^k}$

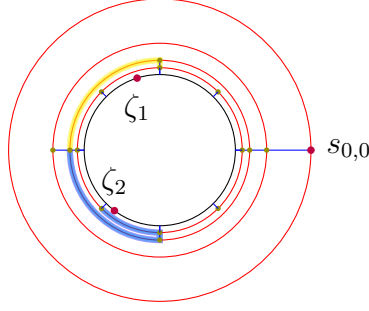


Figure 2: A quasiconvexity certificate between two points  $\zeta_1, \zeta_2$ .

throughout the itinerary. Thus the length of the peripheral track  $\gamma_{\sigma_{2k-1}, \sigma_{2k}}^p$  is either  $r_n \cdot \frac{2\pi}{2^k} = 2^{1/2^k} \frac{2\pi}{2^k}$  or 0 (in case  $\sigma_{2k-1} = \sigma_{2k}$ ), and in any case the length is at most  $\lesssim \frac{1}{2^k}$  for a global hidden constant. The length of the  $k$ -th express track decays exponentially due to the invariance under  $f_0$ . Explicitly it is  $2^{1/2^k} - 2^{1/2^{k+1}} \leq 2^{2^{-k}} - 1 \lesssim 2^{-k}$  since  $\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \log 2$ .

Thus the total length of the itinerary is bounded uniformly in  $\zeta$ .  $\square$

The itinerary between two general stations  $s$  and  $s'$  is obtained by juxtaposing  $\eta_{s_1}$  and  $\eta_{s_2}$ , discarding the part where they coincide. See figure 2. By continuity, this defines the itinerary between two terminals  $\zeta_1$  and  $\zeta_2$ .

In a more precise language:

**Definition.** Let  $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$  be two terminals. The itinerary  $\eta_{\zeta_1, \zeta_2}$  between them is defined as follows. Let  $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^\infty, \eta_{\zeta_2} = (\sigma_n^2)_{n=0}^\infty$  be the corresponding central itineraries. Let  $(\sigma_0, \dots, \sigma_N)$  be the maximal common prefix of  $\eta_{\zeta_1}$  and  $\eta_{\zeta_2}$ . Let  $\eta_{\zeta_i}^{\text{truncated}} = (\sigma_N, \sigma_{N+1}^i, \dots)$  be the truncated paths. By the maximality of  $N$ , we have that  $\eta_{\zeta_1}^{\text{truncated}}$  and  $\eta_{\zeta_2}^{\text{truncated}}$  are two itineraries with a common starting point, so we can concatenate them to obtain a bi-infinite itinerary

$$\eta_{\zeta_1, \zeta_2} = (\dots \sigma_{N+2}^2, \sigma_{N+1}^2, \sigma_N, \sigma_{N+1}^1, \sigma_{N+2}^1, \dots)$$

connecting  $\zeta_1$  and  $\zeta_2$ .

**Theorem 2.2.** *The domain  $\mathbb{D}^*$  is quasiconvex with the itineraries  $\eta_{\zeta_1, \zeta_2}$  as certificates.*

*Proof.* It is enough to show that  $\text{Length}(\eta_{\zeta_1, \zeta_2}) \lesssim |\zeta_1 - \zeta_2|$ .

As  $|\zeta_1 - \zeta_2| \asymp |\theta_1 - \theta_2|$  and  $\text{Arg}(\zeta_i) \propto \theta_i$ , it is equivalent to show

$$\text{Length}(\eta_{\zeta_1, \zeta_2}) \lesssim |\text{Arg}(\zeta_1) - \text{Arg}(\zeta_2)|.$$

By the choice of  $N$ ,

$$|\text{Arg}(\zeta_1) - \text{Arg}(\zeta_2)| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that  $\text{Length}(\eta_{\zeta_1, \zeta_2}) \lesssim 2^{-N}$ . But

$$\text{Length}(\eta_{\zeta_1, \zeta_2}) = \text{Length}(\eta_{\zeta_1}^{\text{truncated}}) + \text{Length}(\eta_{\zeta_2}^{\text{truncated}}),$$

so it is enough to observe that

$$\text{Length}(\eta_{\zeta_i}^{\text{truncated}}) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma. □

### 3 Transporting the Rails

Let  $c \in [-\frac{3}{4}, \frac{1}{4}]$ , and denote by  $\psi$  the Böttcher coordinate of  $f : z \mapsto z^2 + c$  at infinity. This means that  $\psi$  is the unique conformal map  $\mathbb{D}^* \rightarrow \text{Exterior}(\mathcal{J})$  which fixes  $\infty$  and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the Julia set  $\mathcal{J}$  is a Jordan domain, the map  $\psi$  extends to a homeomorphism between the circle  $\partial\mathbb{D}$  and the Julia set  $\mathcal{J}(f)$  by Carathéodory's theorem.

We apply  $\psi$  on the rails that we constructed in  $\mathbb{D}^*$  to obtain corresponding rails in  $\text{Exterior}(\mathcal{J})$ :

**Definition.** 1. The *c-stations* are the points  $s_{n,k,c} = \psi(s_{n,k})$ .

2. The *c-tracks* are the curves of the form  $\psi(\gamma_{s,s'})$ , where  $\gamma_{s,s'}$  is a track. They are classified as express or peripheral according to the corresponding classification of  $\gamma_{s,s'}$ .

Express tracks lie on external rays of the filled Julia set  $\mathcal{K}$ . Peripheral tracks lie on the level sets of  $\psi$ , or equivalently on the equipotentials of  $\mathcal{K}$ .

3. Let  $z_1, z_2 \in \mathcal{J}$ , and let  $\zeta_i = \psi^{-1}(z_i)$  be the corresponding points on  $\partial\mathbb{D}^*$ . The *c-itineraries* are  $\eta_{z,z',c} = \psi(\eta_{\zeta,\zeta'})$ .

These itineraries can equivalently be obtained directly as in the case of  $\mathbb{D}^*$ , in terms of following the central itineraries from the common ancestor in the tree structure.

We observe that the *c*-central station lies on the real line:

**Lemma 3.1.**  $\psi((1, \infty)) \subseteq \mathbb{R}$ . In particular,  $\psi(s_{0,0}) \in \mathbb{R}$ .

*Proof.* This is true by the symmetry of  $\mathbb{D}^*$  and  $\text{Exterior}(\mathcal{J})$  with respect to  $\mathbb{R}$ . Formally,  $\bar{\psi}(\bar{z})$  is another conformal conjugacy between  $f$  and  $f_0$  which fixes infinity, so by uniqueness of the Böttcher coordinate we obtain  $\psi(z) = \bar{\psi}(\bar{z})$ , hence  $\psi(z) \in \mathbb{R}$  for  $z \in \mathbb{R}$ .  $\square$

## 4 Hyperbolic Maps

A rational map is said to be *hyperbolic* if every critical point converges to an attracting cycle.

Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size. More precisely, we have the following:

**Proposition 4.1** (The Principle of the Conformal Elevator). *Let  $f$  be a rational hyperbolic map, let  $z \in \mathcal{J}$  be a point on the Julia set of  $f$  and let  $r > 0$ . Consider the ball  $B = B(z, r)$ . Then there exists some forward iterate  $f^{\circ n}$  of  $f$  which is injective on  $B(z, 2r)$  and such that  $\text{diam} f^{\circ n}(B(z, r))$  is bounded below uniformly in  $z$  and  $r$ .*

See [BLM16] for a stronger version, details and a proof.

We need a corollary of this principle:

**Corollary 4.2.** *Let  $f$  be a rational hyperbolic map. There exists a constant  $\epsilon$  such that for every two points  $z, w \in \mathcal{J}(f)$ , there is a forward iterate  $f^{\circ n}$  for which  $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$ .*

We are now ready to show the analogue of theorem 2.2.

**Theorem 4.3.** *a. Let  $\zeta \in \partial D$ , and decompose its central itinerary into tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 + \dots$$

*Then we have the estimate*

$$\text{Length}(\psi(\gamma_k)) \lesssim \theta^{-k}$$

*uniformly in  $\zeta$ , for some constant  $\theta = \theta(c) > 1$ .*

*b. The domain  $\text{Exterior}(\mathcal{J})$  is quasiconvex with the itineraries  $\eta_{z_1, z_2}$  as certificates..*

*Proof.* a. The map  $f$  has some iterate  $f^{\circ N}$  that is pointwise expanding on the Julia set  $\mathcal{J}(f)$ , where  $N$  is independent of  $z$ . By compactness of  $\mathcal{J}$ , the map  $f^{\circ N}$  is also uniformly expanding there, i.e. there are a constant  $\theta > 1$  and a neighborhood  $\mathcal{U}$  of  $\mathcal{J}(f)$  on which  $|(f^{\circ N})'| = |\prod f'(f^{\circ k})| > \theta$ . Every itinerary is eventually contained in  $\mathcal{U}$ , so for almost all itineraries  $\gamma$  we have  $\text{Length}(f^{\circ N}(\gamma)) \geq \theta \text{Length}(\gamma)$ . For every value  $a \bmod N$ , the peripheral tracks on circles  $C_n$  of index  $n \equiv a \bmod N$  have total length bounded by a geometric series of rate  $\theta$ , hence finite. Express tracks are bounded in the same way.

b. Since we already know that the lengths of tracks in the itinerary decay exponentially with rate  $\theta > 1$ , the same proof of the case  $c = 0$  also shows quasiconvexity (didn't check) in this case.

We give a second proof, relying on corollary 4.2. This proof will better prepare us for the parabolic  $c = 1/4$  case, in which we don't have expansion of  $f$  on its Julia set.

By corollary 4.2, there exists some  $\epsilon$  such that any two points are  $\epsilon$ -apart under some iterate  $f$ . Let  $z_1, z_2 \in \mathcal{J}(f)$ . If  $|z_1 - z_2| \geq \epsilon$  then there is nothing to prove, since we may just concatenate  $\eta_{z_1}$  and  $\eta_{z_2}$  and absorb this bounded length into the quasiconvexity constant  $A$ . Explicitly, if  $\text{Length}(\eta_z) \leq L$  for all  $z \in \mathcal{J}$  then we take  $A \geq \frac{2L}{\epsilon}$  and then automatically  $\text{Length}(\eta_{z_1} + \eta_{z_2}) \leq A|z_1 - z_2|$ .

If, on the other hand,  $|z_1 - z_2| < \epsilon$ , then we use corollary 4.2 to find an iterate  $f^{\circ n}$  such that  $|f^{\circ n}(z_0) - f^{\circ n}(z_1)| \geq \epsilon$ . Then there is a certificate itinerary  $\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}$



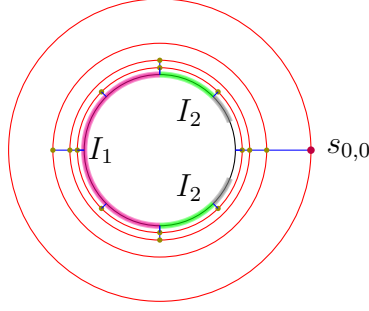


Figure 3: First few parts of the departure decomposition  $I_m$  of the circle.

between them, and we take the certificate  $\eta_{z_0, z_1}$  between the original points to be the component of  $f^{\circ -n}(\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c})$  that connects the points  $z_0, z_1$ .

A distortion estimate:

$$\text{Length}(\eta_{z_0, z_1, c}) \asymp \frac{\text{Length}(\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c})}{|(f^{\circ n})'(\zeta)|}$$

for some point  $\zeta$  on  $\mathcal{J}$ . The denominator grows with  $n$  exponentially at rate  $\theta$ , while the numerator has a bound of the form

$$\text{Length}(\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}) \lesssim |f^{\circ n}(z_0) - f^{\circ n}(z_1)| \lesssim \theta^n |z_1 - z_2|$$

so altogether

$$\text{Length}(\eta_{z_0, z_1, c}) \lesssim \frac{\theta^n |z_1 - z_2|}{\theta^n} = |z_1 - z_2|$$

so  $\eta_{z_0, z_1, c}$  is a quasiconvexity certificate.  $\square$

## 5 The Cauliflower

In this section we denote  $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$ .

We define a decomposition of the circle that is equivariant with  $f_0$  by considering the first "departure": the first time that the central itinerary had to make a turn.

**Definition.** Let  $n \in \mathbb{N}$ . We define the  $n$ -th *departure set*  $I_n$  to be the set of points  $\zeta \in \partial\mathbb{D}^*$  whose central itinerary  $\eta_\zeta$  starts with  $n$  express tracks, followed by a peripheral track. See figure 3.

The decomposition is invariant under  $f_0$  in the sense that  $f_0(I_{n+1}) = I_n$ , because of the invariance of  $\eta_\zeta$ . Thus we apply the Böttcher map  $\psi$  to obtain a corresponding departure decomposition of  $\mathcal{J}$  that is invariant under  $f$ .

## 5.1 Building an Elevator

The map  $f$  has a parabolic fixed point at  $\frac{1}{2}$ , so the usual conformal elevator does not apply. We now develop a substitute.

Let  $z_1, z_2 \in \mathcal{J}$ , and let

$$\text{CriticalOrbit}(f) = \{f^n(0) : n \in \mathbb{N}\}$$

be the forward orbit of the critical point. As long as the distance  $z_1 - z_2$  is small in comparison to the distance of  $z_1, z_2$  from  $\text{CriticalOrbit}(f)$ , we profit from applying  $f$ : By standard distortion estimates,

$$|f(z_1) - f(z_2)| \geq c|z_1 - z_2|$$

for some constant  $c > 1$ .

Indeed, in this case there is an annulus of definite modulus that has both  $z_1, z_2$  in its bounded complementary component and has all points of  $\text{CriticalOrbit}(f)$  in the unbounded complementary component.

Instead of keeping track of the distance to  $\text{CriticalOrbit}(f)$ , it is enough to consider the distance to the fixed point  $1/2$ , which is its only accumulation point on  $\mathcal{J}$ .

Thus by repeatedly applying  $f$  we either manage to separate  $z_1, z_2$  a definite distance apart, in which case the argument for the hyperbolic case works, or we manage to make (WLOG)

$$|f^{\circ m}(z_1) - \frac{1}{2}| \leq c|f^{\circ m}(z_1) - f^{\circ m}(z_2)|$$

for a constant  $c$ .

**Theorem 5.1** (Parabolic Conformal Elevator on  $\mathcal{J}$ ). *Let  $z_1, z_2 \in \mathcal{J}$  and let  $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$  be the corresponding preimages. There exists a forward iterate  $f^{\circ N}$  for which one of the following holds.*

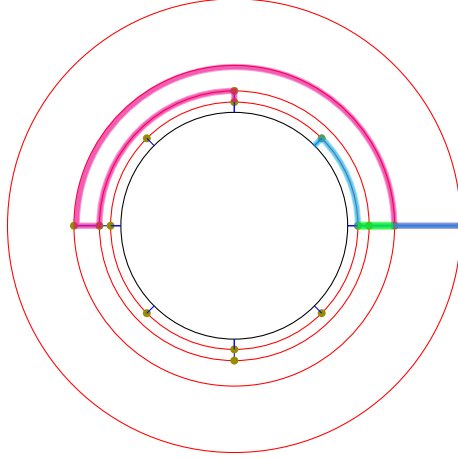


Figure 4: The three parts of the path  $\eta$ .

1.  $|f^{\circ N}(z) - f^{\circ N}(w)| > \epsilon$ , where  $\epsilon > 0$  is an absolute constant.
2.  $f^{\circ N}(z_1) \in \psi(I_n)$  and  $f^{\circ N}(z_2) \in \psi(I_m)$  with  $|m - n| \geq 3$ .

*Proof.* Along the lines of the verbal explanation. □

## 5.2 Climbing the Elevator

By the previous discussion, we reduced to the case where  $f^{\circ N}(z_1) \in \psi(I_n)$  and  $f^{\circ N}(z_2) \in \psi(I_m)$  with  $|m - n| \geq 3$ .

By Koebe's distortion theorem, it is enough to show that for the points

$$w_1 = f^{\circ N}(z_1), \quad w_2 = f^{\circ N}(z_2)$$

the itinerary  $\psi(\eta_{w_1, w_2})$  is a certificate, and then we conclude the same for  $z_1, z_2$ .

Hence we fix  $z_1, z_2$  and suppose WLOG that  $m \geq n$  and  $N = 1$ , i.e. that  $z_1, z_2$  themselves satisfy

$$z_1 \in \psi(I_n), \quad z_2 \in \psi(I_m), \quad m - n \geq 3. \tag{5.1}$$

Condition 5.1 gives us some control from below on  $|z_1 - z_2|$ . We compare it to the length of the itinerary  $\psi(\eta) = \psi(\eta_{z_1, z_2})$  to finish the proof.

We represent  $\eta$  as a concatenation of three paths: The radial segment  $[s_{m,0}, s_{n,0}]$  and the two other components,  $\gamma_m$  and  $\gamma_n$ . See figure 4.

The part in

## References

- [HH08] Hakobyan, Hrant, and Herron, David A.. "Euclidean quasiconvexity.." *Annales Academiae Scientiarum Fennicae. Mathematica* 33.1 (2008): 205-230.
- [BLM16] Mario Bonk, Mikhail Lyubich, Sergei Merenkov, Quasisymmetries of Sierpiński carpet Julia sets, *Advances in Mathematics*, Volume 301, 2016, Pages 383-422, ISSN 0001-8708, <https://doi.org/10.1016/j.aim.2016.06.007>.
- [CJY94] Carleson, L., Jones, P.W. and Yoccoz, JC. Julia and John. *Bol. Soc. Bras. Mat* 25, 1–30 (1994). <https://doi.org/10.1007/BF01232933>
- [KMS09] Koskela P., Miranda M., Shanmugalingam N. (2010) Geometric Properties of Planar BV -Extension Domains. In: Laptev A. (eds) *Around the Research of Vladimir Maz'ya I*. International Mathematical Series, vol 11. Springer, New York, NY.