# 1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ are connected by a rectifiable path $\gamma : [0,1] \to \Omega$ which satisfies

$$\mathrm{Length}(\gamma) \leq A \cdot |z_1 - z_2| .$$

We call such a path $\gamma$ a *quasiconvexity certificate for $z_1$ and $z_2$*.

If $\Omega$ is the interior of a Jordan curve, then by [2, Corollary F], it is enough to find certificates for points $z_1, z_2$ that lie on the boundary curve $\partial\Omega$.

The *cauliflower* is the filled Julia set of the map $f_{1/4}(z) = z^2 + \frac{1}{4}$. We show that its complement, $\mathrm{Exterior}(\mathcal{J}(z^2 + 1/4))$, is quasiconvex. We then adapt our argument to establish that the exterior of the developed deltoid is quasiconvex.

One motivation to study quasiconvexity stems from its connection with the John property: If $\Omega$ is a quasiconvex Jordan domain, then its complement has a John interior. See [2, Corollary 3.4] for a proof. Thus this result is a strengthening of [1, Theorem 6.1], in which it is shown directly that the cauliflower is a John domain.

This result also has a function-theoretic interpretation: By [3, Theorem 1.1], it shows that the cauliflower is a BV-extension domain.

## 1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of certificates connecting any given pair of points on the Julia set. We first build the certificates in the exterior unit disk $\mathbb{D}^*$, then we transport them to the exterior of the cauliflower by the Böttcher coordinate $\psi$ of $f_{1/4}$.

To retain control of the certificates after applying $\psi$, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This is done by only traveling along the boundaries of Carleson boxes in $\mathbb{D}^*$.

The image of a certificate $\eta$ in $\mathbb{D}^*$ under the conjugacy $\psi$ is invariant under $f_{1/4}$. We use this invariance to show that $\psi(\eta)$ is indeed a certificate, by employing a parabolic variant of the principle of the conformal elevator.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps $f_c(z) = z^2 + c$ where $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$. In this case, the usual conformal elevator applies. We subsequently treat the parabolic case of $c = \frac{1}{4}$.

# 2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

**Definition 2.1.** Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{1/2^{n+1}}, 2^{1/2^n}\right], \qquad \arg(z) \in \left(\frac{k}{2^n}\, 2\pi, \frac{k+1}{2^n}\, 2\pi\right] \right\}$$

a *Carleson box*. Observe that for a fixed $n$, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{1/2^{n+1}} < |z| \leq 2^{1/2^n} \right\}$$

into $2^n$ equally-spaced sectors.

The *Carleson box decomposition* is the partition of $\mathbb{D}^*$ into Carleson boxes:

$$\mathbb{D}^* = \{\zeta : \ |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under $f_0$, stemming from the relation

$$f_0\left(B_{n+1,k}\right) = B_{n,k \,(\mathrm{mod}\ 2^n)}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels by train. We now define “stations” and “tracks”.

**Definition 2.2.** A *terminal* is a point $\zeta \in \partial\mathbb{D}^*$ on the unit circle. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : \zeta \mapsto \zeta^2$. We index them as

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\},$$

and refer to $n$ as the *generation* of the station $s_{n,k}$. The $2^n$ stations of generation $n$ are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

**Definition 2.3.** Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of $s$ are the two stations $s_{n,(k\pm1)(\mathrm{mod}2^n)}$ adjacent to $s_{n,k}$ on $C_n$.
2. The *peripheral track* $\gamma_{s,s'}$ from $s$ to a peripheral neighbor $s'$ is the shorter arc of the circle $C_n$ connecting $s$ to $s'$.
3. The *radial successor* of $s$ is $\mathrm{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.
4. The *express track* $\gamma_{s,s'}$ from $s$ to its radial successor $s'$ is the radial segment $[s, s']$.

Notice that the tracks preserve the dynamics: applying $f_0$ to a peripheral track between stations $s, s'$ gives a peripheral track between the parents of $s, s'$ in the tree, and likewise for an express track.

When a passenger travels between two stations $s_1$ and $s_2$, they must follow a particular itinerary from $s_1$ to $s_2$. If $s_1$ is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination $s_2$ in the peripheral distance. This also determines how to travel from the central station to a terminal $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 1 and the next definition.

**Definition 2.4.** Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of $\zeta$ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \dots$ from the central station to $\zeta$, made of tracks between stations $\sigma_0, \sigma_1, \dots$. It is defined inductively as follows:

Start at the central station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \dots, \sigma_k$. If there is a peripheral neighbor $\sigma$ of $\sigma_k$ that is closer peripherally to $\zeta$, meaning that

$$|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma)| < |\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

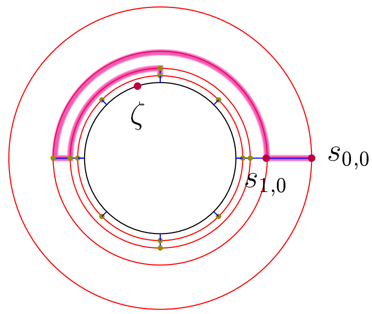


Figure 1: The central journey to a point $\zeta$.

We identify $\eta_\zeta$ with its sequence of stations $(\sigma_0, \ldots)$. We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in $\eta_\zeta$ and thus

$$\text{Generation}(\sigma_k) \geq \frac{k}{2}. \tag{2.1}$$

- Central itineraries are essentially invariant under $f_0$, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

  for every $\zeta \in \partial\mathbb{D}^*$.

**Lemma 2.5.** *Given $\zeta \in \partial\mathbb{D}^*$, decompose the central itinerary $\eta_\zeta$ into its constituent tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 + \ldots .$$

*The lengths of $\gamma_k$ decay exponentially:*

$$\text{Length}(\gamma_k) \lesssim \theta^k,$$

*uniformly in $\zeta$, for some constant $\theta < 1$. In particular, the total length of $\eta_\zeta$ is bounded above by a definite constant independent of $\zeta$.*

*Proof.* The radial distances have size $2^{1/2^n} - 2^{1/2^{n+1}} \asymp 2^{-n}$. By (2.1), the radial tracks in $\eta_\zeta$ satisfy the required bound with $\theta = \sqrt{2}$. The length of a peripheral track of generation $n$ is also $\asymp 2^{-n}$. $\square$

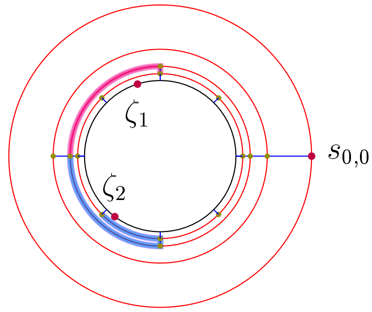


Figure 2: A quasiconvexity certificate between two points $\zeta_1, \zeta_2$.

**Definition 2.6.** Given two distinct terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^{\infty}$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^{\infty}$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both $\eta_{\zeta_1}$ and $\eta_{\zeta_2}$. We define the *itinerary* between $\zeta_1$ and $\zeta_2$ to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\dots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right).$$

This is a simple bi-infinite path connecting $\zeta_1$ and $\zeta_2$, see Figure 2. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{2.2}$$

for every pair of terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$.

**Theorem 2.7.** *The domain $\mathbb{D}^*$ is quasiconvex with the itineraries $\eta_{\zeta_1,\zeta_2}$ as certificates.*

*Proof.* We decompose the itinerary into two paths, so that

$$\mathrm{Length}(\eta_{\zeta_1,\zeta_2}) = \mathrm{Length}\left(\sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right) + \mathrm{Length}\left(\sigma, \sigma_{i+1}^1, \sigma_{i+2}^1, \dots\right), \tag{2.3}$$

and bound each summand using Lemma 2.5. Denoting Generation$(\sigma) = n$, we obtain

$$\mathrm{Length}(\eta_{\zeta_1,\zeta_2}) \lesssim 2\sum_{k=n}^{\infty} \frac{1}{2^k} \lesssim 2^{-n},$$

while

$$\begin{aligned} |\zeta_1 - \zeta_2| &\asymp |\mathrm{Arg}\,(\zeta_1) - \mathrm{Arg}\,(\zeta_2)| \\ &\geq \frac{2\pi}{2^{n+2}}. \end{aligned}$$

□

## 3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$ and denote by $\psi$ the Böttcher coordinate of $f : z \mapsto z^2 + c$ at infinity. Namely, $\psi$ is the unique conformal map $\mathbb{D}^* \to \text{Exterior}(\mathcal{J})$ which fixes $\infty$ and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the Julia set $\mathcal{J}$ is a Jordan curve, the map $\psi$ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and $\mathcal{J}$ by Carathéodory's theorem.

We apply $\psi$ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in Exterior$(\mathcal{J})$:

**Definition 3.1.**

1. The *stations* of $f_c$ are the points $s_{n,k,c} = \psi(s_{n,k})$.

2. The *c-tracks* are the curves of the form $\psi\left(\gamma_{s,s'}\right)$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on *external* rays of the filled Julia set $\mathcal{K}$, while peripheral tracks lie on the equipotentials of $\mathcal{K}$.

3. Let $z_1, z_2 \in \mathcal{J}$ and let $\zeta_i = \psi^{-1}(z_i)$ be the corresponding points on $\partial\mathbb{D}^*$. The *c-itineraries* are $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$.

We omit $c$ from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior$(\mathcal{J})$.

Note that $\psi\left((1,\infty)\right) \subseteq \mathbb{R}$ since $\mathcal{J}$ is symmetric with respect to the real line. In particular $\psi(s_{0,0}) \in \mathbb{R}$, i.e. the central station is real.

## 4 Hyperbolic Maps

A rational map is *hyperbolic* if, under iteration, every critical point converges to an attracting cycle. Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size. More precisely, we have the following:

**Proposition 4.1** (The Principle of the Conformal Elevator). *Let $f$ be a hyperbolic rational map, $z \in \mathcal{J}$ be a point on the Julia set of $f$ and $r > 0$. There exists some forward iterate $f^{\circ n}$ of $f$ which is injective on the ball $B(z, 2r)$ such that* $\operatorname{diam} f^{\circ n}(B(z,r))$ *is bounded below uniformly in $z$ and $r$.*

**Corollary 4.2.** *Let $f$ be a hyperbolic rational map. There exists $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}(f)$ has a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

**Definition 4.3.** A point $z \in \mathcal{J}$ is *rectifiably accessible* from Exterior($\mathcal{J}$) if there is a rectifiable curve $\gamma : [0,1) \to \text{Exterior}(\mathcal{J})$ such that $\gamma(t) \to z$ as $t \to 1$.

We are now ready to show the analog of Theorem 2.7:

**Theorem 4.4.**

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots.$$

*We have the estimate*

$$\text{Length}(\gamma_k) \lesssim \theta^{-k},$$

*uniformly in $z$, for some constant $\theta = \theta(c) > 1$. In particular, any point on $\mathcal{J}$ can be reached from $s_{0,0}$ by a curve of bounded length.*

(ii) *The domain* Exterior($\mathcal{J}$) *is quasiconvex with the itineraries $\eta_{z_1,z_2}$ as certificates.*

*Proof.*

(i) By the Schwarz lemma, any inverse branch $f^{-1} : \text{Exterior}(\mathcal{J}) \to \text{Exterior}(\mathcal{J})$ is a strict contraction in the hyperbolic metric of the domain Exterior($\mathcal{J}$). Hence there is a bound $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$, and we have

$$\begin{aligned} \text{HypLength}(\gamma_k) &\leq \theta \cdot \text{HypLength}(f(\gamma_k)) \\ &\leq \dots \\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ k}(\gamma_k)), \end{aligned} \tag{4.1}$$

and the length of $f^{\circ k}(\gamma_k)$ is uniformly bounded because it is a track.

The hyperbolic metric is equivalent to the Euclidean metric on Exterior($\mathcal{J}$), hence we conclude that Length$(\gamma_k) \lesssim \theta^k$ for all tracks $\gamma_k$.

The corresponding bound Length$(\delta_k) \lesssim \theta^k$ for peripheral tracks follows in an analogous manner. Alternatively, it follows from the corresponding bound on $\gamma_k$ by applying Koebe's distortion theorem on a neighborhood of the given itinerary:

$$\text{Length}(\delta_k) \asymp \frac{\text{Length}(f^{\circ n}(\delta_k))}{\text{Length}(f^{\circ n}(\gamma_k))} \cdot \text{Length}(\gamma_k)$$

for any $n$.

(ii) Since we already know that the lengths of tracks in the itinerary decay exponentially with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case. We give a second proof, relying on Corollary 4.2. This proof will better prepare us for the parabolic $c = 1/4$ case, where we do not enjoy a uniform expansion of $f$ on the Julia set.

By Corollary 4.2, there exists an $\epsilon > 0$ such that any two points are $\epsilon$-apart under some iterate $f$. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of $\eta_{z_1,z_2}$ is bounded above by part (i).

On the other hand, if $|z_1 - z_2| < \epsilon$, then we may use Corollary 4.2 to find an iterate $f^{\circ n}$ such that

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon. \tag{4.2}$$

Koebe's distortion theorem implies that

$$\frac{\text{Length}\left(\eta_{z_1,z_2}\right)}{|z_1 - z_2|} \asymp \frac{\text{Length}\left(\eta_{w_1,w_2}\right)}{|w_1 - w_2|}. \tag{4.3}$$

Since the itineraries $\eta_{w_1,w_2}$ are certificates, the original itineraries $\eta_{z_1 z,_2}$ are also certificates.

□

# 5 The Cauliflower

In this section, $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior($\mathcal{J}$), Theorem 5.11. This is more complicated than the previous hyperbolic case because the postcritical set $\mathcal{P}$ of $f$ accumulates at the parabolic fixed point $p = \frac{1}{2}$. Since one no longer has a uniform bound on the distortion of inverse iterates, we cannot immediately deduce the quasiconvexity of the itinerary $\eta_{z_1,z_2}$ from the quasiconvexity of $\eta_{w_1,w_2}$ using Koebe's distortion theorem. As a substitute, we present an analog of the principle of the conformal elevator in this parabolic setting.

## 5.1 Itineraries have finite length

We first show that each itinerary $\eta_{z_1,z_2}$ has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary $\eta_{\frac{1}{2}}$ have only a quadratic rate of length decay.

**Definition 5.1.** The *reduced decomposition* of an itinerary $\eta$ is the unique decomposition $\eta = \gamma_1 + \delta_1 + \dots$ where each $\gamma_i$ is a concatenation of express tracks and is followed by a single peripheral track $\delta_i$.

**Proposition 5.2.** *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of its itinerary. Then* Length$(\gamma_k) \lesssim \theta^k$ *and* Length$(\delta_k) \lesssim \theta^k$ *for some $\theta < 1$. In particular,* Length$(\eta_z) < \infty$ *and all points $z \in \mathcal{J}$ are rectifiably accessible.*

For the proof, call $s_{-1} := s_{1,1}$ the *pre-central station* and let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-central station and the leftmost one. This domain is constructed so that it contains all itineraries that start at the pre-central station.

**Lemma 5.3.** *Let $\gamma = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of an itinerary $\gamma$. Then for every $k \geq 1$, there exist $k-1$ iterates $n_1 < \dots < n_{k-1}$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

*Proof.* Every station $s \notin (0, \infty)$ has a first iterate $f^{\circ n_s}(s)$ lying on the negative real axis $(-\infty, 0)$. For any $i \in \{2, \dots, k-1\}$, let $s_i$ be the first station of $\gamma_i$ and take

$n_i := n_{s_i}$. By the definition of $\mathcal{U}_{-1}$, the itinerary $f^{\circ n_i}(\gamma)$ is contained in $\mathcal{U}_{-1}$ from the station $f^{\circ n_i}(s_i)$ onwards, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. $\square$

*Proof (Proposition 5.2).* By the Schwarz lemma, any inverse branch $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus \mathcal{P}$ is a contraction in the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$. The contraction is strict as it is a composition of the contraction $\tilde{f}^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus f^{-1}(\mathcal{P})$ and the inclusion $\iota : \hat{\mathbb{C}} \setminus f^{-1}(\mathcal{P}) \to \hat{\mathbb{C}} \setminus \mathcal{P}$. The inclusion is a strict contraction as $f^{-1}(\mathcal{P}) \supsetneq \mathcal{P}$.

As $\mathcal{U}_{-1}$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, there is a bound $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ that holds for both branches $f^{-1} : \mathcal{U}_{-1} \to \mathcal{U}_{\pm i}$. Then, in the notation of Lemma 5.3, we have

$$\begin{aligned} \text{HypLength}(\gamma_k) &\leq \text{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\ &\leq \theta \cdot \text{HypLength}(f^{\circ n_1}(\gamma_k)) \\ &\leq \dots \\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ n_k}(\gamma_k)), \end{aligned} \tag{5.1}$$

and the length of $f^{\circ n_k}(\gamma_k)$ is uniformly bounded because it is a track.

As the hyperbolic metric is locally equivalent to the Euclidean metric, HypLength $\asymp$ Length on $\hat{\mathbb{C}} \setminus \Delta$ where $\Delta$ is a small neighborhood of the point $\frac{1}{2} \in \mathcal{P}$. We conclude that $\text{Length}(\gamma_k) \lesssim \theta^k$ for reduced tracks $\gamma_k$ that are disjoint from $\Delta$. In particular, this holds for all paths that start at the pre-central station $s_{1,1}$.

This concludes the proof for itineraries that start at the pre-central station. The result for a general itinerary follows by Koebe's distortion theorem: Given an itinerary $\gamma_s$ whose first station is $s \neq s_{\text{central}}$, let $\mathcal{U}_s$ be the preimage of $\mathcal{U}_{-1}$ under $f$ corresponding to $s$, then Koebe's distortion on the corresponding iterate $f^{\circ(-m)} : \mathcal{U}_{-1} \to \mathcal{U}_{-s}$ immediately gives that $\text{Length}(\gamma_{s,k}) \lesssim \theta^k$. This proves the bound on express tracks. The analogous bound $\text{Length}(\delta_k) \lesssim \theta^k$ for peripheral tracks is similar. $\square$

## 5.2 Some Estimates and Notation

To estimate the length of express tracks, we introduce the notations $s_n := s_{n,0}$ and

$$\ell_k := \text{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{5.2}$$

**Lemma 5.4.** *The lengths $\ell_n$ satisfy:*

*(i)*

$$\frac{|p - s_n|}{\ell_n} \to \infty, \tag{5.3}$$

*and*

*(ii)*

$$\frac{\ell_n}{\ell_{n+1}} \to 1. \tag{5.4}$$

*In particular, for any $C > 0$, there is a sufficiently large integer $d$ such that*

$$\ell_m + \ldots + \ell_n \geq C(\ell_m + \ell_n)$$

*whenever $|m - n| \geq d$.*

*Proof.* *(i)* This follows from the affine conjugacy of the map $f$ to the map $g : z \mapsto z^2 + z$, sending the fixed point $\frac{1}{2}$ of $f$ to $0$.

*(ii)* For every $n \geq 1$, the ball $B_n = B(s_n, |p - s_n|)$ is disjoint from the post-critical set $\mathcal{P}$, hence for every $m \geq n$ we have a univalent branch of $g_{m,n} = f^{\circ -(m-n)}$ on $B_m$ sending $s_m$ to $s_n$.

Denoting $R_n = |p - s_n|$ and $r_n = \ell_n + \ell_{n+1}$, we apply Harnack's inequality on $|g'_{m,n}|$ in the ball $B_n$ to obtain

$$\begin{aligned}
\frac{\ell_{m+1}}{\ell_m} &\leq \frac{\ell_{n+1}}{\ell_n} \cdot \frac{\max_{z \in [s_n, s_{n+2}]} |g'(z)|}{\min_{z \in [s_n, s_{n+2}]} |g'(z)|} \\
&\leq \frac{\ell_{n+1}}{\ell_n} \cdot \frac{1 + \frac{r_n}{R_n}}{1 - \frac{r_n}{R_n}} \\
&= \frac{\ell_{n+1}}{\ell_n} \cdot (1 + o(1)),
\end{aligned}$$

where the first inequality follows since $g_{m,n}\left([s_m, s_{m+1}]\right) = [s_n, s_{n+1}]$.

Together with the analogous lower bound, we obtain that the sequence $a_n = \frac{\ell_{n+1}}{\ell_n}$ satisfies $c_n \leq \frac{a_m}{a_n} \leq d_n$ for every $m \geq n$, for some sequences $c_n, d_n$ both tending to 1, which forces $a_n$ to converge. Let $L$ be the limit, then we have $L \leq 1$ since $\sum \ell_n < \infty$. Moreover, we cannot have $L < 1$ since this would imply that $\ell_n \asymp \sum_{k=n}^{\infty} \ell_k$, contradicting part (i). Thus $L = 1$, as desired. $\square$

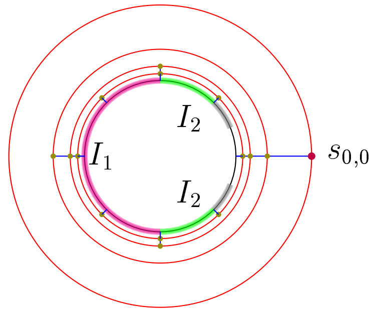


Figure 3: First few parts of the departure decomposition $I_m$ of the circle.

**Definition 5.5.** The *relative distance* of a curve $\gamma$ to the post-critical set $\mathcal{P}$ is

$$\Delta(\gamma, \mathcal{P}) = \frac{\mathrm{dist}(\gamma, \mathcal{P})}{\min(\mathrm{diam}(\gamma), \mathrm{diam}(\mathcal{P}))}.$$

We say that the curve $\gamma$ is *$\eta$-relatively separated* from the post-critical set if $\Delta(\gamma, \mathcal{P}) \geq \eta$.

If an itinerary $\gamma$ is relatively separated from $\mathcal{P}$, then preimages of $\gamma$ under $f$ have bounded distortion. In particular, if $\gamma$ is a quasiconvexity certificate, then Koebe's distortion theorem implies that $f^{-1}(\gamma)$ also is a certificate with a comparable constant.

**Lemma 5.6.** *There exists a constant $k > 0$ such that for any pair of points $z_1, z_2 \in \mathcal{J}$, we have $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$.*

*Proof.* By Lagrange's theorem, we may take $k = \max_{z \in B} |f'(z)|$, where $B$ is any ball containing $\mathcal{J}$. $\square$

## 5.3 Dynamics near the cusp

The purpose of the following definition is to organize points on the Julia set $\mathcal{J}$ according to their distance from the main cusp in an $f$-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure*: the first time that the central itinerary made a turn.

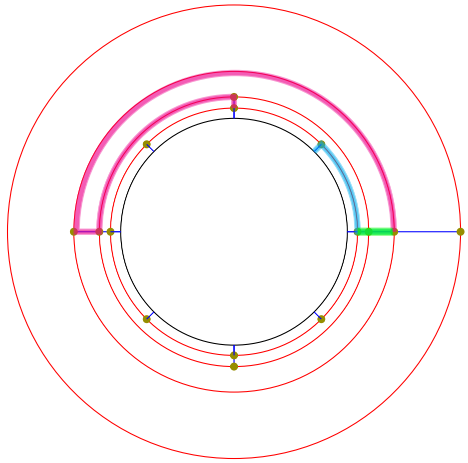

Figure 4: The three parts of an itinerary $\eta$. The green path is $\gamma_{m,n}$, the cyan and magenta are $\gamma_m$ and $\gamma_n$.

**Definition 5.7.** Let $n \in \mathbb{N}$. We define the $n$-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary $\eta_\zeta$ starts with $n$ express tracks, followed by a peripheral track. See Figure 3.

This decomposition is invariant under $f_0$ in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of $\eta_\zeta$. Applying the Böttcher map $\psi$, we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under $f$.

We now use this decomposition to analyze the case where the points $w_1, w_2$ lie in "well-separated cusps". Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \tag{5.5}$$

where $d$ is a large enough integer, to be chosen later. This gives some control from below on $|w_1 - w_2|$. We now bound the length of the itinerary $\eta = \eta_{w_1,w_2}$ from above. We represent $\eta$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, $\gamma_m$ and $\gamma_n$. See Figure 4 for the picture in the exterior unit disk. Thus we have

$$\mathrm{Length}(\eta) = \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_{m,n}) + \mathrm{Length}(\gamma_n). \tag{5.6}$$

The condition $m - n \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to $\gamma_m$ and $\gamma_n$:

**Proposition 5.8.** *There exists an integer $d$ large enough so that whenever $m - n \geq d$, we have:*

$$\mathrm{Length}(\gamma_{m,n}) \lesssim |w_1 - w_2|. \tag{5.7}$$

We henceforth fix a value of $d$ as in the proposition.

*Proof.* The triangle inequality gives

$$\mathrm{Length}(\gamma_{m,n}) \leq |w_1 - w_2| + \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_n), \tag{5.8}$$

and Koebe's distortion theorem on iterates of $f^{-1}$ shows that

$$\mathrm{Length}(\gamma_m) \leq C\ell_m \tag{5.9}$$

for some constant $C \geq 0$. Notice that this holds for $m = 1$ by the proof of Proposition 5.2, which gives a uniform bound on the length of an itinerary.

By Lemma 5.4, there exists an integer $d$ such that

$$C(\ell_m + \ell_n) \leq \mathrm{Length}(\gamma_{m,n}) \tag{5.10}$$

whenever $m - n \geq d$.

which together with (5.8) concludes the proof. □

## 5.4 Quasiconvexity: three special cases

We now show that the itineraries $\eta_{w_1,w_2}$ are certificates in three special cases. To state them, we introduce some notation.

### 5.4.1 Notation

For each $n$, we denote by $\alpha_n$ the union of the two outermost tracks emanating from the station $s_{m,0}$. Explicitly, $\alpha_n = \alpha_{n,\to} \cup \alpha_{n,\leftarrow}$ where $\alpha_{n,\to}$ has after each express track a peripheral track whose pullback to the exterior unit disk is pointing clockwise, and $\alpha_{n,\leftarrow}$ is defined analogously with anti-clockwise turns. Notice that the curves $\alpha_i$ are pairwise disjoint since this holds for their pullbacks to the exterior unit disk.

We define the constants $C_1, C_2, \epsilon$ as follows. We first choose $C_1 \geq 2$, then we let $C_2 = C_1 + d + 2$ and choose $\epsilon > 0$ small enough so that we have

$$\mathrm{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k\epsilon. \tag{5.11}$$

The constant $C_2$ was chosen so that for any pair $(m, n)$ of integers, we have at least one of the following three cases: either $m, n$ are both greater than $C_2$, or both are smaller than $C_1$, or $|m - n| > d$.

### 5.4.2 Three Special Cases

In this section we treat the following special cases:

1. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n < C_2, \quad m, n \geq 2;$
2. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n > C_1;$
3. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

Notice that Case 2 overlaps with Case 1.

We denote the domain enclosed by $\alpha_m, \alpha_n$ and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and denote the domain enclosed by $\mathcal{J}$ and $\alpha_n$ by $\mathcal{K}_n$.

**Lemma 5.9.** *Let $w_1 \in I_m$ and $w_2 \in I_n$, for $n \geq m \geq 2$. Then the itinerary $\eta_{w_1,w_2}$ is contained in the domain $\mathcal{K}_{m,n+1}$.*

**Lemma 5.10.** *Let $w_1, w_2 \in \mathcal{J}$. In each of the three special cases, the itinerary $\gamma_{w_1,w_2}$ is a quasiconvexity certificate. In Cases 1 and 2, $\gamma_{w_1,w_2}$ is relatively separated.*

*Proof. Case 1.* The itinerary, in this case, is contained in the domain $K_{2,C_2+1}$. This domain is $\eta$-relatively separated, for some $\eta > 0$, because $\mathrm{dist}(K_{2,C_2+1}, \mathcal{P}) > 0$.

*Case 2.* Assuming without loss of generality that $n \geq m$, notice that the itinerary is contained in the domain $\mathcal{K}_{m,n+1}$, which has a positive relative distance to the cusp $p$.

*Case 3* is the content of Proposition 5.8. □

### 5.5 Quasiconvexity: general case

We apply a stopping time argument to promote the quasiconvexity of $\eta_{w_1,w_2}$ to quasiconvexity of $\eta_{z_1,z_2}$, thereby proving the following theorem:

**Theorem 5.11.** *The domain* $\mathrm{Exterior}(\mathcal{J})$ *is quasiconvex, with the itineraries* $\eta_{z_1,z_2}$ *as certificates.*

*Proof. (Parabolic Conformal Elevator on* $\mathcal{J}$*).* Let $(z_1, z_2)$ be a pair of points in $\mathcal{J}$. Repeatedly apply $f$ to $(z_1, z_2)$ until either of the three special cases occurs. Denote by $w_i = f^{\circ N}(z_i)$ the resulting points. We have already proved that the itinerary $\eta_{w_1,w_2}$ satisfies

$$\mathrm{Length}(\eta_{w_1,w_2}) \le A|w_1 - w_2|,$$

for some $A > 0$. We deduce that the original pair of points $(z_1, z_2)$ enjoys a similar estimate,

$$\mathrm{Length}(\eta_{z_1,z_2}) \le C|z_1 - z_2|,$$

where $C$ depends only on $A$.

In Cases 1 and 2, we are done by Lemma 5.10. In Case 3, note that the itinerary $\eta_{w_1,w_2}$ is contained in $\mathcal{K}_2$ and let $\mathcal{K}_{-2}$ be the preimage of $\mathcal{K}_2$ under $f$ that contains the negative preimage $f^{-1}(p) = -\frac{1}{2}$ of the cusp $p$. The set $\mathcal{K}_{-2}$ is relatively separated from $\mathcal{P}$ and contains the curve $f^{\circ(N-1)}(\eta_{z_1,z_2})$. We can thus conclude by applying Koebe's distortion theorem on a suitable branch of $f^{\circ(-N)}$:

We take the branch of $f^{\circ-(N-1)}$ on $\mathcal{K}_{-2}$ sending the pair of points $(f^{-1}(w_1), f^{-1}(w_2))$ to $(z_1, z_2)$, and compose it with the branch of $f^{-1}$ that sends $p$ to $-p$. We obtain

$$\frac{\mathrm{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\mathrm{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|}, \tag{5.12}$$

hence $\eta_{z_1,z_2}$ is a certificate. □

# References

[1] Carleson, L., Jones, P. W., and Yoccoz, J.-C. Julia and John. *Bol. Soc. Bras. Mat 25*, 1 (Mar. 1994), 1–30.

[2] Hakobyan, H., and Herron, D. Euclidean quasiconvexity. *Annales Academiae Scientiarum Fennicae Mathematica 33* (Jan. 2008).

[3] Koskela, P., Miranda, M., and Shanmugalingam, N. Geometric Properties of Planar BV -Extension Domains. In *Around the Research of Vladimir Maz'ya I: Function Spaces*, A. Laptev, Ed., International Mathematical Series. Springer, New York, NY, 2010, pp. 255–272.