

# 1 Introduction

Let  $f_c : z \mapsto z^2 + c$  be a quadratic polynomial. Its *filled Julia set* consists of the points in the complex plane with bounded orbit under iteration by  $f_c$ :

$$\mathcal{K}_c = \{z \in \mathbb{C} : \sup_{n \geq 0} f_c^{on}(z) < \infty\}.$$

Its boundary  $\mathcal{J}_c = \partial\mathcal{K}(f_c)$  is known as the *Julia set*, and its complement  $\text{Exterior}(\mathcal{J}_c) = \mathbb{C} \setminus \mathcal{K}(f_c)$  forms the *attracting basin of infinity*.

The set  $\mathcal{K}_c$  is compact, and each of the three sets  $\mathcal{J}_c, \mathcal{K}_c$  and  $\text{Exterior}(\mathcal{J}_c)$  are both forward and backward invariant under the dynamics of  $f$ .

The *main cardioid*

$$\heartsuit = \{c \in \mathbb{C} : c = \lambda/2 - \lambda^2/4, \lambda \in \mathbb{D}\}$$

is the set of parameters  $c \in \mathbb{C}$  for which  $f_c$  has an attracting fixed point. When  $c \in \heartsuit$ , the Julia set  $\mathcal{J}_c$  is a *quasidisk*, the image of a round disk under a quasiconformal map. This intuitively means that  $\mathcal{K}_c$  has no “cusps”.

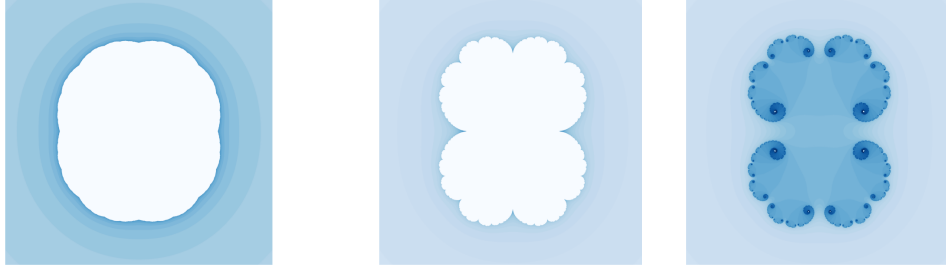
In this work we take  $c = 1/4$ , which lies on the boundary of  $\heartsuit$ . The filled Julia set  $\mathcal{K}_{1/4}$ , also called the *Cauliflower*, is a Jordan domain with an inward-pointing cusp at the point  $p = 1/2$ . However, according to a theorem of Carleson, Jones and Yoccoz [1, Theorem 6.1], the Cauliflower is a *John domain*, a condition which rules out “outward-pointing cusps”. Formally, a domain  $\Omega$  is John if there exists a “center” point  $z_0 \in \Omega$  that can be connected to any other point  $z_1 \in \Omega$  by a curve  $\gamma$  which stays away from the boundary:

$$\text{dist}(z, \partial\Omega) \gtrsim |z_1 - z| \tag{1.1}$$

for all  $z \in \gamma$ . See Figure 1.

A domain  $\Omega \subseteq \mathbb{C}$  is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant  $A \geq 1$  such that every two points  $z_1, z_2 \in \Omega$  are connected by a rectifiable path  $\gamma_{z_1, z_2} : [0, 1] \rightarrow \Omega$  which satisfies

$$\text{Length}(\gamma_{z_1, z_2}) \leq A \cdot |z_1 - z_2|. \tag{1.2}$$



(a)  $c = 0.1$ .

(b)  $c = 1/4$ , the  
Cauliflower.

(c)  $c = 0.3$ .

Figure 1: The Julia set  $\mathcal{J}_c$  of  $f_c$  for different values of  $c$ . When  $c > 1/4$ , the Julia set is no longer connected.

We refer to such a family of paths  $\gamma_{z_1, z_2}$  as *quasiconvexity certificates* for  $\Omega$ .

If  $\Omega$  is a quasiconvex Jordan domain, then its complement has a John interior; see [3, Corollary 3.4] for a proof. In this work, we strengthen the result of [1, Theorem 6.1] by showing:

**Theorem 1.1.** *The exterior of the Cauliflower is quasiconvex.*

Our result also has a function-theoretic interpretation. For a planar domain  $\Omega \subset \mathbb{R}^2$ , the *Sobolev space*  $W^{1,1}(\Omega)$  is the set of functions  $u \in L^1(\Omega)$  for which both weak derivatives  $\partial_1 u, \partial_2 u$  exist and are in  $L^1(\Omega)$ .

We call  $\Omega$  a  $W^{1,1}$  *extension domain* if every  $u \in W^{1,1}(\Omega)$  extends to a function in  $W^{1,1}(\mathbb{C})$ .

In [2, Equation (1.1) and Theorem 1.4], it is shown that a bounded, simply connected domain is a  $W^{1,1}$  extension domain if and only if its complement is quasiconvex. Thus our result can be rephrased as follows:

**Theorem 1.2.** *The Cauliflower is a  $W^{1,1}$  extension domain.*

## 1.1 Sketch of the argument

To show that a Jordan domain  $\Omega$  is quasiconvex, it is enough to find certificates for points  $z_1, z_2$  that lie on the boundary curve  $\partial\Omega$ . For a proof, see [3, Corollary F].

We show quasiconvexity by an explicit construction of certificates connecting any given pair of points on the Julia set. We first build the certificates in the exterior unit disk  $\mathbb{D}^*$ , then we transport them to the exterior of the Cauliflower by the Riemann map  $\psi : \mathbb{D}^* \rightarrow \text{Exterior}(\mathcal{J}_{1/4})$ , which conjugates  $f_0$  with  $f_{1/4}$ .

To retain control of the certificates after applying  $\psi$ , we build the certificates of  $\mathbb{D}^*$  in a manner invariant under the map  $f_0 : z \mapsto z^2$ . This makes the image of a certificate  $\eta$  in  $\mathbb{D}^*$  under the conjugacy  $\psi$  invariant under  $f_{1/4}$ . We use this invariance to show that  $\psi(\eta)$  are indeed certificates for  $\text{Exterior}(\mathcal{J}_{1/4})$ , by employing a parabolic variant of the so-called principle of the conformal elevator.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps  $f_c(z) = z^2 + c$  where  $c \in \heartsuit$ , in which the usual conformal elevator applies, and we subsequently treat the parabolic case of  $c = \frac{1}{4}$ .

## 2 Complex-analytic preliminaries

### 2.1 The Distortion Principle

We record here for convenience a form of Koebe's distortion principle that will be used repeatedly.

**Definition 2.1.** Every topological annulus  $A \subset \hat{\mathbb{C}}$  is biholomorphic to a unique round annulus of the form  $\{1 < |z| < R\}$ . The (conformal) *modulus* of  $A$  is the value  $\text{Mod}(A) = \frac{1}{2\pi} \log R$ .

**Theorem 2.2** (Koebe's Distortion Principle, [4, Theorem 2.9]). *Let  $D \subset U$  be topological disks with  $\text{Mod}(U \setminus D) \geq m > 0$  and let  $f$  be a map univalent in  $U$ , then we have the bound*

$$\frac{|f(y) - f(z)|}{|y - z|} \asymp_m |f'(x)| \tag{2.1}$$

*for all  $x, y, z \in D$ .*

## 2.2 The hyperbolic metric

**Theorem 2.3.** *The Poincaré metric*

$$ds = \frac{|dz|}{1 - |z|^2}$$

*is the unique Riemannian metric on the unit disk  $\mathbb{D}$ , up to multiplication by a positive constant, which is invariant under conformal automorphisms.*

*There exists a Riemannian metric on the unit disk  $\mathbb{D}$  which is invariant under conformal automorphisms, and it is unique up to multiplication by a positive constant.*

**Theorem 2.4.** *Let  $U$  be a planar domain whose complement  $\mathbb{C} \setminus U$  has at least three points. Then the universal covering  $\tilde{U}$  of  $U$  is biholomorphic to  $\mathbb{D}$ . This defines the Poincaré metric on  $U$  as the unique Riemannian metric for which the projection  $\tilde{U} \rightarrow U$  is a local isometry.*

## 3 The exterior disk

We connect any two boundary points  $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$  by a path in  $\mathbb{D}^*$  in a manner that respects the map  $f_0 : \zeta \mapsto \zeta^2$ . We describe these paths using the metaphor of a passenger who travels by train:

**Definition 3.1.** *Stations* are the points in  $\mathbb{D}^*$  of the form

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \quad n \in \mathbb{N}_0, \quad k \in \{0, \dots, 2^n - 1\}.$$

These are the iterated preimages of the *central station*  $s_{0,0} = 2$  under the map  $f_0$ . We refer to  $n$  as the *generation* of the station  $s_{n,k}$ . The  $2^n$  stations of generation  $n$  are equally spaced on the circle  $C_n = \{|\zeta| = 2^{1/2^n}\}$ .

We next lay two types of “rail tracks”, which we use to travel between stations.

**Definition 3.2.** Let  $s = s_{n,k}$  be a station.

1. The *peripheral neighbors* of  $s$  are the two stations  $s_{n,(k\pm 1) \pmod{2^n}}$  adjacent to  $s_{n,k}$  on  $C_n$ .

2. The *peripheral track*  $\gamma_{s,s'}$  from  $s$  to a peripheral neighbor  $s'$  is the shorter arc of the circle  $C_n$  connecting  $s$  to  $s'$ .

3. The *radial successor* of  $s$  is  $\text{RadialSuccessor}(s) = s_{n+1,2k}$ , the unique station of generation  $n+1$  on the radial segment  $[0, s]$ .

4. The *express track*  $\gamma_{s,s'}$  from  $s$  to its radial successor  $s'$  is the radial segment  $[s, s']$ .

Notice that the tracks respect the dynamics: applying  $f_0$  to a track gives a track of the previous generation.

When a passenger travels between two stations  $s_1$  and  $s_2$ , they must follow a particular itinerary from  $s_1$  to  $s_2$ . If  $s_1$  is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination  $s_2$  in the angular distance. This also determines how to travel from the central station to a boundary point  $\zeta \in \partial\mathbb{D}^*$ , by continuity. See Figure 2 and the next definition.

**Definition 3.3.** Let  $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$ . The *central itinerary* of  $\zeta$  is a path  $\eta_\zeta = \gamma_{\sigma_0, \sigma_1} + \gamma_{\sigma_1, \sigma_2} + \dots$  from the central station to  $\zeta$ , made of tracks between the stations  $\sigma_0, \sigma_1, \dots$ . It is defined inductively as follows:

Start at the central station  $\sigma_0 = s_{0,0}$ . Suppose that we already chose  $\sigma_0, \dots, \sigma_k$ . If there is a peripheral neighbor  $\sigma$  of  $\sigma_k$  that is closer peripherally to  $\zeta$ , meaning that

$$|\text{Arg}(\zeta) - \text{Arg}(\sigma)| < |\text{Arg}(\zeta) - \text{Arg}(\sigma_k)|,$$

then take  $\sigma_{k+1} = \sigma$ . Otherwise, take  $\sigma_{k+1} = \text{RadialSuccessor}(\sigma_k)$ .

We identify the central itinerary  $\eta_\zeta$  with its sequence of stations  $(\sigma_0, \dots)$ . We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in  $\eta_\zeta$ , and in particular

$$\text{Generation}(\sigma_k) \geq \frac{k}{2}; \tag{3.1}$$

- Central itineraries are essentially equivariant under  $f_0$ , in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

for every  $\zeta \in \partial\mathbb{D}^*$ .

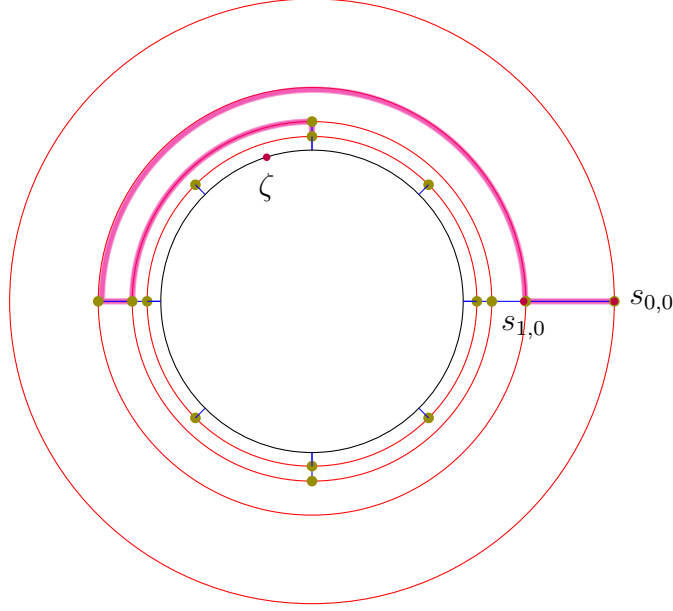


Figure 2: The central itinerary to a point  $\zeta$ .

**Definition 3.4.** Given two distinct boundary points  $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ , form the central itineraries  $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^\infty$  and  $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^\infty$  and let  $\sigma = \sigma_i^1 = \sigma_j^2$  be the last station that is in both  $\eta_{\zeta_1}$  and  $\eta_{\zeta_2}$ . We define the *itinerary* between  $\zeta_1$  and  $\zeta_2$  to be the path

$$\eta_{\zeta_1, \zeta_2} = (\dots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots).$$

This is a simple bi-infinite path connecting  $\zeta_1$  and  $\zeta_2$ , see Figure 3. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1, \zeta_2}) = \eta_{f(\zeta_1), f(\zeta_2)} \quad (3.2)$$

for every pair of boundary points  $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$  with  $|\zeta_1 - \zeta_2| < \sqrt{2}$ .

## 4 Transporting the Rails

Let  $c \in [-\frac{3}{4}, \frac{1}{4}]$ . For these values of  $c$ , the Julia set of  $f_c : z \mapsto z^2 + c$  is a Jordan curve, and  $f_c$  has a Böttcher coordinate  $\psi$  at infinity; namely,  $\psi$  is the unique conformal

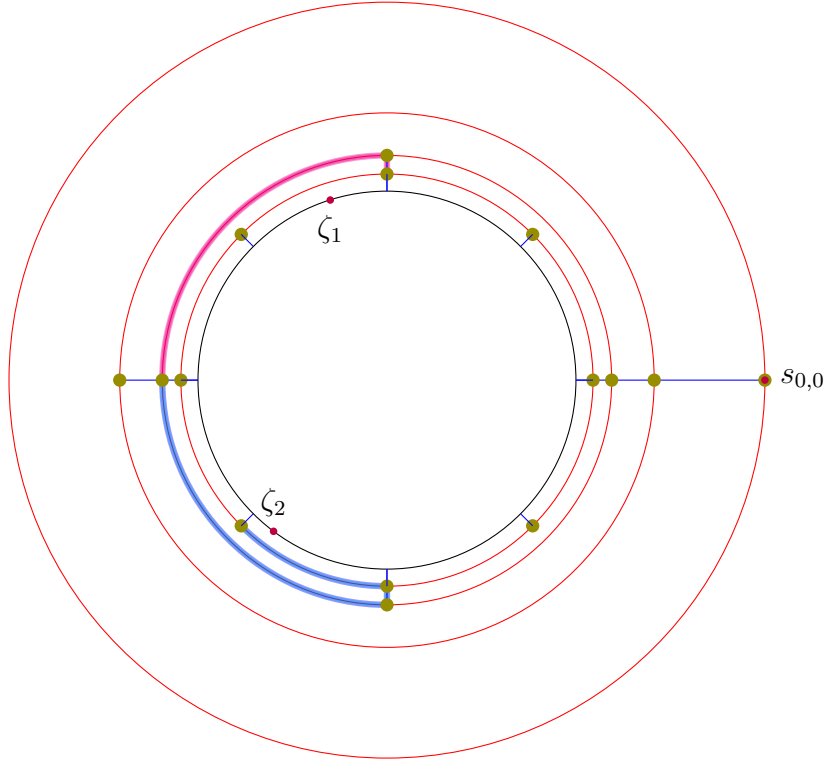


Figure 3: A quasiconvexity certificate between two points  $\zeta_1, \zeta_2$  in  $\mathbb{D}^*$ . Only the first two steps are shown.

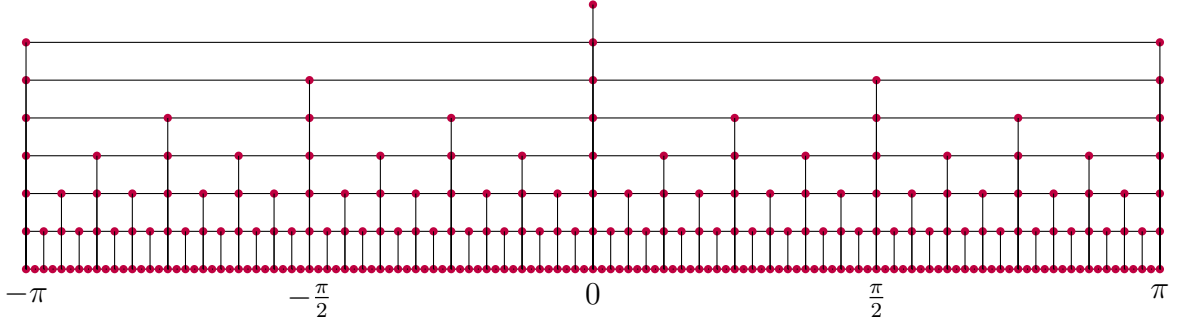


Figure 4: A convenient representation of the dyadic grid in the Böttcher coordinates. The horizontal axis is the external angle  $\text{Arg}(\psi^{-1}(z))$ , and the vertical axis is the equipotential  $|\psi^{-1}(z)|$ , plotted on a log scale. The rightmost edge is glued to the leftmost edge. Stations are marked in red, and the segments connecting adjacent stations are tracks. An express track is a vertical segment, and a peripheral track is horizontal. The magenta contour bounds the region  $\mathcal{U}_{-1}$ .

map  $\mathbb{D}^* \rightarrow \text{Exterior}(\mathcal{J}_c)$  which fixes  $\infty$  and satisfies the conjugacy relation

$$f \circ \psi = \psi \circ f_0.$$

The map  $\psi$  extends to a homeomorphism between the circle  $\partial\mathbb{D}$  and  $\mathcal{J}_c$  by Carathéodory's theorem. Uniqueness is obvious from the uniqueness of the Riemann map up to rotation; see [5, Theorem 9.5] for a proof of existence, relying on the explicit construction

$$\psi(z) = \lim_{n \rightarrow \infty} (f_0)^{\circ(-n)} \circ f^{\circ n} = \lim_{n \rightarrow \infty} (f^{\circ n})^{1/2^n}. \quad (4.1)$$

We apply  $\psi$  to the rails that we constructed in  $\mathbb{D}^*$  to obtain the corresponding rails in  $\text{Exterior}(\mathcal{J}_c)$ :

**Definition 4.1.**

1. The *stations* of  $f_c$  are the points  $\psi(s_{n,k})$ .
2. The *tracks* of  $f_c$  are the curves of the form  $\psi(\gamma_{s,s'})$ , where  $\gamma_{s,s'}$  is a track. They are classified as express or peripheral according to the corresponding classification of  $\gamma_{s,s'}$ . Express tracks lie on the external rays of the filled Julia set  $\mathcal{K}_c$ , while peripheral tracks lie on the equipotentials of  $\mathcal{K}_c$ .



3. The *itinerary* between a pair of points  $(z_1, z_2)$  on  $\mathcal{J}_c$  is  $\eta_{z_1, z_2} = \psi(\eta_{\zeta_1, \zeta_2})$ , where  $\zeta_i = \psi^{-1}(z_i)$  are the corresponding points on  $\partial\mathbb{D}^*$ .

We omit  $c$  and  $\psi$  from the notation for ease of reading. It will be clear from the context whether we work in  $\mathbb{D}^*$  or in  $\text{Exterior}(\mathcal{J})$ .

Note that  $\psi((1, \infty)) \subseteq \mathbb{R}$  since  $\mathcal{J}$  is symmetric with respect to the real line, and in particular the central station  $\psi(s_{0,0})$  lies on the real axis.

## 5 Hyperbolic Maps

In this section we prove quasiconvexity for parameters  $c$  in the main cardioid  $\heartsuit$ .

**Definition 5.1.** The *post-critical set* of  $f_c$  is the closure of the forward orbits of the critical points of  $f_c$  in  $\hat{\mathbb{C}}$ :

$$\mathcal{P} = \overline{\{f^{on}(0) : n \geq 1\} \cup \{\infty\}}.$$

A quadratic map  $f_c$  is called *hyperbolic* if its postcritical set  $\mathcal{P}$  is disjoint from its Julia set  $\mathcal{J}$ . This is equivalent to  $f_c$  being expanding on  $\mathcal{J}$ :

**Theorem 5.2.** Suppose  $f_c : z \mapsto z^2 + c$  is hyperbolic and  $c \neq 0$ . Let  $\|\cdot\|_{\text{hyp}}$  be the norm induced by the hyperbolic metric of the domain  $\hat{\mathbb{C}} \setminus \mathcal{P}$ . Then we have

$$\|Df_z(v)\|_{\text{hyp}} > \|v\|_{\text{hyp}} \tag{5.1}$$

for every  $z \in f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$ , and in particular for every  $z \in \mathcal{J}$ .

For two proofs of this theorem, see [5, Theorem 19.1], which also proves the converse.

**Corollary 5.3.** Let  $f_c$  be a hyperbolic quadratic map. There exists  $\epsilon > 0$  such that every pair of points  $z, w \in \mathcal{J}$  has a forward iterate  $f^{on}$  for which

$$|f^{on}(z) - f^{on}(w)| > \epsilon.$$

*Proof.* There is an iterate  $g = f^{\circ m}$  of  $f$  for which there is a uniform bound  $|g'| > \kappa$  on  $\mathcal{J}$ , for some constant  $\kappa > 1$ . By compactness, there exists  $\epsilon > 0$  such that whenever  $|z - w| < \epsilon$  on  $\mathcal{J}$ , we have  $|g(z) - g(w)| \geq \kappa|z - w|$ . The claim follows by iterating  $g$ .  $\square$

**Definition 5.4.** A point  $z \in \mathcal{J}$  is *rectifiably accessible* from  $\text{Exterior}(\mathcal{J})$  if there is a rectifiable curve  $\gamma : [0, 1) \rightarrow \text{Exterior}(\mathcal{J})$  such that  $\gamma(t) \rightarrow z$  as  $t \rightarrow 1$ .

We are now ready to show quasiconvexity in the hyperbolic case:

**Theorem 5.5.** *Let  $f : z \mapsto z^2 + c$  be a quadratic map with  $c \in \heartsuit$ .*

(i) *Given  $z \in \mathcal{J}$  decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots$$

*We have the estimate*

$$\text{Length}(\gamma_k) \lesssim \theta^k,$$

*uniformly in  $z$ , for some constant  $\theta = \theta(c) < 1$ . In particular, any point on  $\mathcal{J}$  is rectifiably accessible.*

(ii) *The domain  $\text{Exterior}(\mathcal{J})$  is quasiconvex with the itineraries  $\eta_{z_1, z_2}$  as certificates.*

*Proof.* (i) For  $c = 0$  this is a direct computation. Suppose  $c \neq 0$ , and let  $\mathcal{P}$  as before be the post-critical set of  $f$ .

Any inverse branch  $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \rightarrow \hat{\mathbb{C}} \setminus \mathcal{P}$ , potentially defined only in a subdomain, is a strict hyperbolic contraction by Theorem 5.2.

Let  $B(0, R) \subset \mathbb{C}$  be a ball large enough that it contains every central itinerary. By hyperbolicity,  $\hat{\mathbb{C}} \setminus \mathcal{P}$  contains  $\overline{\text{Exterior}(\mathcal{J})}$ . Thus  $\text{Exterior}(\mathcal{J}) \cap B(0, R)$  is compactly contained in  $\hat{\mathbb{C}} \setminus \mathcal{P}$ , and there is a constant  $\theta < 1$  such that  $\|(f^{-1})'\|_{\text{hyp}} < \theta$  on  $\text{Exterior}(\mathcal{J}) \cap B(0, R)$ . Therefore,

$$\begin{aligned} \text{HypLength}(\gamma_k) &\leq \theta \cdot \text{HypLength}(f(\gamma_k)) \\ &\leq \dots \\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ k}(\gamma_k)), \\ &\lesssim \theta^k, \end{aligned}$$

where the last inequality holds since  $f^{\circ k}(\gamma_k)$  lies on the real axis in case  $\gamma_k$  is an express track, or on the equipotential  $\psi(\{|z| = \sqrt{2}\})$  otherwise.

As the hyperbolic metric is equivalent to the Euclidean metric on compact subsets, we conclude that  $\text{Length}(\gamma_k) \lesssim \theta^k$  as well.

Thus any point on  $\mathcal{J}$  can be reached from the central station  $s_{0,0}$  by a curve of bounded length.

(ii) By Corollary 5.3, there exists an  $\epsilon > 0$  such that any two points are  $\epsilon$ -apart under some iterate of  $f$ . Let  $z_1, z_2 \in \mathcal{J}(f)$ . If  $|z_1 - z_2| \geq \epsilon$ , we are done since the length of  $\eta_{z_1, z_2}$  is bounded from above uniformly by part (i). On the other hand, if  $|z_1 - z_2| < \epsilon$ , then there is an iterate  $f^{\circ n}$  for which

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon \quad (5.2)$$

and we have a uniform bound on

$$\frac{\text{Length}(\eta_{w_1, w_2})}{|w_1 - w_2|}$$

as before. Thus we are left with showing that

$$\frac{\text{Length}(\eta_{z_1, z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{w_1, w_2})}{|w_1 - w_2|}, \quad (5.3)$$

which follows from a distortion argument, since  $f^{\circ n}$  sends  $(z_1, z_2)$  to  $(w_1, w_2)$  and sends  $\eta_{z_1, z_2}$  to  $\eta_{w_1, w_2}$ .

Indeed, as  $z_1$  and  $z_2$  are  $\epsilon$ -close, we may find a topological ball  $B$  containing the points  $z_1, z_2$  and the itinerary  $\eta_{z_1, z_2}$  such that  $B$  has a definite modulus inside of  $\hat{\mathbb{C}} \setminus \mathcal{P}$ , and on which the map  $f^{\circ n}$  is univalent.

We apply Theorem 2.2 to deduce that

$$\frac{|w_1 - w_2|}{|z_1 - z_2|} \asymp |(f^{\circ n})'(x)| \quad (5.4)$$

for any point  $x \in B$ . On the other hand, applying  $f^{\circ n}$  on the itinerary  $\eta_{z_1, z_2}$  makes its length grow by the same factor, and we conclude that

$$\frac{|w_1 - w_2|}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{w_1, w_2})}{\text{Length}(\eta_{z_1, z_2})}, \quad (5.5)$$

as needed. □

## 6 The Cauliflower

In this section,  $c = \frac{1}{4}$  and  $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$ . Our goal is to prove the quasiconvexity of  $\text{Exterior}(\mathcal{J})$ , Theorem 6.11. This is more complicated than the hyperbolic case, because the postcritical set  $\mathcal{P}$  of  $f$  accumulates at the parabolic fixed point  $p = \frac{1}{2}$ . One no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce the quasiconvexity of the itinerary  $\eta_{z_1, z_2}$  from the quasiconvexity of  $\eta_{w_1, w_2}$  using Koebe's distortion theorem. As a substitute, we present an analogue of the principle of the conformal elevator in this parabolic setting.

### 6.1 Itineraries have finite length

We first show that each itinerary  $\eta_{z_1, z_2}$  has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary that lies on the real axis,  $\eta_{1/2}$ , have only a quadratic rate of length decay. To fix this, we introduce:

**Definition 6.1.** The *reduced decomposition* of an itinerary  $\eta$  is the unique decomposition  $\eta = \gamma_1 + \delta_1 + \dots$  where each  $\gamma_i$  is a concatenation of express tracks and is followed by a single peripheral track  $\delta_i$ .

**Proposition 6.2.** *Let  $z \in \mathcal{J}$ , and let  $\eta_z = \gamma_1 + \delta_1 + \dots$  be the reduced decomposition of its itinerary. Then  $\text{Length}(\gamma_k) \lesssim \theta^k$  and  $\text{Length}(\delta_k) \lesssim \theta^k$  for some  $\theta < 1$ . In particular,  $\text{Length}(\eta_z) < \infty$  and all points  $z \in \mathcal{J}$  are rectifiably accessible.*

For the proof, let  $\mathcal{U}_{-1}$  be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-central station and the leftmost one. See Figure 4. This domain is constructed so that it contains all itineraries that start at the station  $s_{1,1} = \psi(-1/2)$ , the preimage of the central station under  $f$ . Its crucial property is:

**Lemma 6.3.** *Let  $\gamma = \gamma_1 + \delta_1 + \dots$  be the reduced decomposition of an itinerary  $\gamma$ . Then for every  $k > 1$ , there exist  $k - 1$  iterates  $n_1 < \dots < n_{k-1}$  such that  $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$ .*

*Proof.* Every station  $s \notin (0, \infty)$  has a first iterate  $f^{\circ n_s}(s)$  lying on the negative real axis  $(-\infty, 0)$ . For any  $i \in \{2, \dots, k-1\}$ , let  $s_i$  be the first station of  $\gamma_i$  and take  $n_i := n_{s_i}$ . By the definition of  $\mathcal{U}_{-1}$ , the itinerary  $f^{\circ n_i}(\gamma)$  is contained in  $\mathcal{U}_{-1}$  from the station  $f^{\circ n_i}(s_i)$  onwards, and in particular  $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$ .  $\square$

*Proof (Proposition 6.2).* There is a uniform bound  $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$  on  $\mathcal{U}_{-1}$  with respect to the hyperbolic metric of the domain  $\hat{\mathbb{C}} \setminus \mathcal{P}$ , for both branches  $f^{-1} : \mathcal{U}_{-1} \rightarrow \mathcal{U}_{\pm i}$ . This follows from Theorem 5.2, in the slightly more general formulation of [4, Theorem 3.5], since  $\mathcal{U}_{-1}$  is compactly contained in  $\hat{\mathbb{C}} \setminus \mathcal{P}$ .

In the notation of Lemma 6.3, we then have

$$\begin{aligned} \text{HypLength}(\gamma_k) &\leq \text{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\ &\leq \theta \cdot \text{HypLength}(f^{\circ n_1}(\gamma_k)) \\ &\leq \dots \\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ n_k}(\gamma_k)) \\ &\lesssim \theta^k. \end{aligned} \tag{6.1}$$

As in the hyperbolic case, we infer that  $\text{Length}(\gamma_k) \lesssim \theta^k$  by the equivalence on  $B(0, R) \setminus \mathcal{P}$  of the Euclidean metric and the hyperbolic metric.  $\square$

## 6.2 Some Estimates and Notation

To estimate the length of express tracks, we introduce the notations  $s_n := s_{n,0}$  and

$$\ell_n := \text{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{6.2}$$

**Lemma 6.4.** *The lengths  $\ell_n$  satisfy:*

$$(i) \quad \frac{|p - s_n|}{\ell_n} \rightarrow \infty, \tag{6.3}$$

$$(ii) \quad \frac{\ell_n}{\ell_{n+1}} \rightarrow 1. \tag{6.4}$$

*In particular, for any  $C > 0$ , there is a sufficiently large integer  $d$  such that*

$$\ell_m + \dots + \ell_n \geq C(\ell_m + \ell_n)$$

whenever  $|m - n| \geq d$ .

*Proof.* Using the affine conjugacy of the map  $f$  to the map  $g : z \mapsto z^2 + z$ , which sends the parabolic fixed point  $\frac{1}{2}$  of  $f$  to 0, one can show that

$$\ell_n \asymp \frac{1}{n^2} \quad \text{and} \quad |p - s_n| \asymp \frac{1}{n}.$$

After a little arithmetic, we get (6.3) and (6.4).  $\square$

**Definition 6.5.** The *relative distance* of a curve  $\gamma$  to the post-critical set  $\mathcal{P}$  is

$$\Delta(\gamma, \mathcal{P}) = \frac{\text{dist}(\gamma, \mathcal{P})}{\min(\text{diam}(\gamma), \text{diam}(\mathcal{P}))}.$$

We say that the curve  $\gamma$  is  $\eta$ -*relatively separated* from the post-critical set if  $\Delta(\gamma, \mathcal{P}) \geq \eta$ .

If an itinerary  $\gamma$  is relatively separated from  $\mathcal{P}$ , then the preimages of  $\gamma$  under  $f$  have bounded distortion. In particular, if  $\gamma$  is a quasiconvexity certificate, then Koebe's distortion theorem implies that  $f^{-1}(\gamma)$  is also a certificate with a comparable constant.

**Lemma 6.6.** *There exists a constant  $k > 0$  such that for any pair of points  $z_1, z_2 \in \mathcal{J}$ , we have  $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$ .*

*Proof.* We have

$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} f' \right| \leq k|z_1 - z_2| \tag{6.5}$$

for  $k = \max_{z \in B} |f'(z)|$ , where  $B$  is any ball containing  $\mathcal{J}$ .  $\square$

### 6.3 Dynamics near the parabolic fixed point

The purpose of the following definition is to organize points on the Julia set  $\mathcal{J}$  according to their distance from the main cusp  $z = 1/2$  in an  $f$ -invariant way. We decompose the points of  $\mathcal{J}$  according to the first *departure*: the first time that the central itinerary makes a turn.

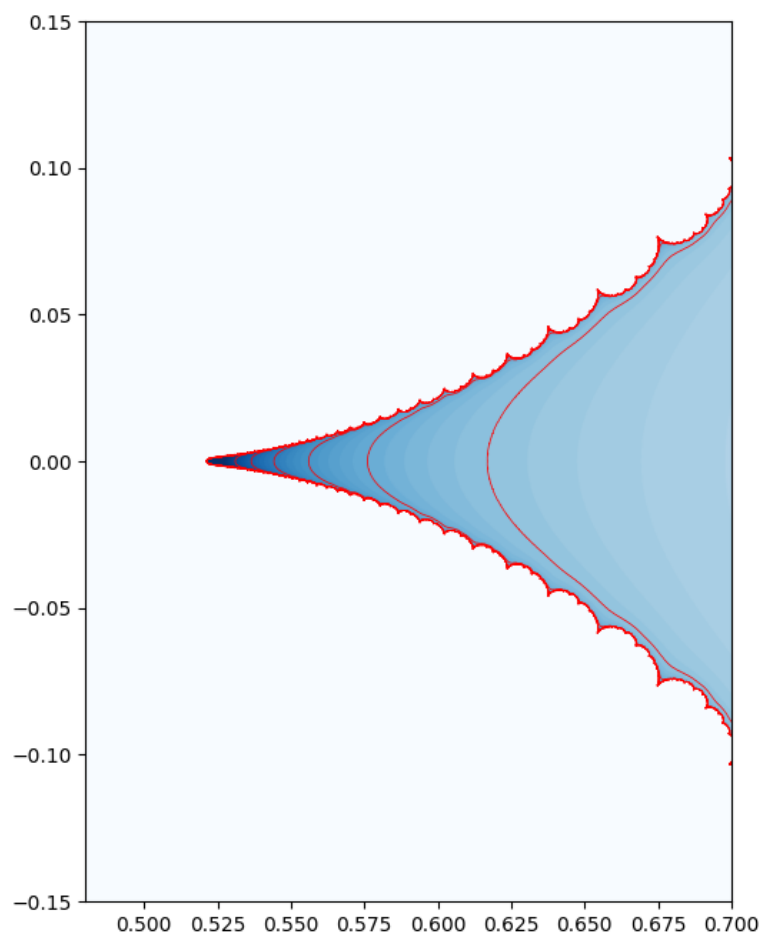


Figure 5: The Cauliflower near the parabolic point  $p = 1/2$ .

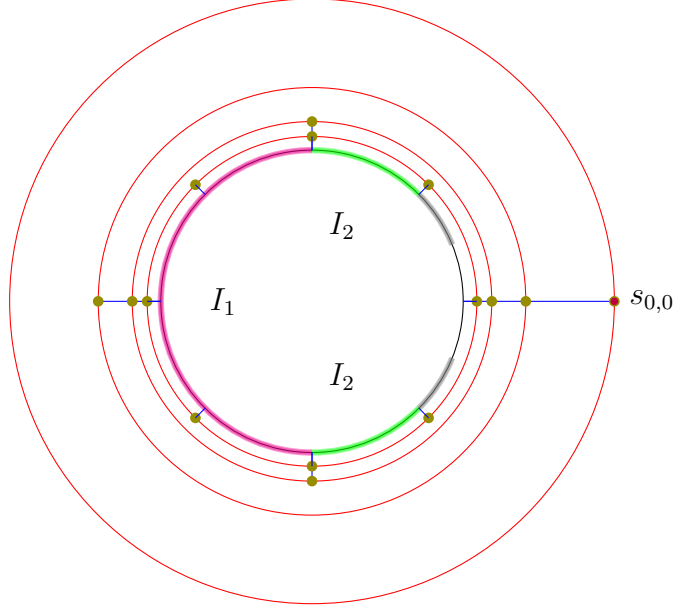


Figure 6: First few parts of the departure decomposition  $I_m$  of the circle.

**Definition 6.7.** Let  $n \in \mathbb{N}$ . We define the  $n$ -th *departure set*  $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$  to be the set of points  $\zeta \in \partial\mathbb{D}^*$  whose central itinerary  $\eta_\zeta$  starts with  $n$  express tracks, followed by a peripheral track. See Figure 6.

This decomposition is invariant under  $f_0$  in the sense that  $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$ , because of the invariance of  $\eta_\zeta$ . Applying the Böttcher map  $\psi$ , we obtain a corresponding departure decomposition  $I_n = \psi(I_{n,\mathbb{D}})$  of  $\mathcal{J}$  that is invariant under  $f$ .

We now use this decomposition to analyze the case where the points  $w_1, w_2$  lie in “well-separated cusps”. Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \quad (6.6)$$

where  $d$  is a sufficiently large integer, to be chosen later. This gives some control from below on  $|w_1 - w_2|$ . We represent the itinerary  $\eta = \eta_{w_1, w_2}$  as a concatenation of three paths: the radial segment  $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$  and the two other components,  $\gamma_m$  and  $\gamma_n$ . See Figure 7 for the picture in the exterior unit disk. Thus we have

$$\text{Length}(\eta) = \text{Length}(\gamma_m) + \text{Length}(\gamma_{m,n}) + \text{Length}(\gamma_n). \quad (6.7)$$



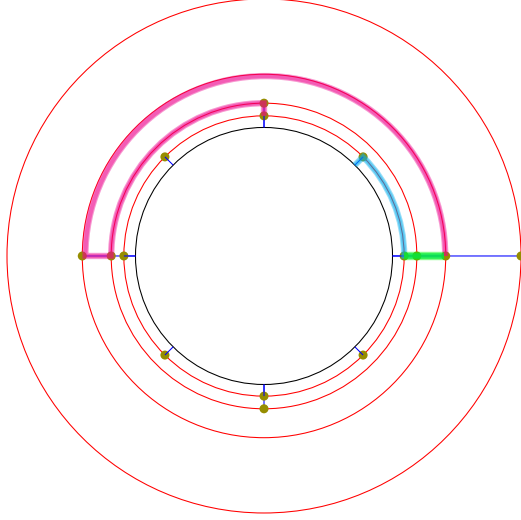


Figure 7: The three parts of an itinerary  $\eta$ . The green path is  $\gamma_{m,n}$ , the cyan and magenta are  $\gamma_m$  and  $\gamma_n$ .

The condition  $m - n \geq d$  prevents the line segment  $\gamma_{m,n}$  from being small in comparison to  $\gamma_m$  and  $\gamma_n$ :

**Proposition 6.8.** *There exists a sufficiently large integer  $d$  so that*

$$\text{Length}(\gamma_{m,n}) \asymp |w_1 - w_2|, \quad (6.8)$$

whenever  $m - n \geq d$ .

We henceforth fix a value of  $d$  as in the proposition.

*Proof.* We first make two elementary observations. Koebe's distortion theorem applied to the iterates of  $f^{-1}$  shows that

$$\text{Length}(\gamma_m) \leq C\ell_m, \quad (6.9)$$

for some constant  $C \geq 0$ . Notice that (6.9) holds for  $m = 1$  by Proposition 6.2, which gives a uniform bound on the length of an itinerary.

Meanwhile, by Lemma 6.4, there exists an integer  $d$  such that

$$C(\ell_m + \ell_n) \leq \frac{\text{Length}(\gamma_{m,n})}{2} \quad (6.10)$$

whenever  $m - n \geq d$ .

By the triangle inequality, we have

$$\begin{aligned} |\text{Length}(\gamma_{m,n}) - |w_1 - w_2|| &\leq \text{Length}(\gamma_m) + \text{Length}(\gamma_n) \\ &\leq \frac{\text{Length}(\gamma_{m,n})}{2}, \end{aligned}$$

which clearly implies (6.8). □

## 6.4 Quasiconvexity: three special cases

We now show that the itineraries  $\eta_{w_1, w_2}$  are certificates in three special cases. To state them, we introduce some notation.

### 6.4.1 Notation

For each  $n$ , we denote by  $\alpha_n$  the union of the two outermost tracks emanating from the station  $s_{n,0}$ . Notice that the curves  $\alpha_n$  are pairwise disjoint since this is true for their pullbacks to the exterior unit disk.

We define the constants  $C_1, C_2, \epsilon$  as follows. We first choose  $C_1 \geq 2$ , then we let  $C_2 = C_1 + d + 2$  and choose  $\epsilon > 0$  small enough so that we have

$$\text{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k\epsilon. \tag{6.11}$$

The constant  $C_2$  was chosen so that for any pair  $(m, n)$  of integers, we have at least one of the following three cases: either  $m, n$  are both greater than  $C_1$ , or both are smaller than  $C_2$ , or  $|m - n| > d$ .

### 6.4.2 Three Special Cases

In this section we treat the following special cases:

1.  $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n < C_2, \quad m, n \geq 2;$
2.  $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n > C_1;$
3.  $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

Notice that Case 2 overlaps with Case 1. We denote the domain enclosed by  $\alpha_m, \alpha_n$  and  $\mathcal{J}$  by  $\mathcal{K}_{m,n}$ , and denote the domain enclosed by  $\mathcal{J}$  and  $\alpha_n$  by  $\mathcal{K}_n$ .

**Lemma 6.9.** *Let  $w_1 \in I_m$  and  $w_2 \in I_n$ , for  $n \geq m \geq 2$ . Then the itinerary  $\eta_{w_1, w_2}$  is contained in the domain  $\mathcal{K}_{m, n+1}$ .*

**Lemma 6.10.** *Let  $w_1, w_2 \in \mathcal{J}$ . In each of the three special cases, the itinerary  $\gamma_{w_1, w_2}$  is a quasiconvexity certificate. In Cases 1 and 2,  $\gamma_{w_1, w_2}$  is relatively separated.*

*Proof. Case 1.* In this case, the itinerary is contained in the domain  $\mathcal{K}_{2, C_2+1}$ . Since  $\text{dist}(\mathcal{K}_{2, C_2+1}, \mathcal{P}) > 0$ ,  $\gamma_{w_1, w_2}$  is  $\eta$ -relatively separated for some  $\eta > 0$ .

*Case 2.* Assuming without loss of generality that  $n \geq m$ , the itinerary is contained in  $\mathcal{K}_{m, n+1}$ . By Koebe's distortion theorem,  $\gamma_{w_1, w_2}$  is also relatively separated.

*Case 3* is the content of Proposition 6.8. □

## 6.5 Quasiconvexity: general case

We apply a stopping time argument to promote the quasiconvexity of  $\eta_{w_1, w_2}$  to the quasiconvexity of  $\eta_{z_1, z_2}$ , thereby proving the following theorem:

**Theorem 6.11.** *The domain  $\text{Exterior}(\mathcal{J})$  is quasiconvex, with the itineraries  $\eta_{z_1, z_2}$  as certificates.*

*Proof. (Parabolic Conformal Elevator on  $\mathcal{J}$ ).* Let  $(z_1, z_2)$  be a pair of points in  $\mathcal{J}$ . Repeatedly apply  $f$  to  $(z_1, z_2)$  until either of the three special cases occurs. Denote by  $w_i = f^{\circ N}(z_i)$  the resulting points. We have already proved that the itinerary  $\eta_{w_1, w_2}$  satisfies

$$\text{Length}(\eta_{w_1, w_2}) \leq A|w_1 - w_2|,$$

for some  $A > 0$ . We deduce that the original pair of points  $(z_1, z_2)$  enjoys a similar estimate,

$$\text{Length}(\eta_{z_1, z_2}) \leq C|z_1 - z_2|,$$

where  $C$  depends only on  $A$ .

In Cases 1 and 2, we are done by Lemma 6.10. In Case 3, the itinerary  $\eta_{w_1, w_2}$  is contained in  $\mathcal{K}_2$ . Let  $\mathcal{K}_{-2}$  be the preimage of  $\mathcal{K}_2$  under  $f$  that contains the negative

preimage  $f^{-1}(p) = -\frac{1}{2}$  of the cusp  $p$ . As the domain  $\mathcal{K}_{-2}$  is relatively separated from  $\mathcal{P}$  and contains the curve  $f^{\circ(N-1)}(\eta_{z_1, z_2}) = \eta_{f^{-1}(w_1), f^{-1}(w_2)}$ , we may use Koebe's distortion theorem to conclude that

$$\frac{\text{Length}(\eta_{z_1, z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{f^{-1}(w_1), f^{-1}(w_2)})}{|f^{-1}(w_1) - f^{-1}(w_2)|} \asymp \frac{\text{Length}(\eta_{w_1, w_2})}{|w_1 - w_2|} \quad (6.12)$$

as desired.  $\square$

## References

- [1] CARLESON, L., JONES, P. W., AND YOCCOZ, J.-C. Julia and John. *Bol. Soc. Bras. Mat* 25, 1 (Mar. 1994), 1–30.
- [2] GARCÍA-BRAVO, M., AND RAJALA, T. Strong  $BV$ -extension and  $W^{1,1}$ -extension domains.
- [3] HAKOBYAN, H., AND HERRON, D. Euclidean quasiconvexity. *Annales Academiæ Scientiarum Fennicæ Mathematica* 33 (Jan. 2008).
- [4] MCMULLEN, C. T. *Complex Dynamics and Renormalization (AM-135)*. Princeton University Press, 1994.
- [5] MILNOR, J. *Dynamics in One Complex Variable. (AM-160): Third Edition. (AM-160)*. Princeton University Press, 2006.

## Nomenclature

- $\alpha_n$  the union of the two outermost tracks emanating from the station  $s_{n,0}$ .
- $\Delta(\gamma, \mathcal{P})$  The relative distance to the post-critical set.
- $\ell_n$   $\text{Length}([s_n, s_{n+1}]) = s_{n,0} - s_{n+1,0}$ .
- $\eta_{z_1, z_2}$  The itinerary connecting two points. When  $z_1$  and  $z_2$  are stations, this is the same as  $\gamma_{z_1, z_2}$ .

$\gamma_{z_1, z_2}$  The track connecting  $z_1$  and  $z_2$ . It can be either angular (“peripheral”) or radial (“express”).

$\mathcal{J}_c$  The Julia set of  $f_c$ .

$\text{Exterior}(\mathcal{J})$  An Alternative notation for  $A_\infty(f_c)$ .

$\psi$  The Bottcher coordinate  $\mathbb{D}^* \rightarrow A_\infty(f_{1/4})$  conjugating  $f_0$  and  $f_{1/4}$ .

$A_\infty(f_c)$  The exterior of the Julia set of  $f_c$ . The complement of  $K_c$ .

$f_c$  The map  $z \mapsto z^2 + c$ .

$I_n$  The  $n$ -th departure set.

$K_c$  The filled Julia set of  $f_c$ .

$s_{n,k}$  A station in  $\mathbb{D}^*$  or its image under  $\psi$ .