1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2| \, .$$

We call such a path γ a *quasiconvexity witness* for z_1, z_2.

If Ω is the interior of a Jordan curve, then by [1, Corollary F] it is enough to find certificates for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

Our interest in quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then the interior of its complement is John. See [1, Corollary 3.4] for details.

We want to show that the exterior of the developed deltoid is quasiconvex.

We show that the exterior of the cauliflower, $\mathcal{J}^{\text{exterior}}(z^2 + 1/4)$, is quasiconvex. We then adapt our argument to the exterior of the developed deltoid.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of paths connecting given points on the boundary.

We use the conjugation of $f_c : z \mapsto z^2 + c$ on the exterior of the Julia set to the map z^2 on the exterior of the unit disk $\mathbb{D}^*$. We decompose $\mathbb{D}^*$ into Carleson boxes invariant under z^2 and connect points on the unit circle by traveling on the boundary of these boxes. Both the boxes and the path on their boundary respect the dynamics of z^2 there, which allow us to transport these paths from $\mathbb{D}^*$ to the exterior of $\mathcal{J}(f_c)$

We construct two collections of curves, which we call ”express” and ”peripheral” tracks. We use the metaphor of a train traveling between the endpoints and switching between tracks. We stitch the quasiconvexity certificates out of these tracks.

The construction of the train tracks is done first in the case of the exterior unit disk ($c = 0$), and then transported to the $c = 1/4$ case using the Böttcher coordinate.

For clarity of exposition, we first show how the method works in the hyperbolic case, in which the conformal elevator makes the argument simpler, and only then prove the parabolic case $c = 1/4$.

2 Power map

The exterior $\mathbb{D}^* = \{|z| > 1\}$ of the unit disk is trivially quasiconvex by connecting points along the perimeter of the circle. However, these paths follow the boundary too closely and their length would blow up if we transport them to the exterior of $\mathcal{J}(f_c)$, $c \neq 0$, via the Riemann map. Instead, we connect points by traveling along the boundaries of Carleson boxes which we now define.

Definition. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{2^{-n}}, 2^{2^{-n-1}} \right], \qquad \arg(z) \in \left(\frac{k}{2^n} 2\pi, \frac{(k+1)}{2^n} 2\pi \right] \right\}$$

an ***f_0-Carleson box***.

Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{2^{-n-1}} < |z| \leq 2^{2^{-n}} \right\}$$

into 2^n equally-spaced sectors.

The ***Carleson f_0-box decomposition*** is the partition of $\mathbb{D}^*$ obtained by f_0-Carleson boxes:

$$\mathbb{D}^* = \{z : |z| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this partition is its invariance under f_0, stemming from the relation

$$f_0 \left(B_{n+1,k} \right) = B_{n, \lfloor \frac{k}{2} \rfloor}.$$

Definition. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : z \mapsto z^2$. We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|z| = 2^{1/2^n}\right\}$.

We next lay two types of "train tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are $s_{n,(k\pm1) \mod 2^n}$, the two stations adjacent to $s_{n,k}$ on C_n.
2. Given a peripheral neighbor s' of s, the *peripheral track* $\gamma^{\text{peripheral}}_{s,s'}$ between these stations is the short arc of the circle C_n connecting s to s'.
3. The *radial successor* of s is $\text{RadicalSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.
4. The *Express track* $\gamma^{\text{express}}_s$ from s is the radial segment $[s, \text{RadicalSuccessor}(s)]$.
5. A *train journey* is a concatenation of tracks. A journey is identified with its sequence of stations.

Notice that the tracks preserve the dynamics: applying $z \mapsto z^2$ on a peripheral track between s, s' gives a peripheral track between the parents of s, s' in the tree, and likewise for an express track.

Lemma 2.1. *There is a family $\{\eta_z : z \in \partial\mathbb{D}\}$ of journeys with the following properties:*

1. *Every η_z is a journey $(\sigma_0, \sigma_1, \ldots)$ from the central station $\sigma_0 = s_{0,0} = 2$ to $\lim \sigma_k = z$.*
2. *There are no two consecutive peripheral tracks in η_z.*
3. $\text{Length}(\sigma_k) \lesssim 2^{-k}$ *uniformly in z.*
4. *The journeys are invariant under f_0, in the sense that*

$$f_0(\eta_z) = \eta_{f_0(z)} \cup [2, 4]$$

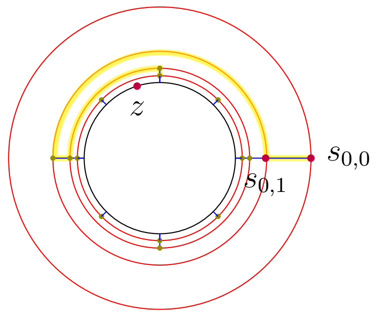

Figure 1: The central journey to a point z.

for every $z \in \partial\mathbb{D}$.

Proof. Let $z = z_1 = \exp(2\pi i\theta) \in \partial\mathbb{D}$. We choose the stations σ_i inductively in pairs, in a greedy manner. In each step we drive peripherally to the station closest to z_1 and then drive to its radial successor. See Figure 1. $\square$

Proof. For the first station σ_1 we have no choice and we drive to the station $\sigma_1 = s_{1,0} = \sqrt{2}$.

Suppose that we already chose the stations $(\sigma_0, \ldots, \sigma_{2k-1})$. Then from σ_{2k-1} we drive to the station σ_{2k} on the same circle, $|\sigma_{2k-1}| = |\sigma_{2k}|$, that minimizes the angular distance $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})|$.

The minimizer σ_{2k} is adjacent peripherally to σ_{2k-1}, since the angular distance between stations on C_n is $\frac{2\pi}{2^n}$ and we maintain the invariant $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})| \leq \frac{2\pi}{2^k}$ throughout the journey. Thus the length of the peripheral track $\gamma^p_{\sigma_{2k-1},\sigma_{2k}}$ is either $r_n \cdot \frac{2\pi}{2^k} = 2^{1/2^k}\frac{2\pi}{2^k}$ or 0 (in case $\sigma_{2k-1} = \sigma_{2k}$), and in any case the length is at most $\lesssim \frac{1}{2^k}$ for a global hidden constant. The length of the k-th express track decays exponentially due to the invariance under f_0. Explicitly it is $2^{1/2^k} - 2^{1/2^{k+1}} \leq 2^{2^{-k}} - 1 \lesssim 2^{-k}$ since $\lim_{x\to 0} \frac{2^x - 1}{x} = \log 2$.

Thus the total length of the journey is bounded uniformly in z. $\square$

We call η_z the *central journey* of z.

Theorem 2.2. *The domain* $\mathbb{D}^*$ *is quasiconvex with quasiconvexity certificates that are journeys.*

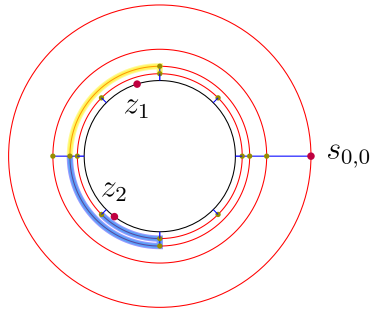

Figure 2: A quasiconvexity certificate between two points z_1, z_2.

Proof. Fix two points ("terminal stations") $z_1, z_2 \in \partial\mathbb{D}$. Let $\eta_{z_1} = (\sigma_n^1)_{n=0}^{\infty}, \eta_{z_2} = (\sigma_n^2)_{n=0}^{\infty}$ be their central journeys, connecting each terminal to the central station.

Let $(\sigma_0, \ldots, \sigma_N)$ be the maximal common prefix of η_{z_1} and η_{z_2}. Let $\eta_{z_i}^{\text{truncated}} = \left(\sigma_N, \sigma_{N+1}^i, \ldots\right)$ be the truncated paths. By the maximality of N, we have that $\eta_{z_1}^{\text{truncated}}$ and $\eta_{z_2}^{\text{truncated}}$ are two journeys with a common starting point, so we can concatenate them to obtain a bi-infinite journey

$$\eta_{z_1,z_2} = \left(\ldots \sigma_{N+2}^2, \sigma_{N+1}^2, \sigma_N, \sigma_{N+1}^1, \sigma_{N+2}^1, \ldots\right)$$

connecting z_1 and z_2.

We conclude the proof by showing that $\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim |z_1 - z_2|$.

As $|z_1 - z_2| \asymp |\theta_1 - \theta_2|$ and $\text{Arg}\left(z_i\right) \propto \theta_i$, it is equivalent to show

$$\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim \left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right|.$$

By the choice of N,

$$\left|\text{Arg}\left(z_1\right) - \text{Arg}\left(z_2\right)\right| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that $\text{Length}\left(\eta_{z_1,z_2}\right) \lesssim 2^{-N}$. But

$$\text{Length}\left(\eta_{z_1,z_2}\right) = \text{Length}\left(\eta_{z_1}^{\text{truncated}}\right) + \text{Length}\left(\eta_{z_2}^{\text{truncated}}\right),$$

so it is enough to observe that

$$\text{Length}\left(\eta_{z_i}^{\text{truncated}}\right) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma.

$\square$

3 Hyperbolic Map

Throughout this section we fix an arbitrary $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$ and denote $f_c : z \mapsto z^2 + c$ by f. Since the critical point 0 is in the filled Julia set of f, there is a conformal map $\psi : \mathbb{D}^* \to \mathcal{J}^{\text{exterior}}(f)$ conjugating f to z^2, i.e. $f \circ \psi(z) = \psi \circ f_0(z)$ for every $z \in \mathbb{D}^*$.

The map ψ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f)$ by Carathéodory's theorem, since $\mathcal{J}$ is a Jordan curve.

All f_0-invariant constructions carry over from $\mathbb{D}^*$ to $\mathcal{J}^{\text{exterior}}(f)$, and now they are f-invariant: We have stations $s_{n,k,c} = \psi(s_{n,k})$ and likewise tracks. The express tracks lie on the external rays of ψ, and the peripheral tracks are on the level sets of ψ, or equivalently on the equipotentials of Green's function.

As an example of how invariance carries other, applying ψ to both sides of the equation

$$f_0\left(B_{n+1,k}\right) = B_{n,\lfloor \frac{k}{2} \rfloor}$$

gives the corresponding relation

$$f\left(B_{n+1,k,c}\right) = B_{n,\lfloor \frac{k}{2} \rfloor,c}.$$

We observe that the central station is still on the real line:

Lemma 3.1. $\psi(\mathbb{D}^* \cap \mathbb{R}) \subseteq \mathbb{R}$. *In particular,* $\psi(s_{0,0}) \in \mathbb{R}$.

Proof. This is true by symmetry of $\mathbb{D}^*$ and $\mathcal{J}^{\text{exterior}}(f)$ with respect to $\mathbb{R}$. Formally, $\overline{\psi}(\overline{z})$ is another conformal map with the same conjugation relation, so by uniqueness of the Böttcher coordinate (with a given derivative at ∞) we obtain $\psi(z) = \overline{\psi}(\overline{z})$, hence $\psi(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. $\square$

Note that this lemma remains true for $c = 1/4$.

As before, we have:

Lemma 3.2. *There is a family* $\{\eta_z : z \in \mathcal{J}(f)\}$ *of journeys with the following properties:*

1. Every η_z *is a journey* $(\sigma_0, \sigma_1, \ldots)$ *from the central station* $\sigma_0 = s_{0,0,c}$ *to* $\lim \sigma_k = z$.

2. There are no two consecutive peripheral tracks in η_z.

3. Length$(\sigma_k) \lesssim \theta^{-k}$ *uniformly in* z, *for a constant* $\theta = \theta(c) > 1$.

4. The journeys are invariant under f, *in the sense that* $f(\eta_z) = \eta_{f(z)} \cup f([2,4])$ *for every* $z \in \partial\mathbb{D}$.

Proof. Let $z \in \mathcal{J}(f)$. Let $\zeta = \psi^{-1}(z)$ be the corresponding point on $\partial\mathbb{D}$,

which has a central journey η_ζ in $\mathbb{D}^*$. We choose the central journey of z to be $\eta_z = f(\eta_\zeta)$.

We take the images of the corresponding journeys from the case of $\mathbb{D}^*$. Parts $1, 2, 4$ are then automatic. We now check part (3).

The map f is pointwise expanding on the Julia set $\mathcal{J}(f)$, so by compactness f is uniformly expanding there, i.e. there are a constant $\theta > 1$ and a neighborhood $\mathcal{U}$ of $\mathcal{J}(f)$ on which $|f'| > \theta$. Since every journey is eventually contained in $\mathcal{U}$, we have for $k \gg 1$ that Length$(f(\gamma)) \geq \theta \cdot$Length$(\gamma)$. Thus the length of peripheral tracks decays exponentially at rate θ, and likewise for express tracks. □

To take advantage of the preceding lemma for showing quasiconvexity, we use the following claim.

Claim 3.3. *Let* f *be a rational map that is expanding on its Julia set* $\mathcal{J}(f)$. *Then there exists a constant* ϵ *such that for every two points* $z, w \in \mathcal{J}(f)$, *there exists* $n \in \mathbb{N}$ *for which* $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.

Proof. (sketch.) This claim follows from the condition that $|f'| > 1$ on $\mathcal{J}$ since as long as two points $z, w \in \mathcal{J}$ are close enough we can approximate f linearly to see that the images $f(z), f(w)$ must be further apart. □

Remark. This claim has some similarity to the *principle of the conformal elevator*, which we now recall.

A rational map is said to be *hyperbolic* if every critical point converges to an attracting cycle, and no critical point is on the Julia set.

The principle of the conformal elevator: Let f be a hyperbolic map, let $\zeta \in \mathcal{J}(f)$ and let $r > 0$ be sufficiently small. Then there exists an iterate $f^{\circ n}(B(\zeta, r))$ of the ball $B(\zeta, r)$ which is a set of diameter bounded below uniformly in ζ, r and which is "almost round". Since we do not need this control on the distortion of the balls, we do not state the precise form of this latter constraint and refer the reader to [2] for details.

Theorem 3.4. *The domain $\mathcal{J}^{exterior}(f)$ is quasiconvex.*

Proof. Let z_1, z_2 be two points on $\mathcal{J}(f)$. We construct a quasiconformality certificate curve connecting z_1 and z_2. We use the obvious candidate: let $\zeta_i = \psi^{-1}(z_i) \in \partial\mathbb{D}$, then we have a quasiconformality certificate for them η_{ζ_1,ζ_2} from the $c = 0$ case. We choose $\eta_{z_1,z_2,c} = \psi(\eta_{\zeta_1,\zeta_2})$ to be the certificates. By the invariance of the construction, this is a journey on the f-Carleson decomposition which can similarly be described directly in terms of a common ancestor in the tree structure, since ψ is a bijective correspondence between the two decompositions. Since we already know that the lengths of tracks in the journey decay exponentially, with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case. □

We give a second proof, relying on the previous claim on separation of points under iteration. This proof will better prepare us to the parabolic $c = 1/4$ case, in which we don't have uniform expansion of f on its Julia set.

Proof. By the claim, there exists some ϵ such that any two points are ϵ-apart under some iteration of f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_0 - z_1| \geq \epsilon$ then there is nothing to prove, since we may just concatenate η_{z_1} and η_{z_2} and absorb this bounded length into the quasiconformality constant A. Explicitly, if $\text{Length}(\eta_z) \leq L$ for all $z \in \mathcal{J}$ then we take $A \geq \frac{2L}{\epsilon}$ and then automatically $\text{Length}(\eta_{z_1} + \eta_{z_2}) \leq A|z_0 - z_1|$.

If, on the other hand, $|z_0 - z_1| < \epsilon$, then we may use the claim to find an iterate $f^{\circ n}$ such that $|f^{\circ n}(z_0) - f^{\circ n}(z_1)| \geq \epsilon$. Then there is a certificate journey $\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}$ between them, and we take the certificate η_{z_0,z_1} between the original points to be the component of $f^{\circ -n}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)$ that connects the points z_0, z_1.

A distortion estimate:

$$\text{Length}(\eta_{z_0,z_1,c}) \asymp \frac{\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)}{\left|(f^{\circ n})'(\zeta)\right|}$$

for some point ζ on $\mathcal{J}$. The denominator grows with n exponentially at rate θ, while the numerator has a bound of the form

$$\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right) \lesssim |f^{\circ n}(z_0) - f^{\circ n}(z_1)| \lesssim \theta^n |z_0 - z_1|$$

so altogether

$$\mathrm{Length}\left(\eta_{z_0,z_1,c}\right) \lesssim \frac{\theta^n \left|z_0 - z_1\right|}{\theta^n} = \left|z_0 - z_1\right|$$

so $\eta_{z_0,z_1,c}$ is a quasiconformality certificate.

$\square$

References

[1] Hakobyan, Hrant, and Herron, David A.. ”Euclidean quasiconvexity..” Annales Academiae Scientiarum Fennicae. Mathematica 33.1 (2008): 205-230.

[2] Mario Bonk, Mikhail Lyubich, Sergei Merenkov, Quasisymmetries of Sierpiński carpet Julia sets, Advances in Mathematics, Volume 301, 2016, Pages 383-422, ISSN 0001-8708, https://doi.org/10.1016/j.aim.2016.06.007.