1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0, 1] \to \Omega$ connecting them which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2| \, .$$

We call such a path γ a *quasiconvexity certificate* for z_1, z_2.

If Ω is the interior of a Jordan curve, then by [HH08, Corollary F] it is enough to find certificates for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

The *cauliflower* is the filled Julia set of the map $z^2 + \frac{1}{4}$. We show that its complement, Exterior$(\mathcal{J}(z^2 + 1/4))$, is quasiconvex. We then adapt our argument to establish that the exterior of the developed deltoid is quasiconvex.

One motivation to study quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then its complement has a John interior. See [HH08, Corollary 3.4] for a proof.

Thus this result is a strengthening of [CJY94, Theorem 6.1], which shows that the cauliflower is a John domain directly.

This result also has a function-theoretic interpretation: By [KMS09, Theorem 1.1], it shows that the cauliflower is a BV-extension domain.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of a certificate connecting two given points on the Julia set.

We first build such certificates for the exterior unit disk $\mathbb{D}^*$, and then transport them by the Böttcher coordinate ψ of $f_{1/4}$ to the exterior of the cauliflower.

In order to retain control on the certificates after applying ψ, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This is done by only traveling along the boundaries of a suitable Carleson box decomposition of $\mathbb{D}^*$.

The image of a certificate c of $\mathbb{D}^*$ under the conjugacy ψ is invariant under $f_{1/4}$. We use this invariance to show that $\psi(c)$ is indeed a certificate, by a parabolic variant of the principle of the conformal elevator: We repeatedly apply $f_{1/4}$ on $\psi(c)$ until either the distance between the endpoints grows to a definite size or one endpoint is attracted sufficiently fast to the parabolic fixed point 1/2. The latter case requires a more delicate treatment.

To facilitate the reading, we separate this latter difficulty by first demonstrating the proof in the hyperbolic case of maps f_c where $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$. In this case the usual conformal elevator applies. We then embark on the $c = \frac{1}{4}$ case.

2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

Definition. Let $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{2^{-n}}, 2^{2^{-n-1}} \right], \qquad \arg(z) \in \left(\frac{k}{2^n} 2\pi, \frac{(k+1)}{2^n} 2\pi \right] \right\}$$

a *Carleson box.*

Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{2^{-n-1}} < |z| \leq 2^{2^{-n}} \right\}$$

into 2^n equally-spaced sectors.

The *Carleson box decomposition* is the partition of $\mathbb{D}^*$ obtained by f_0-Carleson boxes:

$$\mathbb{D}^* = \{\zeta : |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under f_0, stemming from the relation

$$f_0 \left(B_{n+1,k} \right) = B_{n, \lfloor \frac{k}{2} \rfloor}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels by trains. We now define ”stations” and ”tracks”.

Definition. A *terminal* is a point $\zeta \in \partial\mathbb{D}^*$ on the unit circle. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : \zeta \mapsto \zeta^2$. We index them as

$$s_{n,k} = 2^{2^{-n}} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station 2 and the children of a node are its preimages. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are $s_{n,(k\pm1) \mod 2^n}$, the two stations adjacent to $s_{n,k}$ on C_n.
2. Given a peripheral neighbor s' of s, the *peripheral track* $\gamma_{s,s'}$ from s to s' is the short arc of the circle C_n connecting s to s'.
3. The *radial successor* of s is $\mathrm{RadicalSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.
4. Denote $s' = \mathrm{RadicalSuccessor}(s)$, then the *Express track* $\gamma_{s,s'}$ from s to s' is the radial segment $[s, s']$.

Notice that the tracks preserve the dynamics: applying $\zeta \mapsto \zeta^2$ on a peripheral track between s, s' gives a peripheral track between the parents of s, s' in the tree, and likewise for an express track.

When a passenger purchases a ticket between two stations s_1 and s_2, they must follow a particular itinerary to get from s_1 to s_2. If s_1 is the central station, then this compulsory itinerary is determined by the rule that the passenger always stays as close as possible to its destination in the peripheral distance. This also determines how to travel from the central station to a terminal $\zeta \in \partial\mathbb{D}^*$, by continuity. See figure 1 and the next definition.

Definition. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *itinerary* η_ζ from the central station to ζ is a sequence of stations $(\sigma_0, \sigma_1, \ldots)$ where $\sigma_0 = s_{0,0} = 2$ and $\lim_{k\to\infty} \sigma_k = \zeta$, defined as follows.

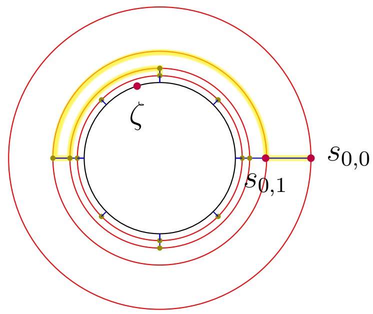

Figure 1: The central journey to a point ζ.

We choose the stations σ_i greedily in pairs. In each step we travel peripherally to the station closest to ζ and then travel to its radial successor. See Figure 1.

For the first station σ_1 we have no choice and we travel to the station $\sigma_1 = s_{1,0} = \sqrt{2}$.

Suppose that we already chose the stations $(\sigma_0, \ldots, \sigma_{2k-1})$. Then from σ_{2k-1} we travel to the station σ_{2k} on the same circle, $|\sigma_{2k-1}| = |\sigma_{2k}|$, that minimizes the angular distance $|\mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(\sigma_{2k})|$.

We call η_ζ the *central itinerary* of ζ. Note that there are no two consecutive peripheral tracks in η_ζ, and that the itineraries are invariant under f_0, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

for every $\zeta \in \partial\mathbb{D}$.

Lemma 2.1. *Let $\zeta \in \partial\mathbb{D}^*$. Decompose the central itinerary η_ζ into its constituent tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 \ldots .$$

Then we have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim 2^{-k}$$

uniformly in ζ.

Proof. The station σ_{2k} is adjacent peripherally to σ_{2k-1}, since the angular distance between stations on C_n is $\frac{2\pi}{2^n}$ and we maintain the invariant $|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_{2k})| \leq \frac{2\pi}{2^k}$

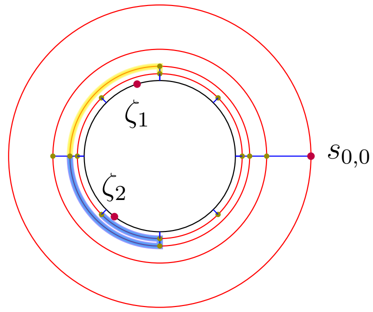

Figure 2: A quasiconvexity certificate between two points ζ_1, ζ_2.

throughout the itinerary. Thus the length of the peripheral track $\gamma^p_{\sigma_{2k-1},\sigma_{2k}}$ is either $r_n \cdot \frac{2\pi}{2^k} = 2^{1/2^k}\frac{2\pi}{2^k}$ or 0 (in case $\sigma_{2k-1} = \sigma_{2k}$), and in any case the length is at most $\lesssim \frac{1}{2^k}$ for a global hidden constant. The length of the k-th express track decays exponentially due to the invariance under f_0. Explicitly it is $2^{1/2^k} - 2^{1/2^{k+1}} \leq 2^{2^{-k}} - 1 \lesssim 2^{-k}$ since $\lim_{x\to 0}\frac{2^x-1}{x} = \log 2$.

Thus the total length of the itinerary is bounded uniformly in ζ. $\square$

The itinerary between two general stations s and s' is obtained by juxtaposing η_{s_1} and η_{s_2}, discarding the part where they coincide. See figure 2. By continuity, this defines the itinerary between two terminals ζ_1 and ζ_2.

In a more precise language:

Definition. Let $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ be two terminals. The itinerary η_{ζ_1,ζ_2} between them is defined as follows. Let $\eta_{\zeta_1} = (\sigma^1_n)^\infty_{n=0}, \eta_{\zeta_2} = (\sigma^2_n)^\infty_{n=0}$ be the corresponding central itineraries. Let $(\sigma_0, \ldots, \sigma_N)$ be the maximal common prefix of η_{ζ_1} and η_{ζ_2}. Let $\eta^{\text{truncated}}_{\zeta_i} = \left(\sigma_N, \sigma^i_{N+1}, \ldots\right)$ be the truncated paths. By the maximality of N, we have that $\eta^{\text{truncated}}_{\zeta_1}$ and $\eta^{\text{truncated}}_{\zeta_2}$ are two itineraries with a common starting point, so we can concatenate them to obtain a bi-infinite itinerary

$$\eta_{\zeta_1,\zeta_2} = \left(\ldots \sigma^2_{N+2}, \sigma^2_{N+1}, \sigma_N, \sigma^1_{N+1}, \sigma^1_{N+2}, \ldots\right)$$

connecting ζ_1 and ζ_2.

Theorem 2.2. *The domain $\mathbb{D}^*$ is quasiconvex with the itineraries η_{ζ_1,ζ_2} as certificates.*

Proof. It is enough to show that Length $(\eta_{\zeta_1,\zeta_2}) \lesssim |\zeta_1 - \zeta_2|$.

As $|\zeta_1 - \zeta_2| \asymp |\theta_1 - \theta_2|$ and $\operatorname{Arg}(\zeta_i) \propto \theta_i$, it is equivalent to show

$$\operatorname{Length}(\eta_{\zeta_1,\zeta_2}) \lesssim |\operatorname{Arg}(\zeta_1) - \operatorname{Arg}(\zeta_2)|\,.$$

By the choice of N,

$$|\operatorname{Arg}(\zeta_1) - \operatorname{Arg}(\zeta_2)| \leq \frac{2\pi}{2^N}.$$

Thus it is enough to prove that Length $(\eta_{\zeta_1,\zeta_2}) \lesssim 2^{-N}$. But

$$\operatorname{Length}(\eta_{\zeta_1,\zeta_2}) = \operatorname{Length}\left(\eta_{\zeta_1}^{\text{truncated}}\right) + \operatorname{Length}\left(\eta_{\zeta_2}^{\text{truncated}}\right),$$

so it is enough to observe that

$$\operatorname{Length}\left(\eta_{\zeta_i}^{\text{truncated}}\right) \lesssim \sum_{k=N}^{\infty} \frac{1}{2^k} \lesssim 2^{-N}$$

by part (3) of the previous lemma.

$\square$

3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$, and denote by ψ the Böttcher coordinate of $f : z \mapsto z^2 + c$ at infinity. This means that ψ is the unique conformal map $\mathbb{D}^* \to \text{Exterior}(\mathcal{J})$ which fixes ∞ and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the Julia set $\mathcal{J}$ is a Jordan domain, the map ψ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f)$ by Carathéodory's theorem.

We apply ψ on the rails that we constructed in $\mathbb{D}^*$ to obtain corresponding rails in Exterior$(\mathcal{J})$:

Definition. 1. The *c-stations* are the points $s_{n,k,c} = \psi(s_{n,k})$.

2. The *c-tracks* are the curves of the form $\psi(\gamma_{s,s'})$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$.

Express tracks lie on external rays of the filled Julia set $\mathcal{K}$. Peripheral tracks lie on the level sets of ψ, or equivalently on the equipotentials of $\mathcal{K}$.

3. Let $z_1, z_2 \in \mathcal{J}$, and let $\zeta_i = \psi^{-1}(z_i)$ be the corresponding points on $\partial\mathbb{D}^*$. The *c-itineraries* are $\eta_{z,z',c} = \psi(\eta_{\zeta,\zeta'})$.

These itineraries can equivalently be obtained directly as in the case of $\mathbb{D}^*$, in terms of following the central itineraries from the common ancestor in the tree structure.

We observe that the c-central station lies on the real line:

Lemma 3.1. $\psi\left((1,\infty)\right) \subseteq \mathbb{R}$. *In particular,* $\psi(s_{0,0}) \in \mathbb{R}$.

Proof. This is true by the symmetry of $\mathbb{D}^*$ and Exterior($\mathcal{J}$) with respect to $\mathbb{R}$. Formally, $\overline{\psi}(\overline{z})$ is another conformal conjugacy between f and f_0 which fixes infinity, so by uniqueness of the Böttcher coordinate we obtain $\psi(z) = \overline{\psi}(\overline{z})$, hence $\psi(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. □

4 Hyperbolic Maps

A rational map is said to be *hyperbolic* if every critical point converges to an attracting cycle.

Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size. More precisely, we have the following:

Proposition 4.1 (The Principle of the Conformal Elevator)**.** *Let f be a rational hyperbolic map, let $z \in \mathcal{J}$ be a point on the Julia set of f and let $r > 0$. Consider the ball $B = B(z, r)$. Then there exists some forward iterate $f^{\circ n}$ of f which is injective on $B(z, 2r)$ and such that* $\operatorname{diam} f^{\circ n}(B(z,r))$ *is bounded below uniformly in z and r.*

See [BLM16] for a stronger version, details and a proof.

We need a corollary of this principle:

Corollary 4.2. *Let f be a rational hyperbolic map. There exists a constant ϵ such that for every two points $z, w \in \mathcal{J}(f)$, there is a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

We are now ready to show the analogue of theorem 2.2.

Theorem 4.3. *a. Let $\zeta \in \partial D$, and decompose its central itinerary into tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 + \dots .$$

Then we have the estimate

$$\mathrm{Length}(\psi(\gamma_k)) \lesssim \theta^{-k}$$

uniformly in ζ, for some constant $\theta = \theta(c) > 1$.

b. The domain Exterior$(\mathcal{J})$ *is quasiconvex with the itineraries η_{z_1,z_2} as certificates..*

Proof. a. The map f has some iterate $f^{\circ N}$ that is pointwise expanding on the Julia set $\mathcal{J}(f)$, where N is independent of z. By compactness of $\mathcal{J}$, the map $f^{\circ N}$ is also uniformly expanding there, i.e. there are a constant $\theta > 1$ and a neighborhood $\mathcal{U}$ of $\mathcal{J}(f)$ on which $|(f^{\circ n})'| = |\prod f'(f^{\circ}k)| > \theta$. Every itinerary is eventually contained in $\mathcal{U}$, so for almost all itineraries γ we have Length $\left(f^{\circ N}(\gamma)\right) \geq \theta$Length$\,(\gamma)$. For every value $a \mod N$, the peripheral tracks on circles C_n of index $n \equiv a \mod N$ have total length bounded by a geometric series of rate θ, hence finite. Express tracks are bounded in the same way.

b. Since we already know that the lengths of tracks in the itinerary decay exponentially with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case. hopefully

We give a second proof, relying on corollary 4.2. This proof will better prepare us for the parabolic $c = 1/4$ case, in which we don't have expansion of f on its Julia set.

By corollary 4.2, there exists some ϵ such that any two points are ϵ-apart under some iterate f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$ then there is nothing to prove, since we may just concatenate η_{z_1} and η_{z_2} and absorb this bounded length into the quasiconvexity constant A. Explicitly, if Length$\,(\eta_z) \leq L$ for all $z \in \mathcal{J}$ then we take $A \geq \frac{2L}{\epsilon}$ and then automatically Length$\,(\eta_{z_1} + \eta_{z_2}) \leq A\,|z_1 - z_2|$.

If, on the other hand, $|z_1 - z_2| < \epsilon$, then we use corollary 4.2 to find an iterate $f^{\circ n}$ such that $|f^{\circ n}(z_0) - f^{\circ n}(z_1)| \geq \epsilon$. Then there is a certificate itinerary $\eta_{f^{\circ n}(z_0), f^{\circ n}(z_1), c}$

between them, and we take the certificate η_{z_0,z_1} between the original points to be the component of $f^{\circ -n}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)$ that connects the points z_0, z_1.

A distortion estimate:

$$\text{Length}\left(\eta_{z_0,z_1,c}\right) \asymp \frac{\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right)}{\left|\left(f^{\circ n}\right)'(\zeta)\right|}$$

for some point ζ on $\mathcal{J}$. The denominator grows with n exponentially at rate θ, while the numerator has a bound of the form

$$\text{Length}\left(\eta_{f^{\circ n}(z_0),f^{\circ n}(z_1),c}\right) \lesssim \left|f^{\circ n}(z_0) - f^{\circ n}(z_1)\right| \lesssim \theta^n \left|z_1 - z_2\right|$$

so altogether

$$\text{Length}\left(\eta_{z_0,z_1,c}\right) \lesssim \frac{\theta^n \left|z_1 - z_2\right|}{\theta^n} = \left|z_1 - z_2\right|$$

so $\eta_{z_0,z_1,c}$ is a quasiconvexity certificate.

5 The Cauliflower

In this section we denote $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. We keep the notations use the same rails and certificates as in the hyperbolic case.

5.1 Parabolic Elevator

The map f has a parabolic fixed point at $\frac{1}{2}$, so the usual conformal elevator does not apply. We proceed to develop a substitute.

Let $z_1, z_2 \in \mathcal{J}$, and let

$$\text{CriticalOrbit(f)} = \{f^n(0) : n \in \mathbb{N}\}$$

be the forward orbit of the critical point. As long as the distance $z_1 - z_2$ is small in comparison to the distance of z_1, z_2 from CriticalOrbit(f), we profit from applying f: By standard distortion estimates,

$$|f(z_1) - f(z_2)| \geq c|z_1 - z_2|$$

for some constant $c > 1$.

Indeed, in this case there is an annulus of definite modulus that has both z_1, z_2 in its bounded complementary component and has all points of CriticalOrbit(f) in the unbounded complementary component.

Instead of keeping track of the distance to CriticalOrbit(f), it is enough to consider the distance to the fixed point $1/2$, which is its only accumulation point on $\mathcal{J}$.

Thus by repeatedly applying f we either manage to separate z_1, z_2 a definite distance apart, in which case the argument for the hyperbolic case works, or we manage to make (WLOG) z_1 closer to $1/2$ than to z_2 by a multiplicative constant.

enlarge the distance by a definite amount by applying f: The distance $|f(z_2) - f(z_1)|$

□

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