1 Introduction

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ have a rectifiable path $\gamma : [0,1] \to \Omega$ connecting them which satisfies

$$\text{Length}(\gamma) \leq A \cdot |z_1 - z_2| \,.$$

We call such a path γ a *quasiconvexity certificate for z_1 and z_2*.

If Ω is the interior of a Jordan curve, then by [2, Corollary F] it is enough to find certificates for points z_1, z_2 that are on the boundary curve $\partial\Omega$.

The *cauliflower* is the filled Julia set of the map $z^2 + \frac{1}{4}$. We show that its complement, Exterior$(\mathcal{J}(z^2 + 1/4))$, is quasiconvex. We then adapt our argument to establish that the exterior of the developed deltoid is quasiconvex.

One motivation to study quasiconvexity stems from its connection with the John property: If Ω is a quasiconvex Jordan domain, then its complement has a John interior. See [2, Corollary 3.4] for a proof. Thus this result is a strengthening of [1, Theorem 6.1], in which it is shown directly that the cauliflower is a John domain.

This result also has a function-theoretic interpretation: By [3, Theorem 1.1], it shows that the cauliflower is a BV-extension domain.

1.1 Sketch of the argument

We show quasiconvexity by an explicit construction of a certificate connecting two given points on the Julia set.

We first build certificates in the exterior unit disk $\mathbb{D}^*$ and then transport them by the Böttcher coordinate ψ of $f_{1/4}$ to the exterior of the cauliflower.

In order to retain control on the certificates after applying ψ, we build the certificates on $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This is done by only traveling along the boundaries of Carleson boxes in $\mathbb{D}^*$.

The image of a certificate η in $\mathbb{D}^*$ under the conjugacy ψ is invariant under $f_{1/4}$. We use this invariance to show that $\psi(\eta)$ is indeed a certificate, by employing a parabolic variant of the principle of the conformal elevator: We repeatedly apply

$f_{1/4}$ on $\psi(\eta)$ until either the distance between the endpoints grows to a definite size or one endpoint becomes sufficiently close to the parabolic fixed point $1/2$.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps f_c where $c \in \left(-\frac{3}{4}, \frac{1}{4}\right)$. In this case the usual conformal elevator applies. We then treat the case of $c = \frac{1}{4}$.

2 The exterior disk

We connect boundary points by moving along the boundaries of Carleson boxes which we now define.

Definition 2.1. Let $n \in \mathbb{N}_0$ and $k \in \{0, \dots, 2^n - 1\}$. We call the set

$$B_{n,k} = \left\{ z : \quad |z| \in \left(2^{1/2^{n+1}}, 2^{1/2^n} \right], \qquad \arg(z) \in \left(\frac{k}{2^n} 2\pi, \frac{k+1}{2^n} 2\pi \right] \right\}$$

a *Carleson box*. Observe that for a fixed n, the union $\bigsqcup_{k=0}^{2^n-1} B_{k,n}$ is a partition of the annulus

$$\left\{ 2^{1/2^{n+1}} < |z| \leq 2^{1/2^n} \right\}$$

into 2^n equally-spaced sectors.

The *Carleson box decomposition* is the partition of $\mathbb{D}^*$ into Carleson boxes:

$$\mathbb{D}^* = \{\zeta : \, |\zeta| > 2\} \sqcup \bigsqcup_{n=0}^{\infty} \bigsqcup_{k=0}^{2^n-1} B_{n,k}.$$

The crucial property of this decomposition is its invariance under f_0, stemming from the relation

$$f_0\left(B_{n+1,k}\right) = B_{n,k \,(\mathrm{mod}\; 2^n)}.$$

We describe the motion along Carleson boxes using the metaphor of a passenger who travels by trains. We now define "stations" and "tracks".

Definition 2.2. A *terminal* is a point $\zeta \in \partial\mathbb{D}^*$ on the unit circle. The *central station* is the point $s_{0,0} = 2$. *Stations* are the iterated preimages of the central station under the map $f_0 : \zeta \mapsto \zeta^2$. We index them as

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

Stations are naturally structured in a binary tree, where the root is the central station and the children of a node are its preimages. The *generation* of a station $s_{n,k}$ is its level n in this tree. The 2^n stations of generation n in the tree are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks" on the boundaries of Carleson boxes, which we use to travel between stations.

Definition 2.3. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are the two stations $s_{n,(k\pm1)(\mathrm{mod} 2^n}$ adjacent to $s_{n,k}$ on C_n.

2. The *peripheral track* $\gamma_{s,s'}$ from s to a peripheral neighbor s' is the short arc of the circle C_n connecting s to s'.

3. The *radial successor* of s is $\mathrm{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.

4. The *Express track* $\gamma_{s,s'}$ from s to its radial successor s'' is the radial segment $[s, s']$.

Notice that the tracks preserve the dynamics: applying f_0 to a peripheral track between stations s, s' gives a peripheral track between the parents of s, s' in the tree, and likewise for an express track.

When a passenger travels between two stations s_1 and s_2, they must follow a particular itinerary from s_1 to s_2. If s_1 is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination s_2 in the peripheral distance. This also determines how to travel from the central station to a terminal $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 1 and the next definition.

Definition 2.4. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \ldots$ from the central station to ζ, made of tracks between stations $\sigma_0, \sigma_1, \ldots$. It is defined inductively as follows:

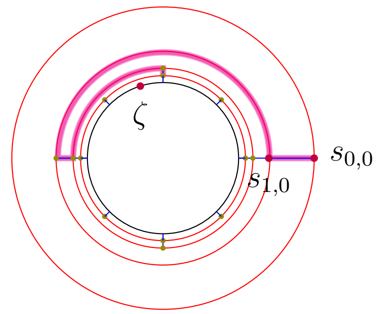

Figure 1: The central journey to a point ζ.

Start at the main station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \ldots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma)| < |\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

We identify η_ζ with its sequence of stations $(\sigma_0, \ldots)$. We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in η_ζ and thus

$$\mathrm{Generation}(\sigma_k) \geq \frac{k}{2}. \tag{2.1}$$

- Central itineraries are essentially invariant under f_0, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

  for every $\zeta \in \partial\mathbb{D}^*$.

Lemma 2.5. *Given $\zeta \in \partial\mathbb{D}^*$, decompose the central itinerary η_ζ into its constituent tracks,*

$$\eta_\zeta = \gamma_1 + \gamma_2 + \ldots\ .$$

The lengths of γ_k decay exponentially:

$$\mathrm{Length}(\gamma_k) \lesssim \theta^k,$$

uniformly in ζ, for some constant $\theta < 1$. In particular, the total length of η_ζ is bounded above by a definite constant independent of ζ.

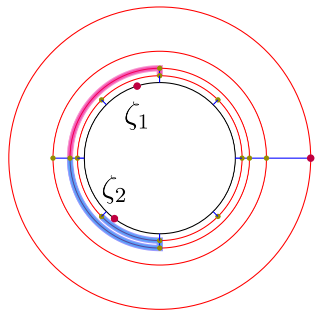

Figure 2: A quasiconvexity certificate between two points ζ_1, ζ_2.

Proof. The radial distances have size

$$2^{1/2^n} - 2^{1/2^{n+1}} = 2^{1/2^{n+1}} \sum_{k=1}^{\infty} \binom{1/2^{n+1}}{k} \asymp 2^{-n}.$$

By (2.1), the radial tracks of η_ζ satisfy the required bound with $\theta = \sqrt{2}$. To conclude, note that a peripheral track of generation n has length $\asymp 2^{-n}$ and reuse (2.1). $\square$

Definition 2.6. Given two distinct terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^\infty$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^\infty$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\dots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2, see Figure 2. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{2.2}$$

for every pair of terminals $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$.

Theorem 2.7. *The domain $\mathbb{D}^*$ is quasiconvex with the itineraries η_{ζ_1,ζ_2} as certificates.*

Proof. We decompose the itinerary into two paths, so that

$$\text{Length}(\eta_{\zeta_1,\zeta_2}) = \text{Length}\left(\sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \dots\right) + \text{Length}\left(\sigma, \sigma_{i+1}^1, \sigma_{i+2}^1, \dots\right), \tag{2.3}$$

and bound each summand using Lemma 2.5. Denoting Generation(σ) $= n$, we obtain

$$\text{Length}(\eta_{\zeta_1,\zeta_2}) \lesssim 2\sum_{k=n}^{\infty} \frac{1}{2^k} \lesssim 2^{-n}.$$

We conclude by noticing that

$$\begin{aligned} |\zeta_1 - \zeta_2| &\asymp |\text{Arg}\,(\zeta_1) - \text{Arg}\,(\zeta_2)| \\ &\geq \frac{2\pi}{2^{n+2}} \\ &\gtrsim \text{Length}\,(\eta_{\zeta_1,\zeta_2})\,. \end{aligned}$$

□

3 Transporting the Rails

Let $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$ and denote by ψ the Böttcher coordinate of $f : z \mapsto z^2 + c$ at infinity. This means that ψ is the unique conformal map $\mathbb{D}^* \to \text{Exterior}(\mathcal{J})$ which fixes ∞ and satisfies the conjugacy

$$f \circ \psi = \psi \circ f_0.$$

Since the Julia set $\mathcal{J}$ is a Jordan curve, the map ψ extends to a homeomorphism between the circle $\partial\mathbb{D}$ and the Julia set $\mathcal{J}(f)$ by Carathéodory's theorem.

We apply ψ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in Exterior($\mathcal{J}$):

Definition 3.1.

1. The *stations* of f_c are the points $s_{n,k,c} = \psi(s_{n,k})$.
2. The *c-tracks* are the curves of the form $\psi\left(\gamma_{s,s'}\right)$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on *external* rays of the filled Julia set $\mathcal{K}$, while peripheral tracks lie on the equipotentials of $\mathcal{K}$.
3. Let $z_1, z_2 \in \mathcal{J}$ and let $\zeta_i = \psi^{-1}(z_i)$ be the corresponding points on $\partial\mathbb{D}^*$. The *c-itineraries* are $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$.

We omit c from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in $\mathrm{Exterior}(\mathcal{J})$.

Note that $\psi\left((1,\infty)\right) \subseteq \mathbb{R}$ since $\mathcal{J}$ is symmetric with respect to the real line. In particular $\psi(s_{0,0}) \in \mathbb{R}$, i.e. the c-central station is real.

4 Hyperbolic Maps

A rational map is *hyperbolic* if under iteration, every critical point converges to an attracting cycle. Hyperbolic maps enjoy the principle of the conformal elevator, which roughly says that any ball centered on the Julia set can be blown up to a definite size. More precisely, we have the following:

Proposition 4.1 (The Principle of the Conformal Elevator)**.** *Let f be a hyperbolic rational map, $z \in \mathcal{J}$ be a point on the Julia set of f and $r > 0$. There exists some forward iterate $f^{\circ n}$ of f which is injective on the ball $B(z, 2r)$ such that* $\mathrm{diam}\, f^{\circ n}(B(z,r))$ *is bounded below uniformly in z and r.*

Corollary 4.2. *Let f be a hyperbolic rational map. There exists $\epsilon > 0$ such that every pair of points $z, w \in \mathcal{J}(f)$ has a forward iterate $f^{\circ n}$ for which $|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon$.*

We are now ready to show the analogue of Theorem 2.7:

Theorem 4.3.

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots.$$

We have the estimate

$$\mathrm{Length}(\gamma_k) \lesssim \theta^{-k},$$

uniformly in z, for some constant $\theta = \theta(c) > 1$. In particular, any point on $\mathcal{J}$ can be reached from $s_{0,0}$ by a curve of bounded length.

(ii) *The domain* $\mathrm{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries η_{z_1,z_2} as certificates.*

Proof.

(i) The map f has some iterate $f^{\circ N}$ such that $|(f^{\circ N})'(z)| > 1$ for all $z \in \mathcal{J}$. By the compactness of $\mathcal{J}$, there is a $\theta > 1$ such that $\left|(f^{\circ N})'\big(z)\right| > \theta$ on some neighborhood $\mathcal{U}$ of $\mathcal{J}$. Since every itinerary is eventually contained in $\mathcal{U}$, for almost all itineraries γ we have

$$\mathrm{Length}(f^{\circ N}(\gamma)) \geq \theta \cdot \mathrm{Length}(\gamma).$$

The peripheral tracks on circles C_n of index $n \equiv k$ modulo N have a total length bounded by a geometric series of rate θ, hence finite. The lengths of the express tracks can be bounded in the same way.

(ii) Since we already know that the lengths of tracks in the itinerary decay exponentially with rate $\theta > 1$, the same proof of the case $c = 0$ also shows quasiconvexity in this case.

We give a second proof, relying on Corollary 4.2. This proof will better prepare us for the parabolic $c = 1/4$ case, where we don't have uniform expansion of f on the Julia set.

By Corollary 4.2, there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of $\eta_{z_1 z,_2}$ is bounded above by part (i).

On the other hand, if $|z_1 - z_2| < \epsilon$, then we may use Corollary 4.2 to find an iterate $f^{\circ n}$ such that

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon. \tag{4.1}$$

Koebe's distortion theorem implies that

$$\frac{\mathrm{Length}\left(\eta_{z_1,z_2}\right)}{|z_1 - z_2|} \asymp \frac{\mathrm{Length}\left(\eta_{w_1,w_2}\right)}{|w_1 - w_2|}. \tag{4.2}$$

Since the itineraries η_{w_1,w_2} are certificates, the original itineraries $\eta_{z_1 z,_2}$ are also certificates.

$\square$

5 The Cauliflower

In this section $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior($\mathcal{J}$), Theorem 5.12. This parabolic case is more complicated than the hyperbolic case because the postcritical set $\mathcal{P}$ of f accumulates at the parabolic fixed point $p = \frac{1}{2}$. Thus one no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce quasiconvexity of the itinerary η_{z_1,z_2} from the quasiconvexity of η_{w_1,w_2} using Koebe's distortion theorem. We present an analogue of the principle of the conformal elevator in this parabolic setting.

5.1 Itineraries have finite length

We first show that each itinerary η_{z_1,z_2} has finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks of the central itinerary $\eta_{\frac{1}{2}}$ have only a quadratic rate of length decay.

Definition 5.1. The *reduced decomposition* of an itinerary η is the unique decomposition $\eta = \gamma_1 + \delta_1 + \ldots$ where each γ_i is a concatenation of express tracks and followed by a single peripheral track δ_i.

Proposition 5.2. *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \ldots$ be the reduced decomposition of its itinerary. Then* $\mathrm{Length}(\gamma_j) \lesssim \theta^j$ *and* $\mathrm{Length}(\delta_j) \lesssim \theta^j$ *for some $\theta < 1$. In particular,* $\mathrm{Length}(\eta_z) < \infty$ *and all points $z \in \mathcal{J}$ are accessible from the main station by a rectifiable curve.*

For the proof, call $s_{-1} := s_{1,1}$ the *pre-main station* and let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-main station and the leftmost one. This domain was constructed so that it contains all itineraries that start at the pre-main station.

Lemma 5.3. *Let $\gamma = \gamma_1 + \delta_1 + \ldots$ be the reduced decomposition of an itinerary γ, and assume that γ is not contained in the positive real axis. Then for every $k \geq 1$, there are k iterates $n_1 < \cdots < n_k$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

Proof (Lemma 5.3). Every station $s \notin [0, \infty)$ is a preimage of a station on the negative real axis. Thus every express reduced track γ_i which is not contained in $[\frac{1}{2}, \infty)$ has a first iterate n_i such that $f^{\circ n_i}(\gamma_i)$ is contained in the negative real axis, and in particular is contained in $\mathcal{U}_{-1}$. By definition of $\mathcal{U}_{-1}$, all tracks of the itinerary $f^{\circ n_i}(\gamma)$ that appear after $f^{\circ n_i}(\gamma_i)$ are contained in $\mathcal{U}_{-1}$ too, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. □

Proof (Proposition 5.2). By the Schwarz lemma, any inverse branch $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus \mathcal{P}$ is a contraction in the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$. The contraction is strict as it is a composition of the contraction $\tilde{f}^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus f^{-1}(\mathcal{P})$ and the inclusion $\iota : \hat{\mathbb{C}} \setminus f^{-1}(\mathcal{P}) \to \hat{\mathbb{C}} \setminus \mathcal{P}$ which is a strict contraction, as $f^{-1}(\mathcal{P}) \supset \mathcal{P}$.

The domain $\mathcal{U}_{-1}$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, hence there is a bound $\|(f^{-1})'\|_{\text{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ that holds for both branches $f^{-1} : \mathcal{U}_{-1} \to \mathcal{U}_{\pm i}$. Then

$$\operatorname{HypLength}(\gamma) \leq \theta \cdot \operatorname{HypLength}(f(\gamma)).$$

By Lemma 5.3 we thus have

$$\operatorname{HypLength}(\gamma_k) \lesssim \theta^k.$$

As the hyperbolic metric is locally equivalent to the Euclidean metric, HypLength $\asymp$ Length on $\hat{\mathbb{C}} \setminus \Delta$ where Δ is a small neighborhood of the point $\frac{1}{2} \in \mathcal{P}$. We conclude that $\operatorname{Length}(\gamma_k) \lesssim \theta^k$ for reduced tracks γ_k that are disjoint from Δ. In particular, this holds for all paths that start at the pre-main station $s_{1,1}$.

For such paths, we deduce a bound $\operatorname{Length}(\delta_i) \lesssim \theta^i$ from the corresponding bound on γ_i by applying Koebe's distortion theorem on a neighborhood of a given itinerary γ:

$$\operatorname{Length}(\delta_i) \asymp \frac{\operatorname{Length}(f^{\circ n}(\delta_i))}{\operatorname{Length}(f^{\circ n}(\gamma_i))} \cdot \operatorname{Length}(\gamma_i)$$

for any n. We choose as before the minimal n for which $f^{\circ n}(\gamma_i)$ lies on the real axis, and then both the numerator $\operatorname{Length}(f^{\circ n}(\delta_i))$ and the denominator $\operatorname{Length}(f^{\circ n}(\gamma_i))$ of the first term are bounded from above and below. Hence, $\operatorname{Length}(\delta_i) \lesssim \operatorname{Length}(\gamma_i) \lesssim \theta^i$.

This concludes the proof for itineraries that start at the pre-main station. The result for a general itinerary follows by Koebe's distortion theorem: Given an itinerary

γ_s whose first station is $s \neq s_{\mathrm{main}}$, let $\mathcal{U}_s$ be the preimage of $\mathcal{U}_{-1}$ under f corresponding to s. Koebe's distortion on the corresponding iterate $f^{\circ(-m)} : \mathcal{U}_{-1} \to \mathcal{U}_{-s}$ shows that $\mathrm{Length}(\gamma_{s,i}) \lesssim \theta^i$ since this bound holds for $\gamma := f^{\circ m}(\gamma_s)$ by the previous case. ☐

5.2 Some Estimates

To estimate the length of express tracks, we introduce the notation

$$\ell_k = \mathrm{Length}(\psi([s_{k,0}, s_{k+1,0}])) = \psi(s_{k,0}) - \psi(s_{k+1,0}). \tag{5.1}$$

Lemma 5.4. *The lengths ℓ_k satisfy:*

(i)

$$\frac{\mathrm{dist}(p, s_k)}{\ell_k} \to \infty, \tag{5.2}$$

and

(ii)

$$\frac{\ell_k}{\ell_{k+1}} \to 1. \tag{5.3}$$

In particular, for any constant $C \geq 0$ there is a sufficiently large integer d such that

$$\begin{aligned}\mathrm{Length}(\gamma_{m,n}) &= \ell_m + \ldots + \ell_n \\ &\leq C(\ell_m + \ell_n)\end{aligned}$$

whenever $|m - n| \geq d$.

Proof.

(ii) For every $k \geq 1$, the ball $B_k = B(s_k, \mathrm{dist}(p, s_k))$ is disjoint from the post-critical set $\mathcal{P}$, hence for every $m \geq k$ we have a univalent branch of $g_{m,k} = f^{\circ-(m-k)}$ on B_k sending s_k to s_m.

Denoting $R_k = \mathrm{dist}(p, s_k)$ and $r_k = \ell_k + \ell_{k+1}$, we apply Harnack's inequality on $|g'_{m,k}|$ in the ball B_k to obtain

$$\frac{\ell_{m+1}}{\ell_m} \leq \frac{\ell_{k+1}}{\ell_k} \cdot \frac{\max_{z \in [s_k, s_{k+2}]} |g'(z)|}{\min_{z \in [s_k, s_{k+2}]} |g'(z)|} \leq \frac{\ell_{k+1}}{\ell_k} \cdot \frac{R_k + r_k}{R_k - r_k}.$$

By part *(i)* we have $\frac{R_k+r_k}{R_k-r_k} \to 1$ as $k \to \infty$. Together with the analogous lower bound, we thus showed that the sequence $a_k = \frac{\ell_{k+1}}{\ell_k}$ satisfies the following: for every $\epsilon > 0$, for every large enough $k \geq 1$ we have $\left|\frac{a_m}{a_k} - 1\right| < \epsilon$ for every $m \geq k$.

This property implies first that the sequence a_k is Cauchy. Let L be the limit. We have $L \leq 1$ since $\sum \ell_k < \infty$. We cannot have $L < 1$ since this would imply that $\ell_k \asymp \sum_{n=k}^{\infty} \ell_n$, contradicting part *(i)*. Thus $\ell_k \asymp \ell_{k+1}$ as desired. □

Definition 5.5. The *relative distance* of a curve γ from the post-critical set is

$$\Delta(\gamma, \mathcal{P}) = \frac{\mathrm{dist}(\gamma, \mathcal{P})}{\min(\mathrm{diam}(\gamma), \mathrm{diam}(\mathcal{P}))}.$$

We say that the curve γ is η*-relatively separated* from the post-critical set if $\Delta(\gamma, \mathcal{P}) \geq \eta$.

If an itinerary γ is relatively separated from $\mathcal{P}$, then preimages of γ under f have bounded distortion. In particular, if γ is a quasiconvexity certificate, then Koebe's distortion theorem implies that $f^{-1}(\gamma)$ also is a certificate with a comparable constant.

Lemma 5.6. *There exists a constant $k > 0$ such that for all $z_1, z_2 \in \mathcal{J}$, we have $|f(z_1) - f(z_2)| < k|z_1 - z_2|$.*

Proof. One may take $k = \|f|_B'\|_\infty$, where $B \subset \mathbb{R}^2$ is the union of all line segments having both endpoints on $\mathcal{J}$. **Not exactly since** $0 \in B$ □

Lemma 5.7. *For any two points z_1, z_2 on the Julia set of $f(z) = z^2 + \frac{1}{4}$, if $f(z_1) = f(z_2)$ then $|z_1 - z_2| \geq c_0$ for a universal constant $c_0 > 0$.*

Proof. Pulling back to the unit disk $\mathbb{D}$, the result holds for $z \mapsto z^2$. □

5.3 Departure Sets

The purpose of the following definition is to organize points according to their distance from the main cusp in an f-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure*: the first time that the central itinerary made a turn.

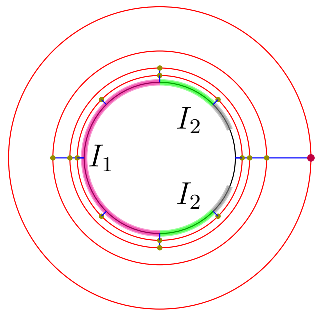

Figure 3: First few parts of the departure decomposition I_m of the circle.

Definition 5.8. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 3.

This decomposition is invariant under f_0 in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of η_ζ. Thus by applying the Böttcher map ψ we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under f.

5.4 Quasiconvexity: three special cases

We define constants C_1, C_2, ϵ as follows. We first choose $C_2 \geq 2$, then we let $C_1 = C_2 + d + 2$ and choose $\epsilon > 0$ small enough so that we have both

$$\mathrm{dist}(\alpha_{C_1}, \alpha_{C_2}) \geq k\epsilon. \tag{5.4}$$

We now show that the itineraries η_{w_1,w_2} are certificates in three special cases:

1. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| < d, \quad m, n < C_1;$
2. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| < d, \quad m, n > C_2;$
3. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

For each n we denote by α_n the union of the two outermost tracks emanating from the station $s_{m,0}$. Namely, α_n has a positively-oriented peripheral track after each express track. We denote the domain enclosed by α_m, α_n and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and denote the domain enclosed by $\mathcal{J}$ and α_n by $\mathcal{K}_n$.

Lemma 5.9. *Let $w_1, w_2 \in \mathcal{J}$ be points, with $w_1 \in I_m$ and $w_2 \in I_n$ for $m, n \geq 2$. Then the itinerary η_{w_1,w_2} is contained in the domain $\mathcal{K}_{m-1,n+1}$.*

Lemma 5.10. *Let $w_1, w_2 \in \mathcal{J}$. In each of the three special cases, the itinerary γ_{w_1,w_2} is a quasiconvexity certificate. In cases 1 and 2, γ_{w_1,w_2} is relatively separated. be as in case 1. Then the itinerary η_{w_1,w_2} is relatively separated from the postcritical set.*

Proof. case 2. Assume $n \geq m$, without loss of generality. Notice that the itinerary is contained in the domain $\mathcal{K}_{m-1,n+1}$, which has a positive relative distance from the cusp p.

case 1. The itinerary in this case is contained in a domain of the form

$$\text{Exterior}(\mathcal{J}) \cap B(0, R) \setminus \mathcal{K}_{C_1+1}.$$

□

In Case 3, the points w_1, w_2 lie in "well-separated cusps". This gives some control from below on $|w_1 - w_2|$. We now bound the length of the itinerary $\psi(\eta) = \psi(\eta_{w_1,w_2})$ from above. We represent $\psi(\eta)$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = \psi([s_{m,0}, s_{n,0}])$ and the two other components, γ_m and γ_n. See Figure 4. Thus we have

$$\text{Length}(\psi(\eta)) = \text{Length}(\gamma_m) + \text{Length}(\gamma_{m,n}) + \text{Length}(\gamma_n). \tag{5.5}$$

The condition $|m - n| \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to γ_m and γ_n:

Proposition 5.11. (i) *We have*

$$\text{Length}(\gamma_m) \lesssim \text{Length}(\gamma_{m,n}). \tag{5.6}$$

(ii) *For $d \gg 1$,*

$$\text{Length}(\gamma_{m,n}) \lesssim |z_1 - z_2|. \tag{5.7}$$

Proof. (i) Koebe's distortion theorem gives that

$$\text{Length}(\gamma_m) \lesssim \ell_m \tag{5.8}$$

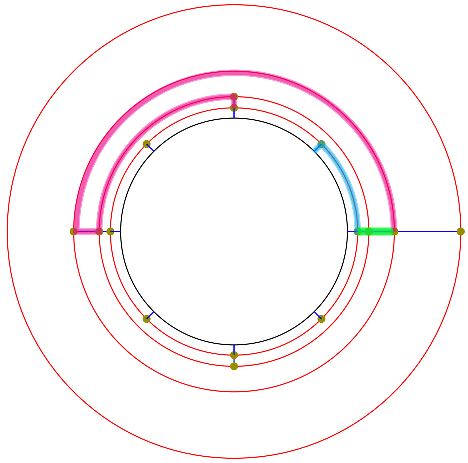

Figure 4: The three parts of an itinerary η. The green path is $\gamma_{m,n}$, the cyan and magenta are γ_m and γ_n.

by the invariance of both sides under f^{-1}.

(ii) By the triangle inequality and part (i),

$$\begin{aligned}\text{Length}(\gamma_{m,n}) &\leq |w_1 - w_2| + \text{Length}(\gamma_m) + \text{Length}(\gamma_n) \\ &\leq |w_1 - w_2| + C(\ell_m + \ell_n),\end{aligned} \tag{5.9}$$

for a constant $C \geq 0$ independent of d. Applying Lemma 5.4 on this constant C we get that for $d \gg 1$,

$$\text{Length}(\gamma_{m,n}) \lesssim \text{Length}(\gamma_{m,n}) - C(\ell_m + \ell_n)$$

which together with (5.9) concludes the proof. □

5.5 Quasiconvexity: general case

We apply a stopping time argument to promote quasiconvexity of γ_{w_1,w_2} to quasiconvexity of $\gamma z_1, z_2$, thereby proving the following:

Theorem 5.12. *The domain* Exterior($\mathcal{J}$) *is quasiconvex with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let $d \gg 1$ be a sufficiently large integer and let (z_1, z_2) be a pair of points in $\mathcal{J}$. Repeatedly apply f on z_1, z_2 until one of the following stopping conditions occurs:

(i) $\left|f^{\circ N}(z_1) - f^{\circ N}(z_2)\right| > \epsilon$, where $\epsilon = \epsilon(d) > 0$ is a constant to be chosen later, or

(ii) $f^{\circ N}(z_1) \in I_n$ and $f^{\circ N}(z_2) \in I_m$ with $|m - n| \geq d$.

Denote $w_i = f^{\circ N}(z_i)$. We already proved that the itinerary η_{w_1,w_2} satisfies

$$\text{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|$$

for some $A > 0$. We now deduce that the original points z_1, z_2 enjoy a similar estimate,

$$\text{Length}(\eta_{z_1,z_2}) \leq C|z_1 - z_2|,$$

where C depends only on A.

In cases 1 and 2, we are done by Lemma 5.10. We therefore focus on case 3. We show that in case 3, the curve $f^{\circ(N-1)}(\gamma)$ is relatively separated.

We fix a small ball $B = B(p, \eta)$ around the main cusp $p = \frac{1}{2}$ of radius $\eta = \eta(\epsilon, d)$. We choose η small enough so that any two points w_1, w_2 with $|w_1 - w_2| \geq \epsilon$ and $w_1 \in B(p, \eta)$ must satisfy criterion *(ii)* with strict inequality. The preimages of B are topological balls centered at cusps q of $\mathcal{J}$, and we index them by B_ζ for $\zeta = \phi^{-1}(q) \in \partial\mathbb{D}$, where ϕ is the Riemann map.

Repeatedly applying f^{-1} to the pair (w_1, w_2) until we reach (z_1, z_2), we get two sequences of balls $B = B_1^1, B_2^1, \ldots$ and $B = B_1^2, B_2^2, \ldots$. We now consider each case separately.

Case (i). Suppose stopping criterion (i) occured, namely $|w_1 - w_2| > \epsilon(d)$. By the stopping criterion we have $|f^{-1}(w_1) - f^{-1}(w_2)| < \epsilon$, so $|w_1 - w_2| < k\epsilon$ where we denote $k = \max_{z \in \mathcal{J}} |f'(z)|$. Thus $\text{Length}(\eta_{w_1,w_2}) \leq k\epsilon A$ where A is the quasiconvexity bound of η_{w_1,w_2}. It folllows that the itinerary η_{w_1,w_2} is contained in the ball $N_1 := B(w_1, k\epsilon A)$. By increasing k if needed, we assume that $kA > 1$.

Sub-case (i.1). Suppose that the ball N_1 is disjoint from the ball $B = B(p, \eta)$ around the cusp $p = \frac{1}{2}$. Choose a neighborhood N_0 of N_1 that is disjoint from B, and a neighborhood $N_2 \subset N_1$ such that for every pair of points w, w' in N_2 the itinerary $\eta_{w,w'}$ is contained in N_1. Applying Koebe's distortion theorem on N_2 shows quasiconvexity in this sub-case directly.

By compactness of the set set of pairs (w_1, w_2) in $\mathcal{J} \backslash B$ with distance $|w_1 - w_2| \geq \epsilon$, these neighborhoods can be chosen from a finite collection, hence the quasiconvexity constant is uniform in the pair (z_1, z_2) and sub-case *(i.1)* is proven.

Sub-case (i.2). We choose η small enough so that we can't have both $w_1, w_2 \in B = B(p, \eta)$. Thus we are left with the case where $w_1 \in B$ and $w_2 \notin B$. Here we use Lemma 5.4.

Case (ii). Suppose that stopping criterion (ii) occurred. We can assume that w_1, w_2 are both inside the ball B, since $m - n > d$. We claim that the two sequences of balls $(B_k^1), (B_k^2)$ coincide. Indeed, suppose otherwise and let j be the first index for which $B_j^1 \neq B_j^2$. The two points $f^{-j}(z_1), f^{-j}(z_2)$ belong to two distinct preimages of the same ball $B_{j-1}^1 = B_{j-1}^2$, hence they are a positive distance apart and stopping criterion i applies. This is a contradiction, hence the two sequences coincide.

There is thus a branch of $f^{\circ -(N-1)}$ sending B_{-1} to the preimage topological ball containing (z_1, z_2) and sending (w_1, w_2) to (z_1, z_2). The necessary bound on η_{z_1, z_2} follows from applying Koebe's distortion theorem on the composition f^{-N} of this branch with the branch of f^{-1} sending p to $-p$. Explicitly, Koebe's distortion theorem gives

$$\frac{\text{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|}. \tag{5.10}$$

$\square$

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