1 Introduction

Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial. Its *filled Julia set* consists of the points in the complex plane with bounded orbit under iteration by f_c:

$$\mathcal{K}_c = \{z \in \mathbb{C} : \sup_{n \geq 0} f_c^{\circ n}(z) < \infty\}.$$

Its boundary $\mathcal{J}_c = \partial\mathcal{K}(f_c)$ is known as the *Julia set*, and its complement $\mathrm{Exterior}(\mathcal{J}_c) = \mathbb{C} \setminus \mathcal{K}(f_c)$ forms the *attracting basin of infinity*.

The set $\mathcal{K}_c$ is compact, and each of the three sets $\mathcal{J}_c, \mathcal{K}_c$ and $\mathrm{Exterior}(\mathcal{J}_c)$ are both forward and backward invariant under the dynamics of f.

The *main cardioid*

$$\heartsuit = \left\{c \in \mathbb{C} : c = \lambda/2 - \lambda^2/4,\ \lambda \in \mathbb{D}\right\}$$

is the set of parameters $c \in \mathbb{C}$ for which f_c has an attracting fixed point. When $c \in \heartsuit$, the Julia set $\mathcal{J}_c$ is a *quasidisk*, the image of a round disk under a quasiconformal map. This intuitively means that $\mathcal{K}_c$ has no "cusps".

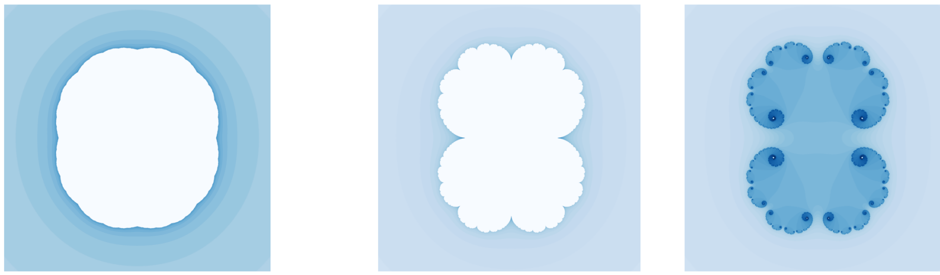
In this work we take $c = 1/4$, which lies on the boundary of $\heartsuit$. The filled Julia set $\mathcal{K}_{1/4}$, also called the *Cauliflower*, is a Jordan domain with an inward-pointing cusp at the point $p = 1/2$. However, according to a theorem of Carleson, Jones and Yoccoz [1, Theorem 6.1], the Cauliflower is a *John domain*, a condition which rules out "outward-pointing cusps". Formally, a domain Ω is John if there exists a "center" point $z_0 \in \Omega$ that can be connected to any other point $z_1 \in \Omega$ by a curve γ which stays away from the boundary:

$$\mathrm{dist}(z, \partial\Omega) \gtrsim |z_1 - z| \tag{1.1}$$

for all $z \in \gamma$. See Figure 1.

A domain $\Omega \subseteq \mathbb{C}$ is called **quasiconvex** if its intrinsic metric is comparable to the ambient Euclidean metric. Explicitly, this means that there exists a constant $A \geq 1$ such that every two points $z_1, z_2 \in \Omega$ are connected by a rectifiable path $\gamma_{z_1,z_2} : [0, 1] \to \Omega$ which satisfies

$$\mathrm{Length}(\gamma_{z_1,z_2}) \leq A \cdot |z_1 - z_2|\,. \tag{1.2}$$



(a) $c = 0.1$.

(b) $c = 1/4$, the Cauliflower.

(c) $c = 0.3$.

Figure 1: The Julia set $\mathcal{J}_c$ of f_c for different values of c. When $c > 1/4$, the Julia set is no longer connected.

We refer to such a family of paths γ_{z_1,z_2} as *quasiconvexity certificates* for Ω.

If Ω is a quasiconvex Jordan domain, then its complement has a John interior; see [3, Corollary 3.4] for a proof. In this work, we strengthen the result of [1, Theorem 6.1] by showing:

Theorem 1.1. *The exterior of the Cauliflower is quasiconvex.*

Our result also has a function-theoretic interpretation. For a planar domain $\Omega \subset \mathbb{R}^2$, the *Sobolev space* $W^{1,1}(\Omega)$ is the set of functions $u \in L^1(\Omega)$ for which the distributional derivatives $\partial_1 u, \partial_2 u$ exist and are in $L^1(\Omega)$. We call Ω a *$W^{1,1}$ extension domain* if every $u \in W^{1,1}(\Omega)$ extends to a function in $W^{1,1}(\mathbb{C})$. In [2, Equation (1.1) and Theorem 1.4], it is shown that a bounded simply connected domain is a $W^{1,1}$ extension domain if and only if its complement is quasiconvex. Thus our result can be rephrased as follows:

Theorem 1.2. *The Cauliflower is a $W^{1,1}$ extension domain.*

1.1 Sketch of the argument

To show that a Jordan domain Ω is quasiconvex, it is enough to find certificates for points z_1, z_2 that lie on the boundary curve $\partial\Omega$. For a proof, see [3, Corollary F].

We show quasiconvexity by an explicit construction of certificates connecting any given pair of points on the Julia set. We first build the certificates in the exterior unit disk $\mathbb{D}^*$, then we transport them to the exterior of the Cauliflower by the Riemann map $\psi : \mathbb{D}^* \to \text{Exterior}(\mathcal{J}_{1/4})$, which conjugates f_0 with $f_{1/4}$.

To retain control of the certificates after applying ψ, we build the certificates of $\mathbb{D}^*$ in a manner invariant under the map $f_0 : z \mapsto z^2$. This translates after applying ψ to an invariance under $f_{1/4}$, which allows us to employ a parabolic variant of the principle of the conformal elevator.

To facilitate the reading, we first demonstrate the proof in the hyperbolic case of maps $f_c(z) = z^2 + c$ where $c \in \heartsuit$), in which the usual conformal elevator applies, and we subsequently treat the parabolic case of $c = \frac{1}{4}$.

2 Complex-analytic preliminaries

2.1 The hyperbolic metric

Even though quasiconvexity is defined using the Euclidean metric, the arguments will involve the hyperbolic metric, which is better-behaved in our setting.

Theorem 2.1. *The hyperbolic metric*

$$\mathrm{ds} = \frac{2}{1 - |z|^2} \, |\mathrm{dz}| \tag{2.1}$$

is the unique Riemannian metric on the unit disk $\mathbb{D}$, up to multiplication by a positive constant, which is invariant under conformal automorphisms.

This defines the hyperbolic metric on topological balls, by requiring that the Riemann map will be an isometry. Notice that this metric is locally equivalent to the Euclidean metric. We will need the hyperbolic metric in more general domains:

Definition 2.2. A domain $U \subset \mathbb{C}$ is *hyperbolic* if its universal covering $\tilde{U}$ of U is biholomorphic to $\mathbb{D}$.

Theorem 2.3. *Let $U \subset \mathbb{C}$ be a domain. If the complement $\mathbb{C} \setminus U$ has at least two points then the domain U is hyperbolic.*

Definition 2.4. Let U be a hyperbolic domain, and equip $\tilde{U}$ with the hyperbolic metric defined in Equation (2.1). The *hyperbolic metric* on U is the unique Riemannian metric for which the projection $\tilde{U} \to U$ is a local isometry.

Theorem 2.5 (The Schwarz-Pick theorem)**.** *Let $f : U_1 \to U_2$ be a holomorphic map between two hyperbolic domains $U_1, U_2 \subset \mathbb{C}$. Then f is a hyperbolic contraction, meaning that*

$$\operatorname{dist}(f(z), f(w)) \leq \operatorname{dist}(z, w) \tag{2.2}$$

for all $z, w \in U_1$. If f is not a covering map, then the inequality is strict for all $z \neq w$.

Proof. ([5], Theorem 2.11). The classical Schwarz lemma is the case $f : \mathbb{D} \to \mathbb{D}$. It implies the general case since the projections from the universal coverings are local isometries. If f is not a covering map, then the lift of f to the universal coverings is a strict contraction $\mathbb{D} \to \mathbb{D}$, again by the classical case. □

Definition 2.6. Let $f : U_1 \to U_2$ be holomorphic map between hyperbolic domains, and let $z \in U_1$. The *hyperbolic derivative* of f at the point z is the operator norm of derivative of f at z:

$$\|f'(z)\|_{hyp} = \frac{\|Df(z)(v)\|_{\mathrm{hyp}(U_2)}}{\|v\|_{\mathrm{hyp}(U_1)}}, \tag{2.3}$$

where v is any nonzero tangent vector at the point z.

2.2 The Distortion Principle

We record here for convenience a form of Koebe's distortion principle that will be used repeatedly.

Definition 2.7. Every topological annulus $A \subset \hat{\mathbb{C}}$ is biholomorphic to a unique round annulus of the form $\{1 < |z| < R\}$. The (conformal) *modulus* of A is the value $\operatorname{Mod}(A) = \frac{1}{2\pi} \log R$.

Theorem 2.8 (Koebe's Distortion Principle, [4, Theorem 2.9])**.** *Let $D \subset U$ be topological disks with* $\operatorname{Mod}(U \setminus D) \geq m > 0$ *and let f be a map univalent in U, then* *we have the bound*

$$\frac{|f(y)-f(z)|}{|y-z|} \asymp_m |f'(x)| \tag{2.4}$$

for all $x, y, z \in D$.

2.3 The relative distance

The following notion will be used for showing the existence of a Koebe space, i.e. lower bounding the conformal modulus between two sets.

Definition 2.9. The *relative distance* between two sets $E, F \subset \mathbb{C}$ is

$$\Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))}, \tag{2.5}$$

and $\Delta(E,F) = \infty$ if the denominator vanishes. If $\Delta(E,F) \geq \eta$, we say that E and F are *η-relatively separated.*

Theorem 2.10. *Let $E, F \subset \mathbb{C}$ be compact connected sets. If E, F are η-relatively separated, for some constant $\eta > 0$, then there exists a topological ball B containing E with a definite modulus inside $\hat{\mathbb{C}} \setminus F$:*

$$\operatorname{Mod}(\hat{\mathbb{C}} \setminus (B \cup F)) \gtrsim_\eta 1. \tag{2.6}$$

Proof. To-do. This might be related to ”the plane is a Loewner space”. □

3 The exterior disk

We connect any two boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ by a path in $\mathbb{D}^*$ in a manner that respects the map $f_0 : \zeta \mapsto \zeta^2$. We describe these paths using the metaphor of a passenger who travels by train:

Definition 3.1. *Stations* are the points in $\mathbb{D}^*$ of the form

$$s_{n,k} = 2^{1/2^n} \exp\left(\frac{k}{2^n} 2\pi i\right), \qquad n \in \mathbb{N}_0, \quad k \in \{0, \ldots, 2^n - 1\}.$$

These are the iterated preimages of the *central station* $s_{0,0} = 2$ under the map f_0. We refer to n as the *generation* of the station $s_{n,k}$. The 2^n stations of generation n are equally spaced on the circle $C_n = \left\{|\zeta| = 2^{1/2^n}\right\}$.

We next lay two types of "rail tracks", which we use to travel between stations.

Definition 3.2. Let $s = s_{n,k}$ be a station.

1. The *peripheral neighbors* of s are the two stations $s_{n,(k\pm 1)(\bmod 2^n)}$ adjacent to $s_{n,k}$ on C_n.

2. The *peripheral track* $\gamma_{s,s'}$ from s to a peripheral neighbor s' is the shorter arc of the circle C_n connecting s to s'.

3. The *radial successor* of s is $\mathrm{RadialSuccessor}(s) = s_{n+1,2k}$, the unique station of generation $n+1$ on the radial segment $[0, s]$.

4. The *express track* $\gamma_{s,s'}$ from s to its radial successor s' is the radial segment $[s, s']$.

Notice that the tracks respect the dynamics: applying f_0 to a track gives a track of the previous generation.

When a passenger travels between two stations s_1 and s_2, they must follow a particular itinerary from s_1 to s_2. If s_1 is the central station, then this itinerary is determined by the rule that the passenger stays as close as possible to its destination s_2 in the angular distance. This also determines how to travel from the central station to a boundary point $\zeta \in \partial\mathbb{D}^*$, by continuity. See Figure 2 and the next definition.

Definition 3.3. Let $\zeta = \exp(2\pi i\theta) \in \partial\mathbb{D}$. The *central itinerary* of ζ is a path $\eta_\zeta = \gamma_{\sigma_0,\sigma_1} + \gamma_{\sigma_1,\sigma_2} + \ldots$ from the central station to ζ, made of tracks between the stations $\sigma_0, \sigma_1, \ldots$. It is defined inductively as follows:

Start at the central station $\sigma_0 = s_{0,0}$. Suppose that we already chose $\sigma_0, \ldots, \sigma_k$. If there is a peripheral neighbor σ of σ_k that is closer peripherally to ζ, meaning that

$$|\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma)| < |\mathrm{Arg}\,(\zeta) - \mathrm{Arg}\,(\sigma_k)|\,,$$

then take $\sigma_{k+1} = \sigma$. Otherwise, take $\sigma_{k+1} = \mathrm{RadialSuccessor}(\sigma)$.

We identify the central itinerary η_ζ with its sequence of stations $(\sigma_0, \ldots)$. We record two properties of central itineraries:

- There are no two consecutive peripheral tracks in η_ζ, and in particular

$$\mathrm{Generation}(\sigma_k) \geq \frac{k}{2}; \tag{3.1}$$

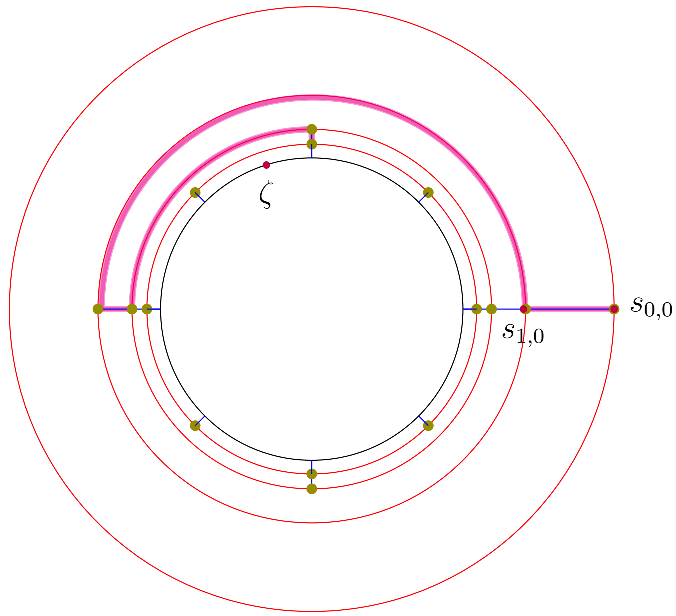


Figure 2: The central itinerary to a point ζ.

- Central itineraries are essentially equivariant under f_0, in the sense that

$$f_0(\eta_\zeta) = \eta_{f_0(\zeta)} \cup [s_{0,0}, f_0(s_{0,0})]$$

  for every $\zeta \in \partial\mathbb{D}^*$.

Definition 3.4. Given two distinct boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$, form the central itineraries $\eta_{\zeta_1} = (\sigma_n^1)_{n=0}^\infty$ and $\eta_{\zeta_2} = (\sigma_n^2)_{n=0}^\infty$ and let $\sigma = \sigma_i^1 = \sigma_j^2$ be the last station that is in both η_{ζ_1} and η_{ζ_2}. We define the *itinerary* between ζ_1 and ζ_2 to be the path

$$\eta_{\zeta_1,\zeta_2} = \left(\ldots, \sigma_{i+2}^1, \sigma_{i+1}^1, \sigma, \sigma_{j+1}^2, \sigma_{j+2}^2, \ldots\right).$$

This is a simple bi-infinite path connecting ζ_1 and ζ_2, see Figure 3. Note that itineraries are equivariant under the dynamics: we have

$$f(\eta_{\zeta_1,\zeta_2}) = \eta_{f(\zeta_1),f(\zeta_2)} \tag{3.2}$$

for every pair of boundary points $\zeta_1, \zeta_2 \in \partial\mathbb{D}^*$ with $|\zeta_1 - \zeta_2| < \sqrt{2}$.

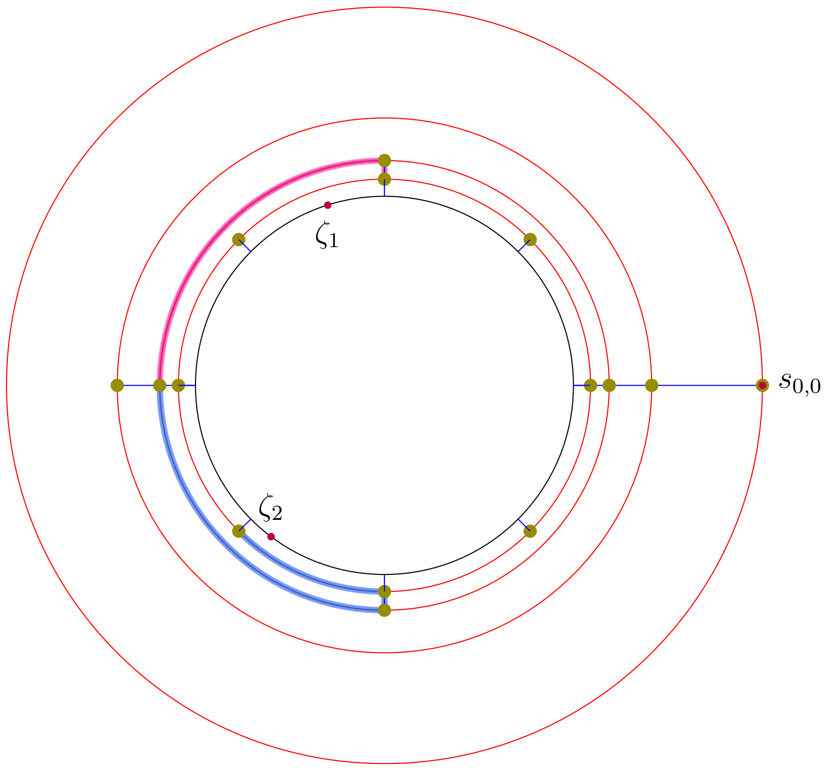


Figure 3: A quasiconvexity certificate between two points ζ_1, ζ_2 in $\mathbb{D}^*$. Only the first two steps are shown.

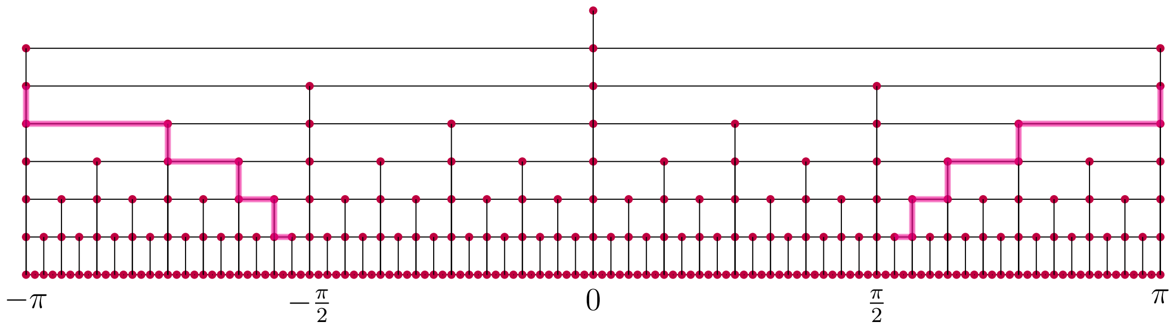


Figure 4: A convenient representation of the dyadic grid in the Böttcher coordinates. The horizontal axis is the external angle $\mathrm{Arg}(\psi^{-1}(z))$, and the vertical axis is the equipotential $|\psi^{-1}(z)|$, plotted on a log scale. The rightmost edge is glued to the leftmost edge. Stations are marked in red, and the segments connecting adjacent stations are tracks. An express track is a vertical segment, while a peripheral track is a horizontal segment. The area under the magenta contour represents the domain $\mathcal{U}_{-1}$.

4 Transporting the Rails

Let $c \in \heartsuit$. For these values of c, the Julia set of $f_c : z \mapsto z^2 + c$ is a Jordan curve, and f_c has a Böttcher coordinate ψ at infinity; namely, ψ is the unique conformal map $\mathbb{D}^* \to \mathrm{Exterior}(\mathcal{J}_c)$ which fixes ∞ and satisfies the conjugacy relation

$$f \circ \psi = \psi \circ f_0.$$

The Böttcher coordinate ψ extends to a homeomorphism between the unit circle $\partial\mathbb{D}$ and $\mathcal{J}_c$ by Carathéodory's theorem. See [5, Theorem 9.5] for a proof of existence, relying on the explicit construction

$$\psi(z) = \lim_{n\to\infty} (f_0)^{\circ(-n)} \circ f^{\circ n} = \lim_{n\to\infty} (f^{\circ n})^{1/2^n}. \tag{4.1}$$

We apply ψ to the rails that we constructed in $\mathbb{D}^*$ to obtain the corresponding rails in $\mathrm{Exterior}(\mathcal{J}_c)$:

Definition 4.1.

1. The *stations* of f_c are the points $\psi(s_{n,k})$.

2. The *tracks* of f_c are the curves of the form $\psi\left(\gamma_{s,s'}\right)$, where $\gamma_{s,s'}$ is a track. They are classified as express or peripheral according to the corresponding classification of $\gamma_{s,s'}$. Express tracks lie on the external rays of the filled Julia set $\mathcal{K}_c$, while peripheral tracks lie on the equipotentials of $\mathcal{K}_c$.

3. The *itinerary* between a pair of points (z_1, z_2) on $\mathcal{J}_c$ is $\eta_{z_1,z_2} = \psi(\eta_{\zeta_1,\zeta_2})$, where $\zeta_i = \psi^{-1}(z_i)$ are the corresponding points on $\partial\mathbb{D}^*$.

We henceforth omit c and ψ from the notation for ease of reading. It will be clear from the context whether we work in $\mathbb{D}^*$ or in Exterior$(\mathcal{J})$.

Note that $\psi\left((1,\infty)\right) \subseteq \mathbb{R}$ since $\mathcal{J}$ is symmetric with respect to the real line, and in particular the central station $\psi(s_{0,0})$ lies on the real axis.

5 Hyperbolic Maps

In this section we prove the quasiconvexity of Exterior$(\mathcal{J})$ for parameters c in the main cardioid $\heartsuit$. Since the argument will involve iterating f and then iterating f^{-1}, we will need to exclude points around which there is no holomorphic inverse for some iterate $f^{\circ n}$:

Definition 5.1. The *post-critical set* of f is the closure of the forward orbits of the critical points,

$$\mathcal{P} = \overline{\{f^{\circ n}(0) : n \geq 1\} \cup \{\infty\}}.$$

For every $c \neq 0$, the post-critical set $\mathcal{P}$ of the map $f : z \mapsto z^2 + c$ contains at least 3 points and consequently its complement $\hat{\mathbb{C}} \setminus \mathcal{P}$ is a hyperbolic domain by Theorem 2.3.

The map f *hyperbolic* if its post-critical set $\mathcal{P}$ is disjoint from its Julia set $\mathcal{J}$. This is equivalent to f being expanding on $\mathcal{J}$:

Theorem 5.2. *Let $c \neq 0$, and view $f : z \mapsto z^2 + c$ as the map*

$$f : f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P}) \to \hat{\mathbb{C}} \setminus \mathcal{P},$$

with the corresponding hyperbolic metrics. Then we have

$$\|f'(z)\|_{\mathrm{hyp}} > 1 \tag{5.1}$$

for every $z \in f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$*, where the hyperbolic derivative was defined in Equation* (2.3)*.*

Proof. (Proof. [5, Theorem 19.1]) The map f is a covering map, hence it is a local isometry as a map

$$f : f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P}) \to \hat{\mathbb{C}} \setminus \mathcal{P}$$

by Theorem 2.5. We conclude by composing with the inclusion $f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P}) \hookrightarrow \hat{\mathbb{C}} \setminus \mathcal{P}$, which is a strict contraction by Theorem 2.5. □

Corollary 5.3. *If* $c \in \heartsuit$ *and* $c \neq 0$*, then we have*

$$\|f'(z)\|_{\mathrm{hyp}} \geq \kappa \tag{5.2}$$

for all $z \in \mathcal{J}$*, for some constant* $\kappa > 1$*.*

Proof. By hyperbolicity $\mathcal{J} \subseteq f^{-1}(\hat{\mathbb{C}} \setminus \mathcal{P})$, and the claim follows by compactness. □

Corollary 5.4. *Let* f *be a hyperbolic quadratic map. There exists* $\epsilon > 0$ *such that every pair of points* $z, w \in \mathcal{J}$ *has a forward iterate* $f^{\circ n}$ *for which*

$$|f^{\circ n}(z) - f^{\circ n}(w)| > \epsilon.$$

Proof. We first convert the bound of Corollary 5.4 from the hyperbolic metric to the Euclidean metric by passing to an iterate of f. Indeed, for every iterate $f^{\circ m}$ we have

$$\|(f^{\circ m})'(z)\|_{\mathrm{hyp}} \geq \kappa^m \tag{5.3}$$

for every point $z \in \mathcal{J}$ and every tangent vector v at z. As the hyperbolic metric and the Euclidean metric are equivalent on $\mathcal{J}$, we may take m large enough so that for $g = f^{\circ m}$ there is a uniform bound $|g'| > \mu$ on $\mathcal{J}$, for some constant $\mu > 1$. By compactness, there exists $\epsilon > 0$ such that whenever $|z - w| < \epsilon$ on $\mathcal{J}$, we have $|g(z) - g(w)| \geq \mu |z - w|$. The claim follows by iterating g. □

Definition 5.5. A point $z \in \mathcal{J}$ is *rectifiably accessible* from $\text{Exterior}(\mathcal{J})$ if there is a rectifiable curve $\gamma : [0, 1) \to \text{Exterior}(\mathcal{J})$ such that $\gamma(t) \to z$ as $t \to 1$.

We are now ready to show quasiconvexity in the hyperbolic case:

Theorem 5.6. *Let $f : z \mapsto z^2 + c$ be a quadratic map with $c \in \heartsuit$.*

(i) *Given $z \in \mathcal{J}$ decompose its central itinerary into tracks,*

$$\eta_z = \gamma_1 + \gamma_2 + \dots .$$

We have the estimate

$$\text{Length}(\gamma_k) \lesssim \theta^k,$$

uniformly in z, for some constant $\theta = \theta(c) < 1$. In particular, any point on $\mathcal{J}$ is rectifiably accessible.

(ii) *The domain* $\text{Exterior}(\mathcal{J})$ *is quasiconvex with the itineraries η_{z_1,z_2} as certificates.*

Proof. (i) For $c = 0$, this is a direct computation. Suppose $c \neq 0$, and let $\mathcal{P}$ be the post-critical set of f.

Any branch of $f^{-1} : \hat{\mathbb{C}} \setminus \mathcal{P} \to \hat{\mathbb{C}} \setminus \mathcal{P}$ is a strict hyperbolic contraction by Theorem 5.2.

Let $B(0, R) \subset \mathbb{C}$ be a ball large enough that it contains every central itinerary. By hyperbolicity, $\hat{\mathbb{C}} \setminus \mathcal{P}$ contains $\overline{\text{Exterior}(\mathcal{J})}$. Thus $\text{Exterior}(\mathcal{J}) \cap B(0, R)$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$, and there is a constant $\theta < 1$ such that $\|(f^{-1})'\|_{\text{hyp}} < \theta$ on $\text{Exterior}(\mathcal{J}) \cap B(0, R)$. Therefore,

$$\begin{aligned} \text{HypLength}(\gamma_k) &\leq \theta \cdot \text{HypLength}(f(\gamma_k)) \\ &\leq \dots \\ &\leq \theta^k \cdot \text{HypLength}(f^{\circ k}(\gamma_k)), \\ &\lesssim \theta^k, \end{aligned}$$

where the last inequality holds since $f^{\circ k}(\gamma_k)$ lies on the real axis in case γ_k is an express track, or on the equipotetial $\psi(\{|z| = \sqrt{2}\})$ otherwise.

As the hyperbolic metric is equivalent to the Euclidean metric on compact subsets of $\hat{\mathbb{C}} \setminus \mathcal{P}$, we conclude that Length$(\gamma_k) \lesssim \theta^k$ as well.

Thus any point on $\mathcal{J}$ can be reached from the central station $s_{0,0}$ by a curve of bounded length.

(ii) By Corollary 5.4, there exists an $\epsilon > 0$ such that any two points are ϵ-apart under some iterate of f. Let $z_1, z_2 \in \mathcal{J}(f)$. If $|z_1 - z_2| \geq \epsilon$, we are done since the length of η_{z_1,z_2} is bounded from above uniformly by part (i). On the other hand, if $|z_1 - z_2| < \epsilon$, then there is an iterate $f^{\circ n}$ for which

$$|w_1 - w_2| := |f^{\circ n}(z_1) - f^{\circ n}(z_2)| \geq \epsilon \tag{5.4}$$

and we have a uniform bound on

$$\frac{\text{Length}\,(\eta_{w_1,w_2})}{|w_1 - w_2|}$$

as before. Thus we are left with showing that

$$\frac{\text{Length}\,(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}\,(\eta_{w_1,w_2})}{|w_1 - w_2|}, \tag{5.5}$$

which we rewrite as

$$\frac{|z_1 - z_2|}{|w_1 - w_2|} \asymp \frac{\text{Length}\,(\eta_{z_1,z_2})}{\text{Length}\,(\eta_{w_1,w_2})}. \tag{5.6}$$

We shall now deduce Equation (5.6) from a distortion argument. Since $\mathcal{P}$ is compactly contained in $\mathcal{K}$, we may enlarge $\mathcal{P}$ to a compact connected set $\tilde{\mathcal{P}} \subset \mathcal{K}$ while maintaining a definite distance from the boundary $\mathcal{J}$. By Theorem 2.10, there is a topological ball $B = B(z_1, z_2)$ containing the itinerary η_{w_1,w_2} and having a definite modulus inside $\hat{\mathbb{C}} \setminus \tilde{\mathcal{P}}$.

Denoting $f^{\circ n}$ by g, we apply Theorem 2.8 on a branch of g^{-1} defined in all of B and sending η_{w_1,w_2} to η_{z_1,z_2}. Such a branch exists because B is disjoint from $\mathcal{P}$ and since we can ensure that B does not encircle the origin by removing an external ray disjoint from η_{w_1,w_2}. (Can I prove it?)

It follows that

$$\frac{|z_1 - z_2|}{|w_1 - w_2|} \asymp |(g^{-1})'(x)| \asymp \frac{\text{Length}\,(\eta_{z_1,z_2})}{\text{Length}\,(g(\eta_{z_1,z_2}))}. \tag{5.7}$$

for any point $x \in B$, as needed. $\square$

6 The Cauliflower

In this section, $c = \frac{1}{4}$ and $f = f_{1/4} : z \mapsto z^2 + \frac{1}{4}$. Our goal is to prove the quasiconvexity of Exterior($\mathcal{J}$), Theorem 6.10. This is more complicated than the hyperbolic case, because the post-critical set $\mathcal{P}$ of f accumulates at the parabolic fixed point $p = \frac{1}{2}$. One no longer has a uniform bound on the distortion of inverse iterates, and we cannot immediately deduce the quasiconvexity of the itinerary η_{z_1,z_2} from the quasiconvexity of η_{w_1,w_2} using Koebe's distortion theorem. As a substitute, we present an analogue of the principle of the conformal elevator in this parabolic setting.

6.1 Itineraries have finite length

We first show that each itinerary η_{z_1,z_2} has a finite length. We will in fact show an exponential decay of the lengths of the constituent tracks. For this to hold it is necessary to glue together consecutive express tracks: for example, the tracks in the central itinerary that lies on the real axis, $\eta_{1/2}$, have only a quadratic rate of length decay. To fix this, we introduce:

Definition 6.1. The *reduced decomposition* of an itinerary η is the unique decomposition $\eta = \gamma_1 + \delta_1 + \dots$ where each γ_i is a concatenation of express tracks and is followed by a single peripheral track δ_i.

Proposition 6.2. *Let $z \in \mathcal{J}$, and let $\eta_z = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of its itinerary. Then* $\mathrm{Length}(\gamma_k) \lesssim \theta^k$ *and* $\mathrm{Length}(\delta_k) \lesssim \theta^k$ *for some $\theta < 1$. In particular,* $\mathrm{Length}(\eta_z) < \infty$ *and all points $z \in \mathcal{J}$ are rectifiably accessible.*

For the proof, let $\mathcal{U}_{-1}$ be the Jordan domain enclosed by the unit circle, the rightmost itinerary starting from the pre-central station and the leftmost one. See Figure 4. This domain is constructed so that it contains all itineraries that start at the station $s_{1,1} = \psi(-1/2)$, the preimage of the central station under f. Its crucial property is:

Lemma 6.3. *Let $\gamma = \gamma_1 + \delta_1 + \dots$ be the reduced decomposition of an itinerary γ. Then for every $k > 1$, there exist $k - 1$ iterates $n_1 < \dots < n_{k-1}$ such that $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$.*

Proof. Every station $s \notin (0,\infty)$ has a first iterate $f^{\circ n_s}(s)$ lying on the negative real axis $(-\infty, 0)$. For any $i \in \{2, \ldots, k-1\}$, let s_i be the first station of γ_i and take $n_i := n_{s_i}$. By the definition of $\mathcal{U}_{-1}$, the itinerary $f^{\circ n_i}(\gamma)$ is contained in $\mathcal{U}_{-1}$ from the station $f^{\circ n_i}(s_i)$ onwards, and in particular $f^{\circ n_i}(\gamma_k) \subset \mathcal{U}_{-1}$. □

Proof (Proposition 6.2). There is a uniform bound $\|(f^{-1})'\|_{\mathrm{hyp}} < \theta < 1$ on $\mathcal{U}_{-1}$ with respect to the hyperbolic metric of the domain $\hat{\mathbb{C}} \setminus \mathcal{P}$, for both branches $f^{-1} : \mathcal{U}_{-1} \to \mathcal{U}_{\pm i}$. This follows from Theorem 5.2, in the slightly more general formulation of [4, Theorem 3.5], since $\mathcal{U}_{-1}$ is compactly contained in $\hat{\mathbb{C}} \setminus \mathcal{P}$.

In the notation of Lemma 6.3, we then have

$$\begin{aligned}
\mathrm{HypLength}(\gamma_k) &\leq \mathrm{HypLength}(f^{\circ(n_1-1)}(\gamma_k)) \\
&\leq \theta \cdot \mathrm{HypLength}(f^{\circ n_1}(\gamma_k)) \\
&\leq \ldots \\
&\leq \theta^k \cdot \mathrm{HypLength}(f^{\circ n_k}(\gamma_k)) \\
&\lesssim \theta^k.
\end{aligned} \tag{6.1}$$

As in the hyperbolic case, we infer that $\mathrm{Length}(\gamma_k) \lesssim \theta^k$ by the equivalence on $B(0,R) \setminus \mathcal{P}$ of the Euclidean metric and the hyperbolic metric. □

6.2 Some Estimates and Notation

To estimate the length of express tracks, we introduce the notations $s_n := s_{n,0}$ and

$$\ell_n := \mathrm{Length}([s_n, s_{n+1}]) = s_n - s_{n+1}. \tag{6.2}$$

Lemma 6.4. *The lengths ℓ_n satisfy:*

(i)
$$\frac{|p - s_n|}{\ell_n} \to \infty, \tag{6.3}$$

(ii)
$$\frac{\ell_n}{\ell_{n+1}} \to 1. \tag{6.4}$$

In particular, for any $C > 0$, there is a sufficiently large integer d such that

$$\ell_m + \ldots + \ell_n \geq C(\ell_m + \ell_n)$$

whenever $|m-n| \geq d$.

Proof. Using the affine conjugacy of the map f to the map $g : z \mapsto z^2 + z$, which sends the parabolic fixed point $\frac{1}{2}$ of f to 0, one can show that

$$\ell_n \asymp \frac{1}{n^2} \qquad \text{and} \qquad |p - s_n| \asymp \frac{1}{n}.$$

After a little arithmetic, we get (6.3) and (6.4). $\square$

Lemma 6.5. *There exists a constant* $k > 0$ *such that for any pair of points* $z_1, z_2 \in \mathcal{J}$*, we have* $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$.

Proof. We have

$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} f(z) \mathrm{dz} \right| \leq k|z_1 - z_2| \tag{6.5}$$

for $k = \max_{z \in B} |f'(z)|$, where B is any ball containing $\mathcal{J}$. $\square$

6.3 Dynamics near the parabolic fixed point

The purpose of the following definition is to organize points on the Julia set $\mathcal{J}$ according to their distance from the main cusp $z = 1/2$ in an f-invariant way. We decompose the points of $\mathcal{J}$ according to the first *departure*: the first time that the central itinerary makes a turn.

Definition 6.6. Let $n \in \mathbb{N}$. We define the n-th *departure set* $I_{n,\mathbb{D}} \subset \partial\mathbb{D}^*$ to be the set of points $\zeta \in \partial\mathbb{D}^*$ whose central itinerary η_ζ starts with n express tracks, followed by a peripheral track. See Figure 6.

This decomposition is invariant under f_0 in the sense that $f_0(I_{n+1,\mathbb{D}}) = I_{n,\mathbb{D}}$, because of the invariance of η_ζ. Applying the Böttcher map ψ, we obtain a corresponding departure decomposition $I_n = \psi(I_{n,\mathbb{D}})$ of $\mathcal{J}$ that is invariant under f.

We now use this decomposition to analyze the case where the points w_1, w_2 lie in "well-separated cusps". Namely, suppose that

$$w_1 \in I_n, \quad w_2 \in I_m, \quad m - n > d, \tag{6.6}$$

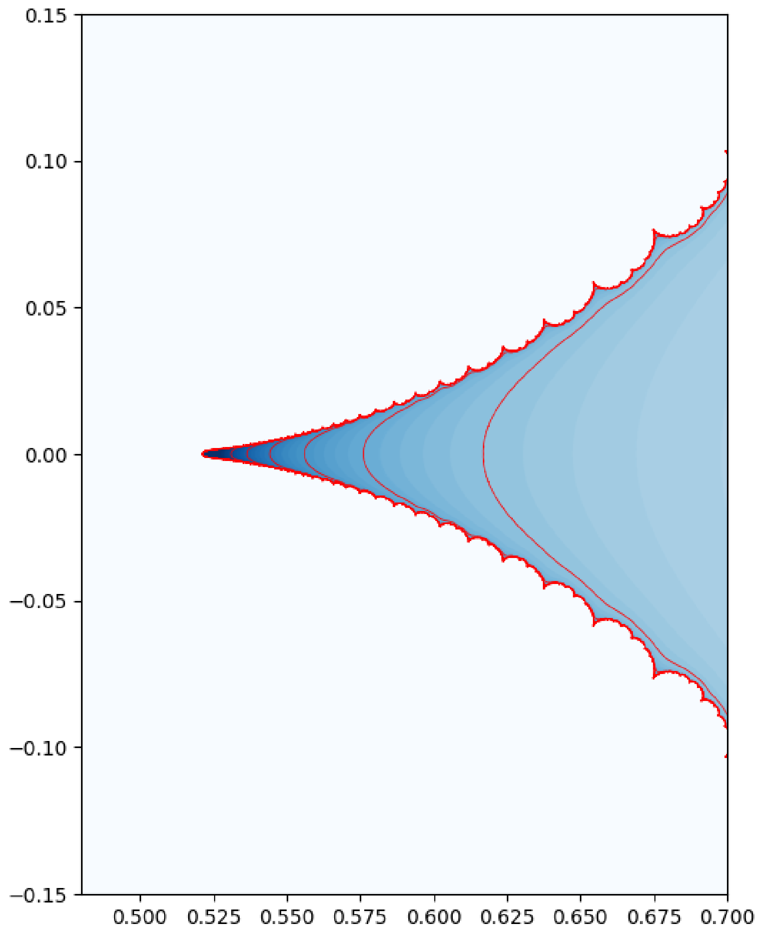


Figure 5: The Cauliflower near the parabolic point $p = 1/2$.

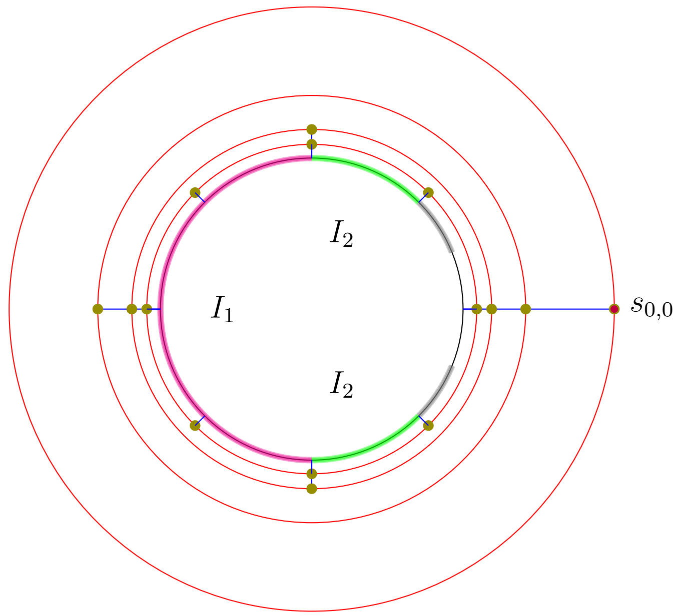


Figure 6: First few parts of the departure decomposition I_m of the circle.

where d is a sufficiently large integer, to be chosen later. This gives some control from below on $|w_1 - w_2|$. We represent the itinerary $\eta = \eta_{w_1,w_2}$ as a concatenation of three paths: the radial segment $\gamma_{m,n} = [s_{m,0}, s_{n,0}]$ and the two other components, γ_m and γ_n. See Figure 7 for the picture in the exterior unit disk. Thus we have

$$\mathrm{Length}(\eta) = \mathrm{Length}(\gamma_m) + \mathrm{Length}(\gamma_{m,n}) + \mathrm{Length}(\gamma_n). \tag{6.7}$$

The condition $m - n \geq d$ prevents the line segment $\gamma_{m,n}$ from being small in comparison to γ_m and γ_n:

Proposition 6.7. *There exists a sufficiently large integer d so that*

$$\mathrm{Length}(\gamma_{m,n}) \asymp |w_1 - w_2|, \tag{6.8}$$

whenever $m - n \geq d$.

We henceforth fix a value of d as in the proposition.

Proof. We first make two elementary observations. Koebe's distortion theorem applied to the iterates of f^{-1} shows that

$$\mathrm{Length}(\gamma_m) \leq C\ell_m, \tag{6.9}$$

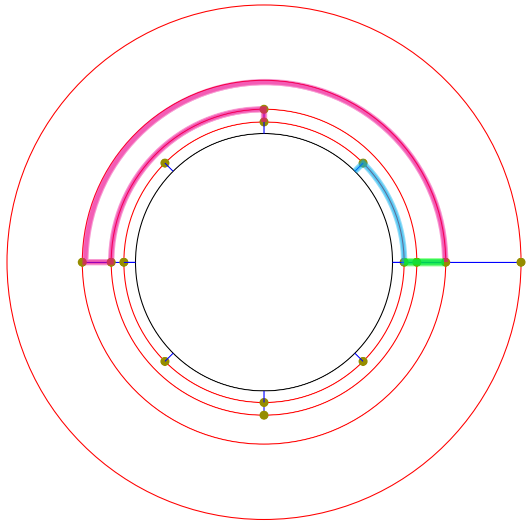


Figure 7: The three parts of an itinerary η. The green path is $\gamma_{m,n}$, the cyan and magenta are γ_m and γ_n.

for some constant $C \geq 0$. Notice that (6.9) holds for $m = 1$ by Proposition 6.2, which gives a uniform bound on the length of an itinerary.

Meanwhile, by Lemma 6.4, there exists an integer d such that

$$C(\ell_m + \ell_n) \leq \frac{\text{Length}(\gamma_{m,n})}{2} \tag{6.10}$$

whenever $m - n \geq d$.

By the triangle inequality, we have

$$\begin{aligned} \big|\text{Length}(\gamma_{m,n}) - |w_1 - w_2|\big| &\leq \text{Length}(\gamma_m) + \text{Length}(\gamma_n) \\ &\leq \frac{\text{Length}(\gamma_{m,n})}{2}, \end{aligned}$$

which clearly implies (6.8). □

6.4 Quasiconvexity: three special cases

We now show that the itineraries η_{w_1,w_2} are certificates in three special cases. To state them, we introduce some notation.

6.4.1 Notation

For each n, we denote by α_n the union of the two outermost tracks emanating from the station $s_{n,0}$. Notice that the curves α_n are pairwise disjoint since this is true for their pullbacks to the exterior unit disk.

We define the constants C_1, C_2, ϵ as follows. We first choose $C_1 \geq 2$, then we let $C_2 = C_1 + d + 2$ and choose $\epsilon > 0$ small enough so that we have

$$\operatorname{dist}(\alpha_{C_2}, \alpha_{C_1}) \geq k\epsilon. \tag{6.11}$$

The constant C_2 was chosen so that for any pair (m, n) of integers, we have at least one of the following three cases: either m, n are both greater than C_1, or both are smaller than C_2, or $|m - n| > d$.

6.4.2 Three Special Cases

In this section we treat the following special cases:

1. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n < C_2, \quad m, n \geq 2;$
2. $|w_1 - w_2| \geq \epsilon, \quad |m - n| < d, \quad m, n > C_1;$
3. $|w_1 - w_2| \leq k\epsilon, \quad |m - n| \geq d.$

Notice that Case 2 overlaps with Case 1. We denote the domain enclosed by α_m, α_n and $\mathcal{J}$ by $\mathcal{K}_{m,n}$, and denote the domain enclosed by $\mathcal{J}$ and α_n by $\mathcal{K}_n$.

Lemma 6.8. *Let $w_1 \in I_m$ and $w_2 \in I_n$, for $n \geq m \geq 2$. Then the itinerary η_{w_1,w_2} is contained in the domain $\mathcal{K}_{m,n+1}$.*

Lemma 6.9. *Let $w_1, w_2 \in \mathcal{J}$. In each of the three special cases, the itinerary γ_{w_1,w_2} is a quasiconvexity certificate. In Cases 1 and 2, γ_{w_1,w_2} is relatively separated.*

Proof. Case 1. In this case, the itinerary is contained in the domain $\mathcal{K}_{2,C_2+1}$. Since $\operatorname{dist}(\mathcal{K}_{2,C_2+1}, \mathcal{P}) > 0$, γ_{w_1,w_2} is η-relatively separated for some $\eta > 0$.

Case 2. Assuming without loss of generality that $n \geq m$, the itinerary is contained in $\mathcal{K}_{m,n+1}$. By Koebe's distortion theorem, γ_{w_1,w_2} is also relatively separated.

Case 3 is the content of Proposition 6.7. □

6.5 Quasiconvexity: general case

We apply a stopping time argument to promote the quasiconvexity of η_{w_1,w_2} to the quasiconvexity of η_{z_1,z_2}, thereby proving the following theorem:

Theorem 6.10. *The domain* Exterior($\mathcal{J}$) *is quasiconvex, with the itineraries* η_{z_1,z_2} *as certificates.*

Proof. (Parabolic Conformal Elevator on $\mathcal{J}$). Let (z_1, z_2) be a pair of points in $\mathcal{J}$. Repeatedly apply f to (z_1, z_2) until either of the three special cases occurs. Denote by $w_i = f^{\circ N}(z_i)$ the resulting points. We have already proved that the itinerary η_{w_1,w_2} satisfies

$$\text{Length}(\eta_{w_1,w_2}) \leq A|w_1 - w_2|,$$

for some $A > 0$. We deduce that the original pair of points (z_1, z_2) enjoys a similar estimate,

$$\text{Length}(\eta_{z_1,z_2}) \leq C|z_1 - z_2|,$$

where C depends only on A.

In Cases 1 and 2, we are done by Lemma 6.9. In Case 3, the itinerary η_{w_1,w_2} is contained in $\mathcal{K}_2$. Let $\mathcal{K}_{-2}$ be the preimage of $\mathcal{K}_2$ under f that contains the negative preimage $f^{-1}(p) = -\frac{1}{2}$ of the cusp p. As the domain $\mathcal{K}_{-2}$ is relatively separated from $\mathcal{P}$ and contains the curve $f^{\circ(N-1)}(\eta_{z_1,z_2}) = \eta_{f^{-1}(w_1),f^{-1}(w_2)}$, we may use Koebe's distortion theorem to conclude that

$$\frac{\text{Length}(\eta_{z_1,z_2})}{|z_1 - z_2|} \asymp \frac{\text{Length}(\eta_{f^{-1}(w_1),f^{-1}(w_2)})}{|f^{-1}(w_1) - f^{-1}(w_2)|} \asymp \frac{\text{Length}(\eta_{w_1,w_2})}{|w_1 - w_2|} \tag{6.12}$$

as desired. □

References

[1] Carleson, L., Jones, P. W., and Yoccoz, J.-C. Julia and John. *Bol. Soc. Bras. Mat 25*, 1 (Mar. 1994), 1–30.

[2] García-Bravo, M., and Rajala, T. Strong BV-extension and $W^{1,1}$-extension domains.

[3] Hakobyan, H., and Herron, D. Euclidean quasiconvexity. *Annales Academiae Scientiarum Fennicae Mathematica 33* (Jan. 2008).

[4] McMullen, C. T. *Complex Dynamics and Renormalization (AM-135)*. Princeton University Press, 1994.

[5] Milnor, J. *Dynamics in One Complex Variable. (AM-160): Third Edition. (AM-160)*. Princeton University Press, 2006.

Nomenclature

α_n the union of the two outermost tracks emanating from the station $s_{n,0}$.

$\Delta(\gamma, \mathcal{P})$ The relative distance to the post-critical set.

ℓ_n Length$([s_n, s_{n+1}]) = s_{n,0} - s_{n+1,0}$.

η_{z_1,z_2} The itinerary connecting two points. When z_1 and z_2 are stations, this is the same as γ_{z_1,z_2}.

γ_{z_1,z_2} The track connecting z_1 and z_2. It can be either angular ("peripheral") or radial ("express").

$\mathcal{J}_c$ The Julia set of f_c.

Exterior$(\mathcal{J})$ An Alternative notation for $A_\infty(f_c)$.

ψ The Bottcher coordinate $\mathbb{D}^* \to A_\infty(f_{1/4})$ conjugating f_0 and $f_{1/4}$.

$A_\infty(f_c)$ The exterior of the Julia set of f_c. The complement of K_c.

f, f_c The map $z \mapsto z^2 + c$.

I_n The n-th departure set.

K_c The filled Julia set of f_c.

$s_{n,k}$ A station in $\mathbb{D}^*$ or its image under ψ.