202 Bonus Theorem 1

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October 25, 2016

Note: The notation $||.||$ is used throughout this document to refer to the 1-norm. It is defined and proven to be a submultiplicative norm in the appendix.

Theorem 1. *The sequence* $(A_n)_{n\in\mathbb{N}}$, $A \in \mathbb{M}^{N\times N}$ *for some* $N \in \mathbb{N}$, *where*

$$A_n = \sum_{k=0}^{n} \frac{M^k}{k!}$$

is absolutely convergent for all $M \in \mathbb{M}^{N\times N}$.

Proof. Consider the sequence $(B_n)_{n\in\mathbb{N}}$ where

$$B_n = \sum_{k=0}^{n} ||\frac{M^k}{k!}||.$$

$$B_n = \sum_{k=0}^{n} ||M^k|| \times |\frac{1}{k!}| \text{ (from the absolute scalability of } ||.||)$$

$$B_n \leq \sum_{k=0}^{n} ||M||^k|\frac{1}{k!}| \text{ (from the submultiplicativity of } ||.||)$$

$B_n \leq e^{||M||}$ (from the definition of the real exponential which is defined on all of $\mathbb{R}$)

Therefore the sequence $(B_n)_{n\in\mathbb{N}}$ is bounded above, which means that

$$A_n = \sum_{k=0}^{n} \frac{M^k}{k!}$$

is absolutely convergent by definition.

$\square$

Theorem 2. *For any $E : \mathbb{R} \to \mathbb{M}^{N\times N}, E(t) = e^{tA}$ with $A \in \mathbb{M}^{N\times N}; E'(t) = AE(t)$.*

Proof. Consider the Frchet definition of the derivative

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||\frac{E(\Delta t + t) - E(t) - \Delta t AE(t)}{\Delta t}|| = 0.$$

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||\frac{e^{(\Delta t+t)A} - e^{tA} - \Delta t Ae^{tA}}{\Delta t}|| = 0.$$

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||\frac{e^{\Delta tA}e^{tA} - e^{tA} - \Delta t Ae^{tA}}{\Delta t}|| = 0. \text{ (since } \Delta tA \text{ and } tA \text{ commute)}$$

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||\frac{e^{\Delta tA} - I - \Delta tA}{\Delta t}e^{tA}|| = 0.$$

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||(\sum_{i=0}^{\infty}(\Delta t^i A^i \frac{1}{i!}) - I - \Delta tA)\frac{1}{\Delta t}e^{tA}|| = 0.$$

$$E'(t) = AE(t) \Leftrightarrow \lim_{\Delta t\to 0} ||\sum_{i=2}^{\infty} \Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| = 0.$$

Consider $||\sum_{i=2}^{\infty} \Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}||$.

$$||\sum_{i=2}^{\infty} \Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq \sum_{i=2}^{\infty} ||\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}||$$

$$||\sum_{i=2}^{\infty} \Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \sum_{i=2}^{\infty} |\Delta t^{i-1}| \times ||A^i|| \times |\frac{1}{i!}|$$

$$||\sum_{i=2}^{\infty} \Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \sum_{i=0}^{\infty} |\Delta t^{i+1}| \times ||A^{i+2}|| \times |\frac{1}{(i+2)!}|$$

$$||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times |\Delta t| \times ||A^2|| \sum_{i=0}^{\infty}|\Delta t^i| \times ||A^i|| \times |\frac{1}{(i+2)!}|$$

$$||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times |\Delta t| \times ||A^2|| \sum_{i=0}^{\infty}|\Delta t|^i ||A||^i \frac{1}{i!}$$

$$||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times |\Delta t| \times ||A^2|| \sum_{i=0}^{\infty}||\Delta t A||^i \frac{1}{i!}$$

$$||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times |\Delta t| \times ||A^2|| e^{||\Delta t A||}$$

Therefore

$$\lim_{\Delta t \to \infty} ||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq \lim_{\Delta t \to \infty} ||e^{tA}|| \times |\Delta t| \times ||A^2|| e^{||\Delta t A||}$$

$$\lim_{\Delta t \to \infty} ||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times ||A^2|| \lim_{\Delta t \to \infty} |\Delta t| e^{||\Delta t A||}$$

$$\lim_{\Delta t \to \infty} ||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| \leq ||e^{tA}|| \times ||A^2|| 0 = 0$$

$$\implies \lim_{\Delta t \to \infty} ||\sum_{i=2}^{\infty}\Delta t^i A^i \frac{1}{i!}\frac{1}{\Delta t}e^{tA}|| = 0 \text{ (squeeze theorem).}$$

Substituting

$$E'(t) = AE(t) \Leftrightarrow 0 = 0.$$

Therefore $E'(t)$ converges, and $E'(t) = AE(t)\ \forall t \in \mathbb{R}$.

$\square$

Lemma 1. *For any differentiable* $f : \mathbb{R} \to \mathbb{M}^{N \times N}$ *and* $A \in \mathbb{M}^{N \times N}$

$$\frac{d}{dt}(Af(t)) = A\frac{d}{dt}f(t).$$

Proof.

$$\lim_{h\to 0}\frac{||Af(t+h)-Af(t)-hA\frac{d}{dt}f(t)||}{|h|}$$

$$=\lim_{h\to 0}\frac{||A(f(t+h)-f(t)-h\frac{d}{dt}f(t))||}{|h|}$$

$$\leq ||A||\lim_{h\to 0}\frac{||(f(t+h)-f(t)-h\frac{d}{dt}f(t))||}{|h|}$$

$$\leq ||A||0=0 \text{ (from the definition of } \frac{d}{dt}f(t))$$

$$\implies \frac{d}{dt}(Af(t))=A\frac{d}{dt}f(t).$$

□

Theorem 3. *Let $E(t)=e^{tA}, t\in\mathbb{R}$. Then $E\in C^{\infty}(\mathbb{R};\mathbb{M}^{N\times N})$ and $E^{(n)}(t)=A^nE(t)$*

Proof. We will proceed via induction.
Case (1) was proven in.

Assume case (k): $E^{(k)}=A^kE(t)$.

Case (k+1): $E^{(k+1)}=(E^{(k)})'=(A^kE(t))'$ (from case (k))
$E^{(k+1)}=A^kE'(t)$ (from Lemma 2)
$E^{(k+1)}=A^kAE(t)=A^{k+1}E(t)$

□

A Appendix

Lemma 2. *The functions*

$$||.||:\mathbb{M}^{N\times N}\to[0,\infty)$$

defined by

$$||M^{N\times N}||=\sum_{i=1}^{N}\sum_{j=1}^{N}|m_{i,j}|$$

are submultiplicative norms for all $N\in\mathbb{N}$

Proof. 1. Absolute scalabilty

$$||\alpha M|| = \sum_{i=1}^{N}\sum_{j=1}^{N}|\alpha m_{i,j}|$$

$$||\alpha M|| = \sum_{i=1}^{N}\sum_{j=1}^{N}|\alpha||m_{i,j}|$$

$$||\alpha M|| = |\alpha|\sum_{i=1}^{N}\sum_{j=1}^{N}|m_{i,j}|$$

$$||\alpha M|| = |\alpha| \times ||M||$$

2. Subadditivity

$$||U+V|| = \sum_{i=1}^{N}\sum_{j=1}^{N}|u_{i,j}+v_{i,j}|$$

$$||U+V|| \leq \sum_{i=1}^{N}\sum_{j=1}^{N}(|u_{i,j}|+|v_{i,j}|)$$

$$||U+V|| \leq \sum_{i=1}^{N}\sum_{j=1}^{N}|u_{i,j}| + \sum_{i=1}^{N}\sum_{j=1}^{N}|v_{i,j}|$$

$$||U+V|| \leq ||U|| + ||V||$$

3. $||M|| = 0 \Rightarrow M = 0^{N\times N}$
Assume that $||M|| = 0$ for some $N \times N$ matrix that is not equal to $0^{N\times N}$.
$M \neq 0^{N\times N}$ so it has at least one element $m_{I,J} \neq 0$.

$$||M|| = \sum_{i=1}^{N}\sum_{j=1}^{N}|m_{i,j}| = |m_{I,J}| + C \text{ for some } C \in [0,\infty).$$

This means that $||M||$ is equal to the sum of a positive value and a non-negative value, implying that it is positive and therefore not equal to 0.

4. Submultiplicitivity

$$||UV|| = \sum_{i=1}^{N}\sum_{j=1}^{N}|\sum_{k=1}^{N}u_{i,k}v_{k,j}|$$

$$||U|| \times ||V|| = (\sum_{i=1}^{N}\sum_{j=1}^{N}|u_{i,j}|)(\sum_{i=1}^{N}\sum_{j=1}^{N}|v_{k,l}|)$$

$$||U|| \times ||V|| = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}|u_{i,j}||v_{k,l}|$$

$$||U|| \times ||V|| = \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}|u_{i,j}||v_{k,l}|$$

$$||U|| \times ||V|| = \sum_{i=1}^{N}\sum_{l=1}^{N}(\sum_{j=1}^{N}|u_{i,j}||v_{j,l}| + \sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}|u_{i,j}||v_{k,l}|)$$

$$||U|| \times ||V|| = \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}|u_{i,j}||v_{j,l}| + \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}|u_{i,j}||v_{k,l}|$$

$$||U|| \times ||V|| \geq \sum_{i=1}^{N}\sum_{l=1}^{N}|\sum_{j=1}^{N}u_{i,j}v_{j,l}| + \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}|u_{i,j}||v_{k,l}|$$

$$||U|| \times ||V|| \geq ||UV|| + \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}|u_{i,j}||v_{k,l}|$$

$$||U|| \times ||V|| \geq ||UV||$$

□