Monte Carlo Simulation

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The purpose of this report is to evaluate the price of an option using several alternatives of the Monte Carlo simulation. To compare the prices I will get from these simulations, I first calculated the price using the Black-Scholes formula, for an American call. The parameters I considered for my analysis are:

$$S_0 = 120;\ \ K = 110;\ \ T = 9 \text{ months};\ \ r = 0.02;\ q = 0;\ \ \sigma = 0.15.$$

In this particular case there is no dividend yield. Since the strike is 110 and the underlying value is 120, I am considering ITM (In-The-Money) option.
The first thing to do is compute the Black-Scholes formula for these parameters:

$$\textit{price}^{\textit{call}}\textit{(S,T)} = \mathrm{N}(\mathrm{d}_1)\mathrm{S} - \mathrm{N}(\mathrm{d}_2)\mathrm{K}\mathrm{e}^{-rt} \tag{1}$$

$$\mathrm{d}_1 = \frac{1}{\sigma\sqrt{\mathrm{t}}}\left[\ln\left(\frac{S}{K}\right) + t\left(r + \frac{\sigma^2}{2}\right)\right] \tag{2}$$

$$\mathrm{d}_2 = \frac{1}{\sigma\sqrt{\mathrm{t}}}\left[\ln\left(\frac{S}{K}\right) + t\left(r - \frac{\sigma^2}{2}\right)\right] \tag{3}$$

Substituting the parameters inside the formulas I got the following result:

$$\textit{B\&S} : 13.4627$$

Now I can use this price for comparison purposes.

1. Vanilla options

In this section I have used an MC approach to estimate the next value of the underlying. Since I considered a maturity of 9 months I took 189 steps, so dt = 1 day. Basically, this means that the new daily value of the underlying is generated from a MC simulation based on the previous (generated) one. If I store all the underlying values for all the steps I can calculate a trajectory for a particular simulation. In the figure below are reported 100 different trajectories generated with this technique. Then I computed the payoff = $max(S_T - K, 0)$, for each trajectories, and the discounted version of them ($payoff * e^{-r*T}$). I finally took the average to get a price from these MC simulations.

MC 'dynamic' price : 15.7436

The price I have obtained is quite far from the one of the B&S formula. The main reason behind this is that 100 simulations are too few to have a good price estimate. To solve the problem I increased the number of simulations (N). Obviously, increasing the number of simulations will increase the time needed to calculate the price:

MC 'dynamic' price : 6.7782 (N = 100)
MC 'dynamic' price : 12.1196 (N = 500)
MC 'dynamic' price : 14.0204 (N = 1000)
MC 'dynamic' price : 13.8519 (N = 10000)

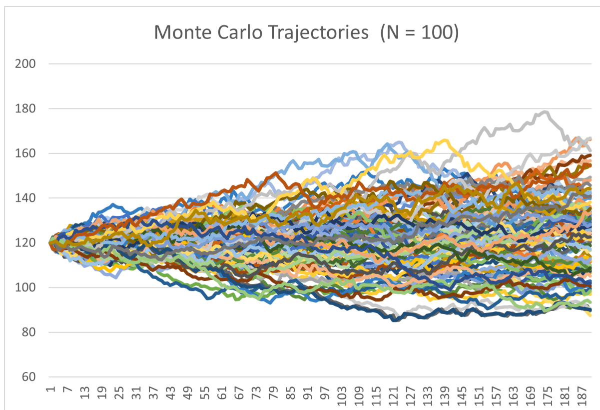

Figure 1: 100 trajectories of the value of the underlying

As we can notice the value for 10000 simulation is much more closer to the one of the B&S formula. Moreover, in order to not have just one result, I repeated the same procedure 20 more times and I take the mean and the standard deviation of them to understand the sensitivity of this procedure. I also considered different step size, 95 and 48, to see the impact of this parameter in the results. The results I have got are reported in the table below.

STEP	189 (dt = 1 day)		95 (dt = 2 days)		48 (dt = 4 days)	
N	mean	*s.d.*	mean	*s.d.*	mean	*s.d.*
100	11.3814	6.8556	12.8259	5.0948	12.0761	3.8267
500	13.7797	1.6804	14.2619	1.4221	13.4945	2.3338
1000	13.7981	1.1788	14.0130	1.3309	13.8352	1.2336
10000	13.6884	0.2971	14.0713	0.3754	13.9089	0.4197

Table 1: Mean and s.d. for 20 simulated prices

From the table we can notice that, as the number of simulations increase, the price get closer to the one of the B&S but most important the standard deviation decrease, this means that we have less variance in the estimated results. Instead for variation in the number of steps the prices doesn't change too much, here the one with 189 steps is closer to the B&S price.

The next thing I have done is consider just one step, so the MC simulation directly generate the final value for the underlying, S_T. As before, taking the average of the discount payoff of the different simulations I get the final price:

MC 'static' price : 15.5294 (N = 100)
MC 'static' price : 13.5153 (N = 500)
MC 'static' price : 12.8211 (N = 1000)
MC 'static' price : 13.3169 (N = 10000)

N	mean	*s.d.*
100	13.8585	1.3368
500	13.4989	0.8473
1000	13.6997	0.7225
10000	13.6652	0.4026

Table 2: Mean and s.d. for 20 simulated prices

From the table we can see that we can get a good price estimate with fewer simulations. In fact, with only 500 simulations I had a good estimate of the price and the standard deviation is not too high, almost half of the one with 189 steps.

2. Euler-scheme based simulation

Here I modified my code in order to use different methods, that will allow me to approximate solutions to differential equations. In this case I used the Euler's Method. With Euler's method you can approximate the curve of the solution by the tangent in each interval, at steps of h:

$$y_{i+1} = y_i + h*f(x_i,y_i)$$

where f(x_i,y_i) is the value of the derivative at the current point. I used this formula to calculate the final value of the underlying S_T. Finally I computed the price of the option using the previous technique for different number of simulations:

Euler-scheme price : 17.0448 (N = 100)
Euler-scheme price : 14.6733 (N = 500)
Euler-scheme price : 13.3043 (N = 1000)
Euler-scheme price : 13.7367 (N = 10000)

I repeated the procedure 20 times and I have obtained the following results.

The mean and standard deviation have the same behavior as the previous methods. In fact, as the number of simulations increases, the average price approaches that of the B&S and the standard deviation decreases.

N	mean	*s.d.*
100	14.5304	4.9925
500	13.9583	1.3668
1000	13.6955	0.7995
10000	13.6503	0.3164

Table 3: Mean and s.d. for 20 simulated prices

3. Asian options

I can apply the MC simulation also for different type of option, for example in this case I used them to price Asian option. The main difference is that the payoff depends on the average price of the underlying asset over a certain period of time, 9 months in this case, as opposed to standard options (American and European) where the payoff depends on the price of the underlying asset at the maturity. Therefore, to price an Asian option, I changed my VBA code in order to consider the mean of the underlying values and not just the final one. As before I considered 189 steps, so dt = 1 day. The results I have obtained are reported below.

Asian opt price : 9.3674 (N = 100)
Asian opt price : 12.1435 (N = 500)
Asian opt price : 11.5023 (N = 1000)
Asian opt price : 11.5627 (N = 10000)

N	mean	*s.d.*
100	10.5698	2.7638
500	11.6983	0.7610
1000	11.1738	0.3288
10000	11.3810	0.1262

Table 4: Mean and s.d. for 20 simulated prices

The price I found for the Asian option is cheaper than the American option, in fact for 10000 simulation I have obtain a price of 11.5627 (or a mean price of 11.3810) compared to 13.4627 of B&S. This is because we are taking the average of the underlying prices, so Asian options reduce the volatility inherent in the option.

4. Lookback options

Finally I consider Lookback options. Here the difference is that the payoff of a Lookback option depends on the minimum of the underlying asset reached during a certain period (9 months). To calculate the price of a Lookback option, I modified my VBA code to consider the minimum value of the underlying as a strike in the payoff calculation. Since I considered the minimum value as new strike the price of a Lookback option should be greater than the price of an American option, because the payoff is equal to zero only if the value at maturity of the underlying (S_T) corresponds to the minimum value. I reported the prices I have obtained below.

Lookback price : 13.1069 (N = 100)
Lookback price : 11.5627 (N = 500)
Lookback price : 13.9478 (N = 1000)
Lookback price : 13.3906 (N = 10000)

If I want to compare these results with the B&S price I have to recalculate it using as new strike K = 120. Basically I am now considering ATM (At-The-Money) option. For this strike I obtained the following price.

B&S : 7.1025 (K = 120)

The results are coherent in fact the price I have got for Lookback option are grater than the one for American option. Also here as the number of simulation increase the standard deviation decrease.

N	mean	*s.d.*
100	12.2029	2.5022
500	12.3206	0.8898
1000	12.6866	0.4648
10000	12.6294	0.2119

Table 5: Mean and s.d. for 20 simulated prices